CHAPTER 5

# Homotopy

同伦　弱于同胚

## 5.1. Homotopy of maps

In physics literature, one encounters phrases like "continuous change of solutions/path", "X is continuously deformed to Y", and etc, which means homotopy in mathematics.

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Definition 5.1 (**Homotopy of maps**). Let $f$ and $g$ be continuous maps from topological space $X$ to $Y$. A *homotopy* between $f$ and $g$ is a continuous map

$$\sigma : X \times [0,1] \to Y, \tag{5.1}$$

such that, for any $x \in X$, $\sigma(x, 0) = f(x)$ and $\sigma(x, 1) = g(x)$, i.e. $\sigma$ defines a path from $f(x)$ to $g(x)$ in $\mathcal{C}[X; Y]$.

Remark 5.2. $f$ and $g$ are called *homotopic* if there exists a homotopy between $f$ and $g$, denoted by $f \simeq g$. A function $f$ who is homotopic to a constant function is also denoted as *null-homotopic*. 0伦.

① $f \simeq f$:
$\sigma(f(x), t) = f(x)$
② $f \simeq g \Rightarrow g \simeq f$
$\exists \sigma_1(f(x), 1) = g(x)$
$\sigma_1(f(x), 0) = f(x)$
$\Rightarrow \sigma_2 = \sigma(f(x), 1-t)$.

Proposition 5.3. *Let $X$ and $Y$ be topological spaces. $f$ is homotopic to $g$ ($f \simeq g$) is an equivalence relation on $\mathcal{C}[X; Y]$.* 和道路连通一样.

Remark 5.4. We denote the quotient set $\mathcal{C}[X; Y]/\simeq$ as $[X; Y]$. The homotopy class (the equivalent class under $\simeq$) of the map $f$ is denoted as $[f]$.

乘空间道路

Example 5.5. Let $P$ be a space with only one point. Then there is a one to one correspondence between continuous functions from $P$ to topological space $X$ and points in $X$. Two homotopic maps $f$ and $g$ from $P$ to $X$ gives a path from $f(P)$ to $g(P)$ on $X$. Homotopic relation on $\mathcal{C}[P; X]$ is the same as the relation "path-connectedness" on $X$. Therefore $\mathcal{C}[P; X]/\simeq$ is the same as $\pi_0(X)$ as a set, where $\pi_0(X)$ is the set of all path-connected components on $X$.

$X/\sim_p$
X所有道路连通分支

Proposition 5.6. *Let $X$, $Y$ and $Z$ be topological spaces. Assume continuous maps $f : X \to Y$, $f' : X \to Y$ are homotopic to each other, and continuous maps $g : Y \to Z$, $g' : Y \to Z$ are homotopic to each other. Then $g \circ f$ is homotopic to $g' \circ f'$.*

Remark 5.7. The composition of maps

$$\mathcal{C}[X; Y] \times \mathcal{C}[Y; Z] \to \mathcal{C}[X; Z] : (f, g) \mapsto g \circ f \tag{5.2}$$

induces a composition of homotopy classes

$$[X; Y] \times [Y; Z] \to [X; Z] : ([f], [g]) \mapsto [g \circ f]. \tag{5.3}$$

### 5.1.1. Homotopy equivalence.

If $f \circ g = \mathrm{Id}_Y$
$g \circ f = \mathrm{Id}_X$
then $X \cong Y$



Definition 5.8 (**Homotopy equivalence**). Topological spaces $X$ and $Y$ are called *homotopy equivalent* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $[f \circ g] = [\mathrm{Id}_Y]$ and $[g \circ f] = [\mathrm{Id}_X]$ (i.e. $f \circ g \simeq \mathrm{Id}_Y$, $g \circ f \simeq \mathrm{Id}_X$). Each of the maps $f$ and $g$ is called a *homotopy equivalence*, and $g$ is said to be a *homotopy inverse* to $f$ (and vice versa). 互为同伦逆

Remark 5.9.
- One can think of homotopy equivalent spaces as spaces, which can be deformed continuously into one another.
- Any homeomorphism $f : X \to Y$ is a homotopy equivalence, with homotopy inverse $f^{-1}$, but converse does not necessarily hold.

Proposition 5.10. *$f : X \to Y$ is a homotopy equivalence. Then for any topological space $T$, natural maps*

$$\begin{aligned} &[T;X] \to [T;Y] : \xi \mapsto [f] \circ \xi, \\ &[Y;T] \to [X;T] : \psi \mapsto \psi \circ [f] \end{aligned} \tag{5.4}$$

*are bijections.*

Remark 5.11. If $T$ is a space with only one point, there is a bijection between $\pi_0(X)$ and $\pi_0(Y)$ following example 5.5.

可缩

Definition 5.12 (**Contractible space**). A space $X$ is called *contractible* if $X$ is homotopic equivalent to a point. $X$ is contractible if and only if $\mathrm{Id}_X$ is null-homotopic.

