The Haldane Phase as a Symmetry-Protected Topological Phase and Quantum Entanglement

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Haldane "Conjecture"

Heisenberg antiferromagnetic chain

$$\mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$$

S=1/2, 3/2, 5/2.......

"massless" = gapless, power-law decay of spin correlations

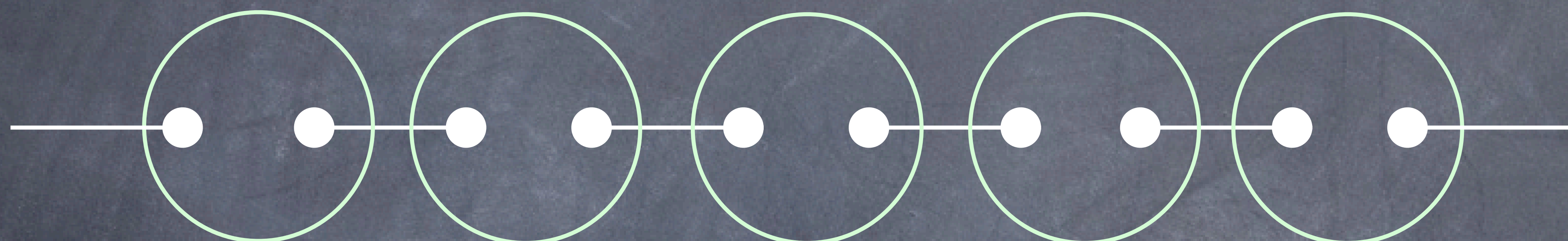
S=1, 2, 3, ...

"massive" = non-zero gap, exponential decay of spin correlations

AKLT model/state

e.g. S=1 $$\mathcal{H} = J\sum_j \left[\vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}(\vec{S}_j \cdot \vec{S}_{j+1})^2\right]$$

Exact groundstate: (Affleck-Kennedy-Lieb-Tasaki 1987)



• S=1/2

Symmetrization (=projection to S=1)

•—• Singlet pair of two S=1/2's -"valence bonds"

✓non-zero gap, exponential decay of correlations

(supporting the Haldane conjecture)

Order in AKLT state?

Groundstate of the AKLT model: UNIQUE

(for periodic boundary condition)

Correlation function of any local operator decays exponentially

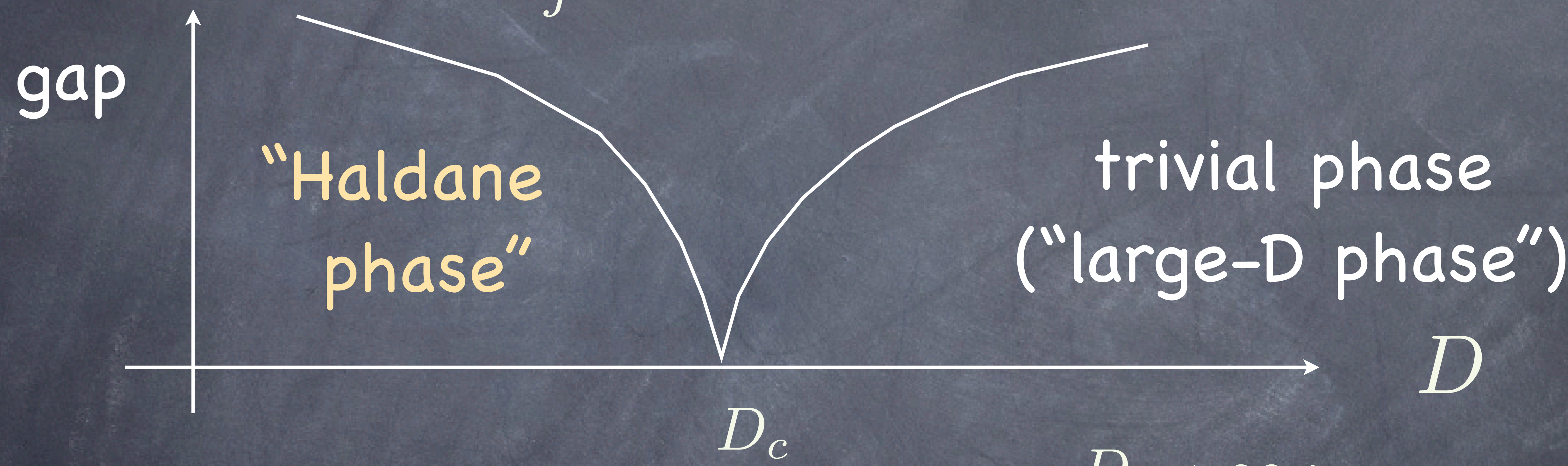
⇓

There is no local order parameter; no symmetry is broken spontaneously

No order, that's it?

“Haldane phase”

$$\mathcal{H} = J \sum_j \left(\vec{S}_j \cdot \vec{S}_{j+1} + D (S^z_j)^2 \right)$$



$D \to \infty$:

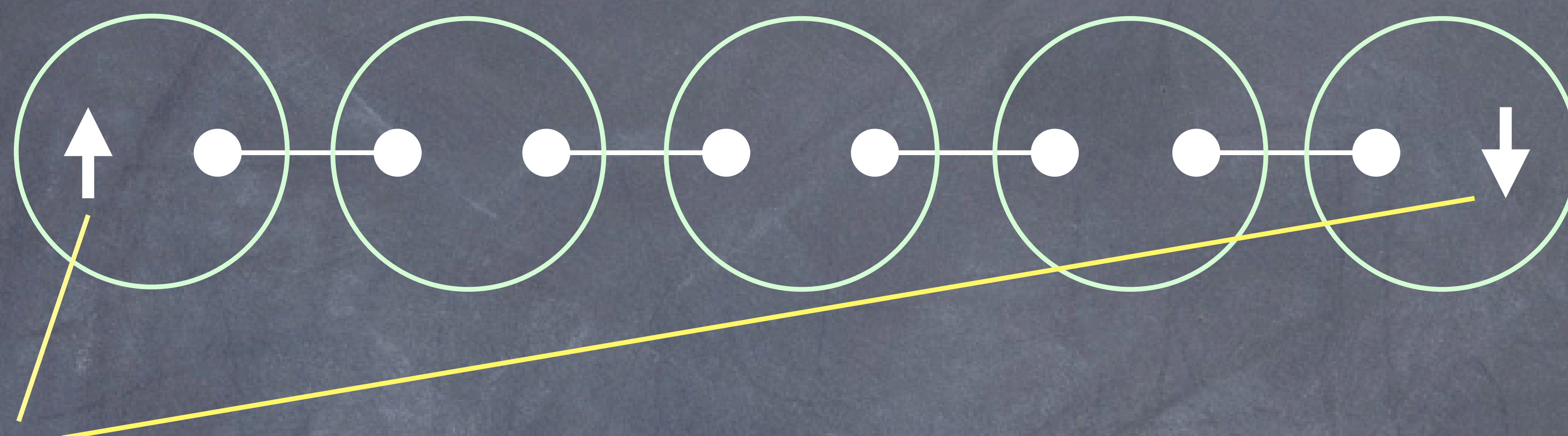
$|\mathcal{D}\rangle = |\ldots 00000 \ldots\rangle$

Why quantum phase transition?

