CHAPTER 6

Homology

6.1. Simplexes and simplicial complexes

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6.1.1. Simplexes. Simplexes are building blocks of a polyhedron. A 0-simplex $\langle p_0 \rangle$ is a point, and a 1-simplex $\langle p_0 p_1 \rangle$ is a line. A 2-simplex $\langle p_0 p_1 p_2 \rangle$ is a triangle with its interior included. A r-simplex is a r-dimensional object, hence its vertices must be *geometrically independent*, that is no $(r-1)$-dimensional hyperplane contains all $r+1$ vertices.

Definition 6.1 (**r-simplex**). Let $\{p_0, p_1, \cdots, p_r\}$ be r *geometrically independent* points in $\mathbb{R}^m$ $(m \geqslant r)$. A r-simplex σ^r (also denoted by $\langle p_0 \cdots p_r \rangle$) is the subset

$$\sigma^r \equiv \{x \in \mathbb{R}^m | x = \sum_{i=0}^{r} c_i p_i, c_i \geqslant 0, \sum_{i=0}^{r} c_i = 1\}. \tag{6.1}$$

Remark 6.2.
- $\{c_i\}$ are *barycentric coordinates* of x.
- σ^r is bounded and closed subset of $\mathbb{R}^m$, hence σ^r is *compact*.
- In some literature, a simplex is defined to include shapes with some vertices *glued* together. We will not call such shape simplex here.

Definition 6.3 (**q-face**). Let q be an integer $0 \leqslant q \leqslant r$. and $\{p_{i_0}, \cdots, p_{i_q}\}$ be $q+1$ points out of $\{p_0, \cdots, p_r\}$. The q-simplex $\sigma^q = \langle p_{i_0}, \cdots, p_{i_q} \rangle$ is called a *q-face* of σ^r. We write $\sigma^q \leqslant \sigma^r$ if σ^q is a face of σ^r. Moreover, if $\sigma^q \neq \sigma^r$, we call σ^q is a *proper face* of σ^r, denoted as $\sigma^q < \sigma^r$.

Remark 6.4.
- A $q-$face σ^q is be definition a subset of σ^r. The barycentric coordinates of points in σ^q are points with $c_i = 0$ for $i \notin \{i_0, \cdots, i_q\}$.
- A r-simplex has $\begin{pmatrix} r+1 \\ q+1 \end{pmatrix}$ q-faces. C^{r+1}_{q+1}

Example 6.5. A tetrahedron is a 3-simplex (figure 6.1).

6.1.2. Simplicial complexes, polyhedra, and triangulation.

Definition 6.6 (**Simplicial complex**). Let K be a set of simplexes. K is a *simplicialcomplex* if it satisfies the following two conditions
- If $\sigma \in K$ and $\sigma' \leqslant \sigma$, then $\sigma' \in K$. In other words, an arbitrary face of a simplex of K is belongs to K.

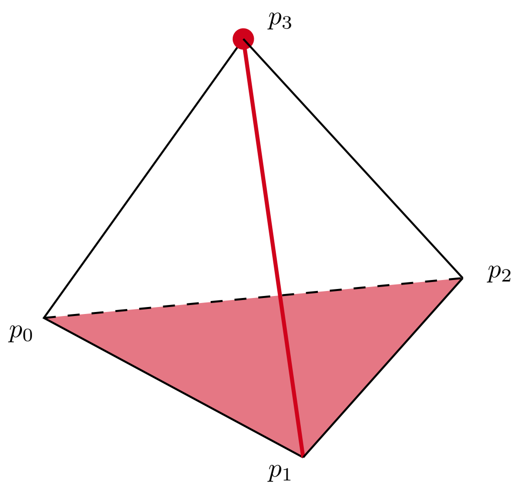


FIGURE 6.1. The 3-simplex $\langle p_0p_1p_2p_3\rangle$ with its 2-face $\langle p_0p_1p_2\rangle$, 1-face $\langle p_1p_3\rangle$ and 0-face p_3.

- If σ, $\sigma' \in K$, then either $\sigma \cap \sigma' = \varnothing$ or $\sigma \cap \sigma' \leqslant \sigma$ and $\sigma \cap \sigma' \leqslant \sigma'$. That is, the intersection of two simplexes of K is either empty or a common face.

REMARK 6.7.
- In our definition of simplicial complex, the intersection of two elements is always a simplex.
- By definition each $\sigma \in K$ can be embedded in some $\mathbb{R}^m$. However, different elements of K may be embedded in $\mathbb{R}^m$ in different ways. This point is important in triangulation.
- The union of all elements of the simplical complex$K$ is called a *polyhedron* $|K|$ of K.