Proof. ("If" part) $\mathrm{Id}_X$ is null-homotopic. Let $P = \{p\}$ be the topological space of one point. Let $c : X \to P : x \mapsto p$ be a constant map from $X$ to $P$. Let $g : P \to X : p \mapsto x_0$. $g \circ c(x) = x_0$ is a constant map from $X$ to itself, then $g \circ c \simeq \mathrm{Id}_X$ by assumption. Clearly $c \circ g = \mathrm{Id}_P$. So $c$ is a homotopy equivalence between $X$ and $P$.

("Only if" part) Let $f : X \to P$ be a homotopy equivalence. Let $g : P \to X$ be the homotopy inverse of $f$. Then $g \circ f \simeq \mathrm{Id}_X$ which means $\mathrm{Id}_X$ is homotopic to the constant function $g \circ f : X \to X$. $\square$

Example 5.13. Any interval in $\mathbb{R}$ is contractible. Actually, any star-shape region in $\mathbb{R}^n$ (see remark 4.33) is contractible.

形变收缩

Definition 5.14 (**Deformation retraction**). Let $X$ be a topological space and $A$ be a subspace of $X$. A *deformation retraction* is a homotopy

$$r : X \times [0,1] \to X \tag{5.5}$$

such that $r(x,0) = x$, $r(x,1) \in A$ and for any point $a \in A$ and $t \in [0,1]$, $r(a,t) = a$ (i.e. $r(x,t)|_A = \mathrm{Id}_A$ for any $t \in [0,1]$).

Remark 5.15.
- The subspace $A$ is called a *deformation retract* of $X$ if there exists a deformation retraction from $X$ to $A$.
- If $A$ is a deformation retract of $X$, then $X$ is homotopy equivalent to $A$, because the deformation retraction $r$ gives the homotopy inverse of the inclusion map $A \hookrightarrow X$.

EXAMPLE 5.16 (**Examples of deformation retractions**).

- $r(\vec{x}, t) = \frac{\vec{x}}{(1-t)+t|\vec{x}|}$ gives a deformation retraction from $\mathbb{R}^{n+1} \backslash \{0\}$ to $\mathbb{S}^n$.
- The central circle is a deformation retract of the Möebius strip.

We are interested in homotopic invariants in this section. Clearly all homeomorphic invariants are also homotopic invariants.

## 5.2. Homotopy of paths and the fundamental group

Recall a path on space $X$ is a continuous map $\gamma : [0,1] \to X$. A *loop* is a path $\gamma$ which $\gamma(0) = \gamma(1)$.

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DEFINITION 5.17. **Loop** $X$ be a topological spaces. A *loop based at* $x \in X$ is a path $\gamma$ with $\gamma(0) = \gamma(1) = x$.

REMARK 5.18. Let $q : [0,1] \to \mathbb{S}^1$ be a quotient map such that $q(0) = q(1) = o \in \mathbb{S}^1$, then a loop based at $x$ is also a continuous map $l : \mathbb{S}^1 \to X$ such that $l(o) = x$.

DEFINITION 5.19 (**Inverse path**). Let $\gamma$ be a path from $x$ to $x'$. Its *inverse path* $\gamma^- : [0,1] \to X$ is

$$\gamma^- : t \mapsto \gamma(1-t), \tag{5.6}$$

which is a path from $x'$ to $x$.

DEFINITION 5.20 (**Constant path**). A *constant path* based at $x \in X$ is the constant map $t \mapsto x$.

Since paths are also continuous maps, we can apply the concept of homotopy on them.

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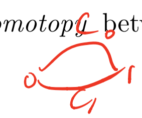
DEFINITION 5.21 (**Path homotopy**). A *path homotopy* between two paths $c_0$ and $c_1$ is a homotopy



$$\sigma : [0,1] \times [0,1] \to X, \tag{5.7}$$

which satisfies $\sigma(0,t) = c_0(0) = c_1(0)$ and $\sigma(1,t) = c_0(1) = c_1(1)$. In particular $c_0$ and $c_1$ have the same start and end. If $c_0$ is path homotopic to $c_1$, we denote as $c_0 \simeq_p c_1$.

REMARK 5.22. Denote all paths from $x \in X$ to $y \in X$ as $\Lambda_{x,y}(X)$. Path homotopy gives an equivalence relation on $\Lambda_{x,y}(X)$.

DEFINITION 5.23 (**Product of paths**). Let $\alpha$ and $\beta$ be two paths in $X$, and $\alpha(1) = \beta(0)$ (i.e. the end of $\alpha$ is the start of $\beta$). Their *product path* $\alpha * \beta$ is

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq 1/2, \\ \beta(2s-1) & \text{if } 1/2 \leq s \leq 1. \end{cases} \tag{5.8}$$

REMARK 5.24. The continuity of $\alpha * \beta$ follows from pasting lemma.

PROPOSITION 5.25. *Let $c$ be a path on $X$. Then $c^- * c$ path homotopic to the constant path based at $c(1)$.*

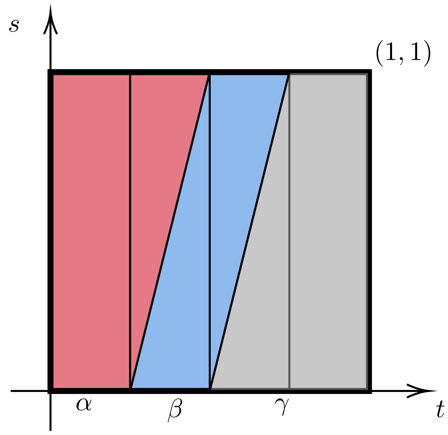



FIGURE 5.1. The associativity of products of $[\alpha]$, $[\beta]$, and $[\gamma]$.