Because there is some kind of order (topological order) in the “Haldane phase”

Edge states

Consider a chain with open boundary condition

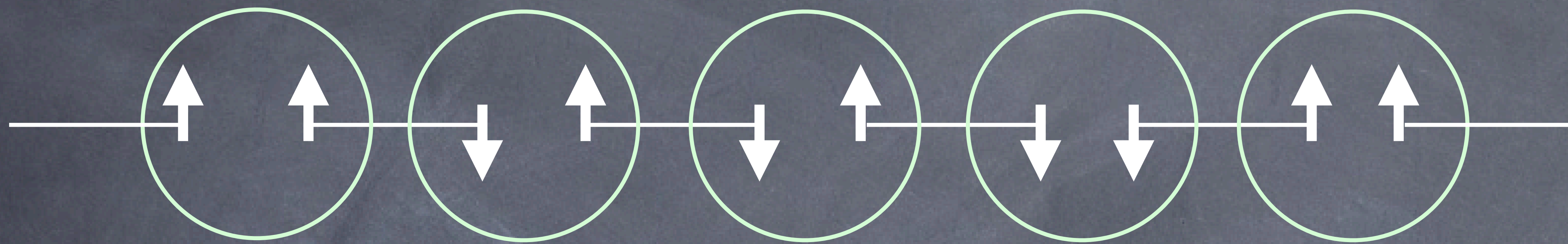


"free" S=1/2 appears at each end, interacting with each other. Effective coupling: $J_{\mathrm{eff}} \sim e^{-L/\xi}$

2x2=4 groundstates below the Haldane gap (nearly degenerate)

Kennedy (1990)

Hidden (string) order



+ 0 0 - +

In S^z basis, + and - alternate, with 0's in between

No long-range order w.r.t. local observables, but a hidden (topological) order measurable by the "string order parameter"

$$\mathcal{O}^{\alpha}_{\mathrm{str}} \equiv \lim_{|j-k|\to\infty} \langle S^{\alpha}_j e^{i\pi \sum_{l=j}^{k-1} S^{\alpha}_l} S^{\alpha}_k \rangle$$

Den Nijs & Rommelse (1989)

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Hidden $Z_2 \times Z_2$ symmetry

$$\mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$$

Kennedy & Tasaki (1992)

non-local unitary transformation

$$U = \prod_{j<k} e^{i\pi S_j^z S_k^x}$$

[simple expression by M.O. (1992)]

[well-defined only for open b.c.]

$$\begin{aligned} \tilde{\mathcal{H}} &= U\mathcal{H}U^{-1} \\ &= J\sum_j \left[S_j^x e^{i\pi S_{j+1}^x} S_{j+1}^x + S_j^y e^{i\pi (S_j^z + S_{j+1}^x)} S_{j+1}^y + S_j^z e^{i\pi S_j^z} S_{j+1}^z \right] \end{aligned}$$

Global discrete symmetry

(π-rotation about x, y, z axes = $Z_2 \times Z_2$)

Spontaneous breaking of hidden $Z_2 \times Z_2$ symmetry

4-fold groundstate degeneracy for $\tilde{\mathcal{H}}$

$$U \mathcal{H} U^{-1} = \tilde{\mathcal{H}}$$

4-fold groundstate degeneracy for $\mathcal{H}$

only with the open b.c.! = edge states

Ferromagnetic order for $\tilde{\mathcal{H}}$

$$U \left(S_j^z e^{i\pi \sum_{l=j}^{k-1} S_l^z} S_k^z \right) U^{-1} = S_j^z S_k^z$$

String order for $\mathcal{H}$

When does it work?

This was not discussed (as far as I know) in 1990s

The Kennedy-Tasaki transformation is nonlocal --
if the transformed Hamiltonian $\tilde{\mathcal{H}}$ is nonlocal, the argument does not work.

Because the transformation is self-dual, for $\tilde{\mathcal{H}}$ to be local, the original Hamiltonian must have **global** $D_2 = Z_2 \times Z_2$ symmetry (π-rotation about x, y, z axes)

This means that S=1 Haldane phase is a topological phase protected by global D_2 symmetry

Pollmann, Berg, Turner, M.O. 2009

Other symmetries?

AKLT model: edge state with S=1/2

Does the edge state survive in more general models?

Consider perturbations to AKLT model

Generic perturbations will lift the edge degeneracy!

However, if the perturbation respect time reversal, it should keep the "Kramers degeneracy" of S=1/2 edge state

i.e. time reversal symmetry protects Haldane phase

cf.) edge state of topological insulator

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Yet another symmetry

$$\mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + B_x \sum_j S_j^x + D \sum_j (S_j^z)^2$$

Gu and Wen, 2009

D_2 symmetry (π-rotation about x,y,z axes): lost

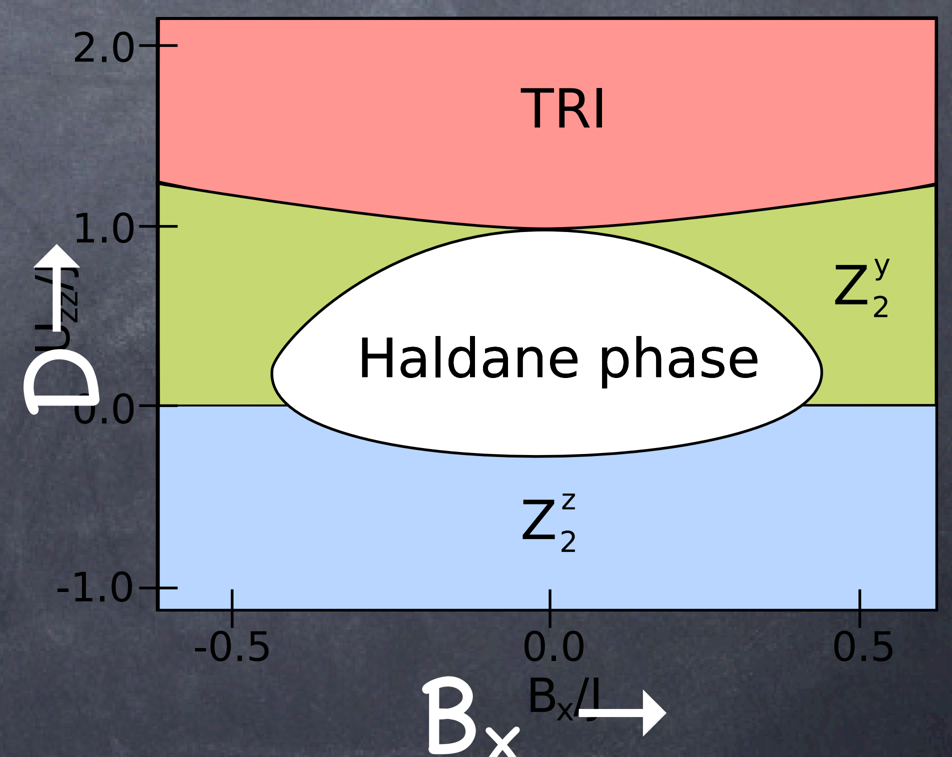
string order does not work as an order parameter