EXAMPLE 6.8.
- A simplex together with all its faces form a simplicial complex. For example, the tetrahedron in figure 6.1 gives a simplicial complex

$$\begin{aligned} K = \{&\langle p_0p_1p_2p_3\rangle, \langle p_0p_1p_2\rangle, \langle p_1p_2p_3\rangle, \langle p_2p_3p_0\rangle, \langle p_3p_0p_1\rangle, \\ &\langle p_0p_1\rangle, \langle p_0p_2\rangle, \langle p_0p_3\rangle, \langle p_1p_2\rangle, \langle p_1p_3\rangle, \langle p_2p_3\rangle, \\ &\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle\}. \end{aligned} \tag{6.2}$$

- The left diagram in figure 6.2 is a simplicial complex while the right is not.
- The simplicial complex may not be connected as in figure 6.3.

DEFINITION 6.9 (**Triangulation**). Let X be a topological space. If there exists a simplicial complex K and a homeomorphism $f : |K| \to X$, then X is said to be *triangulable* and the pair (K, f) is called a *triangulation* of X.

REMARK 6.10. Given a triangulable X, its triangulation is far from unique. However, the minimum triangulation (triangulation with least member) depends on the topology of X as we will see in the following examples.

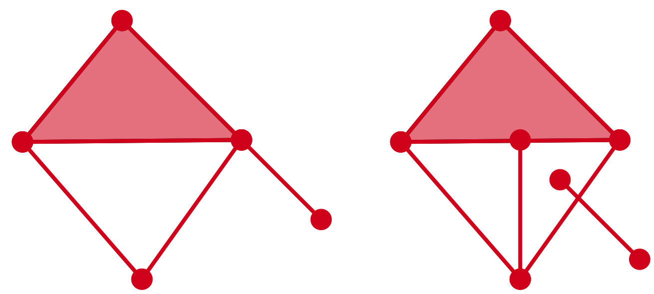


FIGURE 6.2. Left: a simplicial complex with 1 2-simplex, 6 1-simplexes and 5 0-simplexes. Right: not a simplicial complex.

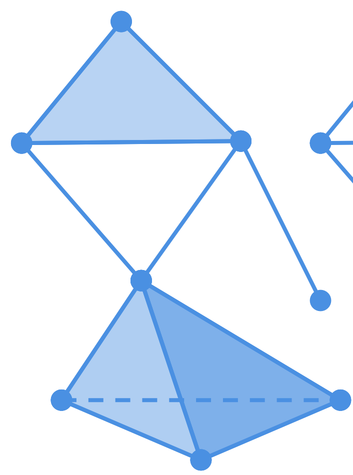


FIGURE 6.3. Simplicial complex can also have multiple connected components.

The allowed triangulation can be quite different for different topological space. For example:

EXAMPLE 6.11.
- Let $X = [0,1]$, then $K_X = \{X, \{0\}, \{1\}\}$ is a triangulation of X. One can also add any points $0 = p_0 < p_1 < \cdots < p_n < p_{n+1} = 1$ and define $K(p_1 \cdots p_n) = \{\langle p_0 p_1 \rangle, \cdots, \langle p_n p_{n+1} \rangle, \{p_0\}, \{p_1\}, \cdots, \{p_n\}, \{p_{n+1}\}\}$, then $K(p_1 \cdots p_n)$ is also a triangulation of X. However, there is no triangulation of X with less elements than K_X (Figure 6.4 a and b).
- Let $Y = \mathbb{S}^1$, the minimal triangulation has at least three vertices (0-simplexes) (Figure 6.4 c). If we mark only two points on $\mathbb{S}^1$, the result is not a simplicial complex in our definition (Figure 6.4 d).

EXAMPLE 6.12. Now look at two dimensional examples. Let $X = \mathbf{I}^2 = [0,1]^2 \subset \mathbb{R}^2$.

- The triangulation of X with minimal triangls contains only two triangles (2-simplexes) as in figure 6.5.

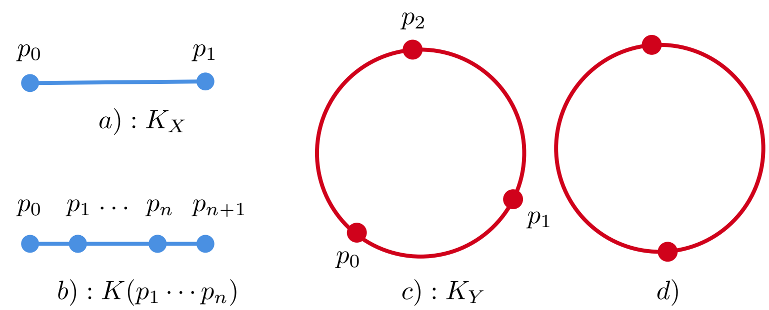


FIGURE 6.4. a): minimal triangulation of $[0, 1]$. b): a generic triangulation of $[0, 1]$. c): minimal triangulation of a circle. d): not a triangulation in our sense because the intersection of left arc and right arc contains two points (0-simplexes).

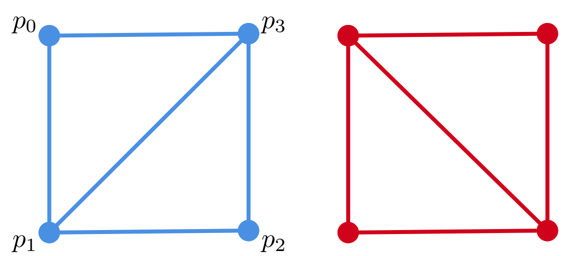


FIGURE 6.5. Two different triangulations of $\mathbf{I}^2$ with minimal triangles.