PROOF. Let $\gamma_s(t) = c((1-s)t+s)$. Then

$$F(t,s) = \gamma_s^-(t) * \gamma_s(t) \tag{5.9}$$

is the path homotopy we need. □

PROPOSITION 5.26. *Let $\alpha$ and $\beta$ be paths on $X$ satisfying $\alpha(1) = \beta(0)$. Assume there exist paths $\alpha'$ and $\beta'$ such that*

$$\alpha' \simeq \alpha, \qquad \beta' \simeq \beta, \tag{5.10}$$

*then $\alpha * \beta \simeq_p \alpha' * \beta'$.*

REMARK 5.27. This proposition tells us that the path product $*$ induces a well-defined product $*$ on path homotopy classes

$$[\alpha] * [\beta] = [\alpha * \beta] \tag{5.11}$$

for any paths $\alpha$ and $\beta$ satisfying $\alpha(1) = \beta(0)$.

PROOF. Let $F$ be the homotopy between $\alpha$ and $\alpha'$, and $G$ be the homotopy between $\beta$ and $\beta'$. Define

$$H(s,t) = \begin{cases} F(2s,t) & \text{if } 0 \le s \le 1/2, \\ G(2s-1,t) & \text{if } 1/2 \le s \le 1. \end{cases} \tag{5.12}$$

$H$ gives the path homotopy we need. □

PROPOSITION 5.28. *Let $\alpha$, $\beta$ and $\gamma$ be three paths on $X$. And $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. Then*

$$\alpha * (\beta * \gamma) \simeq_p (\alpha * \beta) * \gamma. \tag{5.13}$$

REMARK 5.29. Usually $\alpha * (\beta * \gamma)$ is not the same as $(\alpha * \beta) * \gamma$.

PROOF. We illustrate the construction of the homotopy in figure 5.1. The reader should try to write it explicitly. □

From propositions above we see that although the path product is not a "good" operation on paths, it is a "good" operation on path-homotopy classes. Moreover, since any two loops based at $x$ can make path product, it is then natural to consider the set of all loops based at $x$ up to path homotopy.

Homotopy 同伦.
$f \simeq g \quad X \to Y$
$\exists H : X \times I \to Y$

同伦等价
$X \simeq Y \quad$ eg. $S^1 \simeq R^2 \setminus \{0\}$
$\exists \; f: X \to Y$
$g: Y \to X$

道路同伦.
道路起点终点要一致
反例:  $P_1 \not\simeq P_2$

$\begin{cases} H(x,0) = f(x) \\ H(x,t) = g(x) \end{cases}$ $\begin{cases} f\circ g \simeq Id_X \\ g\circ f \simeq Id_Y \end{cases}$

基本群: $\pi_1$ $\Lambda_x(X) = \{\text{loop in } X \text{ based on } x\}$. 所有 $x$ 为基点闭路的集合

$\pi_1(X,x) = \Lambda_x(X)/\sim_p$

$\pi_1(S^1,x) \cong (\mathbb{Z}^*)$

$\pi_1(S^n,x) \cong \{1\}$ $n>1$

$\pi_0(X) = \{\text{Point} \to X\}/\sim$

$\pi_1(X) = \{\text{loop} \to X\}/\sim$

$\pi_n(X,x)$ ?

$I_n = [0,1]^n$

$\partial I_n$ 边界

def $n$-loop.

$\gamma: I_n \to X$

st. $\gamma|_{\partial I_n} = x \in X$



Definition 5.30 (**The fundamental group**). Let $X$ be a topological space. For any point $x \in X$, let $\pi_1(X,x)$ be the quotient set

(5.14) $$\pi_1(X,x) = \Lambda_{x,x}(X)/\simeq_p,$$

i.e. the set of all homotopy classes of loops based at $x$. $(\pi_1(X,x),*)$ forms a group under the path product $*$, which is called the *fundamental group* of $X$ at $x$.

Proof. To show that $(\pi_1(X,x),*)$ forms a group. We need to show the associativity of $*$, the existence of a unit element, and the existence of a inverse of any element.

- Associativity: follows from proposition 5.28.
- Unit: the homotopy class of constant loop based at $x$.
- Inverse: the inverse of $[\gamma]$ is $[\gamma^-]$ because of proposition 5.25.

□

Remark 5.31. A path-connected topological space is called *simply connected by paths* if for any $x \in X$, $\pi_1(X,x) = \{1\}$ is a trivial group.

Proposition 5.32. *The fundamental group is* invariant *under homotopy, i.e. homotopy equivalent topological spaces have isomorphic fundamental groups.*

Example 5.33. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$. Because $\mathbb{R}^2$ with the origin removed is not simply path-connected, while $\mathbb{R}^n$ for $n > 2$ with the origin removed are.

