

Chapter 1

The Symmetry Principle in Condensed Matter Physics (I)

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Symmetry distills the simplicity of natural laws from the complexity of physical phenomena. The symmetry principle is of vital importance in various aspects of modern physics, including analyzing atomic spectra, determining fundamental interactions in the Standard Model, and unifying physics at different energy scales. In this chapter, novel applications of this principle are reviewed in condensed matter physics and cold atom physics for exploring new states of matter.

First, the concept of *space-time group* generalizes crystalline space group symmetries to their dynamic counterparts, including nonsymmorphic space-time symmetries (e.g. *time-screw* rotation, *time-glide* reflection, and *time-shift* rotary reflection). It includes and goes beyond the Floquet theory framework, and applies to a large class of dynamic systems such as laser-driven solid crystals, dynamic photonic crystals, and optical lattices, *etc.* Second, the perspective of high symmetries (e.g. $SU(N)$ and $Sp(N)$) bridges large-spin cold fermion systems with high energy physics. For example, a generic $SO(5)$, or, isomorphically $Sp(4)$ symmetry is proved in spin- $\frac{3}{2}$ systems. Moreover, an exact $SO(7)$ symmetry is identified possessing an extraordinarily unifying power: Its χ -pairing operator extends Yang's η -pairing to a high-rank Lie algebra, integrating 21 orders in both particle-hole and particle-particle channels into a unified framework. Such systems also exhibit multi-fermion orderings, including quartetting superfluidity (charge $4e$) and quartet density wave, which are α -particle-like, or, baryon-like orderings. The resonant quantum plaquette states of $SU(4)$ antiferromagnetism are described by a high-order gauge theory. A quantum phase transition occurs from the Slater region to the Mott region in the $SU(6)$ Hubbard model. A tendency of convergence of itineracy and locality is revealed in 1D $SU(N)$ systems as N goes large. Third, a new mechanism is presented to generate spin-orbit coupling based on "spin-from-isospin" via many-body Fermi surface instabilities of the Pomeranchuk type. In contrast, the conventional wisdom views spin-orbit coupling as a single-body relativistic effect. This mechanism generalizes itinerant ferromagnetism to the unconventional symmetry versions (e.g. *p*-wave), which can also be viewed as magnetic multipolar orderings in momentum space.

Contents

1. Introduction	2
1.1. General backgrounds	3
1.2. Space-time group for dynamic systems	5
1.3. High symmetry perspective to large-spin cold fermion systems	6
1.4. Unconventional magnetism and spontaneous spin-orbit ordering	8
1.5. Outline	10
2. Space-time group for dynamic systems	10
2.1. Space-time unit cell and momentum-frequency Brillouin zone	11
2.2. The generalized Bloch-Floquet theorem	12
2.3. Definition of space-time group	13
2.4. Classifications of space-time group	15
2.5. Spectra degeneracy protected by non-symmorphic space-time symmetry	17
2.6. Discussions	18
3. High symmetry perspective to large-spin cold fermion systems	19
3.1. The generic $SO(5)$ symmetry of spin- $\frac{3}{2}$ cold fermions	19
3.2. The $SO(7)$ unification and the χ -pairing	23
3.3. Quartet (charge- $4e$) superfluidity and quartet density wave	27
3.4. Color magnetism	30
3.5. Half-filled $SU(N)$ Hubbard models: Slater v.s. Mott physics	32
3.6. Discussions	34
4. Unconventional magnetism and spontaneous spin-orbit ordering	35
4.1. Fermi liquid theory and Pomeranchuk instabilities	35
4.2. Unconventional magnetism as multipolar orderings	37
4.3. “Spin from isospin” in non-relativistic systems	39
4.4. Collective excitations in unconventional magnetic states	41
4.5. Spin-orbit coupled Fermi liquid theory	44
4.6. Discussions	46
5. Conclusions	48
6. Acknowledgments	49
References	49

1. Introduction

I feel honored to contribute to this Festschrift for *the Yang Centenary*. Professor C. N. Yang is the role model for Chinese physicists of my generation. Along our careers, we have been inspired by his milestone contributions to theoretical physics, including parity violation in the weak interaction,¹ Yang-Mills gauge theory,² Yang-Baxter equation,³ and monopole gauge theories,^{4,5} *etc.* Among these masterpieces, the symmetry principle is a threading theme, which is also a distinct style of his research.

I learned to appreciate the symmetry principle under the guidance of my Ph. D. advisor Professor Shoucheng Zhang, who himself was deeply influenced by Professor Yang. Symmetries and their applications in condensed matter physics and cold atom physics are my major research directions. Hence, I shall review progresses along this line for this Festschrift.

1.1. General backgrounds

The appreciation of symmetry at a fundamental level has a long history. The ancient Greeks proved the existence of only five types of convex regular polyhedral (the Platonic solids): tetrahedron, cube, octahedron, dodecahedron, and icosahedron. They hypothesized that these regular polyhedral correspond to the classic elements of water, earth, fire, air, and ether, respectively.⁶ Galileo's relativity principle implies the homogeneity of space and time (translational symmetry), the isotropy of space (rotational symmetry), and the equivalence of all the inertial reference frames.⁷ Einstein's relativity is a profound victory of the symmetry principle: The Lorentz symmetry is viewed as a fundamental symmetry of space-time, which is not only a property of Maxwell's equations but also the primary constraint to all physical laws.⁷ In high energy physics, Yang stated, "*Symmetry dictates interaction*", i.e., interactions among fundamental particles in the Standard Model are determined by their fundamental gauge symmetries.⁸

The first application of the symmetry principle in physics actually started in the field of condensed matter. Soon after the establishment of group theory by Galois and Cauchy in the 1830s-1840s, it was applied to analyze crystalline symmetries. In 1890s, Schönflies and Fedorov completed the construction of the 230 space groups.⁹ Each space group corresponds to one type of crystalline structures in three dimensions (3D), which is a subgroup symmetry of 3D flat space containing a discrete translational group as its normal subgroup.

In the 1880s, the concept of group was generalized to continuous groups, i.e., Lie groups, by Sophus Lie, and then calculus and differential equations entered the study of symmetry.¹⁰ Lie group and its generators Lie algebra became the main tools to analyze symmetries. Noether proved that each continuous symmetry gives rise to a local conservation law: Momentum conservation arises from the translational symmetry; angular momentum conservation arises from the rotational symmetry,¹¹ etc.

The application of group theory in quantum physics was pioneered by Wigner¹² and Weyl.¹³ Because of the linear nature of quantum mechanics, the eigenstates of a time-independent Hamiltonian form irreducible representations of its symmetry group G . Its generators commute with the Hamiltonian, and thus are conserved quantities. This principle is extremely successful in classifying the atomic and molecular optical spectra and explaining selection rules for optical transitions.

Two remarkable examples of symmetries of simple systems are the hydrogen atom¹⁴ and the harmonic oscillator.¹⁵ The N -dimensional hydrogen atom possesses the $SO(N+1)$ symmetry due to the conserved Runge-Lenz vectors. Classically, the Runge-Lenz vector specifies the orientation of the elliptical orbit. The N -dimensional harmonic oscillator possesses the $SU(N)$ symmetry which transforms among the complex space spanned by the complex combination of coordinate and momentum $a_i = \frac{1}{\sqrt{2}}(x_i + ip_i)$.

One central theme in modern physics is the unification by the symmetry prin-

principle. Electricity and magnetism are unified by the Lorentz group. The interaction between matter and the electromagnetic field is described by the $U(1)$ gauge theory. In particle physics, the electromagnetic and weak interactions are unified by the $SU_L(2) \otimes U(1)$ gauge theory as electroweak interaction, where L refers to left-handed leptons and quarks.^{16–18} The quantum chromodynamics is described by the $SU(3)$ color gauge theory, and quarks of three colors (R, G, B) form the fundamental representation of the $SU(3)$ group. Mesons are quark-antiquark bound states and baryons are three-quark bound states, both of which are color singlets. In addition, baryons and mesons can be classified as multiplets of the approximate $SU(3)$ flavor symmetry.¹⁹

Spontaneous symmetry breaking is a crucially important concept, which was first proposed by L. Landau for constructing a general framework of phase transitions.^{20–22} Most second order phase transitions are related to certain kinds of symmetry breaking of order parameters (the matter fields). For instance, the magnetic phase transition breaks time-reversal and rotational symmetries; the charge-density-wave breaks translational symmetry; superfluidity breaks the $U(1)$ symmetry. If a continuous global symmetry G is spontaneously broken, the transverse fluctuations of order parameters are gapless, which are the Goldstone mode as a reminiscence of the original symmetry before its breaking.²³ The Goldstone manifold is represented as the coset of G/H , where H represents the residual subgroup symmetry after symmetry breaking.

Even more profound physics occurs when a gauge symmetry is spontaneously broken. For example, superconductivity is a consequence of the $U(1)$ gauge symmetry breaking.^{24,25} The electromagnetic properties of superconductors are characterized by the London equation $\mathbf{j} = -\rho_s \mathbf{A}$, where ρ_s is the superfluid density, giving rise to the celebrated Meissner effect. This is due to the Anderson-Higgs mechanism that the gauge boson (photon) becomes massive and acquires its longitudinal component by absorbing the Goldstone mode of phase fluctuations. Consequently, the electromagnetic field can only enter the superconductor surface at the penetration depth λ with the relation of $\rho_s = c/(4\pi\lambda^2)$.

The Anderson-Higgs mechanism is essential in high energy physics.¹⁹ The gauge bosons become massive, once the corresponding gauge symmetries are broken. This cures the apparent discrepancy between the short-range weak and strong interactions and the massless Yang-Mills gauge fields.² This was the major obstacle to apply the Yang-Mills theory as the paradigm for formulating fundamental interactions. Furthermore, the Higgs field generates masses for fermions of quarks and leptons as shown in the Glashow-Weinberg-Salam theory, which unifies the weak and electromagnetic interactions.^{16–18}

In the context of condensed matter physics, the symmetry principle is employed to unify seemingly unrelated phenomena. For example, Yang's pseudo-spin $SU(2)$ symmetry based on the η -pairing unifies the charge-density-wave ordering and superconductivity.^{26–28} Its extension to the $SO(5)$ theory of high T_c superconductivity

by Zhang views antiferromagnetism and d -wave superconductivity on the equal footing as different components of a 5-vector.^{29,30} The sharp resonance modes of neutron scattering spectroscopy could be interpreted as the pseudo-Goldstone excitations in the superconducting ground state towards the direction of antiferromagnetism.^{29,31}

The new applications of the symmetry principle in condensed matter and ultra-cold atom physics will be reviewed below focusing on exploring novel states of matter. The motivation and outline of the main results for each topic are briefly explained below.

1.2. Space-time group for dynamic systems

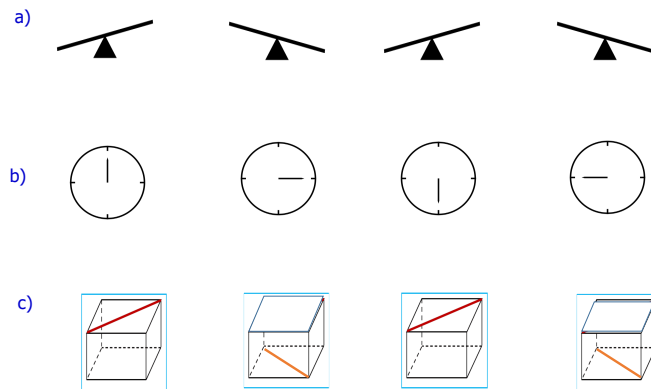


Fig. 1. Time sequence configurations for three representative space-time nonsymmorphic symmetries. *a)* *Time-glide* reflection symmetry. A see-saw is invariant by a reflection followed by a time-shift of half a period. *b)* *Time-screw* rotation symmetry. A clock is invariant by a rotation followed by a fractional time translation. *c)* *Time-shift* rotary reflection symmetry, i.e., rotary reflection followed by a fractional time translation. Time-glide reflection and time-screw rotation are analogies of glide reflection and screw rotation of space group symmetries, respectively, while time-shift rotary reflection has no counterpart in 3D space group operations.

A solid state textbook typically starts with crystalline symmetries, which are classified according to the 230 space groups, and then proceeds with the Bloch theorem setting up the framework of electron's quantum behavior in solids.³² Space group symmetries include the discrete translational symmetry of the underlying Bravais lattice, and point group symmetries (e.g. *rotation*, *reflection*, and *rotary reflection*). Space group possesses non-symmorphic symmetries, which means that under such operations there are no fixed points, including *screw rotation* and *glide reflection*. Screw rotation is the symmetry of a screw: A rotation is insufficient to maintain a screw invariant which needs to be followed by a certain translation along the rotation axis. Glide reflection is a symmetry of a row of footprints, i.e., a reflection followed by a translation of half a period.⁹

Symmetry literally means “balanced proportions”, and thus is commonly viewed as a static concept. However, time dynamics is an important topic in various subjects of physics. The recent experimental progresses, such as the pump-prob measurements^{33,34} and shaken cold-atom optical lattice experiments^{35,36}, have stimulated the study of dynamically driven systems.

A natural question is how to analyze symmetries of dynamic systems? Systems under periodical driving is often denoted as the Floquet ones. In such systems, time translational symmetry is violated while a discrete version still exists, which is the counterpart of the discrete spacial translational symmetry in crystals. However, within the Floquet framework, temporal symmetry is decoupled from the spacial one.^{37–48}

Just like that a 3D crystal is typically not the direct-product between a 2D crystal in the ab -plane with a 1D crystal along the c -axis, a dynamic crystalline system is *not* just the direct-product between a static crystal with a Floquet periodicity. We construct the symmetry group of dynamic systems and dub it *space-time group*, which is a dynamic extension of the crystalline space group.⁴⁹ The Bloch theorem is also generalized accordingly. This concept applies to a large class of dynamic systems beyond the Floquet framework, including laser-driven solid crystals, dynamic photonic crystals, and optical lattices, etc.

There exist nonsymmorphic versions of space-time symmetries as depicted in Fig. 1.⁴⁹ (Please do not confuse them with Lorentz symmetries). For example, a see-saw is invariant by a reflection followed by a time-shift of half a period, and this symmetry is dubbed *time-glide reflection* (Fig. 1(a)). A clock does not exhibit the rotation symmetry but a rotation combined with a suitable time-translation leaves it invariant, and this symmetry is dubbed *time-screw rotation* (Fig. 1(b)). These are actually symmetries of their world lines in analogy to screw rotation and glide reflection of space group. Another space-time nonsymmorphic symmetry, 3D rotary-reflection followed by a time-translation, does not have a space group counterpart (Fig. 1(c)).

A complete classification in 1+1D gives rise to 13 space-time groups in contrast to the 17 wallpaper space groups for the 2D static crystals, and in 2+1D we have found 275 space-time groups.⁴⁹ Space-time group symmetries also protect spectral degeneracies.

Time-screw rotation and time-glide reflection symmetries were also proposed by Morimoto *et. al.* independently for studying novel topological band structures in driven systems,⁵⁰ but the concept of “space-time group” was not proposed there.

1.3. High symmetry perspective to large-spin cold fermion systems

High symmetries (e.g. $SU(N)$ and $Sp(N)$) are essential in high energy physics, nevertheless, their applications in condensed matter physics are often to provide the mathematical tool of the large- N expansion to handle strong correlations.^{51–54} On the other hand, cold atom physics has become a new frontier of condensed matter

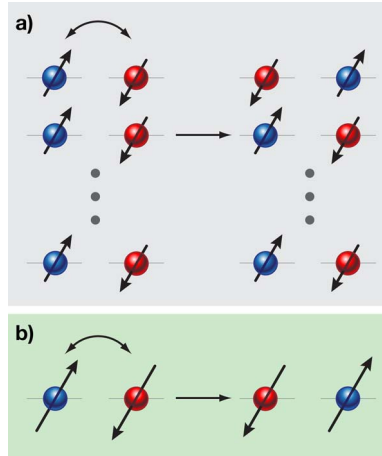


Fig. 2. Superexchange processes in (a) large-spin solid state systems and (b) large hyperfine spin cold fermion systems. In solids, quantum magnetic fluctuations are suppressed by the large- S effect; while quantum fluctuations are enhanced by the large number of spin components $N = 2S+1$. Hence, the appropriate viewpoint for large-spin fermions in the large- N physics of a high symmetry group rather than the large- S physics of the $SU(2)$ group. This feature bridges high energy physics and ultra cold atom physics in spite of hugely different energy scales. From Ref.

physics for creating novel quantum states of matter, particularly those uneasy to access in solids.

Many fermionic atoms possess large-hyperfine-spins. We have been working on exploring new states of large-spin fermions from the new perspective of high symmetries of $SU(N)$ and $Sp(N)$ since 2003.^{55–62} It works as a guiding principle to explore beautiful many-body physics, providing a natural connection between cold atom physics and high energy physics. It is amazing to see that physics at dramatically different energy scales is deeply related. Systematic studies have been performed in exploring high symmetry effects, including the unification of competing orders^{55,57}, novel quantum magnetism,^{59,61} and non-Abelian topological defects.⁶⁰

High-symmetry cold fermions have attracted considerable attentions from various research groups in the cold atom community.^{63–75} This direction has also become an active experiment focus: Takahashi's group realized the $SU(6)$ symmetric alkaline-earth fermions of ^{173}Yb .^{76–79} Fallani's group studied the 1D systems of ^{173}Yb with tunable component numbers.⁸⁰ The 10-component ^{87}Sr systems ($F = I = \frac{9}{2}$) have been studied by Killian's group,^{81,82} Sengstock's group,^{83,84} and Ye's group,^{85–87} etc. For non-technical introductions to the experimental progress, please refer to Refs.^{88,89}

A fundamental difference exists between large-spin cold fermion systems and large-spin solid state systems as shown in Fig. 2.⁸⁸ In solids, quantum magnetic fluctuations are suppressed in the large- S limit: Hund's rule coupling aligns spins of several electrons into a large spin, however, the intersite coupling is dominated by the exchange of a single pair of electrons, hence, spin fluctuations scale as $1/S$

as S goes large. In contrast, this restriction does not occur in cold atom systems because each large-hyperfine-spin fermion moves as an entire object. The exchange of a single pair of atoms completely flip the spin configuration. The large number of spin components actually enhanced quantum fluctuations, and they are actually even stronger than the spin- $\frac{1}{2}$ case. Hence, the large-spin physics of ultra-cold atoms is governed by the large- N physics of a high symmetry group where $N = 2S + 1$.

An exact and generic hidden $\text{Sp}(4)$, isomorphically $\text{SO}(5)$, symmetry is proved for hyperfine-spin- $\frac{3}{2}$ alkali and alkaline fermions without fine-tuning.^{55,57,58,60} The candidate atoms for realizations include ^{132}Cs , ^9Be , ^{135}Ba , ^{137}Ba , and ^{201}Hg . Yang's η -pairing pseudospin $\text{SU}(2)$ symmetry can be generated to the spin-3/2 Hubbard model defined on a bipartite lattice.^{55,58} Such a system could exhibit an $\text{SO}(7)$ symmetry which unifies the singlet superconductivity and the spin-quadruple density-wave order with the 7-vector representation. The adjoint representation of $\text{SO}(7)$ can unify the quintet superconductivity, spin and spin-octupole density-wave order, and charge-density-wave, which are in total 21-dimensional.