Time reversal: lost

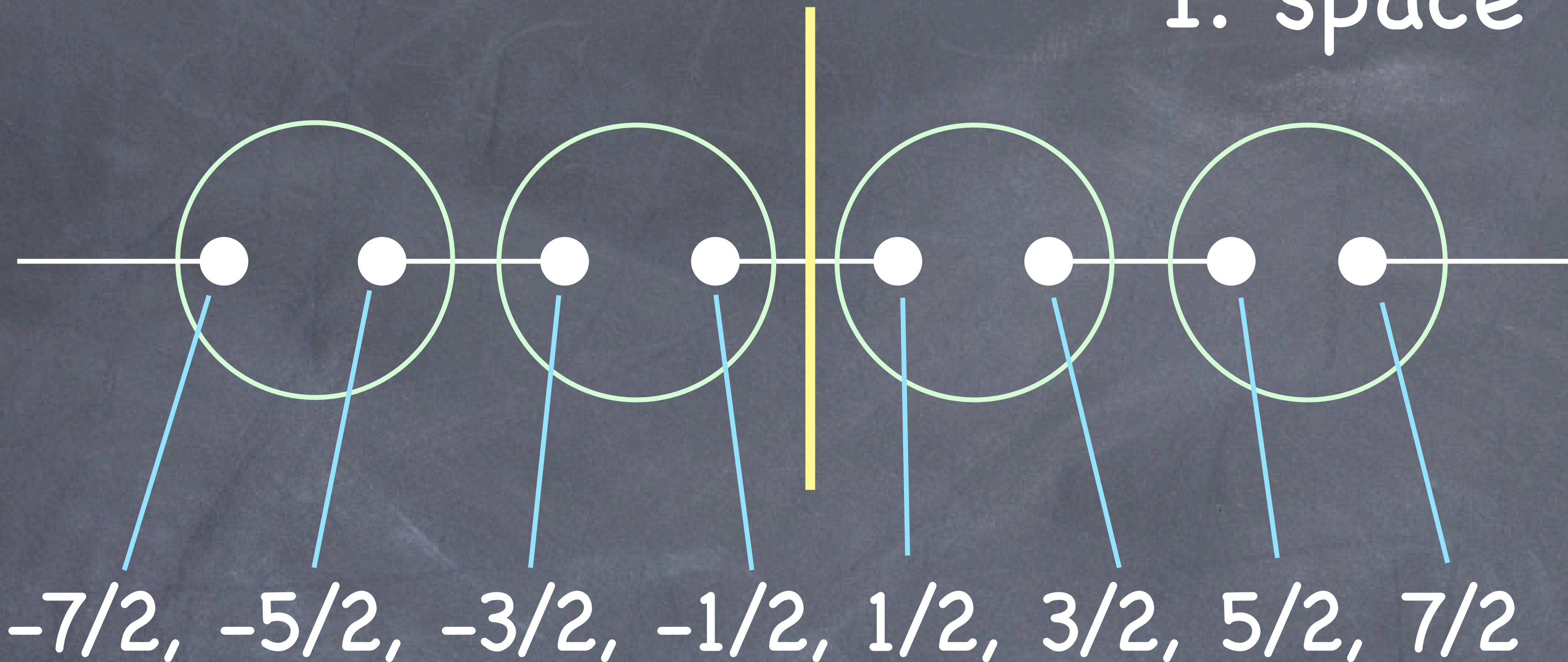
edge state does not characterize the Haldane phase

Nevertheless, Haldane phase is still distinct from other phases by QPTs

Protected by inversion symmetry!



I: space inversion (parity)



valence bond: $$|(j,j+1)\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_j|\downarrow\rangle_{j+1} - |\downarrow\rangle_j|\uparrow\rangle_{j+1}\right)$$

$$\mathcal{I}|(j,j+1)\rangle = |(-j,-j-1)\rangle = -|(-j-1,-j)\rangle$$

$$\mathcal{I}|(-\frac{1}{2},\frac{1}{2})\rangle = -|(-\frac{1}{2},\frac{1}{2})\rangle$$

this vb makes AKLT state P-odd!

$$\mathcal{I}|(-\frac{5}{2},-\frac{3}{2})\rangle|(\frac{3}{2},\frac{5}{2})\rangle = |(-\frac{5}{2},-\frac{3}{2})\rangle|(\frac{3}{2},\frac{5}{2})\rangle$$

each vb pair is P-even

S=1 AKLT state is "P-odd".

Now consider any perturbation, keeping P-invariance. The adiabatically connected state remains P-odd.

There must be a phase transition between the two groundstates (robustness of Haldane phase)

On the other hand, a trivial groundstate is P-even.

$$|\Psi\rangle = \otimes_j |\psi\rangle_j$$

Any adiabatic evolution of the trivial state is also P-even as long as P-invariance is kept.

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Symmetry Protection

S=1 Haldane phase is "protected" by ANY one of

(I) Spontaneous breaking of hidden $Z_2 \times Z_2$ symmetry, robust in the presence of $D_2(=Z_2 \times Z_2)$ symmetry [π-rotation about x,y, and z axes]

(II) Kramers degeneracy of edge spins, robust in the presence of time-reversal

(III) Space Inversion symmetry about a bond center

(Gu-Wen/Pollmann-Berg-Turner-M.O.)

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What about S>1?

The concept of hidden $Z_2 \times Z_2$ symmetry can be generalized to any integer S

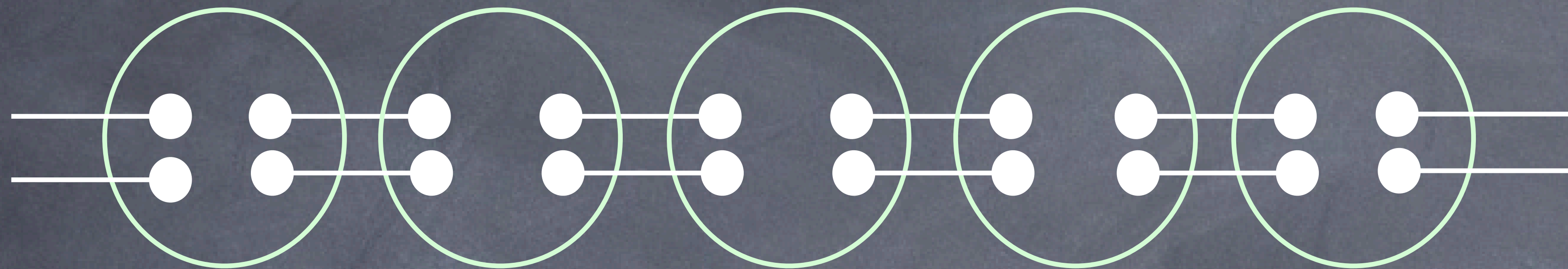
$$U = \prod_{j<k} e^{i\pi S_j^z S_k^x}$$

[M.O. (1992)]

The hidden $Z_2 \times Z_2$ symmetry is unbroken in S=2,4,6,8,.... AKLT state while broken in S=1,3,5,7,..... AKLT state!

"even-odd effect"

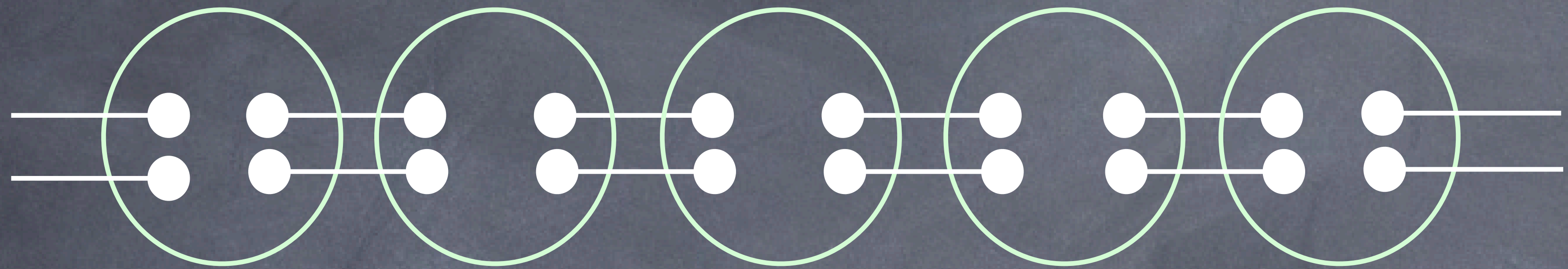
What does it mean?



The hidden $Z_2 \times Z_2$ symmetry is unbroken in the (uniform) S=2 AKLT state.

Q (1992): Is it indistinguishable from a trivial state, or are we just unaware of appropriate hidden order/symmetry?

What does it mean?



The hidden $Z_2 \times Z_2$ symmetry is unbroken in the (uniform) S=2 AKLT state.

Q (1992): Is it indistinguishable from a trivial state, or are we just unaware of appropriate hidden order/symmetry?

A (2009): It is essentially indistinguishable from a trivial state!

Edge state for S=2

S=2 AKLT state: each end has S=1 (3-fold deg.)
The degeneracy will be lifted by perturbations, and generically no degeneracy remains!
(no Kramers degeneracy)

If we keep the SU(2) symmetry, the presence of the S=1 edge state makes the system distinct from trivial states?
In general, the answer is NO.

