# Lieb-Schultz-Mattis Theorem in Open Quantum Systems

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The Lieb-Schultz-Mattis (LSM) theorem provides a general constraint on quantum many-body systems and plays a significant role in the Haldane gap phenomena and topological phases of matter. Here, we extend the LSM theorem to open quantum systems and establish a general theorem that restricts the steady state and spectral gap of Liouvillians based solely on symmetry. Specifically, we demonstrate that the unique gapped steady state is prohibited when translation invariance and U(1) symmetry are simultaneously present for noninteger filling numbers. As an illustrative example, we find that no dissipative gap is open in the spin-$1/2$ dissipative Heisenberg model, while a dissipative gap can be open in the spin-1 counterpart—an analog of the Haldane gap phenomena in open quantum systems. Furthermore, we show that the LSM constraint manifests itself in a quantum anomaly of the dissipative form factor of Liouvillians. We also find the LSM constraints due to symmetry intrinsic to open quantum systems, such as Kubo-Martin-Schwinger symmetry. Our work leads to a unified understanding of phases and phenomena in open quantum systems.

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Spectral gaps are pivotal for characterization of quantum phases of matter [1,2]. While it is generally nontrivial to find out the ground state and spectral gap in quantum many-body systems, universal ingredients such as symmetry can provide their general understanding. The Lieb-Schultz-Mattis (LSM) theorem [3] is a prime example, which generally shows that the symmetry-preserving unique gapped ground state is prohibited for noninteger filling numbers in the simultaneous presence of translation invariance and U(1) symmetry [4,5]. This theorem resolves a part of the Haldane conjecture [6–9], prohibiting the gapped ground state in quantum spin chains with half-integer spins $S = 1/2, 3/2, \ldots$. For integer spins $S = 0, 1, 2, \ldots$, on the other hand, no such general constraints are imposed, and a gap can be open—the Haldane gap. The LSM theorem has been further generalized to higher dimensions [10–13], other symmetries [14–19], and fermionic systems [20–22]. The LSM constraint is also a manifestation of a quantum anomaly in condensed-matter systems [23–29].

Spectral gaps are also crucial for open quantum systems. As a result of coupling to the external environment, open quantum systems are no longer described by Hamiltonians, but instead by Liouvillians that act on density operators [30–32]. In general, Liouvillians are non-Hermitian superoperators and possess the complex-valued spectra. The dissipative gap between the steady state and the first decaying state provides a timescale of the relaxation process and is fundamental for the open quantum dynamics [33–38]. The role of symmetry and topology in open quantum systems has also attracted growing interest [39–51]. Despite the significance and recent interest, the dissipative gap was investigated mainly for specific models. Accordingly, general theories on the steady state and dissipative gap in open quantum systems, akin to the LSM theorem in closed quantum systems, have yet to be established. Such a general theoretical understanding should be relevant to the control of quantum materials and further exploration of quantum technology.

In this Letter, we present a general theorem that restricts the spectral gaps of open quantum systems solely based on symmetry, generalizing the LSM theorem. In particular, we demonstrate that the unique gapped steady state is prohibited in the simultaneous presence of translation invariance and U(1) symmetry for noninteger filling numbers. As an illustrative example, we find that no dissipative gap is open in the $S = 1/2$ dissipative Heisenberg model, while a dissipative gap can be open in the $S = 1$ counterpart—an open quantum analog of the Haldane gap phenomena. As a unique feature with no analogs in closed quantum systems, our analysis elucidates a fundamental difference between strong and weak symmetries in open quantum systems. We also find the LSM constraint arising from symmetry inherent to open quantum systems, Kubo-Martin-Schwinger (KMS) symmetry.