The large-spin fermions also exhibit similar physics to that in quantum chromodynamics – the multi-particle clustering orderings. With attractive interactions, Pauli's exclusion principle allows N -fermions to form an $\text{SU}(N)$ singlet state, a “baryon-like” multiple-fermion instability.^{57,68,90,91} For the super-exchange physics in the Mott-insulating states, if each site is in the fundamental representation, it also needs N sites to form an $\text{SU}(N)$ singlet.^{56,59}

How interaction effects scale with the component number N is also interesting question. For the $\text{SU}(N)$ Hubbard models, systematic quantum Monte Carlo (QMC) simulations free of the sign problem have been performed for the 2D square lattice,^{92,93} square lattice with flux,⁹⁴ and the honeycomb lattice,⁹⁵ and also in 1D.⁹⁶

Spin-orbit coupling plays an important role in the research focus of topological states of matter. Conventionally, it is viewed as a single-particle property inherited from the relativistic Dirac equation, not directly related to many-body physics.³² We have explored another possibility – spontaneously generation of spin-orbit coupling as a many-body effect based on Fermi surface instabilities of the Pomeranchuk type.⁹⁷ This mechanism is essentially itinerant magnetic phase transitions with unconventional symmetries (e.g. p -wave), which is also magnetic multipolar orderings in momentum space.^{98,99}

1.4. *Unconventional magnetism and spontaneous spin-orbit ordering*

In ferromagnetic metals, the rotational symmetry is broken in the spin channel. However, spin polarizes along the same direction around Fermi surfaces independent of the direction of momentum, hence, the orbital rotational symmetry is unbroken as shown in Fig. 3 (a). This is similar to conventional s -wave superconductors whose gap function phase keeps constant over the Fermi surface. Therefore ferromagnetism

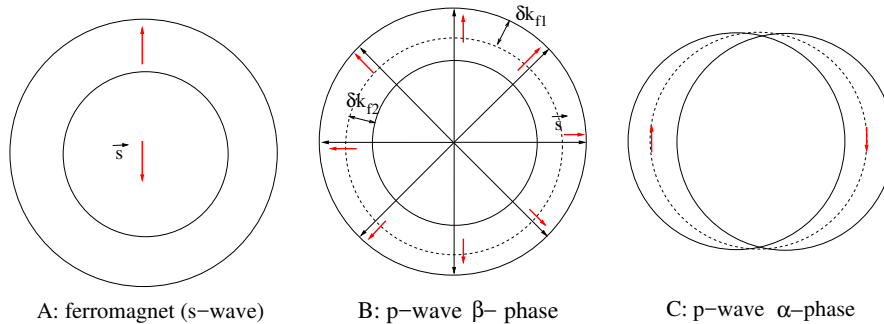


Fig. 3. Fermi surface configurations of the ferromagnetic phase (A) and the unconventional magnetic phases in the p -wave channel (the isotropic β -phase (B) and the anisotropic α -phase (C)). The ferromagnetic state can be viewed as an s -wave type magnetism since it does not break the orbital rotational symmetry. The p -wave itinerant magnetism exhibits dipolar magnetic ordering over the Fermi surface. The β -phase breaks the relative spin-orbit symmetry spontaneously, which is a particle-hole analogy to the superfluid $^3\text{He-B}$ phase. The anisotropic α -phase is the analogy of the superfluid $^3\text{He-A}$ phase. From Ref.⁹⁹

can be viewed as “ s -wave” magnetism.

As for superconductivity (fermion pairing superfluidity), there exist unconventional pairing structures, including the d -wave high T_c cuprates¹⁰⁰ and the p -wave superfluid ^3He .¹⁰¹ In analogy to unconventional superconductivity, we generalize ferromagnetism to cases of unconventional symmetries, in which spin no longer polarizes along a unique direction but varies with momentum. These unconventional magnetic states have close connections to many directions in condensed matter physics, including unconventional superconductivity,¹⁰² spin-orbit coupling and spintronics, and electron liquid crystal states in strongly correlated systems.¹⁰³

The unconventional magnetism includes both isotropic and anisotropic cases, as shown in Fig. 3 (b) and (c), respectively. They are dubbed the β and α -phases analogues to the superfluid $^3\text{He B}$ and A-phases, respectively. The isotropic β -phases still exhibit circular, or, spherical Fermi surfaces developing nontrivial spin-texture configurations in momentum space, providing a mechanism for dynamic generation of spin-orbit coupling independent from relativity. The anisotropic α -phases are electron liquid crystal states with spin degree of freedom, exhibiting anisotropic Fermi surface distortions. Both types of phases arise from the Pomeranchuk instability of Fermi surfaces in the spin channel, which include ferromagnetism as a special example.

The symmetry breaking pattern of the isotropic β -phase is subtle, which breaks the relative spin-orbit symmetry.¹⁰¹ In non-relativistic physics, spin is an internal degree of freedom, i.e., the spin rotational symmetry $SO_S(3)$ is independent from the orbital $SO_L(3)$. The β -phase is invariant only if rotations in the two channels are performed exactly in the same way. In contrast, if there exist a difference between two rotations, i.e., the relative spin-orbit rotation, the system indeed changes. This symmetry breaking pattern is denoted as $[SO_L(3) \otimes SO_S(3)]/SO_{L+S}(3)$. In other

words, the total angular momentum $J = L + S$ in the β -phase is conserved, but $L - S$ is not.

The concept of relative spin-orbit symmetry breaking was first introduced by Leggett¹⁰¹ in the context of superfluid $^3\text{He-B}$ phase, whose Cooper pairing has a p -wave and spin-triplet like structure, i.e. $L = S = 1$. The pair wavefunction in the B-phase is

$$\Psi_{pair}(\mathbf{r}_{12}) = \sum_{i=x,y,z} f_{p_i}(\mathbf{r}_{12})\chi_i, \quad (1)$$

where $f_{p_i}(\mathbf{r}_{12})$ describes the radial wavefunction with the orbital symmetry of p_i ($i = x, y, z$), and $\chi_x = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$, $\chi_y = \frac{1}{\sqrt{2}i}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$, and $\chi_z = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$. The total angular momentum $J = L + S$ of Cooper pairs is zero, and thus the pairing is isotropic. Hence, the β -phase is the particle-hole channel analogy to the $^3\text{He-B}$ phase.

In Sect. 4, we shall review how spin-orbit coupling can be dynamically generated without relativity but from phase transitions, in a similar way to ferromagnetism. We have also extended the Fermi-liquid theory to systems with spin-orbit coupling.

1.5. Outline

The rest part of this chapter is organized as follows: The concept of space-time group for dynamic systems is reviewed in Sec. 2; the high symmetry perspective of ultra-cold fermion physics is reviewed in Sect. 3; unconventional magnetism and spontaneous spin-orbit symmetry is reviewed in Sect. 4. Conclusions are presented in Sect. 5.

2. Space-time group for dynamic systems

The fundamental concept of crystal and band theory based on the Bloch theorem lay the foundation of condensed matter physics.³² In recent years, the study of dynamic systems such as the “pump-prob” systems becomes a new focus direction.^{33,34} The simplest dynamic systems exhibit space-time periodicity, and a natural question is how to classify their symmetries by extending the static crystalline symmetries. There existed previously the framework of Floquet systems, i.e., systems under periodical driving. However, in such a framework, the spacial and temporal symmetries are decoupled, hence, it cannot be the generic case.^{37–48}

We constructed a new framework, dubbed *space-time group*, to describe the general intertwined space-time periodicities in $D + 1$ dimensions, which include both the static crystal and the Floquet crystal as special cases.⁴⁹ Compared to previously known space- and magnetic groups, space-time group is augmented by “time-screw” rotation, “time-glide” reflection, and “time-shift” rotary-reflection, involving fractional translations along the time direction. We have classified that there are 13 space-time groups in 1+1D and 275 space-time groups in 2+1D.

2.1. Space-time unit cell and momentum-frequency Brillouin zone

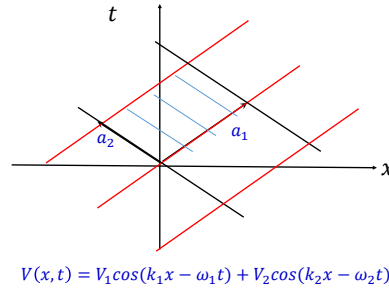


Fig. 4. A simplest space-time crystal in 1+1 D. In the general case, the space-time unit cell is a parallelogram which cannot be decomposed into a direct product between space and time domains. It exhibits neither translational nor time-translational symmetries, but does possess the combined space-time translation symmetries.

Let us begin with a simplest example of space-time crystalline symmetry. Consider a 1 + 1 D system, whose time-dependent potential is the superposition of two plane waves as plotted in Fig. 4,

$$V(x, t) = V_1 \cos(k_1 x - \omega_1 t) + V_2 \cos(k_2 x - \omega_2 t). \quad (2)$$

The wavevectors $k_{1,2}$ and frequencies $\omega_{1,2}$ are supposed to be incommensurate. If we fix a spacial position, say $x = 0$, and look at the time-dependence of $V(0, t)$, there is no temporal periodicity. For Floquet problems, the time-evolution operator $U(T)$ of one period is often used to map them into time-independent problems. Clearly, here this method generally does not apply. Similarly, if we take a snap shot at a fixed time, say $t = 0$, $V(x, 0)$ has no spatial periodicity either. Hence, the ordinary Bloch theorem cannot straightforwardly be applied here.

The periodicity only appears when we extend to space-time. The unit cell is a space-time parallelogram, *not* a direct product between space and time domains. The unit vectors \mathbf{a}_1 , \mathbf{a}_2 are space-time coupled,

$$\mathbf{a}_1 = \begin{pmatrix} \frac{2\pi\omega_2}{k_1\omega_2 - k_2\omega_1} \\ \frac{2\pi k_2}{k_1\omega_2 - k_2\omega_1} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} \frac{2\pi\omega_1}{k_1\omega_2 - k_2\omega_1} \\ \frac{2\pi k_1}{k_1\omega_2 - k_2\omega_1} \end{pmatrix}, \quad (3)$$

which define space-time coupled translation symmetries. For the general case, a potential $V(\mathbf{r}, t)$ exhibiting the intertwined discrete $D + 1$ dimensional space-time translational symmetry satisfies

$$V(\mathbf{r}, t) = V(\mathbf{r} + \mathbf{u}^i, t + \tau^i), \quad i = 1, 2, \dots, D + 1, \quad (4)$$

where $(\mathbf{u}^i, \tau^i) = \mathbf{a}^i$ is the primitive basis vector of the space-time lattice.

We move to the reciprocal space and define reciprocal lattice vectors, which can be done in a similar way to solid state physics. The reciprocal lattice is spanned by

the momentum-energy basis vectors $b^i = (\mathbf{G}^i, \Omega^i)$ defined through

$$b^i \cdot a^j = \sum_{m=1}^D G_m^i u_m^j - \Omega^i \tau^j = 2\pi \delta^{ij}. \quad (5)$$

This minus sign is due to quantum mechanics phase convention. The $D + 1$ dimensional momentum-energy Brillouin zone may not be a direct product between a momentum volume and frequency domain either. The reciprocal lattice vectors contain both momentum and frequency components.

We emphasize that the above framework is already beyond that of Floquet. Floquet systems only have one fundamental frequency, while, in our case each reciprocal lattice vector has an independent frequency. The $D + 1$ dimensional space-time crystals can exhibit at most $D + 1$ incommensurate frequencies, hence, they are related to certain types of quasi-crystals.

2.2. The generalized Bloch-Floquet theorem

For dynamic crystal systems with space-time periodicity, the Bloch and Floquet theorems should be treated at equal footing. Below they are combined and generalized.

Consider the time-dependent Schrödinger equation $i\hbar \partial_t \psi(\mathbf{r}, t) = H(\mathbf{r}, t) \psi(\mathbf{r}, t)$. Its solutions are denoted by the good quantum number of the (lattice) momentum-energy vector $\kappa = (\mathbf{k}, \omega)$, which is defined modulate the reciprocal lattice vectors. The Floquet-Bloch state labeled by κ takes the form of

$$\psi_{\kappa, m}(\mathbf{r}, t) = e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} u_m(\mathbf{r}, t), \quad (6)$$

where m marks different states sharing the common κ . $u_m(\mathbf{r}, t)$ is periodical in the space-time unit cell, which is expanded as Fourier series only involving momentum-energy reciprocal lattice vector as

$$u_m(\mathbf{r}, t) = \sum_b c_{m, b} e^{i(\mathbf{G}\cdot\mathbf{r} - \Omega t)} \quad (7)$$

with $b = (\mathbf{G}, \Omega)$ taking all the momentum-energy reciprocal lattice vectors. The spectra ω_m can be solved through the secular equation,

$$\sum_{b'} \{[\varepsilon_0(\mathbf{k} + \mathbf{G}) - \Omega] \delta_{b, b'} + V_{b-b'}\} c_{m, b'} = \omega_m c_{m, b}, \quad (8)$$

where $\varepsilon_0(\mathbf{k})$ is the free dispersion, and V_b is the momentum-energy Fourier component of the space-time lattice potential $V(\mathbf{r}, t)$.

The above procedure is very similar to the plane-wave expansion method of the band theory in D -dimensions, in which the static lattice potential only has Fourier components in momentum space. The difference is that the effective dimensions become $D + 1$, since the reciprocal lattice vectors lie in the momentum-energy space for space-time crystals. Nevertheless, the Hilbert space of physical states remains the same regardless of whether the potential is time-independent or not. To reconcile

this discrepancy, we notice the gauge-like redundancy in the formalism based on and Eq. (6) and Eq. (8). The solutions in the sector labeled by κ and those by $\kappa + b$ are redundant since the same state in Eq. 6 can also be expressed as $\psi_{\kappa,m}(\mathbf{r}, t) = e^{i[(\mathbf{k}+\mathbf{G})\cdot\mathbf{r}-(\omega+\Omega)t]} u_m(\mathbf{r}, t)$ with $u_m(\mathbf{r}, t) = u_m(\mathbf{r}, t)e^{-i(\mathbf{G}\cdot\mathbf{r}-\Omega t)}$.

The dispersion based on Eq. (8) is generally multiple-valued, represented by a D -dimensional surface in the momentum-energy Brillouin zone which is a $D+1$ dimensional torus. In the static case, the band dispersion only winds around the momentum direction. In space-time crystals, the winding patterns are richer.

Let us take the 1+1D case as a simple example. The dispersion relation $\omega(k)$ forms closed loops in the 2D toroidal momentum-energy Brillouin zone, each of which is characterized by a pair of winding numbers $\mathbf{w} = (w_1, w_2)$ with $w_{1,2}$ integers. In general, nearly all patterns $\mathbf{w} = (w_1, w_2)$ are possible except one constraint explained as follows. Consider a weak lattice potential such that it can be treated as a perturbation. The free dispersion curve $\varepsilon(k)$ is folded into the momentum-energy Brillouin zone with crossings. Two states at a crossing point is connected by a reciprocal vector b before folding. The crossing is lifted if the momentum-energy Fourier component of V_b is nonzero. The total number of states at each k is independent of the potential strength, hence crossing can only be split along the ω -direction by opening a gap of $2|V_b|$, and $d\omega/dk$ is always finite. Hence, the contractible loops with the winding numbers $(0, 0)$ are unallowed.

Nevertheless, the winding number pattern could be constrained by spectral degeneracies protected by symmetries. For example, consider a magnetic group transformation applied to a 1+1 D space-time crystal, whose unit cell is a direct product between spatial and temporal periods a and T , respectively. Define the glide time-reversal operation $g_t(x, t) = (x + \frac{a}{2}, -t)$. It operates on the Hamiltonian as

$$g_t^{-1} H(x, t) g_t = H^*(x + \frac{a}{2}, -t). \quad (9)$$

The corresponding transformation M_{g_t} on the Bloch-Floquet wavefunction $\psi_\kappa(x, t)$ of Eq. 6 is anti-unitary defined as

$$M_{g_t} \psi_\kappa = \psi_\kappa^*(g_t^{-1}(x, t)). \quad (10)$$

Consider two special lines of the momentum-energy Brillouin zone with $\kappa_x = 0$ and $\kappa_x = \pi/a$. $M_{g_t}^2 = 1$ for states with $\kappa_x = 0$, but it becomes a Kramers symmetry $M_{g_t}^2 = -1$ for those of $\kappa_x = \pi/a$,

$$M_{g_t}^2 \psi_\kappa = \psi_\kappa(x - a, t) = e^{-i\kappa_x a} \psi_\kappa = -\psi_\kappa. \quad (11)$$

Then the crossing at $\kappa_x = \pi/a$ cannot be avoided. Hence, the dispersion curve must wind along the momentum direction even times, while its windings along the energy direction cancel. The winding number is constrained to $\mathbf{w} = (2n, 0)$.

2.3. Definition of space-time group

Now we are ready to formally define *space-time* group in analogous to space group describing the static crystalline symmetry. It is the discrete subgroup between the

direct product of the Euclidean group of D spatial dimensions and that along the time-direction $E_D \otimes E_1$. In general, space-time group *cannot* be factorized as a direct product between space and temporal subgroup groups.

In terms of coordinates, a space-time group operation Γ is defined as

$$\Gamma(\mathbf{r}, t) = (R\mathbf{r} + \mathbf{u}, st + \tau), \quad (12)$$

where R here is a point-group operation, including rotation, reflection, rotary reflection. \mathbf{u} is a translation. If \mathbf{u} is not a symmetry by itself, then it is non-symmorphic. Combining R and \mathbf{u} , they span space-groups. Further including $s = -1$, they span a magnetic symmetry group, which are used to describe symmetry properties of magnetic systems. The last term of τ is time translation. Combining all the point group operations, time-reversal, spacial and temporal translations, the algebra is closed. This new symmetry group is dubbed *space-time* group.

If the time translation τ itself is not a symmetry, it should be combined with spatial transformations to form space-time non-symmorphic symmetries as shown in Fig. 1. In 1+1 D, the only available operation to combine is spatial reflection. This is the dynamic symmetry of a see-saw [Fig. 1 (a)]. A see-saw does not possess a static reflection symmetry, but it is invariant by performing reflection and time translation at half a period. This symmetry is the analogy of the glide-reflection symmetry of space group, dubbed *time-glide* reflection symmetry. In 2+1D, a new possibility is to combine τ with spatial rotation to form *time-screw* rotation, which can be intuitively understood as the dynamic symmetry of a clock [Fig. 1 (b)]. Consider a simplified clock with only one pointer rotating. It does not exhibit the rotational symmetry due to the pointer, but a rotation combined with time translation can leave the clock invariant. This is the analogy of screw rotation of space group, dubbed “time-screw” rotation.