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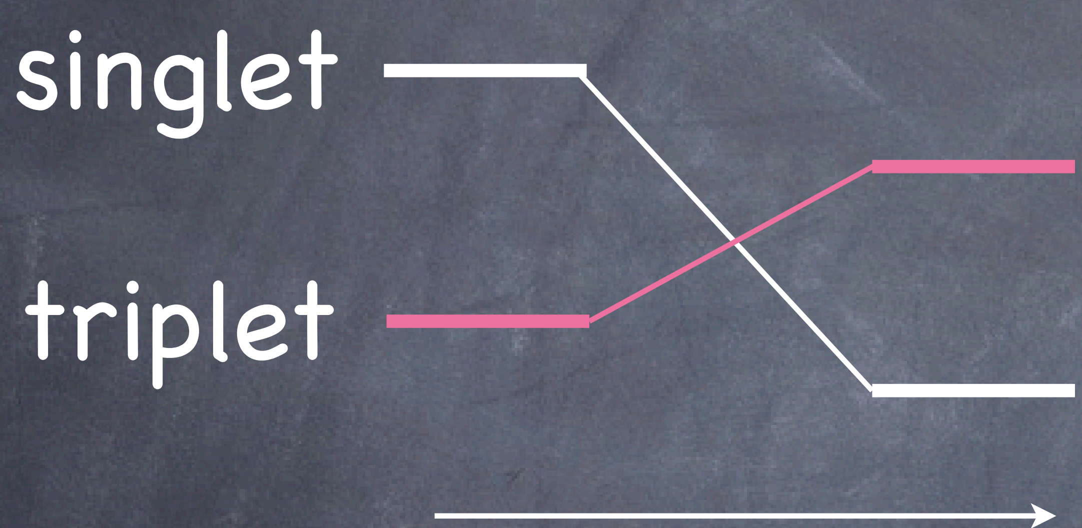
Kramers vs. non-Kramers

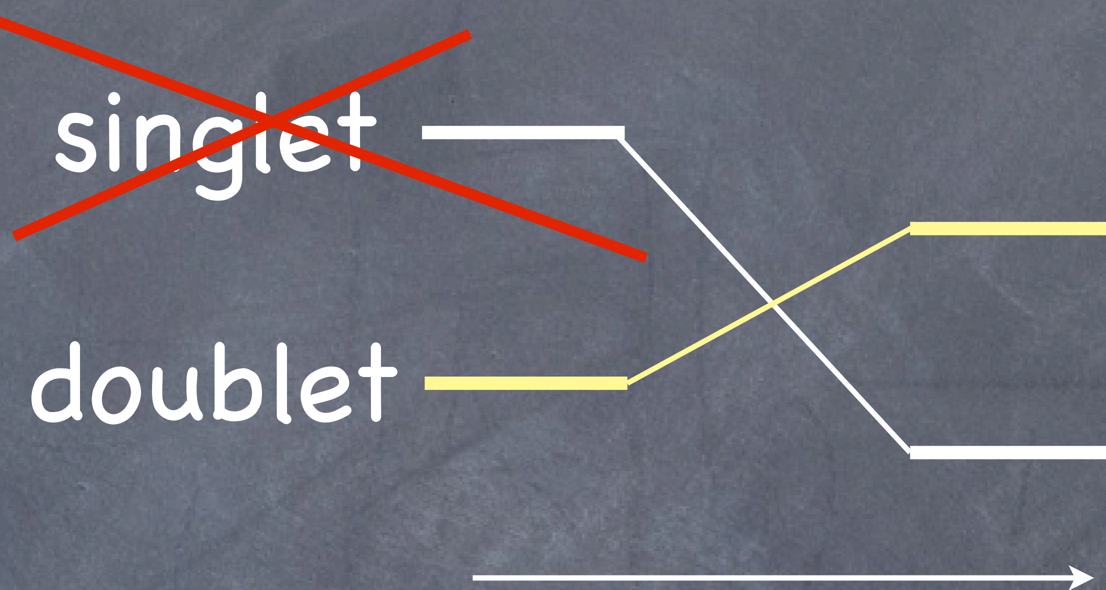
S_b=0 vs. 1



S_b can change by level crossing at the edge (w/o bulk transition)

cf.) Todo et al. (2001)

S_b=0 vs. 1/2

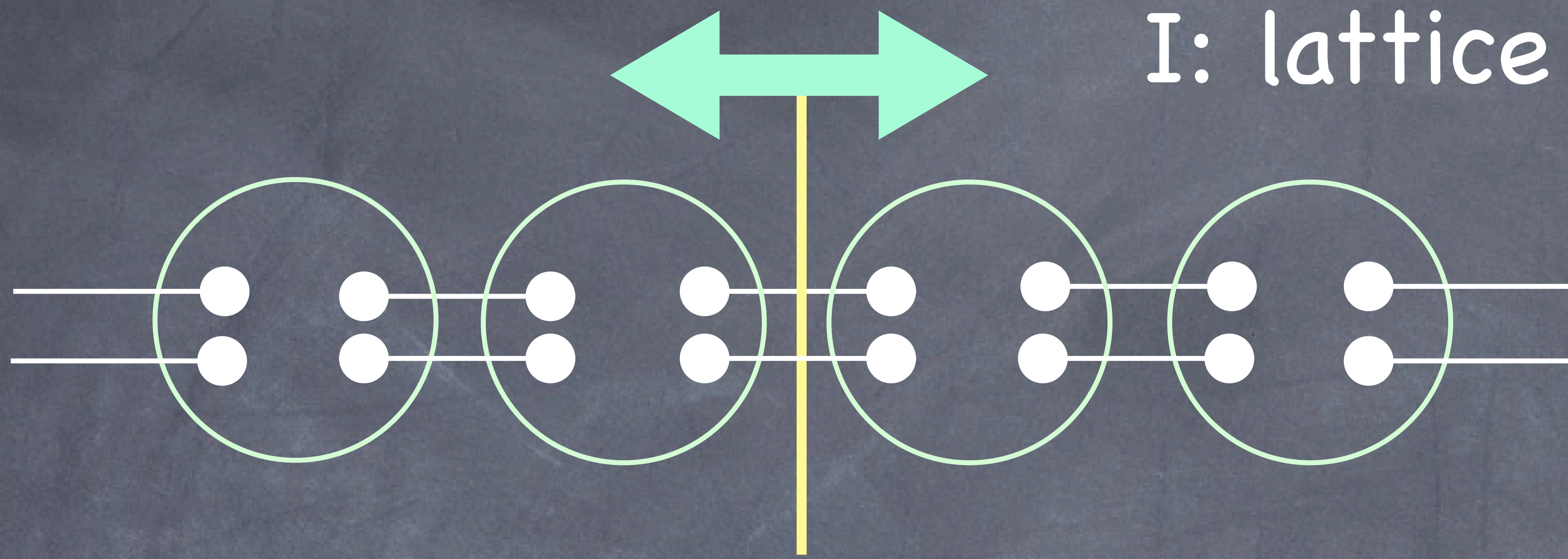


Kramers theorem requires all the edge levels be doubly degenerate!

The degeneracy can be only removed by bulk phase transition.

Intrinsic parity for S>1 chains

I: lattice inversion



The intrinsic parity is even,
because you flip two valence bonds.

In general, intrinsic link parity is even (odd),
if the number of valence bonds is even (odd)!

S=2 Haldane state

None of the 3 symmetries protects the "Haldane phase" as a distinct topological phase!

The S=2 "Haldane state" could be adiabatically connected to a trivial state
Is this really the case?