*Gap and symmetry.*—We consider generic open quantum systems described by the Lindblad master equation $d\rho/dt = \mathcal{L}\rho$ with the Lindbladian [30–32,52,53]

$$\mathcal{L}\rho = -\mathrm{i}[H, \rho] + \sum_n \left[ L_n \rho L_n^\dagger - \frac{1}{2}\{L_n^\dagger L_n, \rho\} \right], \quad (1)$$

where $H$ is a Hermitian Hamiltonian that describes the unitary dynamics, and $L_n$'s are dissipators that describe the

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nonunitary coupling to the environment. To investigate the Lindbladian spectrum, it is useful to double the Hilbert space and map $\mathcal{L}$ and $\rho$ to an operator and a state, respectively. In particular, we map the density operator $\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j|$ to a pure state $|\rho\rangle = \sum_{ij} \rho_{ij} |i\rangle|j\rangle$ in the double Hilbert space. Through this operator-state mapping, the Lindblad equation reduces to $d|\rho\rangle/dt = \mathcal{L}|\rho\rangle$ with

$$\mathcal{L} = -\mathrm{i}H_+ + \mathrm{i}H_- + \sum_n \left[ L_{n,+} L_{n,-}^* - \frac{1}{2} (L_{n,+}^\dagger L_{n,+} + L_{n,-}^T L_{n,-}^*) \right], \quad (2)$$

where $O_\pm$ denotes an operator acting on the ket and bra spaces, respectively, defined from an operator $O$ acting on the original Hilbert space. While Eq. (2) assumes bosonic systems, the operator-state mapping can be similarly carried out for fermionic systems [51,54,55].

Through the operator-state mapping, we calculate the Lindbladian spectrum by diagonalizing the non-Hermitian many-body operator $\mathcal{L}$. The real part of its eigenvalues is constrained to be nonpositive because of the contractive nature of the Lindblad dynamics. The steady state corresponds to the zero eigenvalue of $\mathcal{L}$. We define the dissipative gap $\Delta$ as the negative real part of the second largest eigenvalue (i.e., eigenvalue of the first decaying state). This can be considered as a many-body generalization of a line gap [56–58]. The dissipative gap $\Delta$ gives the relaxation timescale toward the steady state. The Lindbladian is gapped (gapless) if $\Delta$ is nonvanishing (vanishing) in the infinite-size limit $V \to \infty$, and the gapped (gapless) Lindbladian is subject to the exponential (algebraic) relaxation process [33]. The power-law behavior of the dissipative gap also yields the dynamical critical exponent.

We derive a general theorem that constrains the dissipative gap solely by symmetry. First, we assume lattice translation invariance of the Lindbladian, $\mathcal{T}\mathcal{L}\mathcal{T}^{-1} = \mathcal{L}$, with the lattice translation operator $\mathcal{T}$. Additionally, we assume U(1) symmetry in the individual ket and bra spaces, $U_\pm \mathcal{L} U_\pm^{-1} = \mathcal{L}$, which yields a conserved charge $N_\pm$. We focus on the steady-state subspace with $N_+ = N_-$ and define the filling number $\nu \coloneqq N_\pm/V$. The U(1) charge typically corresponds to the total magnetization in spin systems and the total particle number in electron systems. We can also introduce U(1) symmetry in the total Hilbert space, which is referred to as weak symmetry, in contrast to strong symmetry defined above [40,41,50,51]. We later show the different roles of strong and weak symmetries in the LSM constraints.

*Lieb-Schultz-Mattis theorem.*—Now, we demonstrate the theorem:

**Theorem**. In open quantum systems with lattice translation invariance and strong U(1) symmetry, if the Lindbladian is gapped and exhibits a unique steady state in the subspace with the fixed U(1) charge, the filling number $\nu$ is required to be an integer. In other words, if $\nu$ is not an integer, the Lindbladian is gapless or exhibits degenerate steady states.