- Now let Y be X with left and right edge identified, then topologically Y is a cylinder with finite length. A proper triangulation of Y has no less than *six* triangles as shown in figure 6.6.

6.2. Chain group, cycle group and boundary group

6.2.1. Oriented simplexes.

DEFINITION 6.13 (**Orientation of simplexes**). Let $\{i_0, i_1, \cdots, i_r\}$ be a permutation of $\{0, 1, \cdots, r\}$. $\sigma^r(p_{i_0} p_{i_1} \cdots p_{i_r})$ has the same orientation with $\sigma^r(p_0 p_1 \cdots p_r)$ if and only if $\{p_{i_0}, p_{i_1}, \cdots, p_{i_r}\}$ and $\{p_0, p_1, \cdots, p_r\}$ differ by an *even* permutation. If we denote an oriented r-simplex by $(p_0 p_1 \cdots p_r)$, then

(6.3) 排列 $$(p_{i_0} p_{i_1} \cdots p_{i_r}) = \mathrm{sign}(P)(p_0 p_1 \cdots p_r)$$

with P being the permutation from $\{p_0, p_1, \cdots, p_r\}$ to $\{p_{i_0}, p_{i_1}, \cdots, p_{i_r}\}$.

REMARK 6.14.
- When $r = 0$, we formally define the oriented 0-simplex simply as (p_0).
- As a set $(p_{i_0} p_{i_1} \cdots p_{i_r})$ is the same as $(p_0 p_1 \cdots p_r)$.

EXAMPLE 6.15.
- An oriented 1-simplex $(p_0 p_1)$ is a directed line segment pointing from p_0 to p_1, while $(p_1 p_0)$ is pointing from p_1 to p_0, and by definition we have $(p_0 p_1) = -(p_1 p_0)$. See figure 6.7(a).

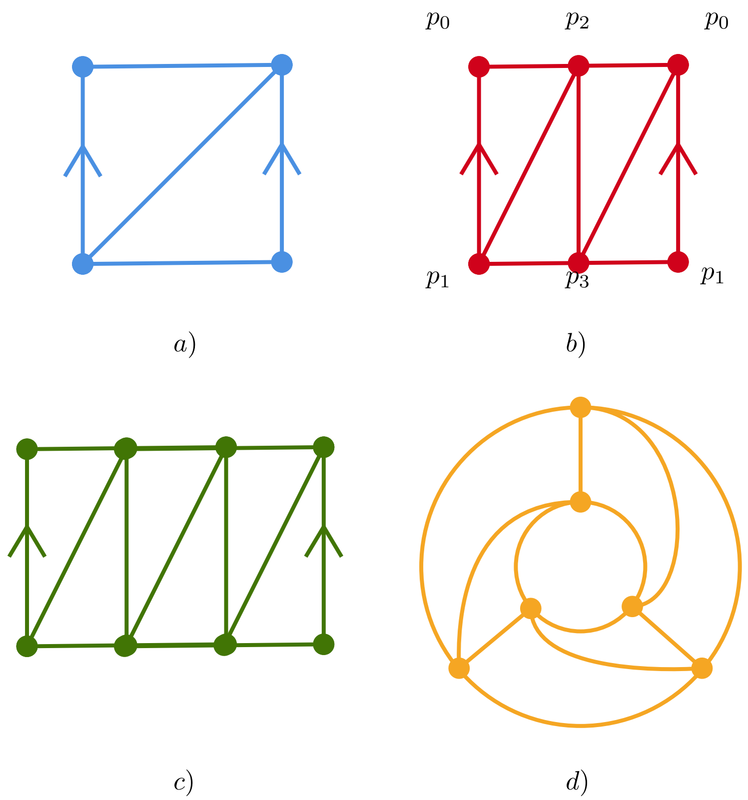


FIGURE 6.6. Triangulation of a cylinder. a): Not a triangulation because two triangles share two edges. b): Not a triangulation because $\langle p_1p_2p_3\rangle \cap \langle p_0p_1p_3\rangle = \{p_1\} \cup \{p_3\}$. c): Minimal triangulation. d): Another visualization of minimal triangulation.

- Similarly the orientation of 2-simplex defines "counterclockwise". By definition we have

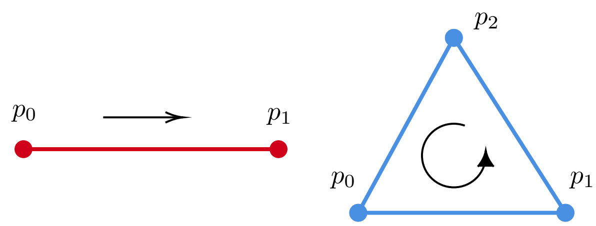
$$\begin{aligned}(p_0p_1p_2) &= (p_1p_2p_0) = (p_2p_0p_1) \\ &= -(p_0p_2p_1) = -(p_2p_1p_0) = -(p_1p_0p_2).\end{aligned} \tag{6.4}$$

  See figure 6.7(b).