Yes! There exists a 1-parameter family of Matrix Product State (and corresponding Hamiltonian) interpolating S=2 AKLT state and large-D state

Pollmann, Berg, Turner, M.O. 2009

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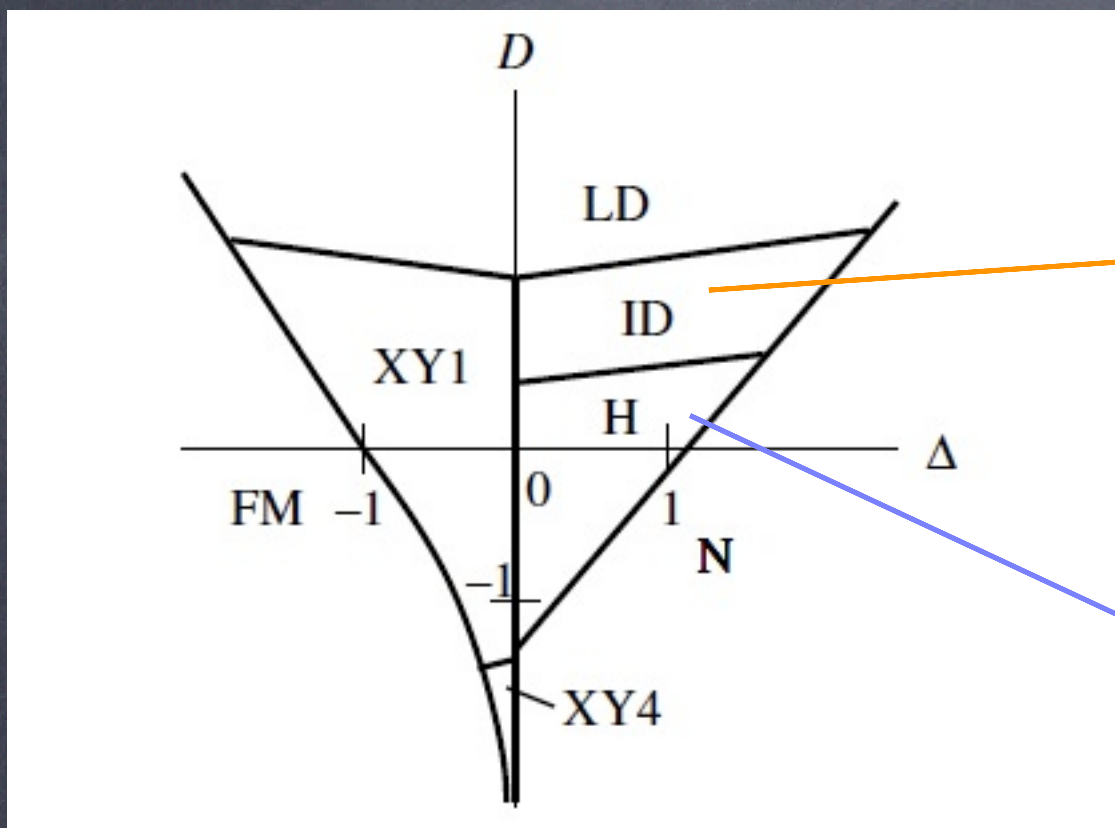
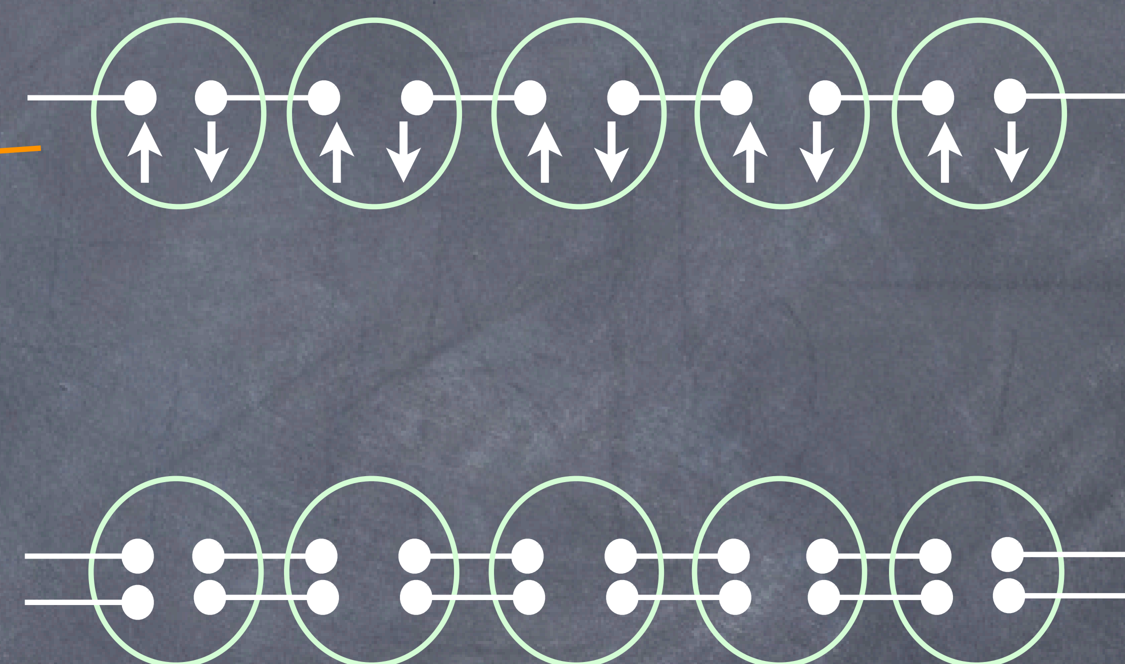
S=2 phase diagram

$$\mathcal{H} = J\sum_j \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z\right] + D\sum_j (S_j^z)^2$$

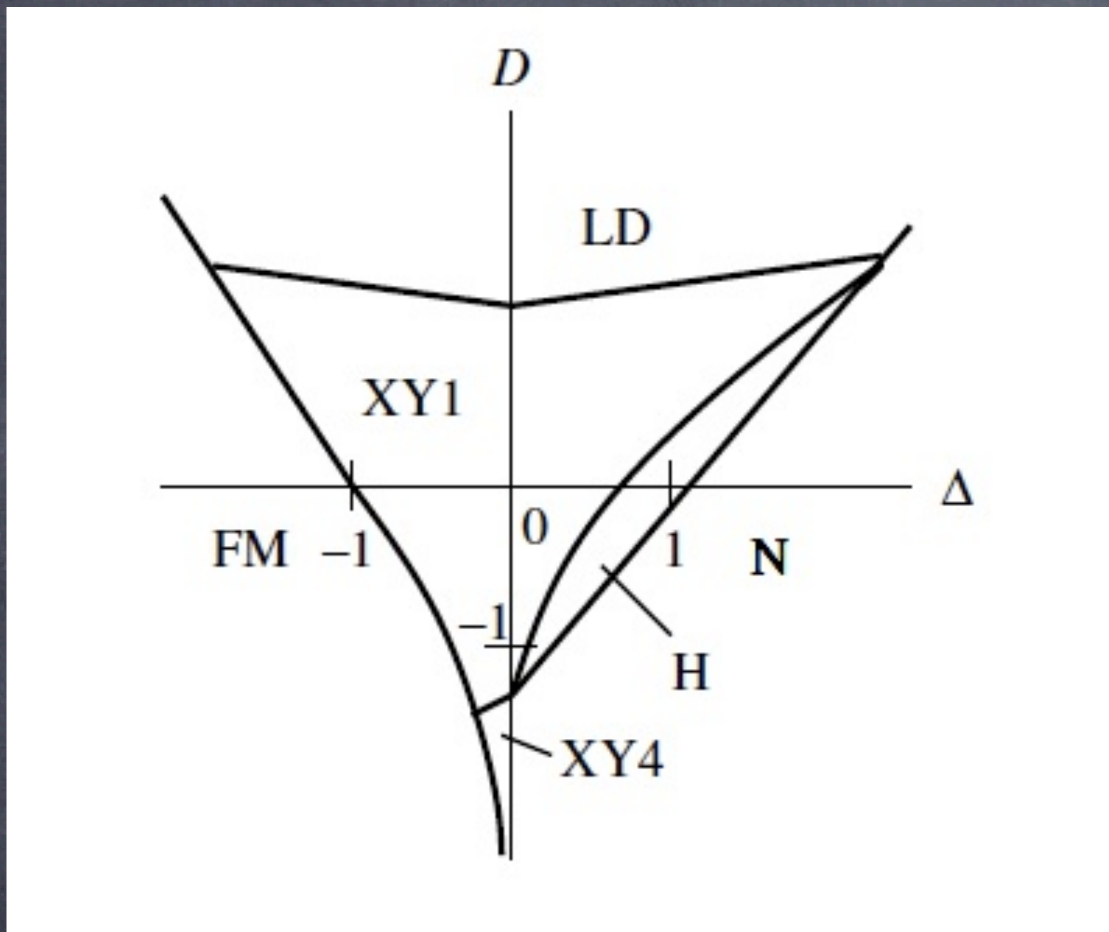




conjecture by M.O. 1992

(figure taken from Tonegawa et al. arXiv:1011.6568)