We prove this theorem, following Oshikawa's argument [10]. We consider a generic Lindbladian in $d$ dimensions with symmetry, where the system length in each direction is denoted by $L_i$ $(i = 1, 2, \ldots, d)$. From U(1) symmetry, we introduce the U(1) flux $\phi_\pm$ in the individual ket or bra space. Such a U(1) flux can be added by the twist operator or twisted boundary conditions and physically corresponds to a magnetic flux in electronic systems. Similar to Ref. [10], we assume that the dissipative gap remains nonvanishing in the presence of the U(1) flux if it is originally open. While this assumption is needed, but has yet to be rigorously justified even for the original LSM theorem [59], it is expected to hold since the thermodynamic quantities including the spectral gaps should not crucially change just by twisting the boundary conditions. The U(1) flux $\phi_\pm$ breaks invariance under modular conjugation, which is required for physical Lindbladians [51]. Still, the adiabatic insertion of $\phi_\pm$ can detect the ingappability of open quantum systems.

Then, let us consider a steady state $|\rho_0\rangle$ in the subspace with the fixed U(1) charge $N_\pm$ and the filling number $\nu = N_\pm/V$, and insert the U(1) flux $\phi_+$ in the ket space perpendicular to the $i = 1$ direction [60]. In the course of the adiabatic insertion of the unit flux $\phi_+ = 2\pi$, the Lindbladian spectrum flows and goes back to the original spectrum. The original steady state $|\rho_0\rangle$ changes to another eigenstate $U_+^{(\mathrm{twist})}|\rho_0\rangle$ of the Lindbladian $\mathcal{L}$ with the twist operator $U_+^{(\mathrm{twist})} \coloneqq e^{2\pi \mathrm{i} \sum_{j=1}^{L_1} j n_{j,+}/L_1}$, where $n_{j,+}$ is the local density of the U(1) charge $N_+ = \sum_{j=1}^{L_1} n_{j,+}$. Importantly, the twist operator evolves by translation as

$$\mathcal{T} U_+^{(\mathrm{twist})} \mathcal{T}^{-1} = U_+^{(\mathrm{twist})} e^{-2\pi \mathrm{i} N_+/L_1}. \quad (3)$$

Hence, when $e^{-\mathrm{i}P_0}$ is the eigenvalue of translation $\mathcal{T}$ for the steady state $|\rho_0\rangle$ (i.e., $\mathcal{T}|\rho_0\rangle = e^{-\mathrm{i}P_0}|\rho_0\rangle$), we have

$$\mathcal{T}(U_+^{(\mathrm{twist})}|\rho_0\rangle) = e^{-\mathrm{i}(P_0 + 2\pi N_+/L_1)} (U_+^{(\mathrm{twist})}|\rho_0\rangle), \quad (4)$$

meaning that $U_+^{(\mathrm{twist})}|\rho_0\rangle$ is an eigenstate of $\mathcal{T}$ with the eigenvalue $e^{-\mathrm{i}(P_0 + 2\pi N_+/L_1)}$ and orthogonal to the original steady state $|\rho_0\rangle$ unless $N_+/L_1 = \nu \sum_{i=2}^{d} L_i$ is an integer [61]. Thus, for noninteger $\nu$, we can always have another steady state $U_+^{(\mathrm{twist})}|\rho_0\rangle$ with the different eigenvalue of $\mathcal{T}$, which implies the gapless Lindblad spectrum or the degeneracy of the steady states. In other words, the gapped Lindbladian with the unique steady state requires the integer filling number $\nu \in \mathbb{Z}$.

*Dissipative Heisenberg XXZ models.*—As an illustrative example that shows the significance of the LSM theorem, we study the dissipative Heisenberg XXZ model in one dimension,

$$H = \sum_{n=1}^{L} [J(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + J_z S_n^z S_{n+1}^z], \tag{5}$$

$$L_n = \sqrt{\gamma} S_n^z \qquad (n = 1, 2, \ldots, L), \tag{6}$$

where $S_n^i$'s ($i = x, y, z$) are quantum spin operators of spin number $S$ at site $n$. The dissipators $L_n$'s describe the dephasing process of spin coherence [31]. The Lindbladian $\mathcal{L}$ respects U(1) symmetry in the individual ket and bra spaces,

$$[\mathcal{L}, S_\pm^z] = 0, \qquad S_\pm^z \coloneqq \sum_{n=1}^{L} S_{n,\pm}^z, \tag{7}$$