## 5.3. Basic properties of the fundamental groups

**5.3.1. Dependence on the base point.** The definition the fundamental group depends on the choice of the base point, the following proposition basically says that the fundamental group is "independent" of the choice of base points for path-connected spaces.

Proposition 5.34. *Let $X$ be a path connected topological space. Then for any $x, y \in X$, the fundamental group $\pi_1(X,x)$ is isomorphic to $\pi_1(X,y)$.*

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Proof. We need to construct a bijection $f : \pi_1(X,x) \to \pi_1(X,y)$ such that $f(\alpha * \beta) = f(\alpha) * f(\beta)$.

Because $X$ is path-connected, let $\delta$ be the path from $x$ to $y$. Define the map

(5.15) $$\tilde{\delta} : \pi_1(X,y) \to \pi_1(X,x) : [\gamma] \mapsto [\delta]^{-1} * [\gamma] * [\delta].$$

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$\tilde{\delta}$ is a group homomorphism because

(5.16) $$\tilde{\delta}([\alpha]*[\beta]) = [\delta]^{-1}*[\alpha]*[\beta]*[\delta] = [\delta]^{-1}*[\alpha]*[\delta]*[\delta]^{-1}*[\beta]*[\delta] = \tilde{\delta}([\alpha])*\tilde{\delta}([\beta]).$$

$\tilde{\delta}$ is an invertible map. Its inverse is

(5.17) $$\tilde{\delta}^{-1} : \pi_1(X,x) \to \pi_1(X,y) : [\alpha] \mapsto [\delta] * [\alpha] * [\delta]^{-1}.$$

Therefore $\tilde{\delta}$ is a bijection. Sinct it is also a group homomorphism, $\tilde{\delta}$ is an isomorphism between $\pi_1(X,x)$ and $\pi_1(X,y)$. □

Corollary 5.35. *Let $X$ be a topological space. The following statements are equivalent.*

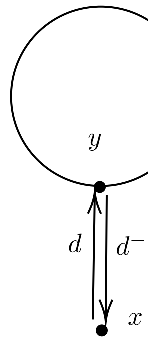


FIGURE 5.2. Constructing a loop based at $x$ which is freely homotopic to the loop $c$.

- $\pi_1(X, x)$ *is* trivial.
- *Any two paths start from $x$ and have the same ends are* homotopy equivalent.
- *Any loop based at $x$ is* homotopy equivalent *to the constant path based at $x$.*

EXAMPLE 5.36. Consider any loop $l : \mathbb{S}^1 \to \mathbb{R}^n$ based at the origin of $\mathbb{R}^n$. Define $L : \mathbb{S}^1 \times [0, 1] \to \mathbb{R}^n$, $L(\theta, t) = (1 - t)l(\theta)$. Then $L$ is a path homotopy between $l$ and the constant path based at the origin. Hence for any $x \in \mathbb{R}^n$, $\pi_1(\mathbb{R}^n, x) = \{1\}$.

### 5.3.2. Freely homotopic loops.

DEFINITION 5.37 (**Freely homotopic loops**). Two loops $c_0$ and $c_1$ on topological space $X$ are *freely homotopic* if there exists a homotopy

$$\sigma : [0, 1] \times [0, 1] \to X, \tag{5.18}$$

such that $\sigma(t, 0) = c_0(t)$, $\sigma(t, 1) = c_1(t)$ and $c(0, s) = \sigma(1, s)$ (i.e. for any $s \in [0, 1]$, $sigma(t, s)$ is a loop).

REMARK 5.38. "freely" means we do not choose a base point. Its relation with homotopy based on a point is discuss in the following proposition.

PROPOSITION 5.39. *Let $X$ be a path-connected space, and $x \in X$. Then any loops on $X$ is freely homotopic to some loop based at $x$.*

PROOF. Assume $c$ is a loop on $X$ with $c(0) = y$. Let $d$ be a path from $x$ to $y$. We need to show that the loop $d * c * d^-$ based on $x$ is freely homotopic to $c$. The homotopy is given by

$$\sigma(t, s) = d((1 - s)t + s) * c * d((1 - s)t + s)^-, \tag{5.19}$$

as illustrated in figure 5.2. □

### 5.3.3. Fundamental group of the product space.

PROPOSITION 5.40. *Let $(X_i)_{i=1,\cdots,n}$ be a collection of topological spaces. $X = \prod_{i=1}^n X_i$. Let $x = (x_1, x_2, \cdots, x_n)$ be a point in $X$, then $\pi_1(X, x)$ is isomorphic to the product group $\prod_{i=1}^n \pi_1(X_i, x_i)$.*

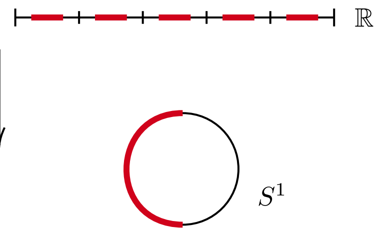


FIGURE 5.3. The covering map $p : \mathbb{R} \to \mathbb{S}^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t)$. The preimage of the left half-circle (red) is the union of disjoint intervals on $\mathbb{R}$.