6.2.2. Chain group. Let K be a simplicial complex. We regard each element in K as an oriented simplex.

DEFINITION 6.16 (**Chain group**). The r*-chain group* $C_r(K)$ of a simplicial complex K is a free Abelian group generated by the oriented r-simplexes of K. If $r > \dim K$, $C_r(K)$ is defined to be 0 (the group containing only the unit element). An element of $C_r(K)$ is called an r*-chain*.

REMARK 6.17. • As a set $C_r(K) = \{\sum_i z_i\sigma_i^r | z_i \in \mathbb{Z}, \sigma_i^r \in K\}$. Let $c = \sum_i z_i\sigma_i^r \in C_r(K)$ and $d = \sum_i w_i\sigma_i^r \in C_r(K)$, $c + d \equiv \sum_i(z_i + w_i)\sigma_i^r$. The identity element of $C_r(K)$ is the element with all $z_i = 0$ ($0 = \sum_i 0\,\sigma_i^r$).



a) *b)*

FIGURE 6.7. An oriented 1-simplex (a) and 2-simplex (b).

The inverse of $c = \sum_i z_i \sigma_i^r \in C_r(K)$ is $(-c) = -\sum_i z_i \sigma_i^r = \sum_i z_i(-\sigma_i^r)$ where $-\sigma_i^r$ is the r-simplex with opposite orientation relative to σ_i^r.

- By the above explanation, one can see that $C_r(K)$ is a free abelian group of rank I_r where I_r is the number of r-simplexes in K, i.e.

$$C_r(K) \simeq \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{I_r} \equiv \mathbb{Z}^{I_r}. \tag{6.5}$$

I_r 是 r-simplexes 数目

- If we treat the collection of r-simplexes of K as a base $\{\sigma_i^r\}$. $C_r(K)$ can be viewed as a subset of $\mathrm{span}_{\mathbb{R}}(\{\sigma_i^r\})$ whose elements have integer coefficients. (i.e. integer points in the real vector (linear) space spanned by $\{\sigma_i^r\}$). Apparently a vector space is an abelian group under the addition of vectors.
- Let $\int_{\sigma^r} f$ and $\int_{\tilde{\sigma}^r} f$ being integrals over σ^r and $\tilde{\sigma}^r$ (f being a r-form). One can understand $\int_{\sigma^r + \tilde{\sigma}^r} f$ as $\int_{\sigma^r} f + \int_{\tilde{\sigma}^r} f$.

EXAMPLE 6.18. We compute some chain groups for examples in figure 6.4.

- Let K_Y be the simplicial complex equal to the triangulation in 6.4a. We have

$$\begin{aligned} C_0(K) &= \{z_0(p_0) + z_1(p_1) | z_i \in \mathbb{Z}\} \simeq \mathbb{Z}^2, \\ C_1(K) &= \{z_0(p_0 p_1) | z_0 \in \mathbb{Z}\} \simeq \mathbb{Z}, \\ C_{n \geqslant 2}(K) &= 0. \end{aligned} \tag{6.6}$$

- Similarly, let K_Y be the simplicial complex equal to the triangulation in 6.4c. We have $C_0(K_Y) = \{z_0(p_0) + z_1(p_1) + z_2(p_2) | z_i \in \mathbb{Z}\} \simeq \mathbb{Z}^3$ and $C_1(K_Y) = \{z_0(p_0 p_1) + z_1(p_1 p_2) + z_2(p_2 p_0) | z_i \in \mathbb{Z}\} \simeq \mathbb{Z}^3$ and $C_{n \geqslant 2}(K_Y) = 0$.

EXAMPLE 6.19. The chain group of the triangulation K in the left plot of figure 6.5 is

$$\begin{aligned} C_0(K) &= \{\sum_{i=0}^{3} z_i(p_i) | z_i \in \mathbb{Z}\} \simeq \mathbb{Z}^4, \\ C_1(K) &= \{\sum_{i=0}^{3} z_i(p_i p_{i+1}) + z_4(p_0 p_2) | z_i \in \mathbb{Z}, p_4 \equiv p_0\} \simeq \mathbb{Z}^5, \\ C_2(K) &= \{z_0(p_0 p_1 p_3) + z_1(p_1 p_2 p_3) | z_i \in \mathbb{Z}\} \simeq \mathbb{Z}^2, \\ C_{n \geqslant 3}(K) &= 0. \end{aligned} \tag{6.7}$$

6.2.3. Boundary operator, cycle group and boundary group. By definition 6.1 of simplexes, we see the *boundary* $\partial_r \sigma^r$ of a r-simplex σ^r is a collection of $(r-1)$-simplexes. ∂_r should be understood as an operator acting on σ^r to and the result is a $(r-1)$-chain in $C_{r-1}(K(\sigma^r))$, where $K(\sigma^r)$ is the simplicial complex containing σ^r and all its faces. In general we have