and conserves the total magnetization $S_\pm^z$ along the $z$ axis. Hence, it is subject to the LSM constraint and cannot exhibit the unique gapped steady state except for the integer filling number $\nu = S_\pm^z / L + S \in \mathbb{Z}$. In particular, for the half filling $S_\pm^z = 0$, we have the half-integer filling number for half-integer spins $S = 1/2, 3/2, \ldots$ and the integer filling number for integer spins $S = 0, 1, 2, \ldots$. Consequently, the LSM theorem prohibits the unique gapped steady state for half-integer spins, whereas no such constraints are imposed for integer spins, akin to the Haldane gap phenomena of the antiferromagnetic Heisenberg model [6–9].

Using the operator-state mapping in Eq. (2), we study the Lindbladian spectra for $S = 1/2$ and $S = 1$ (Fig. 1) [62]. For the half-integer spin $S = 1/2$, the dissipative gap closes in the course of the insertion of the U(1) flux [Fig. 1(b)]. Here, the U(1) fluxes $\phi_\pm$ in the ket and bra spaces are introduced by imposing the twisted boundary conditions,

$$S_{n+L,\pm}^+ = e^{i\phi_\pm} S_{n,\pm}^+, \qquad S_{n+L,\pm}^- = e^{-i\phi_\pm} S_{n,\pm}^- \tag{8}$$

with the spin raising and lowering operators $S_{n,\pm}^+ \coloneqq S_{n,\pm}^x + iS_{n,\pm}^y$ and $S_{n,\pm}^- \coloneqq S_{n,\pm}^x - iS_{n,\pm}^y$. The gap closing due to the flux insertion implies the gapless spectrum, which is compatible with the LSM theorem [65].

For the integer spin $S = 1$, by contrast, the dissipative gap remains open even in the course of the insertion of the U(1) flux [Fig. 1(d)], which is prohibited for $S = 1/2$. The distinct spectral properties between half-integer and integer spins are reminiscent of the Haldane gap phenomena and show the significant predictability of the LSM theorem. The influence of the U(1) flux is merely twisting the boundary conditions and hence intuitively expected to become less significant when we increase the system size $L$. Thus, Fig. 1(d) may imply a nonzero gap even in the infinite-size limit $L \to \infty$. This is in a similar spirit to the assumption made in Oshikawa's argument [10]. However, the gap opening in Fig. 1(d) may be due to a finite-size effect that should be studied carefully in future work. Even if the aforementioned intuitive argument works in closed quantum systems, it may break down in open quantum systems.

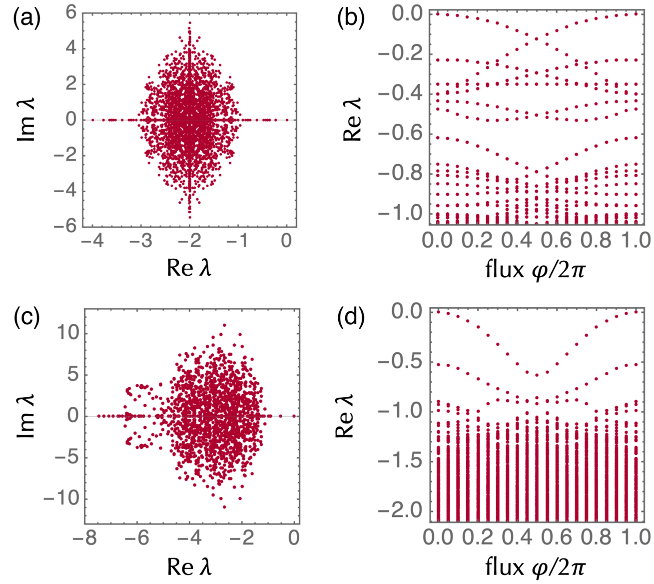


FIG. 1. Dissipative Heisenberg XXZ model ($J = J_z = 1.0$, $\gamma = 1.0$) for (a),(b) spin half $S = 1/2$ ($L = 8$) and (c),(d) spin one $S = 1$ ($L = 5$). The U(1) charge is $S_\pm^z = 0$, i.e., half filling (a),(b) $\nu = 1/2$ for $S = 1/2$ and (c),(d) $\nu = 1$ for $S = 1$. The U(1) flux $\phi$ in the ket space is inserted. (a),(c) Lindbladian spectrum with $\phi = 0$ for (a) $S = 1/2$ and (c) $S = 1$. (b),(d) Real part of the Lindbladian spectrum around the steady state $\lambda = 0$ for (b) $S = 1/2$ and (d) $S = 1$ as a function of the flux $\phi$.