### 5.3.4. Fundamental group of $\mathbb{S}^n$.

PROPOSITION 5.41.
- *Any freely homotopic loops on $\mathbb{R}^2\backslash\{0\}$ have the same winding number.*
- *The winding number of loops gives the group isomorphism $\pi_1(\mathbb{R}^2\backslash\{0\}) \to \mathbb{Z}$.*

COROLLARY 5.42. *The fundamental group of $\mathbb{S}^1$ is isomorphic to the additive group $\mathbb{Z}$.*

PROPOSITION 5.43. *Let $U_1$ and $U_2$ be two path-connected and simply path-connected open sets of $X$. If $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected, then $X$ is path-connected and simply path-connected.*

COROLLARY 5.44. *$\pi_1(\mathbb{S}^n)$ is trivial for $n \geq 2$.*

EXAMPLE 5.45. The fundamental group of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is isomorphic to the additive group $\mathbb{Z}^2$ (proposition 5.40). Hence, the torus is not homeomorphic to $\mathbb{S}^n$.

## 5.4. Covering spaces and lifting lemma

A useful tool to compute the fundamental group is the covering space. Loops of different homotopy type are lifted to different paths in the covering space.

DEFINITION 5.46 (**Covering space**). Let $(E, p)$ be a space over $B$ such that $p : E \to B$ continuous. $p$ is a covering map, or $(E, p)$ is a covering space over $B$, if for any $b \in B$, there exists an open neighborhood $U$ of $b$ such that

$$p^{-1}(U) = \cup_{i \in I} U_i, \tag{5.20}$$

where $U_i$'s are disjoint open subspace of $E$ and $p|_{U_i} : U_i \to U$ is a homeomorphism.

REMARK 5.47. Covering space is the modern way of saying "multi-valued function". In particular if $E$ is a covering space of $\mathbb{C}$, a multi-valued function is $f : E \to \mathbb{C}$.

EXAMPLE 5.48. $p : \mathbb{R} \to \mathbb{S}^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a covering map. The preimage of an arc between $(\theta_1, \theta_2)$ and $\theta_2 - \theta_1 < 2\pi$ is $\cup_{n \in \mathbb{Z}}(n + \frac{\theta_1}{2\pi}, n + \frac{\theta_2}{2\pi})$. Illustrated in figure 5.3.

$$E \xrightarrow{p} B$$

$\forall\, b \in B$

$\forall\, b \subset U$ open $U \subset B$

$U_i \subset E$. disjoint open

s.t. $p^{-1}(U) = U_{i \in I}\, U_i$

EXAMPLE 5.49. $p : \mathbb{S}^2 \to \mathbb{RP}^2 : \vec{x} \mapsto [\vec{x}]$ which maps antipodal points on $\mathbb{S}^2$ to the same point in $\mathbb{RP}^2$ is a covering map. It is also called a double cover because for each $b \in \mathbb{RP}^2$, there exists an open neighborhood of $b$ whose preimage under $p$ is disjoint union of *two* open subspaces of $\mathbb{S}^2$.

Because different points in covering space may map to the same point in base. The image of a path in $E$ under $p$ maybe a loop in $B$, such path is a "lift" of the loop in $E$. One can then discuss homotopy class of loops in $B$ by looking at their lifts in $E$.

提升引理

LEMMA 5.50 (**Lifting lemma**). *Let $p : E \to B$ be a covering map. Let $c : [0,1] \to B$ be a path with $c(0) = b$. Let $\tilde{b} \in E$ such that $p(\tilde{b}) = b$ then there exists a unique path in $E$*

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$$\tilde{c} : [0,1] \to E, \qquad \tilde{c}(0) = \tilde{b}. \tag{5.21}$$ s.t. $p(\tilde{c}) = c$

*$\tilde{c}$ is called a lifting of $c$.*

REMARK 5.51. If $f : E \to \mathbb{C}$ is a multi-valued function. Fixing $\tilde{b}$ is the same as fixing a branch of $f$. If $c$ is a loop ($c(1) = c(0) = b$), $\tilde{c}(1) \in p^{-1}(b)$ but not necessarily $\tilde{c}(1) = \tilde{b}$.

The homotopy between paths on $B$ is the same as the homotopy of their lifting because of the following proposition.

PROPOSITION 5.52. *Let $p : E \to B$ be a covering map. $\sigma : [0,1] \times [0,1] \to B$ is a path homotopy between $c_0 : [0,1] \to B$ and $c_1 : [0,1] \to B$. Let $\tilde{b} \in E$ such that $p(\tilde{b}) = b$. Then lifting $\tilde{c}_0$ and $\tilde{c}_1$ starting from $\tilde{b}$ are also path homotopic. In particular $\tilde{c}_0(1) = \tilde{c}_1(1)$.*

EXAMPLE 5.53. As an application of this proposition, we can show that the identity map on $\mathbb{S}^1$ is not homotopic to the constant map on $\mathbb{S}^1$, because the lifting of the identity map on $\mathbb{S}^1$ is $\tilde{c} : [0,1] \to \mathbb{R} : t \mapsto t$. In particular $\tilde{c}(0) = 0$ and $\tilde{c}(1) = 1$. One can also generalize this result to see that $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$.