There also exist new possibilities that nonsymmorphic space-time symmetries have no analogies in static space groups. In 3+1D, a fractional time translation τ can be combined with the rotary reflection operation R , dubbed *time-shift rotary reflection* with an example depicted in Fig. 1 (c). (Rotary reflection R is a rotation followed by a reflection whose $\det R = -1$ with eigenvalues $\{-1, e^{\pm i\theta}\}$ and $\theta \neq 0$. Another possibility is a space-time translation (\mathbf{u}, τ) followed by a point group operation R . In other words, it is non-symmorphic space group operation followed by a fractional time translation τ .

Naturally, quantum mechanical wavefunctions can be employed to span representations of space-time group. A spacial care needs to be taken is that the representation is anti-unitary when $s = -1$, i.e., time-reversal is involved. The operation of Γ on the Hamiltonian is defined as

$$\Gamma^{-1}H(\mathbf{r}, t)\Gamma = \begin{cases} H(\Gamma(\mathbf{r}, t)) & \text{for } s = 1, \\ H^*(\Gamma(\mathbf{r}, t)) & \text{for } s = -1. \end{cases} \quad (13)$$

Correspondingly, the transformation M_Γ on the Bloch-Floquet wavefunctions $\psi_\kappa(\mathbf{r}, t)$ is

$$M_\Gamma\psi_\kappa = \begin{cases} \psi_\kappa(\Gamma^{-1}(\mathbf{r}, t)) & \text{for } s = 1, \\ \psi_\kappa^*(\Gamma^{-1}(\mathbf{r}, t)) & \text{for } s = -1. \end{cases} \quad (14)$$

2.4. Classifications of space-time group

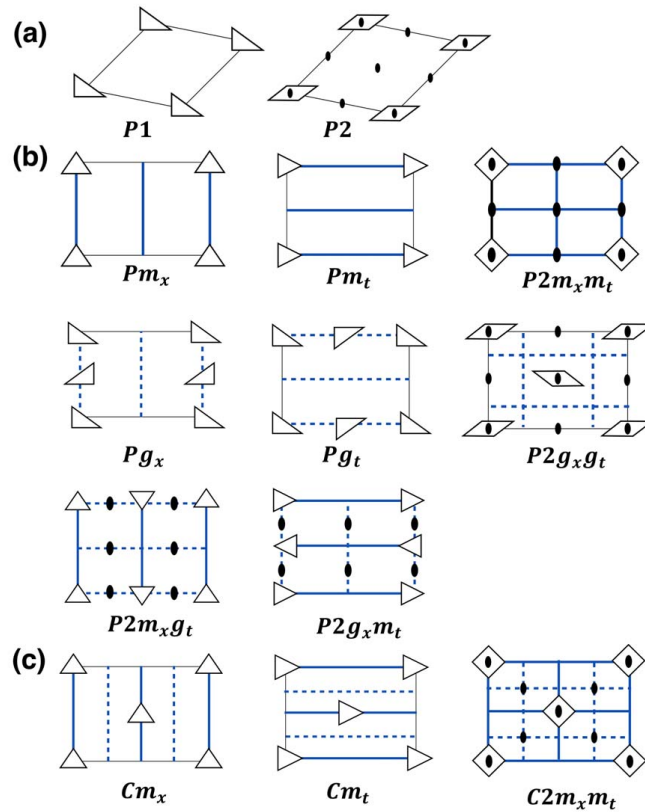


Fig. 5. Crystal configurations of 13 space-time groups in 1+1D. The solid oval marks the 2-fold space-time axis, and the parallelogram means the 2-fold axis without reflection symmetries. The thick solid and dashed lines represent reflection and glide-reflection axes, respectively. Configurations of triangles and the diamond denote the local symmetries under reflections. (a) The oblique Bravais lattice. 2 space-time crystals with (P1) and without (P2) 2-fold axes in this crystal system. They generally do not possess Floquet period, but exhibit space-time mixed translation symmetries. (b) The primitive orthorhombic Bravais lattice with 8 space-time crystals. They are denoted as Pm_x , Pm_t , $P2m_xm_t$, Pg_x , Pg_t , $P2g_xg_t$, $P2m_xg_t$, and $P2g_xm_t$ according to their reflection and glide reflection symmetries. (c) The centered orthorhombic Bravais lattice with 2 space-time crystals denoted as Cm_x , Cm_t , $C2m_xm_t$. Two unit cells are plotted to show crystalline symmetries in this class. Among 13 space-time crystals in 1+1D, 5 of them (Pg_x , Pg_t , $P2g_xg_t$, $P2m_xg_t$, and $P2g_xm_t$) are non-symmorphic, and the other 8 are symmorphic. From Ref.⁴⁹

The 2D space groups are particularly intuitive with a popular name of wallpaper groups. There exist 17 wallpaper groups corresponding to different types of planar patterns. Actually, all these patterns have been already used for ornaments since ancient times.⁶

We classify space-time crystals based on their space-time group symmetry struc-

tures. A natural starting point is to classify 1+1D space-time groups, which is an analogous problem to the 2D wallpaper groups. Due to the non-equivalence between spatial and temporal directions of the non-relativistic Schrödinger equations, we cannot really rotate space and time into each other. Hence, only the 2-fold space-time rotation is allowed, i.e. $(x \rightarrow -x, t \rightarrow -t)$ and 3,4,6-fold rotations are not, which eliminates quite a few possibilities. On the other hand, the non-equivalence between space and time also brings richness. Spatial reflection m_x and time-reversal m_t are of a different nature. The former is a unitary operation, and the latter is anti-unitary. Similarly, as for two glide operations, a time-glide with a spacial reflection g_x is different from a space-glide with a time-reversal g_t .

Taking above considerations into account, in total there are 13 types of space-time crystals as shown in Fig. 5. It is obvious that only two space-time crystal systems are allowed in 1+1D – oblique and orthorhombic. No square and hexagonal space-time crystals exist. Considering the Bravais lattices, the oblique case is monoclinic, and the orthorhombic case has two possibilities: the primitive one and the centered one. The oblique Bravais lattice generates 2 types of space-time crystals, the orthorhombic one generate 8, and the centered orthorhombic one generates 3, as shown in Fig. 5(a, b, c), respectively. For the centered orthorhombic lattices, actually their primitive cells are space-time rhombohedral. To explicitly demonstrate the full symmetries, two unit cells are plotted Fig. 5(c). There are 5 space-time groups are non-symmorphic, and all of them belong to the orthorhombic Bravais lattice. And the rest 8 are symmorphic.

As a concrete example, look at a crystal configuration depicted in Fig. 5(b), the Pg_x one. This is the symmetry group of an array of see-saws, which is actually non-symmorphic. Such a system does have a Floquet period, but it is insufficient to show its complete space-time symmetries. In contrast, space-time group goes inside the Floquet period and extract all the space-time symmetries. In the case of Pg_x , it shows the symmetry between the first and second halves of the Floquet period.

The classifications of the space-time groups in higher dimensions are generally complicated. The simple method of enumeration is cumbersome. We have classified 2+1D space-time groups based on the method of group cohomology, and the details will be presented elsewhere. This is an analogous problem to the 3D space groups. There exist 275 space-time groups with 72 of them symmorphic and the rest 203 non-symmorphic.

There are still 7 crystalline systems and 14 Bravais lattices for 2+1D space-time groups, whose numbers are the same as in the 3D case, but the situation is different. The cubic space-time crystal does not exist in 2+1D since we cannot compare the length along the time direction with that in the ab -plane, i.e., there does not exist a universal velocity like the light velocity in non-relativistic physics. Instead, there exist two different types of monoclinic space-time crystals. “Monoclinic” here means that the c -axis is perpendicular to the ab -plane, but the a and b axes are non-perpendicular to each other. The c -direction could be chosen as time,

or, one of the spatial direction corresponding to the T - and R - monoclinic space-time crystals. Other crystal systems such as tetragonal, orthorhombic, trigonal, hexagonal, triclinic ones can be similarly constructed.

2.5. Spectra degeneracy protected by non-symmorphic space-time symmetry

For static crystals, it has been extensively studied that non-symmorphic space-group symmetries can protect spectral degeneracies and enrich topological phases.^{104–108} In this subsection, we show that the intertwined space-time nonsymmorphic symmetries also protect non-trivial spectral degeneracies of the driven system.

We express a general space-time group element,

$$g = T_{\mathbf{r}}(\mathbf{u})T_t(\tau)Rm_t^s, \quad (15)$$

where $T_{\mathbf{r}}(\mathbf{u})$ and $T_t(\tau)$ are spatial and temporal translations, respectively. R is a point group operator; m_t is the time-reversal operation; $s = 1$ or 0 determines whether g is anti-unitary or not, respectively. If two operations g_1 and g_2 are in the little group of a high symmetric point $\kappa = (\mathbf{k}, \omega)$, whose point group operations commute, then

$$g_1g_2 = Tg_2g_1 \quad (16)$$

with T a translation of integer unit vectors. T is decomposed into spatial and temporal parts as $T = T_{\mathbf{r}}(\tilde{\mathbf{u}})T_t(\tilde{t})$, where

$$\tilde{\mathbf{u}} = (\mathbf{I} - R_2)\mathbf{u}_1 - (\mathbf{I} - R_1)\mathbf{u}_2, \quad \tilde{t} = 2s_2t_1 - 2s_1t_2. \quad (17)$$

Below we review the spectra degeneracies protected by this symmetry.

The representation matrices $M_{g_{1,2}}$ acting on Floquet-Bloch wavefunctions with κ satisfy

$$M_{g_1}M_{g_2} = e^{i\mathbf{k}\cdot\tilde{\mathbf{u}} - i\omega\tilde{t}}M_{g_2}M_{g_1}. \quad (18)$$

The following three cases need to be examined depending on whether $g_{1,2}$ are unitary or not.

First, if neither g_1 nor g_2 reverses the direction of time. In this case, $\tilde{t} = 0$, and then the phase factor in Eq. (18) does not involve time. If $\mathbf{k} \cdot \mathbf{u} = 2\pi p/q$ with p and q coprime, then the Bloch-Floquet wavefunctions exhibit a q -fold degeneracy at $\kappa = (\mathbf{k}, \omega)$ proved as follows. Since g_1 belongs to the little group, $\psi_{\kappa}(\mathbf{r}, t)$ can be chosen to satisfy $M_{g_1}\psi_{\kappa,1} = \mu\psi_{\kappa,1}$, then

$$\psi_{\kappa}, M_{g_2}\psi_{\kappa}, M_{g_2}^2\psi_{\kappa}, \dots, M_{g_2}^{q-1}\psi_{\kappa} \quad (19)$$

are the common Bloch-Floquet eigenstates sharing the same κ but exhibiting a set of different eigenvalues of g_1 as $\eta, \mu\eta, \mu\eta^2, \dots, \mu\eta^{q-1}$ with $\eta = e^{i\pi p/q}$. Then they are orthogonal to each other forming a q -fold degeneracy.

Second, we consider the case that only one of $g_{1,2}$ involves time-reversal. Without loss of generality, M_{g_1} is assumed to be unitary while M_{g_2} is anti-unitary. Then

the prefactor in Eq. (18) exhibits frequency dependence. Again since g_1 is in the little group, the Floquet-Bloch eigen state ψ_κ can be chosen as an M_{g_1} eigenstate, expressed as

$$M_{g_1}\psi_\kappa = e^{i\mathbf{k}\cdot\mathbf{u}_1 - i\omega t_1} e^{i\theta} \psi_\kappa, \quad (20)$$

in which θ is extracted to be *only* dependent on the point group operation R_1 . Based on Eq. (18), $M_{g_2}\psi_\kappa$ is also an eigenstate of g_1 as

$$M_{g_1}(M_{g_2}\psi_\kappa) = e^{i\mathbf{k}\cdot(-\mathbf{u}_1+\tilde{\mathbf{u}})+i\omega(-\tilde{t}+t_1)} e^{-i\theta} (M_{g_2}\psi_\kappa). \quad (21)$$

Plugging in $\tilde{t} = 2t_1$, we arrive at if $e^{i\mathbf{k}\cdot(2\mathbf{u}_1-\tilde{\mathbf{u}})+2i\theta} \neq 1$, then two phases in Eqs. (20) and (21) do not equal. Hence, the degeneracy is protected. Nevertheless, further applying high powers of M_{g_2} does not bring new phases.

The last case is when both g_1 and g_2 flip the time-direction, i.e., both M_{g_1} and M_{g_2} are anti-unitary. By defining $g = g_1g_2$ which is unitary again and combining g and g_2 , we have

$$gg_2 = T_{\mathbf{r}}(\mathbf{u})T_t(\tau)g_2g, \quad (22)$$

which is reduced to case 2.

We emphasize that in none of the above three cases, the degeneracy condition depends on the frequency component of κ . This is expected since one can always shift the frequency of the spectrum by adding a constant to the time-dependent Hamiltonian.

2.6. Discussions

So far the concept of *space-time group* has received considerable attentions.^{109–118} We expect it would serve as a guiding principle for quantum dynamic studies, in analogy to the role of space group to static crystalline symmetries.⁹ The classification of space-time group in 3+1D would be of a fundamental importance if it is completed successfully, which is currently under investigation.

Actually the lattices in solids are dynamic, and the quantized lattice vibrations are known as phonons. However, phonons are typically thermally driven and incoherent. If a certain type phonon mode is coherently excited, say, optically, or, by other pumping methods, it cannot be treated perturbatively.^{34,119} Instead, the time-dependent motions of lattice ions should be treated at the zeroth order, i.e., we should include them as the time-dependent lattice potential in the Schrödinger equation. In artificial lattices, such as the phononic, photonic crystals, and optical lattice for ultra-cold atoms, the lattice potential could be manipulated on purpose,^{36,109,116} In these cases, space-time group should replace space group as the symmetry guidance of quantum dynamics in such systems.

Certainly, the semi-classic transport in dynamic crystals is of importance. When the periodicity of lattice potential is weakly broken by slowly varying external fields both spatially and temporally, semi-classic equations of motion for a quantum

particle could be developed.³² We should distinguish two different types of dynamics: the fast changing periodical lattice potential which should be taken by the band structure calculation, and the slowly changing external field which should be treated in an adiabatic way. A challenging problem is how to generalize the Berry curvature formalism to the dynamic version and incorporate it into equations of motion.¹²⁰ The work along this direction would provide a general framework for further studying topological properties in dynamic systems.¹²¹

Another direction to explore is the connection to time crystal.^{43,122–131} The current study of time crystal is concentrated on the spontaneous breaking of the discrete time-translational symmetries, which is an profound interaction effect. Nevertheless, the symmetry breaking pattern typically is just the Floquet type. It would be interesting to combine these two research directions together, for example, to consider how to spontaneously generate dynamic crystals with non-symmorphic space-time symmetries. More philosophically, we could ask the problem of discrete subgroups of different types of space-time symmetries, including Galilean, Poincaré, anti-de Sitter symmetries, etc.

3. High symmetry perspective to large-spin cold fermion systems

The study of ultra-cold atom systems has become a new frontier for condensed matter physics as a way of creating novel quantum states of matter.^{132,133} We have proposed a new perspective of high symmetries (e.g. $SU(2N)$ and $Sp(2N)$) since 2003 to study the alkali and alkaline-earth fermion systems,^{55–62} where $2N$ is the fermion component number and hence typically even. It is exciting to explore, in atomic systems, complex and beautiful many-body physics difficult to realize in usual solids.^{88,89} It also significantly enriches the physics of large- N quantum magnetism by providing a realistic system.

3.1. The generic $SO(5)$ symmetry of spin- $\frac{3}{2}$ cold fermions

In this subsection, we review the proof of an exact and generic hidden symmetry of $Sp(4)$, or, isomorphically $SO(5)$ symmetry in spin- $\frac{3}{2}$ fermion systems (e.g., ¹³²Cs, ⁹Be, ¹³⁵Ba, ¹³⁷Ba, ²⁰¹Hg).^{55,58} ^a It plays the role of $SU(2)$ in electron systems since its exactness is independent of dimensionality, lattice geometry, and particle filling. Such a high symmetry without fine-tuning is rare, which can be used as a guiding principle to exploring novel quantum phases.

Let us begin with the standard s -wave scattering interactions of spin- $\frac{3}{2}$ ^b fermion-

^a $Sp(4)$ and $SO(5)$ share the same Lie algebra. Rigorously speaking, $Sp(4)$ has spinor representations while $SO(5)$ has not. $Sp(4)$ is the double covering group of $SO(5)$, and the relation between them is the same as that between $SU(2)$ and $SO(3)$. For simplicity, we will use $Sp(4)$ and $SO(5)$ interchangeably neglecting their minor difference.

^bThe total spin of atom is often called “hyperfine spin” including contributions from the nuclear spin, electron spins and electron orbital angular momentum. Below we follow the convention of atomic physics to use F to denote atom’s hyperfine spin, and for simplicity, spin and hyperfine

ic atoms.^{132,133} Since the orbital partial wave channel is symmetric, the total spin wavefunction of two atoms is constrained by Pauli's exclusion principle to be anti-symmetric, which must be either singlet or quintet. The corresponding interaction parameters are denoted g_0 and g_2 , respectively. The Hamiltonian reads as

$$H = \int d^d \vec{r} \left\{ \sum_{\alpha=\pm 3/2, \pm 1/2} \psi_{\alpha}^{\dagger}(\vec{r}) \left(\frac{-\hbar^2}{2m} \nabla^2 - \mu \right) \psi_{\alpha}(\vec{r}) + g_0 P_{0,0}^{\dagger}(\vec{r}) P_{0,0}(\vec{r}) + g_2 \sum_{m=\pm 2, \pm 1, 0} P_{2,m}^{\dagger}(\vec{r}) P_{2,m}(\vec{r}) \right\}, \quad (23)$$

with d the space dimension, μ the chemical potential, and $P_{0,0}^{\dagger}, P_{2,m}^{\dagger}$ the singlet and quintet pairing operators defined through the Clebsh-Gordan coefficient for two indistinguishable particles as

$$P_{F,m}^{\dagger}(\vec{r}) = \sum_{\alpha\beta} \left\langle \frac{3}{2} \frac{3}{2}; F, m \middle| \frac{3}{2} \frac{3}{2} \alpha\beta \right\rangle \psi_{\alpha}^{\dagger}(\vec{r}) \psi_{\beta}^{\dagger}(\vec{r}), \quad (24)$$

where $F = 0, 2$ and $m = -F, -F + 1, \dots, F$. Its lattice version is the spin- $\frac{3}{2}$ Hubbard model,

$$H = -t \sum_{\langle ij \rangle, \sigma} (\psi_{i\sigma}^{\dagger} \psi_{j\sigma} + h.c.) - \mu \sum_{i\sigma} \psi_{i\sigma}^{\dagger} \psi_{i\sigma} + U_0 \sum_i P_0^{\dagger}(i) P_0(i) + U_2 \sum_{i, -2 \leq m \leq 2} P_{2m}^{\dagger}(i) P_{2m}(i), \quad (25)$$

where t is the hopping integral, $U_{0,2}$ are the onsite Hubbard interaction parameters in the singlet and quintet channels, respectively.