The gapped ground state of the $S = 1$ antiferromagnetic Heisenberg model further exhibits the symmetry-protected topological phase [66–69]. By contrast, the steady state of the dissipative $S = 1$ Heisenberg model is given as the identity (i.e., infinite-temperature state) and does not exhibit nontrivial topological properties. In other dissipative spin models, the steady state can exhibit topological phases. The LSM theorem does not necessarily enforce the unique gapped steady state for $S = 1$; a gapless steady state can appear for different dissipators even for $S = 1$ [70]. Away from the integer filling $\nu \neq 1$, the gap opening is prohibited even for $S = 1$. In the absence of U(1) symmetry, the LSM theorem is inapplicable, and the dissipative gap can be open even for $S = 1/2$. In other words, if we are to open the dissipative gap for $S = 1/2$ without degenerate steady states, we need to break U(1) symmetry or translation invariance. Consistently, the dissipative Ising model without U(1) symmetry was shown to exhibit the unique gapped steady state [33,34]. Importantly, the LSM theorem is not only applicable to bosonic systems but also fermionic systems [70].

*Quantum anomaly.*—The original LSM theorem is related to a quantum anomaly in closed systems [23–29]. Similarly, we find that the LSM theorem developed in this

Letter is a manifestation of a quantum anomaly in open systems. For clarity, we study generic one-dimensional Lindbladians. Let us first introduce the U(1) fluxes $\phi_{\pm}$ by twisting the boundary conditions as in Eq. (8). Because of the twisted boundary conditions, translation invariance $\mathcal{T}$ is broken. However, the twisted Lindbladian is invariant under the generalized translation, $\mathcal{T}(\phi_{\pm})\mathcal{L}(\phi_{\pm})\mathcal{T}^{-1}(\phi_{\pm})=\mathcal{L}(\phi_{\pm})$, for $\mathcal{T}(\phi_{\pm}) \coloneqq e^{\mathrm{i}\phi_{+}(n_{1,+}-\nu_{+})}e^{\mathrm{i}\phi_{-}(n_{1,-}-\nu_{-})}\mathcal{T}$ with the local density $n_{j,\pm}$ of the strong U(1) charge (i.e., $N_{\pm}=\sum_{j=1}^{L}n_{j,\pm}$). This twisted translation operator satisfies $\mathcal{T}(\phi_{+}+2\pi,\phi_{-})=e^{-2\pi\mathrm{i}\nu_{+}}\mathcal{T}(\phi_{+},\phi_{-})$ and

$$\mathcal{T}^{L}(\phi_{\pm})=e^{\mathrm{i}\phi_{+}(N_{+}-L\nu_{+})}e^{\mathrm{i}\phi_{-}(N_{-}-L\nu_{-})}. \tag{9}$$

While the choice of a partition function is nontrivial in open quantum systems, we study the twisted dissipative form factor

$$Z(T,L,\phi_{\pm},\psi_{\pm},l)\coloneqq\mathrm{tr}\left[e^{T\mathcal{L}(\phi_{\pm})}e^{\mathrm{i}\psi_{+}N_{+}}e^{\mathrm{i}\psi_{-}N_{-}}\mathcal{T}^{l}(\phi_{\pm})\right] \tag{10}$$

with $T,\psi_{\pm}\in\mathbb{R}$. This is considered as the form factor of the real-time open quantum dynamics and should capture the dissipative gap [74,75]. To characterize the anomaly in a fixed filling sector of strong U(1) symmetry, we Fourier transform $Z(T,L,\phi_{\pm},\psi_{\pm},l)$ and obtain the projected dissipative form factor