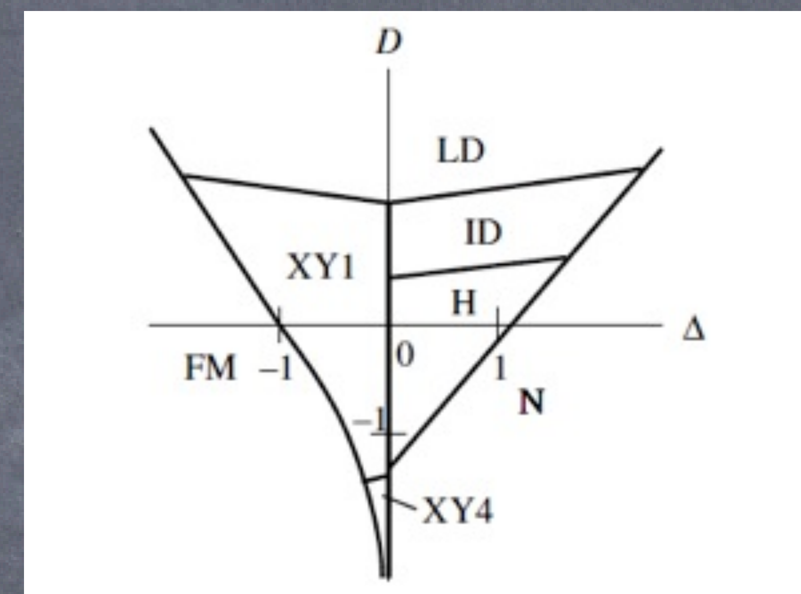
S=2 phase diagram



DMRG result

Schollwock et al.

1995～1996



conjecture

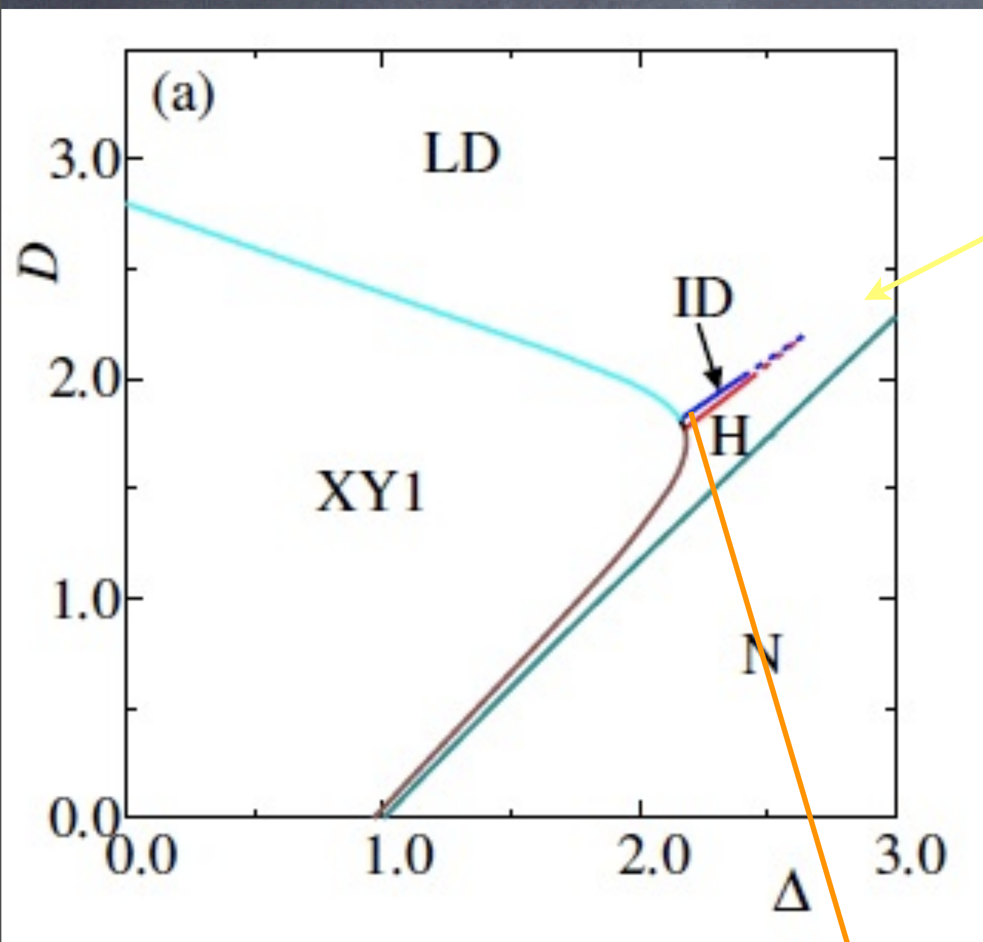
M.O. 1992

(figures from arXiv:1011.6568)

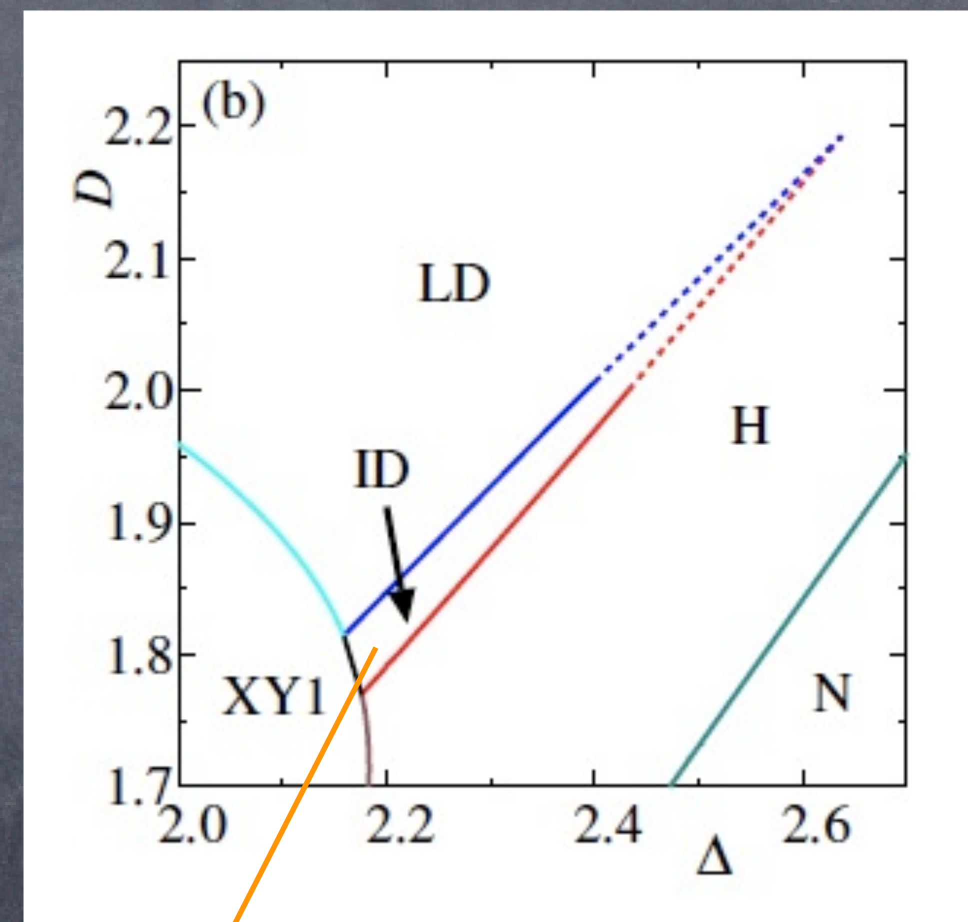
S=2 phase diagram

Tonegawa et al. arXiv:1011.6568

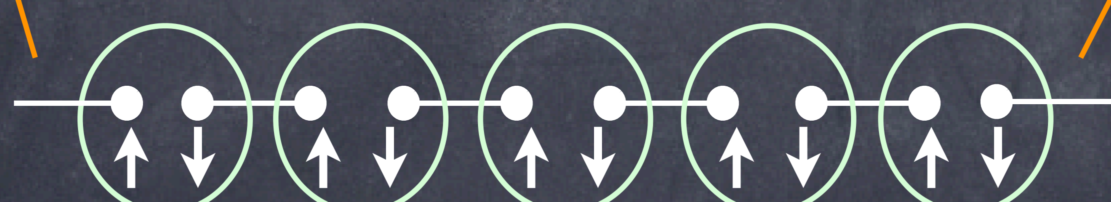
exact diagonalization + "level spectroscopy"
(finite size scaling using CFT+perturbation)