So far, the perspective in Eqs. (23) and (25) is the usual spin $SU(2)$. The 4-component spinor, singlet and quintet interaction channels are of the spin quantum number $\frac{3}{2}, 0$, and 2 , respectively. Below we will show that this degeneracy pattern equally well fits in a high symmetry group of $Sp(4)$, or, isomorphically, $SO(5)$, which provides a whole new perspective on spin- $\frac{3}{2}$ fermion systems.

For this purpose, we construct the $Sp(4)$ algebra by extending the typical charge and spin sectors. For spin- $\frac{1}{2}$ systems, charge and spin form a complete set for the particle-hole (p-h) channel physical observable. However, they are incomplete for spin- $\frac{3}{2}$ systems since there are $4^2 = 16$ bilinears,

$$M^I = \psi_{i,\alpha}^{\dagger} M_{\alpha\beta}^I \psi_{i,\beta} \quad (I = 1 \sim 16). \quad (26)$$

To systematically decompose the 16 matrix kernels of $M_{\alpha\beta}^I$, high rank spin tensors are employed in terms of the product of spin- $\frac{3}{2}$ matrices:

$$\begin{aligned} \text{particle number:} & \quad I; \\ \text{spin:} & \quad F^i, \quad i = 1, 2, 3; \\ \text{spin quadrupole:} & \quad \xi_{ij}^a F_i F_j, \quad a = 1, \dots, 5; \\ \text{spin octupole:} & \quad \xi_{ijk}^L F_i F_j F_k, \quad L = 1, \dots, 7, \end{aligned} \quad (27)$$

spin are used interchangeably.

where ξ 's are the typical fully symmetric, traceless tensors converting 3-vector into spherical tensors.

The five spin quadrupole matrices are denoted $\Gamma^a = \xi_{ij}^a F_i F_j$, which remarkably anticommute with each other forming a basis of the Dirac Γ matrices satisfying

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}. \quad (28)$$

Explicitly, they are

$$\Gamma^1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \Gamma^{2,3,4} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (29)$$

where I and $\vec{\sigma}$ are the 2×2 unit and Pauli matrices. They are explicitly expressed by the spin matrices as

$$\begin{aligned} \Gamma_1 &= \frac{1}{\sqrt{3}}(F_x F_y + F_y F_x), & \Gamma_2 &= \frac{1}{\sqrt{3}}(F_z F_x + F_x F_z), & \Gamma_3 &= \frac{1}{\sqrt{3}}(F_z F_y + F_y F_z), \\ \Gamma_4 &= \frac{1}{\sqrt{3}}(F_z^2 - \frac{5}{4}), & \Gamma_5 &= \frac{1}{\sqrt{3}}(F_x^2 - F_y^2). \end{aligned} \quad (30)$$

Moreover, the 3 spin and 7 spin octupole matrices together can be organized into the 10 commutators of Γ -matrices defined as

$$\Gamma^{ab} = -\frac{i}{2}[\Gamma^a, \Gamma^b] \quad (1 \leq a, b \leq 5). \quad (31)$$

Consequently, the 16 particle-hole channel bilinear operators are classified according to their properties under the $\text{Sp}(4)$ transformations as scalar, vector, and anti-symmetric tensors (generators) as

$$\begin{aligned} n(\vec{r}) &= \psi_\alpha^\dagger(\vec{r})\psi_\alpha(\vec{r}), & n_a(\vec{r}) &= \frac{1}{2}\psi_\alpha^\dagger(\vec{r})\Gamma_{\alpha\beta}^a\psi_\beta(\vec{r}), \\ L_{ab}(\vec{r}) &= -\frac{1}{2}\psi_\alpha^\dagger(\vec{r})\Gamma_{\alpha\beta}^{ab}\psi_\beta(\vec{r}). \end{aligned} \quad (32)$$

The $\text{SO}(5)$ generators L_{ab} and its vectors n_a together span the $\text{SU}(4)$ algebra. They satisfy the commutation relations as

$$\begin{aligned} [L_{ab}, L_{cd}] &= -i(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad}), \\ [L_{ab}, n_c] &= -i(\delta_{ac}n_b - \delta_{bc}n_a), \\ [n_a, n_b] &= -iL_{ab}. \end{aligned} \quad (33)$$

It is well known that the $\text{SU}(4)$ algebra is isomorphically to $\text{SO}(6)$, and $\text{SU}(4)$ is the double covering group of $\text{SO}(6)$.

In order to further study the pairing operators in the particle-particle (p-p) channel and time-reversal transformation, we introduce the charge conjugation matrix R , which renders the creation operators transform in the same way as the annihilation operators. Concretely, $R_{\alpha\beta}\psi_\beta^\dagger$ transforms the same as ψ_α under the $\text{Sp}(4)$ transformation. The existence of R is based on the pseudoreality of $\text{Sp}(4)$ spinor representation. It satisfies

$$R^2 = -1, \quad R^\dagger = R^{-1} = {}^t R = -R, \quad R\Gamma^a R = -{}^t\Gamma^a, \quad R\Gamma^{ab} R = {}^t\Gamma^{ab}, \quad (34)$$

where ${}^t\Gamma^{a,ab}$ are the transposed matrices of $\Gamma^{a,ab}$. In the representation of Eq. 29, it is expressed as $R = \Gamma_1\Gamma_3$.

Under the assistance of R , the fermion pairing operators be expressed as⁶⁰

$$\begin{aligned}\eta^\dagger(\vec{r}) &= \text{Re}\eta + i \text{Im}\eta = \frac{1}{2}\psi_\alpha^\dagger(\vec{r})R_{\alpha\beta}\psi_\beta^\dagger(\vec{r}), \\ \chi_a^\dagger(\vec{r}) &= \text{Re}\chi_a + i \text{Im}\chi_a = -\frac{i}{2}\psi_\alpha^\dagger(\vec{r})(\Gamma^a R)_{\alpha\beta}\psi_\beta^\dagger(\vec{r}).\end{aligned}\quad (35)$$

Clearly, $\eta^\dagger(\vec{r})$ is Sp(4) scalar, and $\chi_a^\dagger(\vec{r})$'s are Sp(4) vector. They are related to the spin SU(2) representation via $P_{0,0}^\dagger = -\frac{1}{\sqrt{2}}\eta^\dagger$, $P_{2,\pm 2}^\dagger = \frac{1}{2}(\mp\chi_1^\dagger + i\chi_5^\dagger)$, $P_{2,\pm 1}^\dagger = \frac{1}{2}(-\chi_3^\dagger \pm i\chi_2^\dagger)$, $P_{2,0}^\dagger = -i\frac{1}{\sqrt{2}}\chi_4^\dagger$.

The anti-unitary time-reversal transformation $T^2 = -1$ is expressed as

$$T = R C, \quad (36)$$

where C denotes complex conjugation. L_{ab} 's consist of spin and spin-octupole operators.^{55,134} Since they are odd rank spin tensors, they are time-reversal odd. n_a 's and N are time-reversal even. It is also straightforward to check that they transform differently under the T transformation

$$TnT^{-1} = n, \quad Tn_aT^{-1} = n_a, \quad TL_{ab}T^{-1} = -L_{ab}. \quad (37)$$

Now we are ready to prove the generic SO(5) symmetry of Eq. 23 and Eq. 25. The kinetic energy part has an explicit SU(4) symmetry which is the unitary transformation among four spin components. The singlet and quintet interactions are proportional to $\eta^\dagger(\vec{r})\eta(\vec{r})$ and $\chi_a^\dagger(\vec{r})\chi_a(\vec{r})$ respectively, thus reducing the symmetry group from SU(4) to SO(5). When $g_0 = g_2$, the SU(4) symmetry is restored because $\chi_a^\dagger, \eta^\dagger$ together form its 6 dimensional antisymmetrical tensor representation.

For the continuum model, the odd partial wave scatterings include the spin triplet and septet channels, whose interactions are denoted as g_1 and g_3 , respectively. The SO(5) symmetry is broken if $g_1 \neq g_3$, and restored at $g_1 = g_3$ since the triplet and septet together could form the 10-d adjoint representation of SO(5). However, to the leading order, p -wave scattering is weak for neutral atoms, and can thus be safely neglected. For the lattice model, the onsite interaction does not allow the triplet and septet interactions.

For later convenience, the lattice Hubbard model of Eq. 25 is rewritten in another manifestly Sp(4) invariant form as

$$\begin{aligned}H_0 &= -t \sum_{\langle i,j \rangle} (\psi^\dagger(i)\psi(j) + h.c.), \\ H_I &= \sum_{i,1 \leq a \leq 5} \left[\frac{3U_0 + 5U_2}{16} (n(i) - 2)^2 - \frac{U_2 - U_0}{4} n_a^2(i) \right] - (\mu - \mu_0) \sum_i n(i),\end{aligned}\quad (38)$$

where the SU(4) symmetry appears at $U_0 = U_2$ as before. At half-filling^c $\mu_0 = (U_0 + 5U_2)/2$ to ensure the particle-hole symmetry.

^cHere half-filling means the average particle number per site equals 2, half of the component number.

3.2. The $SO(7)$ unification and the χ -pairing

The spin- $\frac{1}{2}$ Hubbard model defined in a bipartite lattice in any dimensions actually possesses a pseudospin $SU(2)$ symmetry spanned by the η -pairing operators and particle number as discovered by Yang.²⁶⁻²⁸ This symmetry plays an important role to unify competing orders (e.g. superconductivity and charge-density-wave). In this subsection, we review the extension of the pseudo-spin symmetry to the $SO(7)$ symmetry in the spin- $\frac{3}{2}$ Hubbard model, and define the quintet χ -pairing operators. It exhibits much richer unifying power in treating a variety of competing orders at equal footing.

The η -pairing operator in spin- $\frac{1}{2}$ systems sums over the onsite singlet pairing operators with opposite signs on two sublattices. The pseudospin $SU(2)$ algebra is particularly useful for unifying competing orders in the negative- U Hubbard model.²⁶⁻²⁸ The order parameters of superconductivity, whose number of components is 2 since it is complex, and charge-density-wave are unified forming a 3D representation. The η -pairing generator transforms superconductivity into charge-density-wave and vice versa. At half-filling, the pseudospin $SU(2)$ symmetry is exact, and these two orders are degenerate. Away from half-filling, this symmetry is explicitly broken, and the superconducting ground state is selected. However, when applying the η -pairing operator on the ground state, it creates well-defined excitations, which are the pseudo Goldstone modes rotating superconductivity into charge-density-wave.

Before considering unifications, let us fully explore the symmetry structure of spin- $\frac{3}{2}$ systems. The largest algebra formed by 4-component fermions is actually $SO(8)$,¹³⁵ including 16 p-h channel fermion bilinears and the other 12 ones in the p-p channel. On each site, the local $SO(8)$ generators $M_{ab}(i)$ ($0 \leq a < b \leq 7$) are organized as follows,^{55,58}

$$M_{ab}(i) = \begin{pmatrix} 0 & \text{Re}\chi_1(i) \sim \text{Re}\chi_5(i) & N(i) & \text{Re}\eta(i) \\ & L_{ab}(i) & \text{Im}\chi_1(i) & n_1(i) \\ & & \sim & \sim \\ & & \text{Im}\chi_5(i) & n_5(i) \\ & & 0 & -\text{Im}\eta(i) \\ & & & 0 \end{pmatrix}, \quad (39)$$

with $N(i) = (n(i) - 2)/2$. For $1 \leq a < b \leq 5$, they are just $L_{ab}(i)$ forming its $SO(5)$ subalgebra. The global $SO(8)$ generators are defined as

$$M_{ab} = \sum_i M_{ab}(i), \quad \text{or,} \quad M_{ab} = \sum_i (-)^i M_{ab}(i), \quad (0 \leq a < b \leq 7), \quad (40)$$

depending on M_{ab} lying in the p-h or p-p channels, respectively. More explicitly, we write

$$L_{ab} = M_{ab}, \quad n_a = M_{a7}, \quad \chi_a^\dagger = M_{0a} + iM_{a6}, \quad N = M_{06}, \quad \eta^\dagger = M_{06} - iM_{67}, \quad (41)$$

with $1 \leq a < b \leq 5$. L_{ab} , n_a and N lie in the p-h channel, and χ_a^\dagger and η lie in the p-p channel. The η^\dagger operator is the spin- $\frac{3}{2}$ generalization of Yang's, both of

which are spin singlet. In contrast, the χ_a^\dagger pairing operator is a non-trivial quintet generalization.

It is easy to check that the H_0 part of the Hamiltonian Eq. 38 satisfies $[H_0, M_{ab}] = 0$. However, H_{int} typically breaks the $SO(8)$ symmetry unless it vanishes within the framework of 4-fermion interactions.

The next highest algebra is $SO(7)$ spanned by M_{ab} with $0 \leq a < b \leq 6$, which is the high-rank Lie algebra generalization of Yang's η -pairing. Explicitly, they include the $SO(5)$ generators L_{ab} , the χ -pairing operators $\text{Re}\chi_a, \text{Im}\chi_a$, and the particle number N . This $SO(7)$ symmetry becomes exact at $U_0 = -3U_2$, where the interacting part of the Hamiltonian Eq. 38 is reduced to

$$H_I = \sum_{i, 0 \leq a < b \leq 6} \left\{ \frac{2}{3} U_2 [M_{ab}(i)]^2 - (\mu - \mu_0) n(i) \right\}. \quad (42)$$

At half-filling, $\mu = \mu_0$, then the global $SO(7)$ symmetry becomes exact.

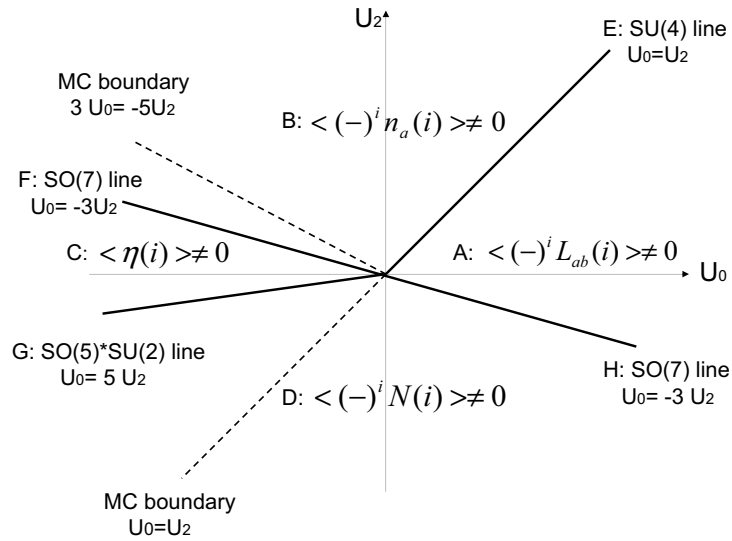


Fig. 6. Competing phases of spin- $\frac{3}{2}$ Hubbard model unified by high symmetries. (A) and (B): Antiferromagnetism in the $Sp(4)$ adjoint and vector representations; (C): the singlet superconductivity; (D): CDW; (E), (F), (G), and (H): exact phase boundaries with higher symmetries of $SU(6)$, $SO(7)$, $SO(5)SU(2)$ and $SO(7)$, respectively. From Ref.⁵⁵

The $SO(7)$ symmetric spin- $\frac{3}{2}$ Hubbard model can be further divided into two cases. In the case of $U_0 = -3U_2$ with $U_2 > 0$, the system in the weak coupling regime exhibits the competition between the singlet superconductivity and the density-wave

of spin quadrupole orders, whose order parameters are organized as

$$V_a = \sum_i (-)^i M_{a6}, \quad \text{or,} \quad V_a = \sum_i M_{a6}, \quad (0 \leq a \leq 7) \quad (43)$$

depending on V_a lying in the p-h channel or the p-p channel, respectively. More explicitly,

$$\Delta_s = V_0 - iV_6, \quad \text{SDW}_a = V_a \quad (1 \leq a \leq 5). \quad (44)$$

They are unified by a 7-vector V_a with $0 \leq a \leq 6$ with $V_0 = \text{Re}\Delta_s$, $V_a = \text{SDW}_a$ for $1 \leq a \leq 5$, and $V_6 = -\text{Im}\Delta_s$, and transform according to the vector representation of SO(7),

$$[M_{ab}, V_c] = i(\delta_{bc}V_a - \delta_{ac}V_b). \quad (45)$$

Hence, the Goldstone manifold is S^6 . Away from half-filling, the SO(7) is broken and the singlet superconductivity is selected as the ground state ordering. The χ -pairing operators remain the eigen-operators as

$$[H, \chi_a^\dagger] = -(\mu - \mu_0)\chi_a^\dagger, \quad \text{and,} \quad [H, \chi_a] = (\mu - \mu_0)\chi_a. \quad (46)$$

At $\mu < \mu_0$, applying χ_a^\dagger on the superconducting ground state $|\Omega\rangle$ creates a quintet excitation,

$$H(\chi_a^\dagger|\Omega\rangle) = (\mu_0 - \mu)(\chi_a^\dagger|\Omega\rangle), \quad (47)$$

which carries the lattice momentum $\mathbf{Q} = (\pi, \pi)$. In other words, the χ_a -pairing operator behaves like a quasi-Goldstone mode, which rotates the singlet superconductivity to the density-wave state of the a -th component spin quadrupole.

Yang's η -pairing operators were generalized to the SO(5) theory of high T_c superconductivity, which unifies the 2-component superconductivity and 3-component antiferromagnetism into a 5-vector. Nevertheless, the SO(5) algebra is not exact. The celebrated neutron resonance modes in the superconducting states were interpreted as the pseudo-Goldstone modes rotating from superconductivity to antiferromagnetism, denoted as π -modes. The χ -modes here are just analogs of the π modes SO(5) theory.²⁹ However, the SO(5) algebra is not exact in high T_c cuprate systems. In contrast, here the SO(7) symmetry can be constructed to be exact.