Definition 6.20 (**Boundary operator**). Let K be a simplicial complex. The *boundary operator* $\partial_r : C_r(K) \to C_{r-1}(K)$ is a group homomorphism, i. e. ∂_r is linear in coefficients z_i

$$\partial_r \left(\sum_i z_i \sigma_i^r \right) = \sum_i z_i \partial_r (\sigma_i^r). \tag{6.8}$$

For an oriented r-simplex $\sigma^r = (p_0 p_1 \cdots p_r)$ its boundary $\partial_r \sigma^r$ is defined to be the $(r-1)$-chain

$$\partial_r (p_0 p_1 \cdots p_r) \equiv \sum_{i=0}^{r} (-1)^i (p_0 p_1 \cdots \hat{p}_i \cdots p_r), \tag{6.9}$$

where $\hat{p}_i$ means the point p_i is omitted. And we formally define $\partial_0 \sigma^0 = 0$.

Remark 6.21. We only have to define the action of ∂_r on the generators of the r-chain group. The action on other elements are obtained from linearity.

Example 6.22. $\hat{p}_i$ means removing p_i from the simplex $(p_0 p_1 \cdots\cdots p_r)$ and the result is a $(r-1)$-face of $(p_0 p_1 \cdots\cdots p_r)$ with the orientation determined by how many vertices on the left of p_i. For example

$$\begin{aligned} \partial_2 (p_0 p_1 p_2) &= (p_1 p_2) - (p_0 p_2) + (p_0 p_1) \\ &= (p_1 p_2) + (p_2 p_0) + (p_0 p_1), \\ \partial_3 (p_0 p_1 p_2 p_3) &= (p_1 p_2 p_3) - (p_0 p_2 p_3) + (p_0 p_1 p_3) - (p_0 p_1 p_2). \end{aligned} \tag{6.10}$$

Example 6.23. We again go back to examples in figure 6.4.

- For the triangulation K_X in figure 6.4a,

$$\partial_1 : C_1(K_X) \to C_0(K_X) : (p_0 p_1) \mapsto (p_1) - (p_0). \tag{6.11}$$

- For the triangulation K_X in figure 6.4c,

$$\partial_1 (p_0 p_1) = (p_1) - (p_0), \qquad \partial_1 (p_2 p_1) = (p_2) - (p_1), \qquad \partial_1 (p_2 p_0) = p_0 - p_2. \tag{6.12}$$

Remark 6.24. Here we see the reason for the minus sign in the definition of ∂_r: $(p_0 p_1) + (p_1 p_2)$ is the arc from p_0 to p_2, hence its boundary is two points p_0 and p_2. By linearity, we have

$$\partial_1 ((p_0 p_1) + (p_1 p_2)) = (p_1) - (p_0) + (p_2) - (p_1) = (p_2) - (p_0), \tag{6.13}$$

which is consistent with the fact that when gluing two connected arcs, their shared boundary is not a boundary in the new arc. Similar idea holds for r-simplexes. Notice the shared boundary of two simplexes always have different orientation in either one. $\partial_1 (p_0 p_1) = p_1 - p_0$ is also related to the Newton-Leibniz formula

$$\int_{(p_0 p_1)} f(t) dt = F(p_1) - F(p_0). \tag{6.14}$$

We will return to this when talking about cohomology.

$Z_r(K) \equiv \{ c \in C_r(K) \mid \partial_r c = 0 \}$

Since ∂_r is a group homomorphism, it is also interested to consider its *image* $\text{Im}\partial_r$ and *kernel* $\text{Ker}\partial_r$. They also have geometric interpretations.

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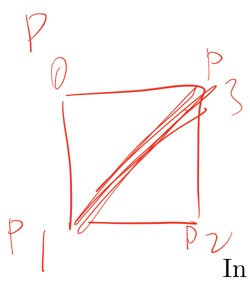
Definition 6.25 (r-cycle group). The *kernel* ∂_r $\text{Ker}\partial_r \equiv \{c \in C_r(K) | \partial_r c = 0\}$ is a *subgroup* of $C_r(K)$, also called the *r-cycle group* and denoted by $Z_r(K)$. $c \in \text{Ker}\partial_r$ is called an r-cycle.

Remark 6.26.
- If $r = 0$, $\partial_0 c = 0$ automatically.
- Elements in $Z_r(K)$ have no boundary. When $r = 1$, they are closed loops.

Definition 6.27 (r-boundary group). The *image* of ∂_{r+1} $\text{Im}\partial_{r+1} \equiv \{c \in C_r(K) | \exists d \in C_{r+1}(K), c = \partial_{r+1} d\}$ is a *subgroup* of $C_r(K)$, also called the *r-boundary group* and denoted by $B_r(K)$.

Remark 6.28.
- Elements in $B_r(K)$ are boundaries of certain elements in $C_{r+1}(K)$.
- Let n be the dimension of K, $B_n(K)$ is defined to be 0.