Haldane state connected to Large-D (triviall) phase?



"ID phase" (topological)?

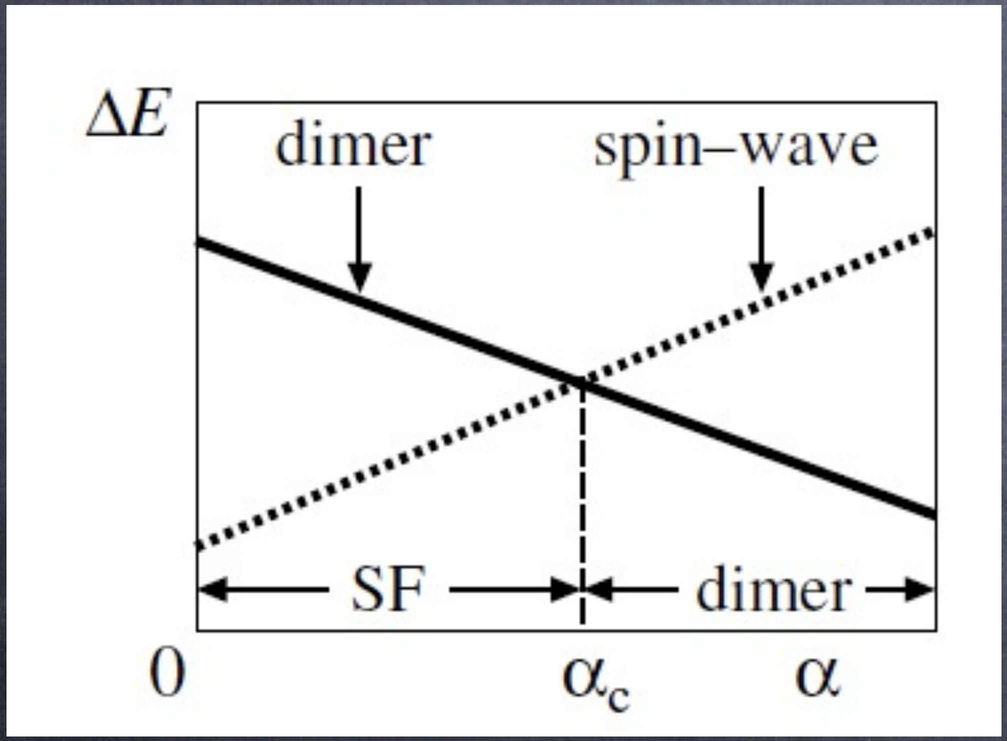


Level spectroscopy

Okamoto and Nomura 1992

e.g. $$\mathcal{H} = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \alpha \vec{S}_j \cdot \vec{S}_{j+2}$$ S=1/2

KT transition between fluid (TL liquid) phase and spontaneously dimerized phase



crosing of excited states

⇔ phase transition

size dependence of crossing point is weak, despite KT!

$\alpha_c \sim 0.2411$

crossing point is α=0.25 already for 4 spins!

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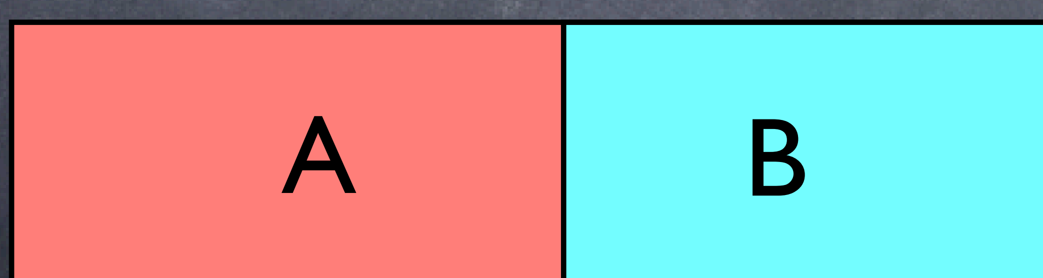
Haldane phase (odd S)

Topological phase protected by (any one of) 3 symmetries

Different mechanism for each symmetry?

Is there a "universal feature" of the Haldane phase?

Entanglement!



$$|\Psi\rangle \neq |\psi_a\rangle_A |\psi_b\rangle_B$$

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Entanglement Spectrum

$$|\Psi\rangle = \sum_{a=1,N_A} \sum_{b=1,N_B} \Psi_{ab} |\psi_a\rangle_A |\psi_b\rangle_B$$

Ψ_{ab} N_A x N_B matrix

Singular Value Decomposition

unitary matrices

$$\Psi = U D V^\dagger$$

N_A x N_B diagonal matrix

Schmidt decomposition

$$|\Psi\rangle = \sum_j \Lambda_j |\phi_j\rangle_A |\phi'_j\rangle_B$$

$$\rho_A = \sum_j {\Lambda_j}^2 |\phi_j\rangle_{AA} \langle \phi_j|$$

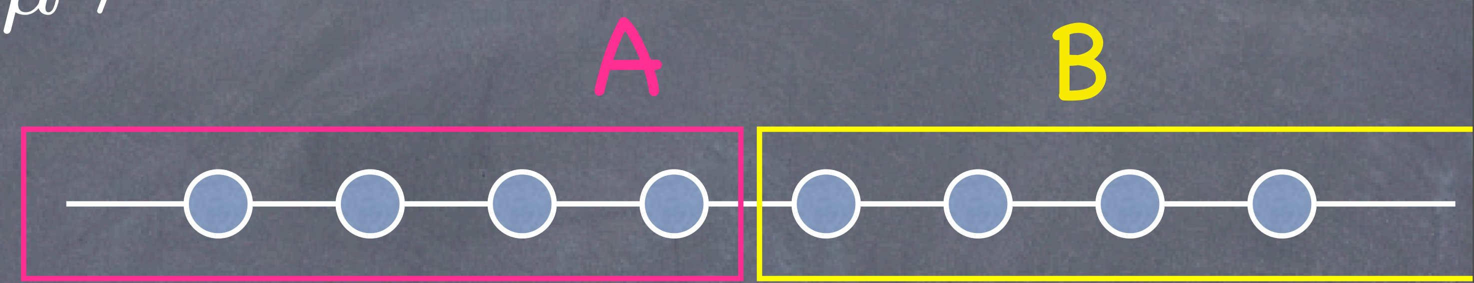
Entanglement Entropy

$\{\Lambda_j\}$ Entanglement Spectrum $\underset{\cancel{\Longleftarrow}}{\Longrightarrow}$ $S_E = -\sum_j {\Lambda_j}^2 \log {\Lambda_j}^2$

Entanglement spectrum contains more information than entanglement entropy!

Entanglement Spectrum

$$|\Psi\rangle = \sum_{\mu} \Lambda_{\mu} |\phi_{\mu}^{A}\rangle_{A} |\phi_{\mu}^{B}\rangle_{B}$$



The entire entanglement spectrum has exact double degeneracy in the Haldane phase!