How do we construct covering maps? One method is to use the homogeneous action of a group.

DEFINITION 5.54 (**Covering space action**). Let $X$ be a topological space, $G$ be a group. The left action of $G$

$$\mu : G \times X \to X : (g, x) \mapsto g \cdot x \tag{5.22}$$

is a *covering space action* if for any $y \in X$, there exists a neighborhood $U$ of $y$ such that

$$\{g \in G | g \cdot U \cap U \neq \phi\} = \{e\}, \tag{5.23}$$

where $e$ is the identity element of $G$ (i.e. $g_1 \cdot U \cap g_2 \cdot U = \phi$ if and only if $g_1 \neq g_2$).

EXAMPLE 5.55.
- The action of $\mathbb{Z}$ on $\mathbb{R}$ by $(n, x) \mapsto x + n$ (translation by $n$) is a covering space action.
- The action of $\mathbb{Z}_2$ on $\mathbb{S}^2$ by $(\pm 1, x) \mapsto \pm x$ is a covering space action.
- The action of the rotation group on $\mathbb{R}^2$ by $(\theta, (x, y)) \mapsto (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$ is *not* a covering space action.

This action is called covering map action because it generates a covering map.

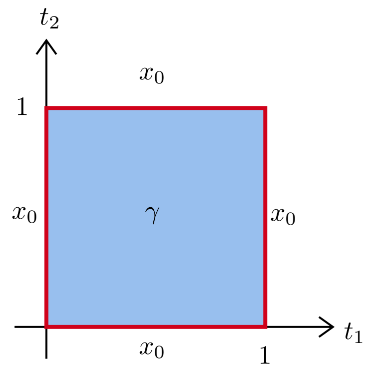


FIGURE 5.4. Illustration of a $2-$loop. The red part is $\partial\mathbf{I}_2$. The red and blue part together is $\mathbf{I}_2$. Every point on $\partial\mathbf{I}_2$ is mapped to the base point $x_0$.

PROPOSITION 5.56. *If the left action of $G$ on $X$ is a covering map action, then the quotient map*

$$p : X \to {}_G\backslash X \tag{5.24}$$

*is a covering map. Here ${}_G\backslash X$ is the quotient space of $X$ under the equivalence relation $x \sim g \cdot x$ for $g \in G$.*

One can then use the following theorem to compute the fundamental group.

THEOREM 5.57. *Let $X$ be a simply path-connected topological space. $G$ be a group, and $\mu : G \times X \to X$ is a covering map action on $X$. Then the quotient map $p : X \to {}_G\backslash X$ is a covering map, moreover, $\pi_1({}_G\backslash X, p(x))$ is isomorphic to $G$.*

EXAMPLE 5.58.
- $(n, x) \mapsto x + n$ is a covering map action of $\mathbb{Z}$ on $\mathbb{R}$, and ${}_{\mathbb{Z}}\backslash\mathbb{R} \cong \mathbb{S}^1$, therefore $\pi_1(\mathbb{S}^1, x) \simeq \mathbb{Z}$.
- $(\pm 1, x) \mapsto \pm x$ is a covering map action of $\mathbb{Z}_2$ on $\mathbb{S}^2$ and ${}_{\mathbb{Z}_2}\backslash\mathbb{S}^2 \cong \mathbb{R}\mathrm{P}^2$. Since $\pi_1(\mathbb{S}^2)$ is trivial, $\pi_1(\mathbb{R}\mathrm{P}^2) \simeq \mathbb{Z}_2$. In general we have

$$\pi_1(\mathbb{R}\mathrm{P}^n) \simeq \mathbb{Z}_2, \qquad n \geq 2. \tag{5.25}$$

## 5.5. Higher homotopy groups

We generalize $\pi_0(X)$ and $\pi_1(X, x)$ to higher dimensions in this section. Firstly, we define the generalization of loops in higher dimension. We denote the $n-$cube $\mathbf{I}_n \equiv [0,1]^n$ by $\mathbf{I}_n$ and its boundary by $\partial\mathbf{I}_n$ ($\mathbf{I}_2$ is illustrated in figure 5.4).

$$\begin{aligned} \mathbf{I}_n &= \{(t_1, \cdots, t_n) \in \mathbb{R}^n | 0 \leq t_i \leq 1\}, \\ \partial\mathbf{I}_n &= \{(t_1, \cdots, t_n) \in \mathbb{R}^n | 0 \leq t_i \leq 1, \text{ some } t_i = 0 \text{ or } 1\}. \end{aligned} \tag{5.26}$$

DEFINITION 5.59 ($n-$**loop**). An *$n-$loop* based at $x_0 \in X$ is a continuous map $\gamma : \mathbf{I}_n \to X$ such that $\gamma|_{\partial\mathbf{I}_n} = x_0$.

REMARK 5.60.
- Figure 5.4 is an illustration of a $2-$loop based at $x_0$. We will denote a generic point in $\mathbf{I}_n$ by $t = (t_1, \cdots, t_n) \in \mathbf{I}_n$.