The power of the SO(7) unification exhibits even more explicitly at $U_0 = -3U_2$ with $U_0 > 0$, in which the 21D adjoint representation of SO(7) plays the role. The order parameter manifold includes the quintet superconductivity, the 10-fold density-wave of spin and spin octupole orders, and charge-density-wave order, which are organized as

$$T_{ab} = \sum_i (-)^i M_{ab}, \quad \text{or,} \quad T_{ab} = \sum_i M_{ab}, \quad (0 \leq a < b \leq 6), \quad (48)$$

depending on T_{ab} lying in the p-h channel or the p-p channel, respectively. Explicitly, they are

$$\Delta_{q,a} = T_{0a} + iT_{a6}, \quad \text{SDW}_{ab} = T_{ab}, \quad \text{CDW} = T_{06}, \quad (49)$$

where $1 \leq a < b \leq 6$. They transform according to the $SO(7)$ algebra as

$$[M_{ab}, T_{cd}] = i(\delta_{ac}T_{bd} + \delta_{bd}T_{ac} - \delta_{ad}T_{bc} - \delta_{bc}T_{ad}). \quad (50)$$

It is amazing to realize such a “grand unification” in a non-relativistic model. The Goldstone manifold is $SO(7)/[SO(5) \times SO(2)]$, which is 10-dimensional. When away from half-filling, the $SO(7)$ symmetry is broken into $SO(5)$, and the ground state exhibit the quintet superconductivity. Eq. 47 still applies. Assuming $\langle \Omega | \Delta_{a,b} | \Omega \rangle \neq 0$, $\chi_a^\dagger | \Omega \rangle$ remains a quasi-Goldstone mode which rotates to the density-wave state of spin/spin octupole SDW_{ab} if $a \neq b$, or the charge-density-wave state if $a = b$.

The pseudo-spin $SU(2)$ symmetry of the spin- $\frac{3}{2}$ version occurs at $U_0 = 5 U_2$. In this case, H_I is rewritten as

$$H_I = \sum_{i, 1 \leq a, b \leq 5} \{-U_2 L_{ab}^2(i) - (\mu - \mu_0)n(i)\}, \quad (51)$$

which only involve the $SO(5)$ generators. Then M_{06} , M_{07} , M_{67} span an $SU(2)$ algebra commuting with all the $Sp(4)$ generators L_{ab} . More explicitly, they are just the pseudo-spin $SU(2)$ algebra spanned by the η -pairing and particle number operators. At $U_0 = 5 U_2 < 0$, this pseudospin $SU(2)$ symmetry unifies the singlet pairing and charge-density-wave order parameters in a similar way to the spin- $\frac{1}{2}$ negative- U Hubbard model. Again way from half-filling, this symmetry is broken, and the ground state is the singlet pairing state. In this case, the η -pairing operators remain eigen-operators

$$[H, \eta^\dagger] = -(\mu - \mu_0)\eta^\dagger, \quad \text{and}, \quad [H, \eta] = (\mu - \mu_0)\eta. \quad (52)$$

We emphasize that the pseudo-spin $SU(2)$ symmetry in the spin- $\frac{3}{2}$ system still exhibits difference from Yang’s η -pairing for the spin- $\frac{1}{2}$ case. In spin- $\frac{1}{2}$ case, the empty and doubly occupied states form a pseudospin- $\frac{1}{2}$ representation for the pseudo-spin $SU(2)$. In our case, there are three onsite singlet state: empty, 2-particle singlet, and the 4-occupied states forming a pseudo-spin-1 representation.

Based on the above analysis and assisted by mean-field calculations, the weak-coupling phase diagram in a bipartite lattice at half-filling in two dimensions and above is shown in Fig. 6. The higher symmetries lines are as follows: The $SU(4)$ symmetry appears along line E with $U_0 = U_2$; the $SO(7)$ symmetry appears along lines F and H with $U_0 = -3U_2$; and the $SO(5) \otimes SU(2)$ symmetry appears long line G with $U_0 = 5U_2$. These lines are phase boundaries separating phases A, B, C, and D. Phase A and B are regimes where repulsive interactions dominate, hence, they are density-wave states of spin tensors. In phase A, the onsite singlet energy is smaller than the quintet energy, leading to the density-wave of spin quadrupoles forming the 5-vector representation of the $Sp(4)$ group. On the other hand, the lowest onsite states in phase B form quintet, leading to the density-wave of spin/spin octupoles for the 10-dimensional adjoint representation of $Sp(4)$. The Goldstone manifold in phase A is $SO[5]/SO(4) = S^4$, while that in phase B is $SO(5)/[SO(3) \otimes SO(2)]$. On line E, the $SU(4)$ symmetry unifies all the 15 dimensional density-wave orders in

the spin channel forming SU(4) adjoint representation, whose Goldstone manifold is $U(4)/[U(2) \otimes U(2)]$.

Phase C is the singlet pairing state, and phase D is the charge-density-wave state. Orders in phases B and C are unified along the SO(7) line F, and orders in phases A, D, and the quintet pairing are unified along the SO(7) line H. Orders in phase C and D are unified along the pseudospin SU(2) symmetry line G.

At last, let us mention an interesting point that SO(7) possesses a subgroup of G_2 , which is the smallest exceptional Lie group and also the automorphism group of non-associative algebra of octonions. A G_2 symmetric spin- $\frac{3}{2}$ Hubbard model is constructed which is the common subgroup of two different SO(7) algebra connected by the structure constant of octonions.⁶² This model exhibits various interesting symmetry breaking pattern: The G_2 symmetry can be spontaneously broken into SU(3), or, $SU(2) \otimes U(1)$, both of which are essential in high energy physics. In the quantum disordered states, quantum fluctuations generate the effective SU(3) or $SU(2) \otimes U(1)$ gauge theories.

3.3. Quartet (charge-4e) superfluidity and quartet density wave

Superconductivity arises from the coherent condensation of Cooper pairs, which is the central content of the celebrated Bardeen-Cooper-Schrieffer (BCS) theory. Moreover, there exist multi-fermion clustering instabilities in strong correlation systems in various disciplines of modern physics. The SU(3) gauge symmetry requires three quarks to form a color singlet bound state of baryon;¹⁹ α -particles are 4-body bound states of two protons and two neutrons, and biexcitons are bound states of two electrons and two holes. These states go beyond the framework of the BCS theory since they cannot be reduced to a 2-body problem. The competitions among the quartetting (charge $4e$) and pairing ($2e$) superfluidities, quartet and pair density wave orderings are investigated in 1D 4-component fermion systems.^{57,68} In recent years, charge- $4e$ superconductivity has been proposed as a consequence of strong fluctuations of the pair-density-wave state in high T_c cuprates.^{136,137} Competitions of 4-fermion orderings in the context of superconducting phase fluctuations have also recently received attentions.¹³⁸⁻¹⁴⁰ Excitingly experimental evidence of Little-Parks oscillations at the periods of $hc/(4e)$ and $hc/(6e)$ have been observed in the Kagome superconductor CsV_3Sb_5 .¹⁴¹

Spin- $\frac{3}{2}$ systems allow the quartetting order, i.e., 4-fermions forming clustering instability, which is also called “charge- $4e$ ” in condensed matter physics. A quartet in the strong coupling limit is a 4-body maximally entangled EPR state with all the spin components forming an SU(4) singlet, whose order parameter is expressed as

$$Q(x) = \psi_{\frac{3}{2}}^\dagger(x) \psi_{\frac{1}{2}}^\dagger(x) \psi_{-\frac{1}{2}}^\dagger(x) \psi_{-\frac{3}{2}}^\dagger(x). \quad (53)$$

Furthermore, spin- $\frac{3}{2}$ systems could support 6 different types of Cooper pairing states including an Sp(4) singlet and a set of Sp(4) quintet whose order parameters are

presented in Eq. (35). It would be interesting to investigate their competitions.

Assisted by the strong coupling methods for 1D problems, we are able to analyze the competition between the quartetting and pairing formations, or, charge- $4e$ and $2e$ states. Quartets and pairs can undergo either superfluidity or density-wave transitions depending on the charge channel interactions. Only the quartetting states are $SU(4)$ invariant, and the 6 pairing operators presented in Eq. (35) form the rank-2 antisymmetric tensor representation of $SU(4)$.^d Due to the strong quantum fluctuations, non-Abelian Lie group symmetries cannot be spontaneously broken in 1D. Hence, only quartet orderings, either superfluidity or density wave, are allowed by the $SU(4)$ symmetry. Nevertheless, if the symmetry is reduced to $Sp(4)$, the $Sp(4)$ singlet pairing could survive, while the quintet pairing still cannot since it is not an $Sp(4)$ singlet. Naturally, their exist competitions between $Sp(4)$ singlet (charge- $2e$) pairing orders and quartteting (charge- $4e$) orders. The transition between them is Ising order-disorder transition in the spin channel.

Blow we briefly outline the procedure of the bosonization and renormalization group (RG) analysis, and the details were presented in Ref.⁵⁸ The $Sp(4)$ currents include scalar (charge), vector (spin quadrupole), and tensor (spin plus spin octupole) ones,

$$\begin{aligned} J_{R,L}(z) &= \psi_{R,\alpha}^\dagger(z)\psi_{R,\alpha}(z), \quad J_{R,L}^a(z) = \frac{1}{2}\psi_{R,\alpha}^\dagger(z)\Gamma_{\alpha\beta}^a\psi_{R,\beta}(z) \quad (1 \leq a \leq 5), \\ J_{R,L}^{ab}(z) &= \frac{1}{2}\psi_{R,\alpha}^\dagger(z)\Gamma_{\alpha\beta}^{ab}\psi_{R,\beta}(z) \quad (1 \leq a < b \leq 5), \end{aligned} \quad (54)$$

where R and L refer to right and left-movers. The low energy effective Hamiltonian density $H = H_0 + H_{int}$ is written as,

$$\begin{aligned} H_0 &= v_f \left\{ \frac{\pi}{4} J_R J_R + \frac{\pi}{5} (J_R^a J_R^a + J_R^{ab} J_R^{ab}) + (R \rightarrow L) \right\}, \\ H_{int} &= \frac{g_c}{4} J_R J_L + g_v J_R^a J_L^a + g_t J_R^{ab} J_L^{ab}, \end{aligned} \quad (55)$$

where the chiral couplings are neglected at one-loop level since it only renormalizes Fermi velocities. At the tree level, these dimensionless coupling constants are related by the pair interaction parameters g_0, g_2 defined in Eq. (23) as $g_c = (g_0 + 5g_2)/2$, $g_v = (g_0 - 3g_2)/2$, $g_t = -(g_0 + g_2)/2$. Certainly, they are renormalized significantly under the RG process. At $g_v = g_t$, or, $g_0 = g_2$, the $SU(4)$ symmetry is restored.

The phase diagram at incommensurate fillings are presented at Fig. 7. The charge sector remains gapless and decouples with the spin sectors. In the spin sector, three phases are identified: Phase A is the gapless Luttinger liquid phase lying in the repulsive interaction region where $0 < g_2 < g_0$, which is controlled by the non-interacting fixed point. Phase B is the quartetting phase controlled by the strong coupling fixed point along the $SU(4)$ line with $g_v = g_t \rightarrow +\infty$, or, $g_0 = g_2 \rightarrow -\infty$. It lies in the regime where attractive interactions dominate. Phase C is the spin singlet pairing phase controlled by the strong coupling point along the

^dIn terms of $SO(6)$, which equals $SU(4)/Z_2$, they form the 6-vector representation.

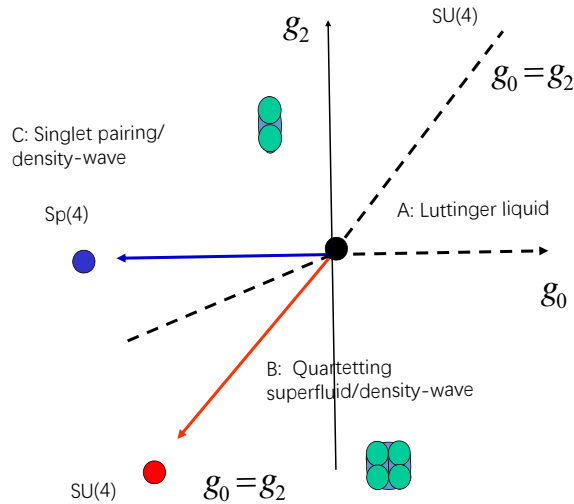


Fig. 7. Competition between quarettling (charge- $4e$) and singlet pairing (charge- $2e$) phases in 1D spin- $\frac{3}{2}$ systems at incommensurate fillings. Combined with the charge channel Luttinger parameter K_c , three phases are identified with phase boundaries marked by dashed lines. A) the gapless Luttinger liquid phase controlled by the non-interacting fixed point (the black spot). B) Quarettling superfluidity at $K_c > 2$ or quartet density-wave ($2k_f$) at $K_c < 2$. They are controlled by the strong coupling fixed point along the SU(4) line (the red spot); C) singlet pairing superfluidity at $K_c > 1/2$ or pair density wave ($4k_f$) at $K_c < 1/2$. They are controlled by the strong coupling fixed point along $g_2 = 0$ (the blue spot). Phases B and C are both gapped in spin channels, and the transition between them is an Ising order-disorder transition. From Ref.⁵⁷

line of $-g_v = g_t \rightarrow +\infty$ corresponding to $g_0 \rightarrow -\infty$ and $g_2 \rightarrow 0$. The pairing phase even covers the regime with purely repulsive interaction regime.

Within quarettling phase B, there also exist two competing orders, the quarettling superfluidity and quartet density wave. By checking the periodicity, the quartet density wave corresponds to the $2k_f$ CDW. Four fermions first form quartets, and then they either undergo superfluidity, or, density wave instability. As for the charge sector, their bosonic expressions are

$$Q = \psi_{\frac{3}{2}}^\dagger \psi_{\frac{1}{2}}^\dagger \psi_{-\frac{1}{2}}^\dagger \psi_{-\frac{3}{2}}^\dagger \propto e^{2i\sqrt{\pi}\theta_c}, \quad O_{qdw} = \psi_{R\alpha}^\dagger \psi_{L\alpha} \propto e^{i\sqrt{\pi}\phi_c}. \quad (56)$$

The scaling dimensions for quartet superfluidity and density-wave orders are $1/K_c$ and $K_c/4$, respectively. Hence, the quartet superfluidity wins at $K_c > 2$, while the quartet density wave wins at $K_c < 2$.

Similarly in phase C, Cooper pairs can either undergo superfluidity, or, pair density wave instabilities. The pair density wave corresponds to the $4k_f$ charge

density wave. As for the charge sector, their bosonic expressions are

$$\Delta_s = \eta^\dagger = \psi_{\frac{3}{2}}^\dagger \psi_{-\frac{3}{2}}^\dagger - \psi_{\frac{1}{2}}^\dagger \psi_{-\frac{1}{2}}^\dagger \propto e^{i\sqrt{\pi}\theta_c}, \quad O_{pdw} = \psi_{R\alpha}^\dagger \psi_{R\beta}^\dagger \psi_{L\beta} \psi_{L\alpha} \propto e^{2i\sqrt{\pi}\phi_c}. \quad (57)$$

The scaling dimensions for the singlet pairing and pair density wave orders are $1/(4K_c)$ and K_c , respectively. Hence, the pairing superfluidity dominates over the pair-density-wave at $K_c > 1/2$. In the region of $1 > K_c > 1/2$, pairing superfluidity is the leading instability in an overall repulsive interaction environment.

The boundary between phase B quartetting (charge- $4e$) and phase C singlet pairing (charge- $2e$) is determined by the unstable fixed point ($g_v = 0, g_t \rightarrow \infty$), which is approximately plotted in Fig. 7. The competitions between these two phases can be mapped to a phase-locking problem of two-band superconductivity. The first component is $\Delta_1 = \psi_{\frac{3}{2}}^\dagger \psi_{-\frac{3}{2}}^\dagger$, and the second one $\Delta_2 = \psi_{\frac{1}{2}}^\dagger \psi_{-\frac{1}{2}}^\dagger$, whose bosonic representations are

$$\Delta_1 \propto e^{i\sqrt{\pi}\theta_1} = e^{i\sqrt{\pi}(\theta_c + \theta_r)}, \quad \Delta_2 \propto e^{i\sqrt{\pi}\theta_2} = e^{i\sqrt{\pi}(\theta_c - \theta_r)}, \quad (58)$$

where the charge channel θ_c is the average phase and θ_r is the relative phase. In fact, θ_r and its vortex, or, dual field ϕ_r , are of the spin quadrupole channel. The bosonic expressions of the pairing and quartetting order parameters are expressed as

$$\Delta_s = \Delta_1 - \Delta_2 \propto e^{i\sqrt{\pi}\theta_c} \cos \sqrt{\pi}\theta_r, \quad Q = \Delta_1 \Delta_2 = e^{i2\sqrt{\pi}\theta_c} \cos 2\sqrt{\pi}\phi_r. \quad (59)$$

θ_c is power-law fluctuating, and does not play a role in the transition between quartetting and pairing. It is the relative phase fluctuations that control the transition as described by the sine-Gordon theory,

$$H_{eff} = \frac{1}{2} \{ (\partial_x \theta_r)^2 + (\partial_x \phi_r)^2 \} + \frac{1}{2\pi a^2} (\lambda_1 \cos 2\sqrt{\pi}\theta_r + \lambda_2 \cos 2\sqrt{\pi}\phi_r), \quad (60)$$

which contains cosine terms of both θ_r and ϕ_r . If $\lambda_1 > \lambda_2$, the relative phase θ_r is pinned leading to the pairing order; otherwise if $\lambda_1 < \lambda_2$, the vortex (dual) field ϕ_r is pinned giving rise to the quartetting order. The transition occurs at $\lambda_1 = \lambda_2$. Eq. 60 can be mapped to two free majorana fermions with masses $m_{\pm} = |\lambda_1 \pm \lambda_2|$. One channel becomes massless at the transition, which is the Ising critical point.

As a difference between the pairing and quartetting orders, there exist quartet breaking processes of $4 \rightarrow 1 + 3 \rightarrow 1 + 1 + 2$ and $4 \rightarrow 2 + 2$, which can be used to distinguish quartetting and pairing. The vortex lattice configurations is also different for quartetting superfluidity. In the quartetting superfluid, the flux quantization is $hc/(4e)$, hence, the number of vortices should be doubled compared to those of pairing superfluidity.

3.4. Color magnetism

The prominent multi-particle correlations also manifest in the $SU(N)$ quantum antiferromagnetism in the Mott insulating states at $1/N$ -filling, i.e., one fermion per

site. The superexchange favors the tendency that every N sites form an $SU(N)$ singlet as dubbed “color magnetism” due to its similarity to the $SU(3)$ gauge theory of quantum chromodynamics in which 3 quarks form a color singlet.

In the one-dimensional $Sp(4)$ Heisenberg chain in the fundamental spinor representation, it has been found the ground state exhibit a period of four sites oscillation.^{57,61} The plaquette tendency was investigated in the $SU(4)$ symmetric Kugel Khomskii model by diagonalization on a 4×4 sites.¹⁴² The Majumdar-Ghosh model were generalized to the $SU(4)$ case in a ladder system whose ground state is solvable as a direct product state of $SU(4)$ plaquettes.⁵⁶ The 4-site $SU(4)$ singlet plaquette wavefunction can be written as

$$\frac{1}{\sqrt{4!}} \epsilon_{\alpha\beta\gamma\delta} \psi_{\alpha}^{\dagger}(1) \psi_{\beta}^{\dagger}(2) \psi_{\gamma}^{\dagger}(3) \psi_{\delta}^{\dagger}(4) |\Omega\rangle, \quad (61)$$

which is a 4-particle maximally entangled EPR state.