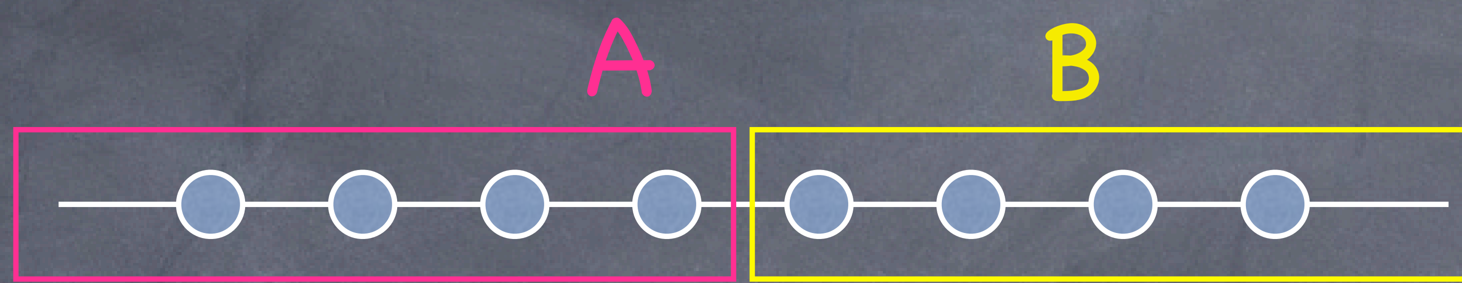
$$\Lambda_1 = \Lambda_2, \Lambda_3 = \Lambda_4, \Lambda_5 = \Lambda_6, \ldots$$

This degeneracy is protected by any one of the three symmetries.

Minimal entanglement entropy log(2)

when $\Lambda_1 = \Lambda_2 = 1/\sqrt{2}, \Lambda_\alpha = 0 (\alpha \geq 3)$

“Odd parity” state



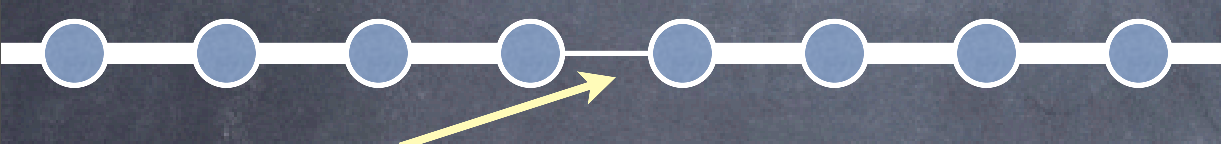
$$|\Psi\rangle \sim \sum_{\alpha} \lambda^{(\alpha)} \left(|\alpha, 1\rangle_A |\overline{\alpha, 2}\rangle_B - |\alpha, 2\rangle_A |\overline{\alpha, 1}\rangle_B \right)$$

Exact two-fold degeneracy in the entire entanglement spectrum

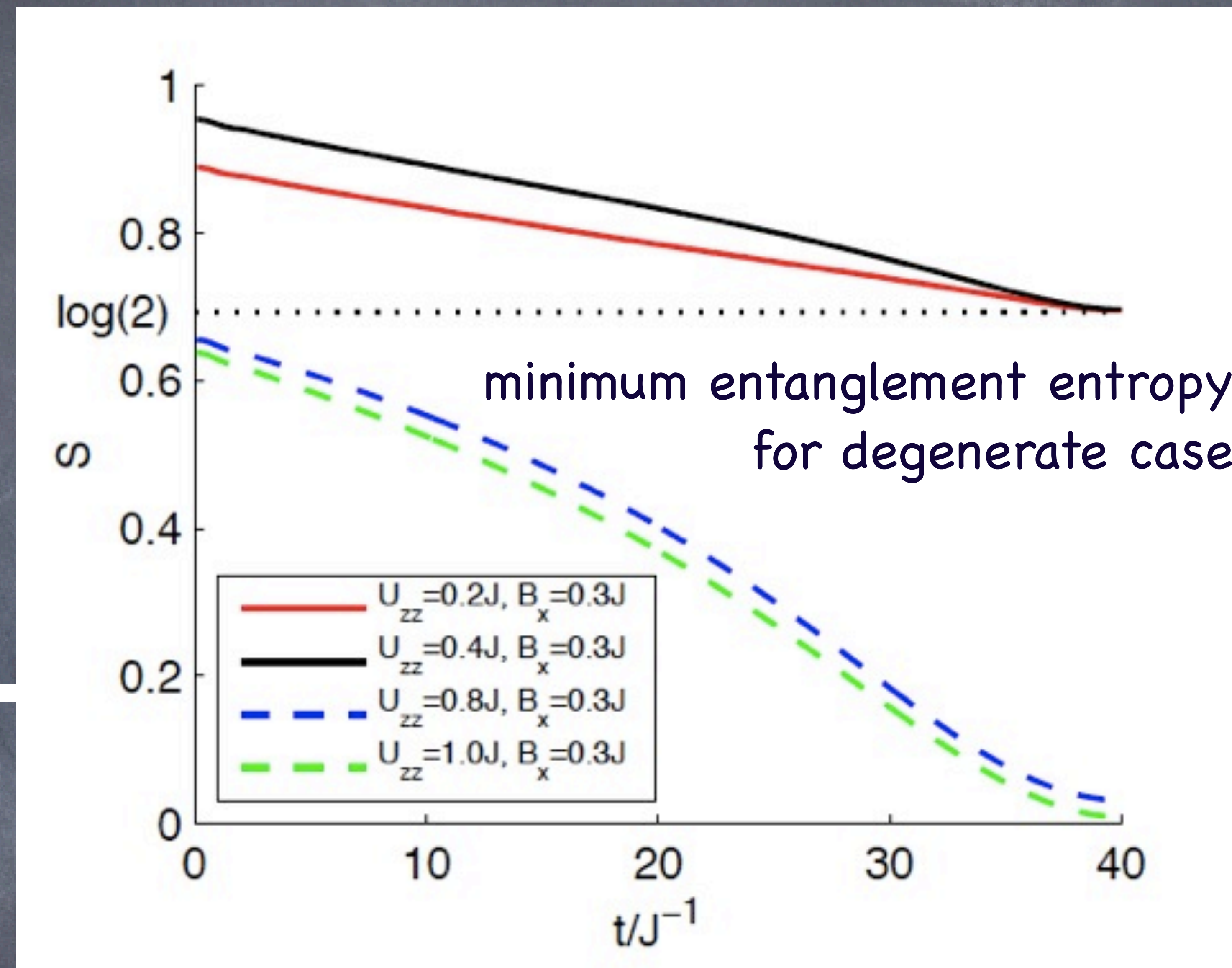
Time evolution

We introduced adiabatic weakening of one bond

$$J_0(t)\vec{S}_0 \cdot \vec{S}_1 \qquad J_0(t) \to 0$$



weakened bond



Inversion symmetry of the Hamiltonian is kept
(although translation symmetry is lost)

The degeneracy of the spectrum survives
in the Haldane phase!

Degeneracy is universal

TABLE I. The different symmetries which can stabilize the Haldane phase. For each class of symmetries, the table shows whether string order, edge states, or the degeneracy of the entanglement spectrum are necessarily present. The symmetry under π rotations about a pair of orthogonal axes is represented by the dihedral group D_2.

Symmetry	String order	Edge states	Degeneracy
D_2 ($=Z_2 \times Z_2$)	Yes	Yes	Yes
Time reversal	No	Yes	Yes
Inversion	No	No	Yes

Summary

Odd # of valence bonds

[D_2 symmetry]

Hidden $Z_2 \times Z_2$ symmetry breaking (dNR string order)

[inversion symmetry]

[time-reversal invariance]

Kramers degeneracy of edge spins

odd intrinsic link parity

Exact double degeneracy of entire entanglement spectrum

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