Example 6.29. Let K be the triangulation in the left plot of figure 6.5
- $\text{Ker}\partial_2 = \{0\}$.
- $\text{Im}\partial_2 = \text{span}_{\mathbb{Z}}\{(p_2p_3) - (p_0p_3) + (p_0p_2), (p_1p_2) - (p_0p_2) + (p_0p_1)\}$.
- $\text{Ker}\partial_1 = \text{span}_{\mathbb{Z}}\{(p_2p_3) - (p_0p_3) + (p_0p_2), (p_1p_2) - (p_0p_2) + (p_0p_1)\}$.
- $\text{Im}\partial_1 = \text{span}_{\mathbb{Z}}\{p_0 - p_1, p_0 - p_3, p_0 - p_2\}$.

In this example we see $\text{Im}\partial_2 = \text{Ker}\partial_1$ in particular. ★

An important property is that the boundary of a simplex has no boundary, hence the following proposition.

Proposition 6.30. *Let K be a simplicial complex. The composition $\partial_r \circ \partial_{r+1} : C_{r+1}(K) \to C_{r-1}(K)$ is a zero map, i.e. $\partial_r(\partial_{r+1}c) = 0$ for any $c \in C_{r+1}(K)$.*

Therefore an r-boundary is always an r-cycle

Corollary 6.31. *Let K be a simplicial complex. Then its r-boundary group $B_r(K) \equiv \text{Im}\partial_{r+1}$ is always a subset of its r-cycle group $Z_r(K) \equiv \text{Ker}\partial_r$*

$$B_r(K) \subset Z_r(K) \subset C_r(K). \tag{6.15}$$

6.3. Simplicial homology

A simplicial complex K and its chain groups $C_r(K)$ of a topological space is not a topological invariant. However, it leads to homology groups which are topological invariants. Firstly, we consider a general situation.

Definition 6.32 (Chain complex). A *chain complex* $(C_\bullet, d_\bullet)$ (or $(C_\bullet, d)$) of abelian groups $(C_n)_{n \in \mathbb{Z}}$ and homomorphisms of abelian groups

$$d_n : C_n \to C_{n-1}, \tag{6.16}$$

subject to the condition $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Remark 6.33. We also represent a chain complex $(C_\bullet, d_\bullet)$ graphically as

$$\cdots \xleftarrow{d_{-1}} C_{-1} \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots \tag{6.17}$$

and the condition $d_n \circ d_{n+1} = 0$ simplified as $d^2 = 0$.

r 个边界 $\Leftrightarrow$ $B_r = 0$

★ 极面 $B_r = 2$

One can also consider maps between chain complexes.

Definition 6.34 (Chain maps). Let $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d_\bullet)$,A *chain map* is a collection of homomorphisms $(f_n : C_n \to C'_n)_{n\in\mathbb{Z}}$ such that

$$f_n \circ d_{n+1} = d'_n \circ f_{n+1}. \tag{6.18}$$

Graphically

$$\begin{tikzcd}
\cdots & C_{-1} \arrow[l, "d_{-1}"'] \arrow[d, "f_{-1}"] \arrow[r, phantom, "\circlearrowleft"] & C_0 \arrow[l, "d_0"'] \arrow[d, "f_0"] \arrow[r, phantom, "\circlearrowleft"] & C_1 \arrow[l, "d_1"'] \arrow[d, "f_1"] & \cdots \arrow[l, "d_2"'] \\
\cdots & C'_{-1} \arrow[l, "d_{-1}"'] & C'_0 \arrow[l, "d_0"'] & C'_1 \arrow[l, "d_1"'] & \cdots \arrow[l, "d_2"']
\end{tikzcd} \tag{6.19}$$

From the definition 6.32 of chain complex, one can a associate a chain complex to a simplicial complex K and its chain groups $C_r(K)$ by formally adding elements identified to 0.

Definition 6.35 (Chain complex of a simplicial complex). Let K be a simplicial complex and n be its dimension. The chain complex $C(K) = (C_\bullet, d_\bullet)$ associated with K is defined as the following

- $C_r = \begin{cases} C_r(K) & 0 \leqslant r \leqslant n \\ 0 & \text{otherwise} \end{cases}$.
- $d_r = \begin{cases} \partial_r & 0 \leqslant r \leqslant n \\ 0 \mapsto 0 & \text{otherwise} \end{cases}$.

$C(K)$ can be represented graphically as

$$\cdots 0 \xleftarrow{d_{-1}:0\mapsto 0} 0 \xleftarrow{\partial_0} C_0(K) \xleftarrow{\partial_1} C_1(K) \xleftarrow{\partial_2} C_2(K) \cdots \xleftarrow{\partial_n} C_n(K) \xleftarrow{0\mapsto 0} 0 \cdots \tag{6.20}$$

Remark 6.36. $d^2 = 0$ follows from proposition 6.30 $\partial_r \circ \partial_{r+1} = 0$.