Consider the $SU(4)$ antiferromagnetism with each site in the fundamental representation in a 3D cubic lattice. We construct the $SU(4)$ resonating plaquette model in 3D in analogous to the Rokhsar-Kivelson quantum dimer model in 2D square lattice.^{59,143} There exist three resonant configurations are the left-right, front-back, up-bottom plaquette coverings in a cube as shown in Fig. 8.

The Rokhsar-Kivelson (RK) type Hamiltonian is constructed as:¹⁴⁴

$$H = -t \sum_{\text{each cube}} \{ |A\rangle\langle B| + |B\rangle\langle C| + |B\rangle\langle C| + h.c. \} \\ + V \sum_{\text{each cube}} \{ |A\rangle\langle A| + |B\rangle\langle B| + |C\rangle\langle C| \}, \quad (62)$$

where t is assumed to be positive and V is the plaquette flipping amplitude. Similarly to the RK point of the quantum dimer model, here at $V/t = 2$, the ground state is the equal weight superposition of all plaquette configurations connected by local flips.¹⁴³

The low energy physics of the quantum plaquette model can be mapped to a gauge theory model, actually, it is a high order gauge theory. We assign each face with an integer number n only taking values of 1 and 0: 1 corresponds to that the plaquette is an $SU(4)$ singlet, and 0, otherwise. The “electric field” at site i is defined as a rank-2 symmetric traceless tensor

$$E_{i,\mu\nu} = \eta(i) \left(n_{i+\frac{1}{2}\hat{\mu}+\frac{1}{2}\hat{\nu}} - \frac{1}{2} \right), \quad (63)$$

where $\eta(i) = \pm 1$ depending on the sublattice, and $i + \frac{1}{2}\hat{\mu} + \frac{1}{2}\hat{\nu}$ refers to the location of a face center. Since each site can only join one singlet, the sum of n over all the twelve faces sharing the same site is constrained to be 1, which can be represented as

$$\nabla_x \nabla_y E_{xy} + \nabla_y \nabla_z E_{yz} + \nabla_z \nabla_x E_{zx} = 5\eta(i), \quad (64)$$

where ∇ is the lattice derivative. According to the standard electrodynamics, E is conjugate to the vector potential $A_{i,\mu\nu}$, which is also a rank-2 tensor, as

$$[E_{i,\mu\nu}, A_{j,\rho\sigma}] = i\delta_{ij} (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}). \quad (65)$$

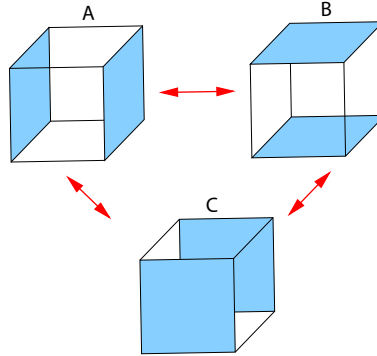


Fig. 8. In the SU(4) Mott-insulator at quarter-filling, i.e., one particle per site in the fundamental representation of SU(4). The superexchange interaction favors four sites of a plaquette form an SU(4) singlet in analogous to the dimer formation in SU(2) antiferromagnetism. For a 3D cube, there exist three flappable plaquette configurations, based on which a quantum plaquette model can be constructed. It can be described by an effective high-rank gauge theory with conserved electric dipoles instead of charges. From Ref.⁵⁹

Since E is like angular momentum taking integer numbers, A should behave as an angular variable $A_{i,\mu\nu} = \eta(i) \theta_{i+\frac{1}{2}\hat{\mu}+\frac{1}{2}\hat{\nu}}$, which is compact with the period of 2π . Then

$$[E_{i,\mu\nu}, e^{iA_{j,\nu\sigma}}] = (\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})e^{iA_{j,\nu\sigma}}. \quad (66)$$

Then the plaquette flipping term in Eq. 62 is represented as

$$H_t = -t\{\cos(\nabla_z A_{xy} - \nabla_x A_{yz}) + \cos(\nabla_x A_{yz} - \nabla_y A_{zx}) + \cos(\nabla_y A_{zx} - \nabla_z A_{xy})\}. \quad (67)$$

The associated gauge invariant transformation is,

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \nabla_\mu \nabla_\nu f, \quad (68)$$

where f an arbitrary scalar function. The corresponding Gauss's law becomes

$$\partial_i \partial_j E_{ij} = \rho. \quad (69)$$

Its physical meaning has recently been revealed in the context of the “fracton” physics, which is a recent focus in the condensed matter community for exotic states of matter and has the potential of applications for topological quantum memory.¹⁴⁵

3.5. Half-filled SU(N) Hubbard models: Slater v.s. Mott physics

How interactions drive a partially filled band into an insulating state is an outstanding problem. There exist two basic physical pictures - the Slater physics (Fermi surface nesting) at weak coupling, and the Mott physics at strong coupling. For the SU(2) case, the antiferromagnetic (AFM) order increases monotonically and smoothly, no phase transition between the Slater and Mott regions.^{146–148}

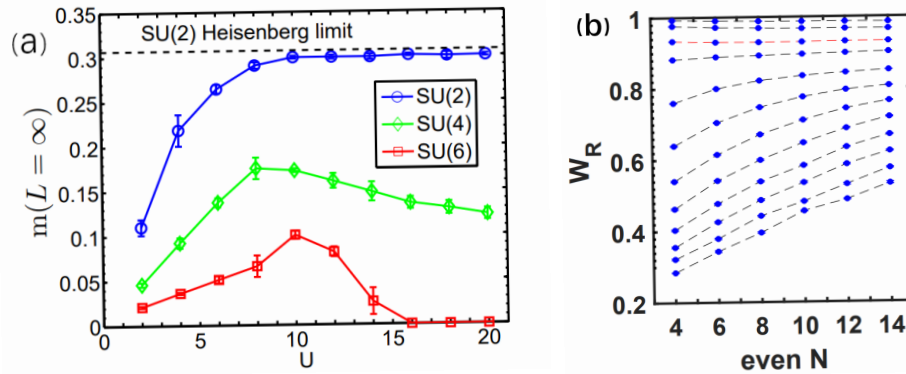


Fig. 9. QMC simulations on $SU(N)$ Hubbard models to reveal how interaction effects scale with fermion component number N . *a*) The AFM orders of the $SU(N)$ Hubbard models in the square lattice as varying U and N . The AFM order shows a monotonically increase at $N = 2$ but behaves different at $N = 4$ and 6 . For the latter case, it starts to grow and then drop as increasing U . In particular, the AFM orders is completely suppressed for the $SU(6)$ case at $U > U_c \approx 15$ at $N = 6$, after which the VBS order appears. This shows a quantum phase transition as evolving from the Slater physics to the Mott physics region. From Ref.⁹² *b*) QMC simulations of the relative band width W_R for the 1D $SU(N)$ Hubbard model at half-filling. The results shown the convergence from the itineracy and the local moment site as $N \rightarrow \infty$. The dashed lines shown as a guide from top to bottom correspond to $U/t = 0.5; 1.0; 2.0; 3.0; 5.0; 7.0; 9.0; 11.0; 13.0; 15.0; 17.0; 19.0$, respectively. The cross-over lines with $U/t = 2$ (marked red) separating the weak and strong interaction regions are nearly N -independent. from Ref.⁹⁶

There exist qualitative differences between the Slater and Mott regimes for the two-dimensional $SU(N)$ Hubbard models arising from the enhanced spin and charge fluctuations at $N > 2$. Previous large- N studies in the literature mostly focus on the antiferromagnetic Heisenberg models.^{54,149} In contrast, the interplay between charge and spin physics is even more challenging, which could be investigated via the sign-problem free quantum Monte-Carlo (QMC) simulations. The following fermionic $SU(N)$ Hubbard model at half-filling is employed,

$$H = -t \sum_{\langle ij \rangle} \{c_{i\alpha}^\dagger c_{j\alpha} + h.c.\} + U \sum_i \left(n(i) - \frac{N}{2} \right)^2, \quad (70)$$

where N is an even number. The U -term is written in the particle-hole symmetric form, which pins the average particle number per site at $N/2$, i.e., half-filling.

QMC simulations indicate the fundamental difference between the $SU(N)$ case and $SU(2)$ case as shown in Fig. 9 (a) for a square lattice. The AFM orders in both $SU(4)$ and $SU(6)$ cases start to appear at small U in agreement with the Slater physics, where the single-particle gaps are exponentially small. As U further increases, the AFM orders reach the maxima and then decrease. Meanwhile, the single-particle gaps scale linearly with U , marking the onset of Mott physics. For the $SU(6)$ case, the AFM order is completely suppressed as at a large value of $U_c \approx 15$. Fitting the simulation data shows that the critical exponents of the AFM

order with $\nu = 0.60$ and $\eta = 0.44$.⁹³ At $U > U_c$, the transition to the valence-bond-solid (VBS) state is found, which can be interpreted as the transition from the Slater regime to the Mott regime where the local-moment super-exchange dominates.

How do interaction effects scale with N with fixing the filling level and the interaction U ? Sign-problem free QMC simulations have been performed for the half-filled $SU(N)$ Hubbard models in 1D to address this problem.⁹⁶ Based on simulation results, we conjecture the existence of a universal interacting state as $N \rightarrow \infty$ explained as follows: The relative bandwidth is defined to reflect the correlation strength,

$$W_R(U, N) = E_{K,N}(U)/E_K(0), \quad (71)$$

where $E_{K,N}(U)$ is the kinetic energy per component with the interaction parameter U and component number N , and $E_K(0)$ is that at $U = 0$. Hence $W_R(U = 0, N) = 1$ for the free system, and it becomes 0 in the strong coupling limit at $U = \infty$. At small values of U/t , say $U/t \sim 1$, fermions are nearly itinerant, and correlations manifest through inter-component collisions. Hence, $W_R(U, N)$ decreases monotonically as N increases which enhances the collision possibility resulting in the amplification of correlations. In contrast, at large values of U/t , say, when $U/t > 10$, increasing N softens the Mott insulating background. The kinetic energy gain from virtual hoppings scales as $N^2 t^2/U$, hence, $W_R(U, N)$ increases linearly as increasing N . In the crossover region which lies around $U/t \approx 2$, $W_R(U, N) \approx 0.9$ nearly independent on N . Although the simulation data are still inconclusive, we conjecture that

$$\lim_{N \rightarrow \infty} (1 - W_R(U, N)) \approx 0.9, \quad (72)$$

which means an interacting large- N limit. It means that weak and strong interacting systems are driven to a crossover region as $N \rightarrow \infty$, but from opposite directions exhibiting a convergence of itinerancy and Mottness. This shows the difference from the fact that $\lim_{N \rightarrow \infty} \lim_{U \rightarrow 0} (1 - W_R(U, N)) = 0$, i.e., there exists a singularity at $U \rightarrow 0$ and $N \rightarrow \infty$. Other physical quantities, including Fermi distribution, and the spin structure factor, also exhibit nearly N -independent behavior. More analytic and numeric works are needed to further check if there exists a universal strongly interacting limit with vanishing charge gaps as $N \rightarrow \infty$, and its possible connection to non-Fermi liquid states.

3.6. Discussions

The perspective of high symmetries ($SU(N)$, $Sp(N)$) brings much richness and novelty in studying large-spin ultra-cold fermions. The large numbers of spin components render the system in the quantum large- N regime instead of the semi-classical large- S regime. We have reviewed systematically the hidden $Sp(4)$ symmetries in spin- $\frac{3}{2}$ systems, the unification based on the χ -pairing which is an $SO(7)$ generalization of Yang's η -pairing. Quartet superfluidity, quartet density wave state, and plaquette singlet formation in the Mott insulating state exhibit similar features

of multi-particle clustering correlations analogous to the color singlet in quantum chromodynamics. Interaction effects are varying the value of N are investigated, which show a tendency of convergence of itinerancy and locality as $N \rightarrow \infty$.

On the experimental side, there have been significant progresses in the past decades. The SU(6) symmetric ^{173}Yb ^{76,79} and SU(10) symmetric ^{87}Sr fermionic⁸¹ atom systems have been experimentally realized. The electron-orbital degrees of freedom as well as the nuclear spin leads to rich physics.^{63,64} Various SU(N) symmetric quantum degenerate gases and Mott insulators in optical lattices have been realized.^{63,76,77,79–81} As for spin- $\frac{3}{2}$ systems, there are a few candidate ^{132}Cs , ^9Be , ^{135}Ba and ^{137}Ba , ^{201}Hg . Considering the rapid development in this field, we also expect that the exotic Sp(4) physics could also be experimentally investigated.

4. Unconventional magnetism and spontaneous spin-orbit ordering

In the non-relativistic Fermi liquid theory, spin is an internal symmetry independent of orbital rotations, which rigorously speaking should be denoted as “isospin” instead of spin from the relativistic perspective. In the mechanism of unconventional magnetic transitions, “isospin” develops entanglement with momentum orientation and genuinely becomes the physical spin, hence, it shares the same spirit of “spin-from-isospin” in gauge theories.¹⁵⁰ The consequential states can be viewed as “non- s -wave” generalizations of ferromagnetic metals in which spin no longer polarizes along a unique direction but varies with momentum forming a non-trivial representation of the rotation group.^{99,151}

In other words, effective spin-orbit coupling is generated as an order parameter through the Pomeranchuk type of Fermi surface instabilities, which is tunable by external parameters such as temperature and pressure. Furthermore, similar to magnetic fluctuations in ferromagnets, this effective spin-orbit coupling possesses its dynamics as collective modes. This gives rise to a conceptually new mechanism to generate spin-orbit coupling dynamically without involving relativity.^{98,99} Due to the richness of many-body physics, unconventional magnetism potentially provides a new way to engineer spin-orbit couplings and to control electron spins.

4.1. Fermi liquid theory and Pomeranchuk instabilities

The unconventional magnetism arises from the Fermi liquid instabilities of the Pomeranchuk type in the spin channel. In this subsection, we briefly review the concept of the non-relativistic Fermi liquid theory and Pomeranchuk instability.⁹⁷

A large part of our current understanding on interacting electronic systems are based on the Landau Fermi liquid theory, which was designed originally for the normal state ^3He and also applies for most metals.^{101,152} The central assumption is the existence of the well-defined Fermi surface and fermionic quasi-particles, which exist as long-lived states at very low energies. Interactions among quasi-particles are describe by the Landau interaction function, which are forward scattering processes

of quasi-particles near the Fermi surface. The Landau interaction function can be classified into the density (particle-hole singlet) and spin (particle-hole triplet) channels, which are also traditionally denoted as symmetric (s) and asymmetric (a) channels, respectively,

$$f_{\alpha\beta,\gamma\delta}(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2) = f^s(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2) + f^a(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2)\vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\gamma\delta}, \quad (73)$$

where $\hat{\mathbf{k}}$ is the direction of quasi-particle momentum close to the Fermi surface. Each of them can be further decomposed into different orbital partial wave channels as

$$f^{s,a}(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \sum_l f_l^{s,a} P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \quad (74)$$

where P_l is the l -th Legendre polynomial and l denotes the orbital angular momentum of the partial wave channel.

In the Landau Fermi liquid theory, the interactions among quasi-particles are captured by a few dimensionless Landau parameters

$$F_l^{s,a} = N_0 f_l^{s,a}, \quad (75)$$

where N_0 is the density of states on the Fermi surface. Physical susceptibility in each channel acquires significant renormalizations by the Landau interactions,

$$\chi_{FL,l}^{s(a)} = \chi_{0,l}^{s(a)} \frac{1 + F_1^s/3}{1 + F_l^{s(a)}/(2l+1)}. \quad (76)$$

For example, spin susceptibility χ_0^a lies in the F_0^a channel, and compressibility χ_0^s lies in the F_0^s channel.

Pomeranchuk instabilities refer to a large class of Fermi surface instabilities of Fermi surface distortions in both the density and spin channels.⁹⁷ In order for the Fermi surface to be stable, Landau parameters $F_l^{s(a)}$ cannot be too negative. Otherwise, Fermi surface distortions will occur. The Fermi surface could be viewed as an elastic membrane in momentum space. Consider to perturb the Fermi surface and expand the energy cost in different partial-wave channels. We arrive at

$$\frac{\Delta E}{V} = \frac{2\pi}{N_0} \sum_{lm} \left\{ \left(1 + \frac{F_l^{s,a}}{2l+1} \right) |\delta n_{lm}^{s,a}|^2 \right\}, \quad (77)$$

where $\delta n_{lm}^{s(a)}$ are amplitudes of Fermi surface distortions in partial-wave channels, and V the system volume. The first term is the kinetic energy cost which is always positive, while the second term is the interaction contribution, which can be either positive or negative. When $F_l^{s,a} < -(2l+1)$, the surface tension of the Fermi surface goes negative, and develops instability in the corresponding channels, which is consistent with the divergence of susceptibility in Eq. (76) at $F_l^{s,a} = -(2l+1)$.

The most familiar Pomeranchuk instabilities are found in the s -wave channel, i.e., ferromagnetism at $F_0^a < -1$ and phase separation at $F_0^s < -1$. Pomeranchuk instabilities in non- s -wave channels ($l \geq 1$) have been attracting a great deal of attention in recent years.^{98,99,153–165} The density channel instabilities result in

uniform but anisotropic liquid (nematic) phases,¹⁶⁶ which have been investigated in the context of doped Mott insulators,¹⁶⁷ high T_c materials.^{160,167} Experimental evidence has also been found in ultra-high mobility two-dimensional electron gases (2DEG) in $\text{Al}_x\text{Ga}_{1-x}\text{As}$ -GaAs heterostructures and quantum wells in large magnetic fields, in nearly half-filled high Landau levels at very low temperatures,^{168,169} and near the metamagnetic transition of the ultra-clean samples of the bilayer ruthenate $\text{Sr}_3\text{Ru}_2\text{O}_7$.^{170–172}

Unconventional magnetism corresponds to Pomeranchuk instabilities in the spin channel with $l \geq 1$.^{98,99,153,154,162–164} In Ref.,⁹⁸ these states are classified by the author and Zhang as isotropic and anisotropic phases dubbed β and α -phases, respectively. The α -phases was studied by many groups before: The p -wave phase was first studied by Hirsch^{153,154} under the name of the “spin-split” state, and was also proposed by Varma *et al.*^{162,163} as a candidate for the hidden order phenomena in the heavy fermion compound URu_2Si_2 ; the d -wave phase was studied by Oganessian *et al.*¹⁶⁶ under the name of “nematic-spin-nematic” phase. Systematic studies of ground state properties and collective excitations in both the anisotropic α and isotropic β -phases have been performed.^{99,151} Chubukov and Maslov found that when approaching the ferromagnetic quantum critical point, the p -wave channel spin Pomeranchuk instability develops before the ferromagnetic instability.¹⁷³

4.2. Unconventional magnetism as multipolar orderings

The unconventional magnetic order parameters are defined as multipolar parameters in momentum space but not in coordinate space.^{98,99} For simplicity, we first take the 2D p -wave case as an example. Its order parameters are the x and y -spatial components of spin-dipole moments defined as

$$\mathbf{n}_1 = \frac{|f_1^a|}{V} \sum_{\mathbf{k}} \mathbf{s}(\mathbf{k}) \hat{k}_x, \quad \mathbf{n}_2 = \frac{|f_1^a|}{V} \sum_{\mathbf{k}} \mathbf{s}(\mathbf{k}) \hat{k}_y, \quad (78)$$

where f_1^a is the Landau interaction parameter defined in Eq. (74); $\hat{k}_{x,y} = k_{x,y}/|k|$ are the p -wave angular form factors; $\mathbf{s}(\mathbf{k}) = \langle c_{\mathbf{k}\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{\mathbf{k}\beta} \rangle$ is the spin-moment of momentum \mathbf{k} , and $\langle \rangle$ means ground state expectation value. Each of $\mathbf{n}_{1,2}$ is a 3-vector in spin space. This is a natural generalization of the ferromagnetic moment $\mathbf{S} = \sum_{\mathbf{k}} \mathbf{s}(\mathbf{k})$ whose s -wave angular form factor is just a constant.