- Another point of view of $n-$loop is to identify $\partial \mathbf{I}_n$ as one point, under this identification $\partial \mathbf{I}_n / \sim \cong \mathbb{S}^n$. Therefore an $n-$loop based at $x_0$ can also be defined as a continuous map $\gamma : \mathbb{S}^n \to X$ which maps a point $b \in \mathbb{S}^n$ to $x_0$.
- We define $\mathbf{I}_0$ as a set of a single point, and $\partial \mathbf{I}_0 = \emptyset$, therefore we can not say its base point.
- $\mathbf{I}_1 = [0,1]$ and $\partial \mathbf{I}_1 = \{0,1\}$, therefore a (1-)loop $\gamma$ requires $\gamma(0) = \gamma(1) = x_0$.

Similarly we can define homotopy of $n-$loops.

Definition 5.61 (**Homotopy of $n-$loops**). Two loops $\gamma$ and $\sigma$ based at $x_0 \in X$ are *homotopic* if there exists a continuous map $H : \mathbf{I}_n \times \mathbf{I} \to X$ such that

$$\begin{aligned} &H(t;0) = \gamma(t), \\ &H(t;1) = \sigma(t), \\ &H(t;s) = x_0, \text{ if } t \in \partial \mathbf{I}_n, \ t \in \mathbf{I}. \end{aligned} \tag{5.27}$$

Remark 5.62.
- A homotopy $H$ is a path in $\mathcal{C}[\mathbf{I}_n; X]$.
- Homotopy of $n-$loops is an equivalent relation.

All operations of loops can also be generalized to $n-$loops.

Definition 5.63 (**Constant $n-$loop**). A *constant* $n-$loop $e : \mathbf{I}_n \to$ based at $x_0$ is the constant map $e(\mathbf{I}_n) = x_0$.

One can also define the product of $n-$loops.

Definition 5.64 (**Product of $n-$loops**). Let $\gamma$ and $\sigma$ be two $n-$loops at $x_0 \in X$. Then their *product loop* $\rho = \sigma * \gamma$ is an $n-$loop at $x_0 \in X$ defined as

$$\rho(t_1, \cdots, t_n) = \begin{cases} \gamma(2t_1, t_2, \cdots, t_n), & 0 \le t_1 \le 1/2, \\ \sigma(2t_1 - 1, t_2, \cdots, t_n), & 1/2 \le t_1 \le 1. \end{cases} \tag{5.28}$$

Remark 5.65.
- A product of $2-$loops is illustrated in figure 5.5.
- Actually one can pick any direction $t_i$ in the definition, and each definition is homotopic to each other.

Definition 5.66 (**Inverse $n-$loop**). The *inverse $n-$loop* $\gamma^- : \mathbf{I}_n \to X$ of an $n-$loop $\gamma$ at $x_0$ is the map

$$\gamma^-(t_1, t_2, \cdots, t_n) = \gamma(1 - t_1, t_2, \cdots, t_n). \tag{5.29}$$

The product and inverse of $n-$loops are also well-defined on homotopy classes.

Now we can define the higher dimensional generalization of the fundamental group.

Definition 5.67 ($n-$**homotopy group**). Let $\pi_n(X, x_0)$ be the set of homotopy classes of all $n-$loops based at $x_0$. $(\pi_n(X, x_0), *)$ forms a group under the product $*$ of $n-$loops.

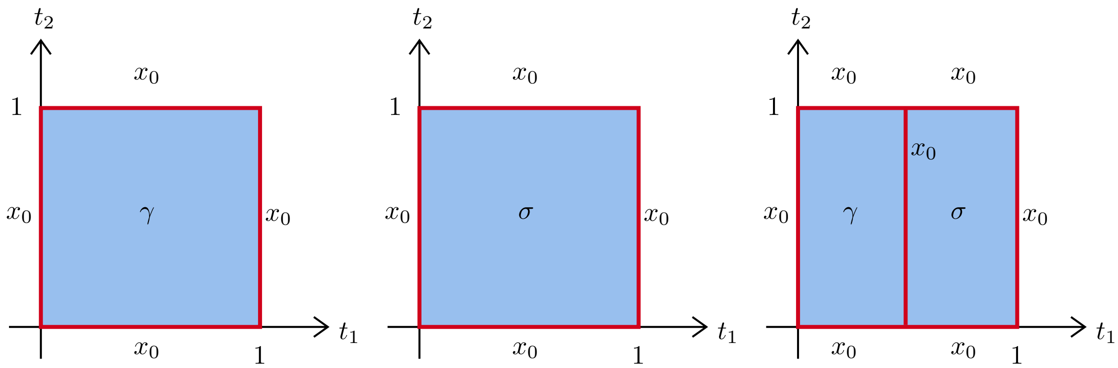


FIGURE 5.5. Left: the 2−loop $\gamma$. Middle: the 2−loop $\sigma$. Right: the product 2−loop $\rho = \gamma * \sigma$.