$$\begin{aligned}&\tilde{Z}_{q_{\pm}}(T,L,\phi_{\pm},l)\\&\quad\coloneqq\int_{0}^{2\pi}\frac{d\psi_{+}d\psi_{-}}{(2\pi)^{2}}e^{-\mathrm{i}q_{+}\psi_{+}-\mathrm{i}q_{-}\psi_{-}}Z(T,L,\phi_{\pm},\psi_{\pm},l).\end{aligned} \tag{11}$$

Then, we have

$$\tilde{Z}_{q_{\pm}}(T,L,\phi_{+}+2\pi,\phi_{-},l)=e^{-2\pi\mathrm{i}l\nu_{+}}\tilde{Z}_{q_{\pm}}(T,L,\phi_{+},\phi_{-},l), \tag{12}$$

and

$$\begin{aligned}&\tilde{Z}_{q_{\pm}}(T,L,\phi_{\pm},l+L)\\&\quad=e^{\mathrm{i}\phi_{+}(q_{+}-L\nu_{+})+\mathrm{i}\phi_{-}(q_{-}-L\nu_{-})}\tilde{Z}_{q_{\pm}}(T,L,\phi_{\pm},l)\end{aligned} \tag{13}$$

from Eq. (9). Thus, for the noninteger filling $\nu_{\pm}\notin\mathbb{Z}$, the projected dissipative form factor $\tilde{Z}_{q_{\pm}}(T,L,\phi_{\pm},l)$ acquires the nontrivial quantum phases, signaling a mixed anomaly between the strong U(1) symmetry and weak translation symmetry—the LSM anomaly. For the integer filling $\nu_{\pm}\in\mathbb{Z}$, on the other hand, such nontrivial phases do not appear, consistent with no LSM constraints. Notably, our discussion is also applicable to discrete quantum channels $\mathcal{E}$, for which the untwisted dissipative form factor can be introduced as $\mathrm{tr}\mathcal{E}^{m}$. In contrast to strong U(1) symmetry, no anomaly arises for weak U(1) symmetry [70].

*Kubo-Martin-Schwinger symmetry.*—The LSM theorem can be generalized to other symmetry including discrete symmetry in a similar manner to closed quantum systems [14–22]. For example, the unique gapped steady state is prohibited also in dissipative quantum spin models with $\mathbb{Z}_2\times\mathbb{Z}_2$ spin-flip symmetry [70].

Notably, KMS symmetry—symmetry inherent in thermal equilibrium [76–80]—also yields the LSM constraint. For illustration, let us consider a translation-invariant Lindbladian $\mathcal{L}$ in one dimension that consists of Majorana fermions $\lambda_n$'s $(n=1,2,\ldots,L)$, satisfying $\{\lambda_m,\lambda_n\}=2\delta_{mn}$. We assume that the system length $L$ is even. Because of the Hermiticity-preserving nature, $\mathcal{L}$ is generally invariant under modular conjugation $\mathcal{J}$, defined by an antiunitary operator $\mathcal{J}$ satisfying [51,81,82]

$$\mathcal{J}\lambda_{n,\pm}\mathcal{J}^{-1}=\lambda_{n,\mp}. \tag{14}$$

To introduce KMS symmetry, we also consider another antiunitary operation,

$$\mathcal{R}\lambda_{n,\pm}\mathcal{R}^{-1}=\lambda_{n,\pm}, \tag{15}$$

with an antiunitary operator $\mathcal{R}$. If the Lindbladian $\mathcal{L}$ is invariant under $\mathcal{R}$, it is also invariant under the combination of $\mathcal{R}$ and $\mathcal{J}$. Thus, we introduce KMS symmetry $U_{\mathrm{KMS}}\mathcal{L}U_{\mathrm{KMS}}^{-1}=\mathcal{L}$ by the unitary operator $U_{\mathrm{KMS}}\coloneqq\mathcal{J}\mathcal{R}$ with $U_{\mathrm{KMS}}^{2}=1$. In contrast to ordinary symmetry, KMS symmetry $U_{\mathrm{KMS}}$ accompanies the exchanges of the ket and bra degrees of freedom.