In the anisotropic p -wave α -phase depicted in Fig. 3 (C), the order parameter configuration is equivalent to only one of \mathbf{n}_1 and \mathbf{n}_2 is nonzero, or more generally, $\mathbf{n}_1 \parallel \mathbf{n}_2$. Their orientation in spin space is arbitrary. The order parameter configuration in the p -wave β phase depicted in Fig. 3 (B) shows that $\langle n_1^x \rangle = \langle n_2^y \rangle \neq 0$. More generally, this is equivalent to both $\mathbf{n}_{1,2} \neq 0$ and their orientations are perpendicular to each other as $\mathbf{n}_1 \perp \mathbf{n}_2$. Using an optics analogy, the spin configuration over the Fermi surface in the α -phase is linearly polarized, while that in the β -phase is circularly polarized.

This order parameter definition can be easily generalized into other partial wave channels in 2D and 3D systems by using the corresponding multipolar angular form factors. For example, the 2D d -wave channel order parameters can be defined as components of spin-quadrupole moments

$$\mathbf{n}_1^d = \frac{|f_2^a|}{V} \sum_{\mathbf{k}} \mathbf{s}(\mathbf{k}) \cos 2\phi_{\mathbf{k}}, \quad \mathbf{n}_2^d = \frac{|f_2^a|}{V} \sum_{\mathbf{k}} \mathbf{s}(\mathbf{k}) \sin 2\phi_{\mathbf{k}}, \quad (79)$$

where $\phi_{\mathbf{k}}$ is the azimuthal angle of \mathbf{k} . We could also combine them as a matrix form $n^{\mu,b}$ with each column representing a 3-vector \mathbf{n}_b . Below we will use both the matrix and vector forms of order parameters according to which one is more convenience in the context.

The 3D counterpart of these expressions can be written in terms of spherical harmonic functions. Hence, in 3D the Latin label b of the order parameter $n^{\mu b}$ take $2l + 1$ values, while the Greek index μ still takes x, y, z .

We consider a 2D Fermi-liquid system focusing on a general partial-wave channel- l . Since there is no spin-orbit coupling, the symmetry is the direct product $SO_L(2) \otimes SO_S(3)$, where L and S refer to the orbit and spin channels, respectively. The Landau interaction function f_a^l could depend on the total momentum \mathbf{q} of the particle-hole excitations with the assumption that

$$f(\mathbf{q}) = \frac{f_l^a}{1 + \kappa |f_l^a| q^2}, \quad (80)$$

which gives rise to an interaction range $\xi \approx \sqrt{\kappa |f_l^a|}$. Mean-field theory is valid when $\xi \gg d \approx 1/k_F$, where d is the inter-particle distance. After the mean-field decomposition, the mean-field Hamiltonian becomes

$$H_{MF} = \sum_{\mathbf{k}} \psi_{\alpha}^{\dagger}(\mathbf{k}) [\epsilon(\mathbf{k}) - \mu - (\mathbf{n}_1 \cos(l\theta_{\mathbf{k}}) + \mathbf{n}_2 \sin(l\theta_{\mathbf{k}})) \cdot \vec{\sigma}] \psi_{\beta}(\mathbf{k}) + \frac{|n_1|^2 + |n_2|^2}{2|f_l^a|}. \quad (81)$$

The validity of mean-field theory at quantum criticality requires an analysis of the effects of quantum fluctuations which are not included in mean field theory.^{174,175}

To determine the ground state configuration of $\vec{n}_{1,2}$, the Ginzburg-Landau (GL) free energy is constructed as,

$$F(\mathbf{n}_1, \mathbf{n}_2) = \gamma_1 \partial_a \mathbf{n}_b \cdot \partial_a \mathbf{n}_b + r(\mathbf{n}_1^2 + \mathbf{n}_2^2) + v_1[\mathbf{n}_1^2 + \mathbf{n}_2^2]^2 + v_2 |\mathbf{n}_1 \times \mathbf{n}_2|^2. \quad (82)$$

The coefficients $r, v_{1,2}$ are calculated from mean-field free energy in Ref.,⁹⁹ whose expressions are omitted here.

When $l = 1$, a new gradient term can appear which contains the linear order spatial derivative and the cubic order of order parameters as

$$\Delta F(\mathbf{n}_1, \mathbf{n}_2) = \gamma_2 \epsilon_{\mu\nu\lambda} n^{\mu a} n^{\nu b} \partial_a n^{\lambda b} = \gamma_2 \{(\partial_x \mathbf{n}_2 - \partial_y \mathbf{n}_1) \cdot (\mathbf{n}_1 \times \mathbf{n}_2)\}, \quad (83)$$

where γ_2 can be calculated approximately as $\gamma_2 \approx \frac{N_0}{v_f k_f^2}$ with v_f the Fermi velocity and k_f the Fermi momentum. Such a term is allowed because $\mathbf{n}_{1,2}$ are odd under

parity transformation and even under time-reversal transformation, i.e., $P\mathbf{n}_{1,2}P^{-1} = -\mathbf{n}_{1,2}$, and $T\mathbf{n}_{1,2}T^{-1} = \mathbf{n}_{1,2}$. It does not bring much effect in the normal phase because it is at the cubic order of the order parameter. However, we will see in Sect. 4.4.2, this term becomes important in the ordered p -wave β -phase, which drives a Lifshitz transition spontaneously developing a chiral pitch.

Both α and β -phases are driven by the negative value of r , i.e., $F_l < -2$ in 2D. Whether the ground state takes the β or α -phase depends on the sign of v_2 . If $v_2 < 0$, Eq. (82) favors $\mathbf{n}_1 \perp \mathbf{n}_2$, thus gives rise to the β -phase. On the other hand, α -phase appears at $v_2 > 0$, which favors $\mathbf{n}_1 \parallel \mathbf{n}_2$.

4.3. “Spin from isospin” in non-relativistic systems

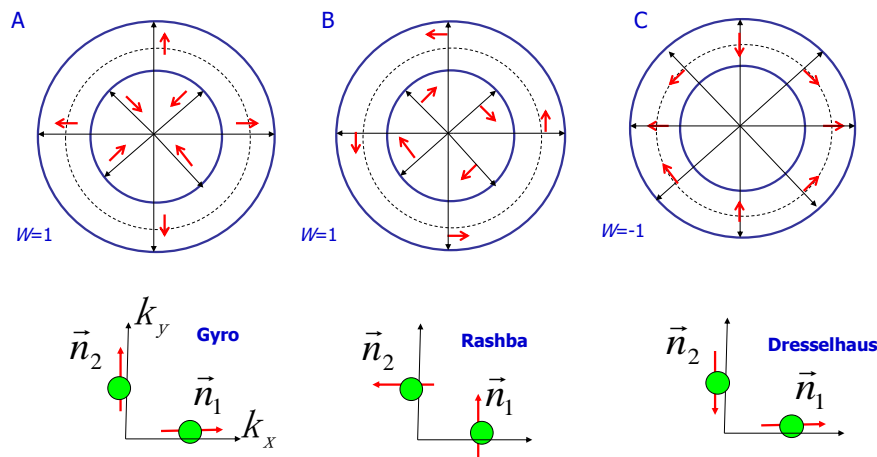


Fig. 10. Spontaneous spin-orbit orderings in the β -phases. Order parameter configurations and the momentum space vortices with the winding numbers $w = \pm 1$. (A) Gyrotropic ($w = 1$), (B) Rashba ($w = 1$), (C) Dresselhaus ($w = -1$). From Ref.⁹⁹

Spin in the relativistic theory, by definition, is part of the generators of the rotation transformation, hence, is always coupled to momentum as required by the Lorentz invariance. While in the non-relativistic theory, it decouples from momentum, and becomes an “isospin” type internal degree of freedom. The α and β -phases entangle spin with momentum together via order parameters. In this sense, spin genuinely changes from the status of “isospin” into spin. As we explained before, this effective spin-orbit coupling arises from many-body interaction instead of from the single-particle relativistic physics.

In the isotropic β -phase with $l \geq 1$, spin winds around the Fermi surface exhibiting a vortex-like structure in momentum space. For the 2D p -wave β -phase

depicted in Fig. 10 A, its mean-field single particle Hamiltonian reads

$$H_{MF,\beta} = \sum_{\mathbf{k}} \psi^\dagger(\mathbf{k}) \left[\epsilon_0(\mathbf{k}) - \mu - \bar{n}(\hat{k}_x \sigma_x + \hat{k}_y \sigma_y) \right] \psi(\mathbf{k}), \quad (84)$$

where $|n_1| = |n_2| = \bar{n}$. It exhibits a $\vec{\sigma} \cdot \mathbf{k}$ type spin-orbit coupling, which is called the gyrotropic spin-orbit coupling. The fermion spectrum is isotropic as $\epsilon(\mathbf{k}) = \epsilon_0(\mathbf{k}) \pm \bar{n}$ in the β -phase. Similarly to the ferromagnet, Fermi surfaces in the β -phase split into large and small circles. However, they are characterized by helicity, the spin projection along its momentum, not by spin polarization.

The symmetry breaking in the β -phases is particularly interesting. The normal Fermi liquid state has both spin and orbital rotational symmetries. The state depicted in Fig. 10 A is still isotropic where the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ remains conserved although \mathbf{L} and \mathbf{S} are no longer separately conserved. If we fix momentum and only rotate spin, the configuration in Fig. 10 A changes. In other words, the relative spin-orbit symmetry is broken, a concept first proposed by Leggett in superfluid ^3He systems.¹⁰¹

In solid state physics, Rashba and Dresselhaus are two familiar spin-orbit couplings whose spin configurations in momentum space are depicted in Fig. 10 B and C, which corresponds to order parameter configurations of $(\mathbf{n}_1 \parallel \hat{y}, \mathbf{n}_2 \parallel -\hat{x})$, and $(\mathbf{n}_1 \parallel \hat{x}, \mathbf{n}_2 \parallel -\hat{y})$, respectively. These two spin-orbit couplings are equivalent to the gyrotropic in Eq. (84) up to a global spin rotation. Starting from the configuration depicted Fig. 10 A, we can arrive at the Rashba configuration by fixing \mathbf{k} unchanged and rotating electron spin around z -axis at 90° of each \mathbf{k} . Similarly the Dresselhaus configuration can be obtained by the rotation around x -axis at 180° degree. These ground state spin configurations exhibit, in momentum space, the vortex structures with the winding numbers $w = \pm 1$. In principle, we can perform an arbitrary spin rotation to obtain all the equivalent states, thus the ground state Goldstone manifold is $[SO_L(2) \otimes SO_S(3)]/SO(2)_J = SO(3)$.

This vortex picture in momentum space can be generalized into a general F_l^a channel with the winding numbers $\pm l$. In fact, the consequential generated spin-orbit coupling pattern is beyond the relativity framework. In particular, for even values of l , the dynamic spin-orbit orders break time-reversal symmetry, while the relativistic spin-orbit coupling is time-reversal invariant. The mean field Hamiltonian $H_{\beta,l}$ for the β -phase in angular momentum channel l can be expressed through a d -vector, defined by $\mathbf{d}(\mathbf{k}) = (\cos(l\theta_{\mathbf{k}}), \sin(l\theta_{\mathbf{k}}), 0)$, as follows

$$H_{\beta,l} = \sum_{\mathbf{k}} \psi^\dagger(\mathbf{k}) \left[\epsilon(\mathbf{k}) - \mu - \bar{n} \vec{d}(\theta_{\mathbf{k}}) \cdot \vec{\sigma} \right] \psi(\mathbf{k}), \quad (85)$$

where $\mathbf{d}(\theta_{\mathbf{k}})$ is the spin quantization axis for single particle state at \mathbf{k} . Each Fermi surface is characterized by the eigenvalues ± 1 of the helicity operators $\vec{\sigma} \cdot \mathbf{d}(\hat{\mathbf{k}})$.

The mean field Hamiltonian in the anisotropic α -phase (Fig. 3 C) can be written as

$$H_{MF,\alpha} = \sum_{\mathbf{k}} \psi^\dagger(\mathbf{k}) \{ \epsilon_0(\mathbf{k}) - \mu - \bar{n} \hat{k}_x \sigma_z \} \psi(\mathbf{k}). \quad (86)$$

The fermion spectra read $\epsilon(k) = \epsilon_0(k) \pm \bar{n}\hat{k}_x$, and the spin up and down Fermi surfaces shift to left and right, respectively. This configuration is equivalent to the combination of Rashba and Dresselhaus spin-orbit couplings with a equal strength. It is anisotropic phase in both spin and orbit channels. Generally the Fermi surface shift can along any in-plane direction, and the spin axis can pick any 3D direction. In other words, we can perform either orbital rotation operation or spin rotation operation to reach all other equivalent configurations, thus the ground state Goldstone manifold is $[SO_L(2) \otimes SO_S(3)]/[SO_S(2)] = SO_L(2) \otimes S_2$.

These β and α -phases are particle-hole channel analogies to the triplet p -wave pairing superfluid ${}^3\text{He}$ - B and A phases, respectively. The order parameters in ${}^3\text{He}$ are defined as x, y and z -spatial components of the dipole-moment of the Cooper pairing amplitude over the Fermi surfaces.^{101,176} They are defined as

$$\Delta_i = \sum_{\mathbf{k}} \Delta(\mathbf{k}) \hat{k}_i \quad (i = x, y, z), \quad (87)$$

where $\Delta(\mathbf{k}) = \langle c_{\alpha}^{\dagger}(\mathbf{k})(i\sigma_2\vec{\sigma})_{\alpha\beta}c_{\beta}^{\dagger}(-\mathbf{k}) \rangle$. Each one of Δ_x, Δ_y and Δ_z is a 3-vector in spin space. In the B -phase, $\Delta_{x,y,z}$ are perpendicular to each other forming a triad. In the A -phase only two of them are nonzero with a phase difference of $\frac{\pi}{2}$, and they are parallel to each other in spin space. As a result, the B -phase is essentially isotropic with a constant gap over the Fermi surface, while the A -phase is anisotropic with nodes.

From the symmetry point of view, the unconventional magnetic β and α phases exhibit the similar properties to the ${}^3\text{He}$ - B and A phases under spatial rotations. The angular form factor of the pairing gap functions in the ${}^3\text{He}$ - B and A phases are very similar to the Fermi surface splittings in the p -wave magnetic β and α -phases, respectively.

4.4. Collective excitations in unconventional magnetic states

As a result of spontaneous symmetry breaking, unconventional magnetic states exhibit low energy excitations, *i.e.*, the Goldstone modes. In this subsection, we review the collective Goldstone modes in both α and β -phases. Such modes are absent in the conventional spin-orbit coupling systems.

4.4.1. Goldstone modes in the α -phase

we first comment on the stability of the p -wave α -phase. The Ginzburg-Landau energy of Eq. (83) contains a cubic term linear in spatial derivatives. It might induce a linear derivative coupling between the massless Goldstone modes at the quadratic level via the non-vanishing order parameter, leading to a Lifshitz instability in the ground state. However, as will be shown below the Goldstone modes in the α -phase either share the same orbital or spin indices as the condensed mode, hence, they cannot be coupled together by Eq. (83). Instead, the Goldstone modes couple to

other gapped modes at the quadratic level through linear derivative terms, which renormalizes the stiffness of the Goldstone modes.

The α -phases break rotational symmetries in both orbital and spin channels, hence, the Goldstone modes can be classified into density and spin channel modes, respectively. Without loss of generality, we assume the ordered configuration as shown in Fig. 3 C,

$$\langle n_{\mu b} \rangle = \bar{n} \delta_{\mu z} \delta_{b1}, \quad \text{i.e.}, \quad \mathbf{n}_1 = \bar{n} \hat{e}_z, \quad \mathbf{n}_2 = 0. \quad (88)$$

In other words, spin configuration is along $\pm \hat{\mathbf{z}}$, and Fermi surface distortion is along the x -axis. Among the 6-dimensional order parameter space, three are Goldstone modes as described by the manifold $S^2 \otimes SO_L(2)$, including one branch in the density channel, and two branches in the spin channel.

The density channel Goldstone mode is the oscillation of the distorted Fermi surface. The density channel Goldstone mode is associated with the field $n_{z,2}$,

$$n_{z,2}^z(\mathbf{q}) = -\frac{f_1^a(\mathbf{q})}{V} \sum_{\mathbf{k}} \psi_{\mathbf{k}+\mathbf{q}}^\dagger \vec{\sigma} \psi_{\mathbf{k}} k_y, \quad (89)$$

which describes the Fermi surface oscillation in the y -direction while keeping the spin configuration unchanged. Calculations at the random-phase approximation (RPA) level show the following effective Lagrangian,

$$L_{FS}^\alpha(\mathbf{q}, \omega) = N_0 \left[\frac{q\xi^2}{|F_l^a|} - i \frac{\omega}{2v_f q} (1 + 2 \cos 2\phi_q) \right] \quad (90)$$

This Goldstone mode is overdamped because of the Landau damping, and the damping depends on propagation directions.

The spin channel Goldstone modes $n_{sp,x\pm iy}$ describe spin oscillations while keeping the Fermi surface unchanged, which are spin-dipole precession modes. In contrast, they exhibit nearly isotropic underdamped dispersion relation at small propagating wavevectors with the dispersion relation

$$\omega_{x\pm iy}^2 = \frac{\bar{n}^2}{|F_1^a|} (q\xi)^2. \quad (91)$$

Different from spin-waves in the ferromagnets, the dispersion relation here is linear with momentum, which is a consequence of time-reversal symmetry.

4.4.2. Goldstone modes in the β -phase

We further study the Goldstone modes in the β -phase. For simplicity, we consider the 3D β -phase with the isotropic ground state exhibiting

$$n^{\mu a} = \bar{n} \delta_{\mu a}. \quad (92)$$

In other words, $\mathbf{n}_{1,2,3}$ form an orthogonal triad. The total angular momentum \mathbf{J} remains conserved, such that fluctuations of $\delta n^{\mu a}$ are classified into eigenstates of \mathbf{J} as $O_{jj_z}(\mathbf{q}, \omega)$. $j = 0, 1, 2$ mean the singlet, triplet and quintet channels respectively,

and j_z is the SO(2) quantum number rotating around the the propagation direction \mathbf{q} .