Definition 6.37 (Homology group). Let $(C_\bullet, d)$ be a chain complex of abelian groups. Define the r-th *homology group* of $(C_\bullet, d)$, denoted by $H_r(C_\bullet, d)$ (or simplfy $H_r(C_\bullet)$, $H_r(C)$), to be the quotient group

$$H_r(C_\bullet, d) = \frac{\mathrm{Ker}(d_r : C_r \to C_{r-1})}{\mathrm{Im}(d_{r+1} : C_{r+1} \to C_r)} = \mathrm{Ker}d_r/\mathrm{Im}d_{r+1}. \tag{6.21}$$

Remark 6.38.
- Each element in H_r is called a *homology class*.
- Elements in $\mathrm{Ker}d_r$ is also called *d_r-closed*, while elements in $\mathrm{Im}d_{r+1}$ is also called *d_{r+1}-exact*. Two d_r-closed elements belong to the same equivalence class in H_r if their difference is d_{r+1}-exact,

$$[c_1] = [c_2] \in H_r, \text{if } \partial_r c_1 = \partial_r c_2 = 0,\ c_1 - c_2 = \partial_{r+1} d. \tag{6.22}$$

- Let K be a simplicial complex. When $0 \leqslant r \leqslant n$, by definition $\mathrm{Ker}d_r = Z_r(K)$ and $\mathrm{Im}d_{r+1} = B_r(K)$, then $H_r(C(K)) = Z_r(K)/B_r(K)$. Elements of $H_r(C(K))$ are r-cycles which are *not* boundaries. Two elements of $H_r(C(K))$ belong to the same equivalent class if their difference is a boundary.

Although triangulations and chain groups are not, homology groups are topological invariants.

$H = \frac{Z}{B}$

Z_2 计算: c ① $= \sum k c_2$

② $\partial_2 c_2 = 0 \Rightarrow k \Rightarrow c$, 数 c 个数

Theorem 6.39. *Let X and Y be two homeomorphic topological spaces, and let (K, f) and (L, g) be triangulations of X and Y respectively, then*

$$H_r(K) \simeq H_r(L). \tag{6.23}$$

Example 6.40. Homology groups of $\mathbf{I} = [0, 1]$. Let K be the triangulation of figure 6.4a. Previous example tells us

$$\begin{aligned} C_0(K) &= \{z_0(p_0) + z_1(p_1) | z_i \in \mathbb{Z}\} \simeq \mathbb{Z}^2, \\ C_1(K) &= \{z_0(p_0p_1) | z_0 \in \mathbb{Z}\} \simeq \mathbb{Z}. \end{aligned} \tag{6.24}$$

and

$$\partial_1 : C_1(K) \to C_0(K) : (p_0p_1) \mapsto p_1 - p_0. \tag{6.25}$$

$C(K)$ is then

$$\begin{array}{ccccccc} \cdots C_{-1} & \xleftarrow{0 \mapsto 0} & C_0 & \xleftarrow{(p_0p_1) \mapsto p_1 - p_0} & C_1 & \xleftarrow{0 \mapsto 0} & C_2 \cdots . \\ \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathrm{span}_{\mathbb{Z}}\{p_0, p_1\} & & \mathrm{span}_{\mathbb{Z}}\{(p_0p_1)\} & & 0 \end{array} \tag{6.26}$$

Next we compute the $Z_r(K)$ and $B_r(K)$. The cycle groups are

$$Z_r(K) = \begin{cases} \mathrm{span}_{\mathbb{Z}}\{p_0, p_1\} \simeq \mathbb{Z}^2 & r = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{6.27}$$

and the boundary groups are

$$B_r(K) = \begin{cases} \mathrm{span}_{\mathbb{Z}}\{p_1 - p_0\} \simeq \mathbb{Z} & r = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{6.28}$$

In the end we get the homology groups of $\mathbf{I}$

$$H_0(\mathbf{I}) = \mathrm{span}_{\mathbb{Z}}\{[p_0]\} \simeq \mathbb{Z}, \quad H_{r \neq 0}(\mathbf{I}) = 0. \tag{6.29}$$

p_0 and p_1 belong to the same homology class because

$$p_1 = p_0 + (p_1 - p_0) = p_0 + \partial_1(p_0p_1). \tag{6.30}$$

One can also compute homology groups of $\mathbf{I}$ using the triangulation in figure 6.4b and the result is the same. The homology groups of $\mathbf{I}$ is the same as that of a point.