We find that the interplay of KMS symmetry $U_{\mathrm{KMS}}$ and translation invariance $\mathcal{T}$ also leads to the LSM constraint. The key is the nontrivial algebra [70]

$$\mathcal{T}U_{\mathrm{KMS}}\mathcal{T}^{-1}=-U_{\mathrm{KMS}}. \tag{16}$$

Consequently, all the eigenvalues of the Lindbladian $\mathcal{L}$, including the zero eigenvalue of the steady states, are at least twofold degenerate. To see this, let $|\rho\rangle$ be an eigenstate of $\mathcal{L}$, and let $\lambda$ and $k\in\{+1,-1\}$ be their eigenvalues (i.e., $\mathcal{L}|\rho\rangle=\lambda|\rho\rangle$, $U_{\mathrm{KMS}}|\rho\rangle=k|\rho\rangle$). Owing to translation invariance $\mathcal{T}\mathcal{L}\mathcal{T}^{-1}=\mathcal{L}$ and Eq. (16), we also have $\mathcal{L}(\mathcal{T}|\rho\rangle)=\lambda(\mathcal{T}|\rho\rangle)$ and $U_{\mathrm{KMS}}(\mathcal{T}|\rho\rangle)=-k(\mathcal{T}|\rho\rangle)$, which implies that $\mathcal{T}|\rho\rangle$ is another eigenstate of $\mathcal{L}$ that belongs to the same eigenvalue $\lambda$ but has the different eigenvalue $-k$ of $U_{\mathrm{KMS}}$, i.e., degeneracy of the Lindbladian spectrum. This is the LSM constraint in Majorana Lindbladians, an open quantum analog of that in Majorana Hamiltonians [20–22]. This LSM constraint makes the dissipative form factor vanish, which also signals a quantum anomaly. Such degeneracy of the Lindbladian spectrum should affect the steady-state properties and dynamics of open quantum systems.

*Discussions.*—Spectral gaps are crucial for understanding closed and open quantum systems. In this Letter, we establish the LSM theorem in open quantum systems, which provides a general constraint on their steady state and dissipative gap solely by symmetry. As a consequence of the LSM constraint, we discover a fundamental distinction between half-integer and integer spins—an open quantum analog of the Haldane gap phenomena. It merits further research to investigate the dissipative Haldane gap in various analytical and numerical approaches. Since our discussions on quantum anomaly rely solely on symmetry and the structure of the double Hilbert space, they should also be relevant to non-Markovian Liouvillians, which we leave for future work.

The LSM theorem developed in this Letter gives a guiding principle to understand quantum phases of open systems. Specifically, when a dissipative gap is open in the presence of symmetry for noninteger filling, the LSM theorem ensures the nontrivial degeneracy of the steady states, typically originating from spontaneous symmetry breaking or topological order. In closed quantum systems, such LSM-type constraints also prohibit short-range-entangled states. Similarly, our LSM theorem should prohibit short-range-entangled states in the double Hilbert space. It merits further study to clarify its connection with entanglement properties of mixed states [83]. It should also be noted that the mere presence of a dissipative gap does not necessarily lead to short-range correlations of the steady state in contrast with closed quantum systems [84].

Our formalism based on the double Hilbert space may find applications to other physical systems, such as disordered systems [85]. A quantum-channel formulation of average-symmetry-protected topological phases has recently been developed [86]. It is worthwhile to further explore a relationship between disorder and dissipation. Moreover, a unique feature of open quantum systems is the non-Hermitian skin effect [87–91], which makes the spectral properties under the open boundary conditions distinct [70]. The skin effect is captured by the complex-spectral winding [92–95] and the concomitant quantum anomaly [96], thereby having a potential connection with the LSM theorem. The role of pseudospectra [97] in the LSM theorem is also worth studying.

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