REMARK 5.68.
- The identity $[e]$ of $\pi_n(X, x_0)$ is the homotopy class of the constant $n$−loop based at $x_0$.
- The inverse of $[\gamma]$ is $[\gamma^-]$.
- Since $\partial \mathbf{I}_0$ is empty, we do not specify a base point $x_0$ in $\pi_0(X)$.

PROPOSITION 5.69. *If two path-connected topological spaces $X$ and $Y$ are homotopy equivalent, then*

$$\pi_n(X, x_0) \simeq \pi_n(Y, y_0), \tag{5.30}$$

*i.e. $\pi_n(X, x_0)$ is a homotopy invariant.*

**5.5.1. Properties of higher homotopy groups.** A lot of properties of fundamental groups can be generalized to higher homotopy groups. For example

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PROPOSITION 5.70. *If $X$ is a path-connected topological space, then*

$$\pi_n(X, x_0) \simeq \pi_n(X, y_0), \ \forall \ x_0, y_0 \in X. \tag{5.31}$$

REMARK 5.71. We sometimes omit $x_0$ in $\pi_n(X, x_0)$ when $X$ is path-connected.

PROPOSITION 5.72. *If $A \subset X$ is a deformation retract of $X$, then*

$$\pi_n(X, a) \simeq \pi_n(A, a), \forall \ a \in A. \tag{5.32}$$

PROPOSITION 5.73. *Let $X$ and $Y$ be path-connected topological spaces,*

$$\pi_n(X \times Y, x_0 \times y_0) \simeq \pi_n(X, x_0) \times \pi_n(Y, y_0). \tag{5.33}$$

However, there is a crucial difference between higher homotopy groups and the fundamental group.

PROPOSITION 5.74. *The $n$−homotopy group $\pi_n(X, x)$ is abelian for $n > 1$.*

EXAMPLE 5.75. $\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$ for $n \geq 1$.

EXAMPLE 5.76 (**Homotopy groups of $\mathbb{S}^n$**). We list some homotopy groups of $\mathbb{S}^n$ in table 1.

EXAMPLE 5.77. Homotopy groups of $\mathbb{R}\mathrm{P}^n$.
- $\pi_1(\mathbb{R}\mathrm{P}^2) \simeq \mathbb{Z}_2$.
- $\pi_n(\mathbb{R}\mathrm{P}^n) \simeq \pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$ for $n \geq 2$.

$\pi_n(X, x)$ $n \geq 2$ 时, Abel Group.
乘法交换

$\pi_n(S^m) \simeq 0$ if $m > n$

2 > 1：在球面套绳子

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ |
|---|---|---|---|---|---|---|
| $\mathbb{S}^0$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbb{S}^1$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 |
| $\mathbb{S}^2$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ |
| $\mathbb{S}^3$ | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ |
| $\mathbb{S}^4$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\mathbb{S}^5$ | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ |

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TABLE 1. Homotopy groups of $\mathbb{S}^n$. $\mathbb{Z}/p$ is the cyclic group of order $p$.

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ |
|---|---|---|---|---|---|---|
| $U(1) \cong \mathbb{S}^1$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 |
| $SU(2) \cong \mathbb{S}^3$ | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ |
| $SU(3)$ | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_6$ |
| $SU(n)$, $n > 3$ | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| $SO(3) \cong \mathbb{RP}^3$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ |
| $SO(4)$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{12}^2$ |
| $SO(5)$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 |
| $SO(6)$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | 0 |
| $SO(n)$, $n > 6$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| $E_6$ | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| $E_7$ | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| $E_8$ | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| $G_2$ | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}_3$ |
| $F_4$ | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

TABLE 2. Some homotopy groups of Lie groups.

- $\pi_n(\mathbb{RP}^3) \simeq \pi_n(SO(3)) \simeq \pi_n(\mathbb{S}^3) \simeq \pi_n(SU(2))$ for $n \geq 2$.

In general we summarize the homotopy groups of $\mathbb{RP}^n$ as

$$\pi_k(\mathbb{RP}^n) = \begin{cases} \{e\} & k = 0, \\ \mathbb{Z} & k = 1,\ n = 1, \\ \mathbb{Z}_2 & k = 1,\ n > 1, \\ \pi_k(\mathbb{S}^n) & k > 1,\ n > 0. \end{cases} \tag{5.34}$$

EXAMPLE 5.78 (**Homotopy groups of Lie groups**). We list homotopy groups of some Lie groups in table 2. There are some interesting facts regarding the table 2.

- The *Bott periodicity theorem* states that

$$\pi_k(U(n)) \simeq \pi_k(SU(n)) \simeq \begin{cases} \{e\} & k \text{ even}, \\ \mathbb{Z} & k \text{ odd}, \end{cases} \tag{5.35}$$

  for $n \geq (k+1)/2$. Similarly,

$$\pi_k(O(n)) \simeq \pi_k(SO(n)) \simeq \begin{cases} \{e\} & \text{if } k \equiv 2, 4, 5, 6 \pmod 8, \\ \mathbb{Z}_2 & \text{if } k \equiv 0, 1 \pmod 8, \\ \mathbb{Z} & \text{if } k \equiv 3, 7 \pmod 8, \end{cases} \tag{5.36}$$

  for $n \geq k + 2$.