Example 6.41. Homology groups of $\mathbb{S}^1$. Let K be the triangulation of figure 6.4c. Then $C(K)$ is

$$\begin{array}{ccccccc} \cdots C_{-1} & \xleftarrow{0 \mapsto 0} & C_0 & \xleftarrow{\partial_1} & C_1 & \xleftarrow{0 \mapsto 0} & C_2 \cdots \\ \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathrm{span}_{\mathbb{Z}}\{p_0, p_1, p_2\} & & \mathrm{span}_{\mathbb{Z}}\{(p_0p_1), (p_1p_2), (p_2p_0)\} & & 0 \end{array} \tag{6.31}$$

with

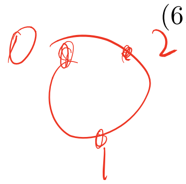
$$\partial_1(p_0p_1) = (p_1) - (p_0), \ \partial_1(p_2p_1) = (p_2) - (p_1), \ \partial_1(p_2p_0) = p_0 - p_2. \tag{6.32}$$

The cycle groups are

$$Z_r(K) = \begin{cases} \mathrm{span}_{\mathbb{Z}}\{p_0, p_1, p_2\} \simeq \mathbb{Z}^3 & r = 0 \\ \mathrm{span}_{\mathbb{Z}}\{(p_0p_1) + (p_1p_2) + (p_2p_0)\} & r = 1 \\ 0 & \text{otherwise} \end{cases}, \tag{6.33}$$

and the boundary groups are

$$B_r(K) = \begin{cases} \mathrm{span}_{\mathbb{Z}}\{p_1 - p_0, p_2 - p_1\} \simeq \mathbb{Z}^2 & r = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{6.34}$$



$C_0 \simeq \mathbb{Z}^3$ $\quad C_1 \simeq \mathbb{Z}^3$

$Z_0 \simeq \mathbb{Z}^3$ $\quad Z_1 \simeq \mathbb{Z}^1$

$B_0 \simeq \mathbb{Z}^2$

$H_0 \simeq \mathbb{Z}$

$B_r = \exists\, d \in C_{r+1}$

$B_0 = \; d \in C_? \quad \partial_1 c_1$

The homology groups of $\mathbb{S}^1$ are

$$H_0(\mathbb{S}^1) = \text{span}_{\mathbb{Z}}\{[p_0]\} \simeq \mathbb{Z}, \tag{6.35}$$
$$H_1(\mathbb{S}^1) = \text{span}_{\mathbb{Z}}\{[(p_0p_1) + (p_1p_2) + (p_2p_0)]\} \simeq \mathbb{Z}, \qquad H_{r\neq 0,1}(\mathbb{S}^1) = 0.$$

6.4. More examples

6.5. Properties of homology groups

6.5.1. Connectness and homology groups.

PROPOSITION 6.42. *Let K be a connected simplicial complex, then*

$$H_0(K) \simeq \mathbb{Z}. \tag{6.36}$$

PROPOSITION 6.43. *Let K be a simplicial complex. If K is a disjoint union of N connected components, $K = K_1 \cup K_2 \cup \cdots \cup K_N$ where $K_i \cap K_j = \varnothing$, then*

$$H_r(K) \simeq K_r(K_1) \times H_r(K_2) \times \cdots \times H_r(K_N). \tag{6.37}$$

COROLLARY 6.44. *Let K be a simplicial complex. If K is a disjoint union of N connected components, $K = K_1 \cup K_2 \cup \cdots \cup K_N$ where $K_i \cap K_j = \varnothing$, then*

$$H_0(K) \simeq \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n}. \tag{6.38}$$

$H_0 \simeq \mathbb{Z}$

COROLLARY 6.45. *Let K be a simplicial complex. K is connected if and only if $H_0(K) \simeq \mathbb{Z}$.*

6.5.2. Structure of homology groups. Since $H_r(K)$ is abelian. By group theory the most general form of an abelian group is

$$H_r(K) \simeq \underbrace{\mathbb{Z} \times \cdot \times \mathbb{Z}}_{f \text{ free part}} \times \underbrace{\mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_p}}_{p \text{ torsion part}}. \tag{6.39}$$

If we change coefficients in chain groups from integers to real numbers, we will not see the torsion part of the homology group, then

$$H_r(K;\mathbb{R}) \simeq \mathbb{R}^f. \tag{6.40}$$

6.5.3. Betti numbers and the Euler characteristics.

DEFINITION 6.46 (**Betti number**). Let K be a simplicial complex. The rth *Betti number* $b_r(K)$ is

$$b_r(K) \equiv \dim H_r(K;\mathbb{R}) \tag{6.41}$$

which is also the rank of the free abelian part of $H_r(K;\mathbb{Z})$.

One also defines the Euler characteristics as

DEFINITION 6.47 (**Euler characteristics**). Let K be an n-dimensional simplicial complex and I_r be the number of r-simplexes in K. The *Euler characteristics* χ is

$$\chi(K) \equiv \sum_{r=0}^{n} (-1)^r I_r. \tag{6.42}$$

Betti numbers and the Euler characteristics are related by the following theorem.

$\chi(k) \equiv \sum_{r=0}^{n} (-1)^r I_r$

$0 - 1 + 2 - 3 - \cdots n$

THEOREM 6.48 (**Euler-Poincare theorem**). *Let K be an n-dimensional simplicial complex and I_r be the number of r-simplexes in K, then*

$$\chi(K) = \sum_{r=0}^{n}(-1)^r b_r(K). \tag{6.43}$$

$b_r = \{k \mid H_r \simeq \mathbb{Z}^k\}$

$\chi(K) = \sum (-1)^r b_r(K)$