The Goldstone modes belong to the triplet channel ($j = 1$), which are the small relative spin-orbit rotations as

$$O_{1,0}(\mathbf{q}, \omega) = \frac{1}{\sqrt{2}} \epsilon_{z\mu a} \delta n^{\mu a}(\mathbf{q}, \omega), \quad O_{1,\pm 1}(\mathbf{q}, \omega) = \frac{1}{2} (\epsilon_{x\mu a} \pm i \epsilon_{y\mu a}) \delta n^{\mu a}(\mathbf{q}, \omega). \quad (93)$$

The RPA approximation gives their dispersion relations in the low energy and small momentum regime $\omega, v_f q \ll \bar{n} \ll v_f k_f$ as

$$\omega^2 = 4\bar{n}^2 |F_1^a| \left(\frac{\kappa q^2}{N_f} + \frac{j_z |q|x}{18k_f} \right) \quad (x = \frac{\bar{n}}{v_f k_f}, j_z = 0, \pm 1). \quad (94)$$

The linear dependence on q in the dispersion Eq. (94) is due to the broken parity in the ordered β phase. Consequently, ω^2 becomes negative for the branch with helicity $j_z = -1$ at small q . This means that the uniform ground state in the β -phase is unstable, instead it exhibits a Lifshitz-like instability. This behavior is a general feature in systems with broken parity such as the spiral order in helical magnets, and the cholesteric liquid crystals.¹⁷⁷ The true ground state configuration in the β phase is complicated with preliminary analysis presented in Ref.⁹⁹

4.4.3. Resonances in inelastic neutron scattering spectroscopy

The unconventional magnetic orders are spin-multipole moment in momentum space and cannot couple to neutron magnetic moments statically. Even in ordered unconventional magnetic states, there should be no elastic Bragg peaks. The spin-channel Goldstone modes in the α -phase do not couple to neutron moments directly, nevertheless, they carry spin quantum numbers and thus couple to spin-wave modes dynamically. Consider the following commutation relations,

$$[S_x, n_1^y] = i n_1^z, \quad [S_y, n_1^x] = -i n_1^z. \quad (95)$$

In the p -wave α -phase with the configuration given in Eq. (88), n_1^z can be replaced by the constants of $\pm i \bar{n}$. As a result, the Goldstone modes n_1^x and n_1^y become conjugate to spin polarization and couple to neutron spin moment dynamically.

More formally, we can write down the following coupling Lagrangian,

$$L = (\vec{n}_1 \times \partial_t \vec{n}_1 + \vec{n}_2 \times \partial_t \vec{n}_2) \cdot \vec{S}. \quad (96)$$

In the ordered state of Eq. (88), it is reduced to

$$L = \bar{n} (S_y \partial_t n_{1x} - S_x \partial_t n_{1y}). \quad (97)$$

The RPA approximation shows that the dynamic spin-spin correlation function behaves as

$$\chi_s(\mathbf{q}, \omega) = \langle S_+(\mathbf{q}, \omega) S_-(-\mathbf{q}, -\omega) \rangle = \frac{N_0 \frac{\omega^2}{\bar{n}^2}}{\frac{\kappa q^2}{N_0} - \frac{2}{|F_1^a|} \frac{\omega^2}{\bar{n}^2} - i\delta}. \quad (98)$$

Hence, it induces a resonance part in the transverse spin wave-excitations. The spectral functions exhibit the δ -peak at the excitation energy of the Goldstone mode,

$$\text{Im}\chi_s(\mathbf{q}, \omega) = \kappa\pi v_f^2 \bar{n}^2 q^2 |F_1^a|^2 \delta(\omega^2 - \omega_q^2), \quad (99)$$

which can be detected in the inelastic neutron scattering experiments. This is very similar to the interpretation from the SO(5) theory to the neutron resonance mode: the π -mode lies in the particle-particle channel which decouples from spin in the normal state, but becomes conjugate to spin in the superconducting state giving rise to the spin resonances.²⁹⁻³¹

Such a resonance peak only exhibits in the ordered phase, and vanishes in the disordered phase. As shown in Eq. (97), in the anisotropic α -phases this resonances only occur in spin-flip channels. Similar analysis can also be performed in the isotropic β -phases, in which the resonates occur in both spin-flip and non-flip channels.⁹⁹

4.5. Spin-orbit coupled Fermi liquid theory

So far we have considered the dynamic generation of spin-orbit coupling, or, “spin from isospin”, in non-relativistic Fermi liquid theory. Nevertheless in many materials with heavy elements, there does exist the relativistic spin-orbit coupling.

If a system does not exhibit inversion symmetry, the relativistic spin-orbit coupling leads to Fermi surface splitting, say, the Rashba type. In this case, the relativistic spin-orbit coupling behaves like an external field which would round off the unconventional phase transition and pin down a particular spin-orbit ordering configuration. This situation is similar to cool a magnet below the transition temperature in an external magnetic field.

On the other hand, if a system Hamiltonian still preserves both parity and time-reversal symmetries, the Fermi surface should remain doubly degenerate. Spin-orbit coupling does not manifest itself in the Fermi surface splitting but should exhibit in the Landau Fermi liquid theory. Such a situation also occurs in the presence of prominent magnetic dipolar interactions, which is only invariant under simultaneous spin-orbit rotations. Landau-Fermi liquid theory has been extended to this simulation,^{178,179} and Pomeranchuk instabilities have also been studied in this context.

In the inversion invariant spin-orbit coupling systems, the fermion distribution function is reorganized in the spin-orbit coupled bases as

$$\delta n_{\alpha\alpha'}(\hat{k}) = \sum_{JJ_z;LS} \delta n_{JJ_z;LS} Y_{JJ_z;LS}(\hat{k}, \alpha\alpha'), \quad (100)$$

where $Y_{JJ_z;LS}(\hat{k}, \alpha\alpha')$ is the spin-orbit coupled spherical harmonic functions

$$Y_{JJ_z;LS}(\hat{k}, \alpha\alpha') = \sum_{ms_z} \langle LmSs_z | JJ_z \rangle Y_{Lm}(\hat{k}) \chi_{Ss_z, \alpha\alpha'}, \quad (101)$$

and $\chi_{Ss_z, \alpha\alpha'}$ is the bases for the particle-hole singlet (density) channel and triplet (spin) channel, respectively. The Landau interaction function is generalized to the interaction matrix,

$$\frac{N_0}{4\pi} f_{\alpha\alpha'; \beta\beta'}(\hat{k}, \hat{k}') = \sum_{JJ_z LL'} Y_{JJ_z; L1}(\hat{k}, \alpha\alpha') F_{JJ_z L1; JJ_z L'1} Y_{JJ_z; L'1}^\dagger(\hat{k}, \beta\beta'), \quad (102)$$

where we only keep the particle-hole triplet component. The Landau matrix is diagonal with respect to the total angular momentum J and its z -component J_z , but may have off-diagonal elements with $L \neq L'$. Constrained by the parity invariance, $L - L' = 0, 2$.

Similarly to the non-relativistic case, when there exists an eigenvalue of the Landau interaction matrix is negatively large, i.e. $\lambda < -1$, it triggers the Pomeranchuk instability in the corresponding channel. For example, the instability in the channel with $J = 1^-$, $L = S = 1$, where “ $-$ ” means odd parity, generates the 3D analogy of the Rashba spin-orbit coupling,

$$H_{so, 1^-} = |n| \sum_{\mathbf{k}} \psi^\dagger(\mathbf{k}) (\mathbf{k} \times \vec{\sigma}) \cdot \hat{\mathbf{I}} \psi(\mathbf{k}), \quad (103)$$

where $\hat{\mathbf{I}}$ is a 3D unit direction, $|n|$ is the magnitude of the spin-orbit order parameter. The Pomeranchuk instability promotes it to the single particle level by breaking the rotational symmetry and parity.

Let us still use the order parameter n^{ub} defined in Sect. 4.2 to represent the order parameters in the sector $L = S = 1$ for a 3D inversion invariant spin-orbit coupled Fermi liquid theory. The 3×3 matrix of n include three sectors of $J = 0, 1, 2$, which corresponds to pseudo-scalar (gryotropic), vector (Rashba), and tensor (Dresselhaus) type spin-orbit coupling, respectively.

The Ginzburg-Landau free energy can be constructed as $F = F_0 + \Delta F$,

$$\begin{aligned} F_0 &= r_0 \text{tr}(n^T n) + \beta_1 \left(\text{tr}(n^T n) \right)^2 + \beta_2 \text{tr} \left(n^T n \right)^2, \\ \Delta F &= \frac{r_1}{3} (\text{tr} n)^2 + \frac{r_2}{4} \text{tr}(n^T - n)^2. \end{aligned} \quad (104)$$

Under $SO_L(3)$ and $SO_S(3)$ rotations, n is transformed as $n \rightarrow T_S n T_L^\dagger$, where $T_{L,S}$ is the rotation matrix in the orbit and spin channels, respectively. F_0 is invariant under independent T_L and T_S , and ΔF is only invariant under simultaneous spin-orbit rotations. The $r_{1,2}$ terms are analogy to magnetic anisotropy for magnetic phase transitions, which lead to different types of universal classes. Then at the quadratic level, the eigenvalues of the pseudo-scalar, vector, tensor channels are determined by $r_0 + r_1$, $r_0 + r_2$, and r_0 , respectively. The actual ordering depends on which eigenvalue is negatively most dominant. If the pseudo-scalar channel instability dominates, the phase transition only breaks parity, which is an Ising type transition without the Goldstone mode. If the vector channel instability dominates, there is also rotational symmetry breaking with the Goldstone manifold S^2 . The symmetry breaking pattern for the tensor ordering channel is more involved, the Goldstone manifold is formally denoted as $SO(3)/G$, where G is the residual symmetry group

in the ordered state. The nature of G depends on at which tensor component the symmetry breaking takes place. Nevertheless, when β_2 is included, the situation is complicated, and the analysis of the phase diagram is deferred to another work.[?]

4.6. Discussions

We are not aware of conclusive evidence for the existence of the unconventional magnets. Taking into account the great discoveries of the unconventional superconductivity and pairing superfluidity in high T_c cuprates and ^3He , respectively, we are optimistic that unconventional magnetic phases also exist in Nature. We propose to systematically search for these new phases in ^3He , ultra-cold atomic systems, semiconductors, heavy fermion materials and ruthenates, both in experiments and in numerical simulations.

Unconventional magnetic orders are natural generalizations of itinerant ferromagnetism, whose driving force is still the exchange interaction. But it needs to be in the non-local version, i.e., a non- s -wave channel. Nevertheless, interactions in the high angular momentum channels are typically weak. In Ref.,¹⁸⁰ a heuristic argument is provided to employ the orbital hybridized band structure to promote the Landau interaction to high partial-wave channels. Consider a d_{xz}/d_{yz} hybridized orbital band. Around the Fermi surface, the Bloch wavefunction takes the orbital configuration as

$$|\Psi_\alpha(\mathbf{k})\rangle = e^{i\mathbf{k}\cdot\mathbf{r}} (\cos \phi_k |d_{xz}\rangle + \sin \phi_k |d_{yz}\rangle) \otimes \chi_\alpha, \quad (105)$$

where χ_α is the spin eigenstate. The Landau interaction at the Hartree-Fock level is

$$\begin{aligned} f_{\uparrow\uparrow}(\mathbf{k}_1\mathbf{k}_2) &= V(\mathbf{q}=0) - \frac{1}{2}(1 + \cos 2\theta_{\mathbf{k}_1\mathbf{k}_2})V(\mathbf{k}_1 - \mathbf{k}_2), \\ f_{\uparrow\downarrow}(\mathbf{k}_1\mathbf{k}_2) &= V(\mathbf{q}=0). \end{aligned} \quad (106)$$

The appearance of the d -wave form factor $\cos 2\theta_{\mathbf{k}_1\mathbf{k}_2}$ is due to the orbital hybridization, i.e., even though two electrons possess the same spin, they can still be distinguished by their orbital components. Hence, although $V(\mathbf{k}_1 - \mathbf{k}_2)$ could be dominated by the s -wave component, the angular form factor shifts a significant part of the spectral weight into the d -wave channel. Based on this formalism, a possible explanation of the nematic transition observed in $\text{Sr}_3\text{Ru}_2\text{O}_7$ was provided.

Below we summarize several possible direction for searching unconventional magnetism. Ferromagnetic fluctuations in the normal state of ^3He are strong. The values of F_{1a} of ^3He are measured as negative via normal-state spin diffusion constant, spin-wave spectrum, and the temperature dependence of the specific heat.¹⁸¹⁻¹⁸⁴ It varies from around -0.5 to -1.2 with increasing pressures to the melting point, reasonably close to the instability point $F_1^a = -3$. We conjecture that ^3He could support unconventional magnetism under certain conditions or exhibit strong fluctuations of these orders.

An important direction to search for unconventional magnetism is the so-called “hidden-order” systems. Hidden orders typically mean that thermodynamic quantity measurements exhibit a transition to a low temperature ordered state. However, the nature of the orders remains unknown since they do not exhibit themselves in typical detections. Unconventional magnetic orders neither break translation symmetry nor exhibit magnetic orderings in real space. They are multipolar orderings in momentum space, hence, they are difficult to detect via typical experimental methods. Hence, they are natural candidates for hidden orders. In fact, multipolar orderings in real space are also popular candidates for hidden orders in literature.

For example, the well-known system of heavy fermion compound URu₂Si₂, which exhibits a mysterious phase transition at 17K by showing a large anomaly in specific heat. It also exhibits a jump in the non-linear magnetic susceptibility at the transition. However, even with efforts after a few decades, the nature of this transition remains elusive.^{185,186} Varma proposed an order, which is essentially the *p*-wave α -phase in our language.^{162,163} Calculations for thermodynamic quantities fit in experiment measurements reasonably well. Another hidden order compound Cd₂Re₂O₇ exhibits heat capacity anomaly and a kink of DC resistivity around 200K. Recently, it has been discovered that the hidden order phase exhibits inversion symmetry breaking via the optical 2nd harmonic generation measurements.^{187,188} Since this is a heavy element compound, Pomeranchuk instabilities of spin-orbit coupled Fermi liquid theory may be a promising candidate.^{179,189}

An obstacle to identify unconventional magnetism is the lack of definitive experimental signatures and detection methods. We know that antiferromagnetism is very common among transition metal oxides, more common than itinerant ferromagnetism. However, the experimental identification of the antiferromagnetic ordering is only possible after the detection method of neutron scattering spectroscopy became available.

Maybe unconventional magnetism already exists somewhere, but we need to think how to detect them. In addition to the inelastic neutron scattering resonances (Sect. 4.4.3), we outline the following possible methods.

The β phases exhibit effective spin-orbit coupling, hence, standard methods to detect spin-orbit coupling still apply. The distinctive feature is that the spin-orbit coupling effects should turn on and off at a phase transition.

Transport measurements can be used to detect the dynamic generation of spin-orbit coupling. For example, the existence of the anomalous Hall effect (AHE) relies on spin-orbit coupling. Therefore, detecting the AHE signal turning on at a phase transition would be an evidence of the onset of the entanglement of spin and momentum. As for the *d*-wave α -phase, i.e., spin- \uparrow and spin- \downarrow Fermi surfaces exhibit opposite quadrupolar distortions. Taking the principle axes of the quadrupolar distortion as *x* and *y*-axis, it is straightforward to show that the spin and charge

currents satisfy

$$\begin{pmatrix} j_x^{sp} \\ j_y^{sp} \end{pmatrix} \propto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} j_x^c \\ j_y^c \end{pmatrix}. \quad (107)$$

An verification of this transport relation would be a signature of the d -wave α -phase.⁹⁹

Methods can detect Fermi surface splitting are useful. The angular resolved photon emission spectroscopy (ARPES) can be used to detect the band splitting. In fact, such experiment has been performed in the system with relativistic spin-orbit coupling. In the unconventional magnetic phases, ARPES in principle can measure temperature dependent Fermi surface splittings. Fermi surface splitting also shows up in quantum oscillation experiments (e.g. Shubnikov-de Haas (SdH) oscillations) as beat patterns. Hence, a temperature dependent beat pattern in this kind of experiments would signature the developing of unconventional magnetism.

5. Conclusions

We presented several novel applications of the symmetry principle in condensed matter and cold atomic systems.

First, we reviewed the construction of the concept of “space-time” group, which provides a symmetry framework for studying transport and topological properties in a variety of dynamic systems beyond the Floquet framework, such as laser-driven solid state lattices, dynamic photonic crystals, and optical lattices. Various fundamental concepts are generalized, including space-time unit cell, momentum-energy Brillouin zone, Bloch-Floquet theory. Novel non-symmorphic space-time transformations are identified including “time screw” rotation, “time glide” reflection, and “time shift” rotary reflection. 13 space-time groups are classified in 1+1D with 5 of them non-symmorphic, and 275 space-time group are classified in 2+1D. We expect that space-time group will play an important role for studying dynamic systems, in a similar way to space group for studying static crystals.

Second, we reviewed the progress of studying large-spin ultra-cold fermions from the perspective of high symmetries. We show that due to enhanced quantum spin fluctuations from the large fermion components, large spin cold fermions naturally lie in the large- N region instead of the large- S region which is typically studied in solids. A generic $\text{Sp}(4)$, or, isomorphically, $\text{SO}(5)$ symmetry is proved in spin- $\frac{3}{2}$ systems, which plays the similar role of $\text{SU}(2)$ in spin- $\frac{1}{2}$ systems. This symmetry can be upgraded to $\text{SO}(7)$ under certain conditions which extends Yang’s η -pairing to χ -pairing as its high rank Lie algebra counterpart. The 7D vector and 21D adjoint representations of $\text{SO}(7)$ unify a variety of competing orders in both particle-particle and particle-hole channels. Large-spin systems can exhibit multi-particle clustering orderings or correlations both in the superfluid state with attractions and in the super-exchange physics with repulsions, which is similar to 3-quark baryon (color singlet) formation in high energy physics. The competitions among quarteting

superfluidity/density-wave and pairing superfluidity/density-wave are investigated. The $SU(4)$ singlet plaquette states in a 3D cubic lattice can be described by a quantum plaquette model, whose effective description is mapped to a high order gauge theory. We anticipate that research along this direction can bridge cold atom physics, condensed matter matter, and high energy physics together. Along with the experiment progress, even more exotic strong coupling physics that is not easily accessible in usual solid state systems could be investigated.

At last, we reviewed the unconventional magnetism as a mechanism of “spin from isospin” to generate spin-orbit coupling in non-relativistic Fermi liquids. They are also novel states of itinerant electrons generalizing ferromagnetism to unconventional symmetries based on the Fermi surface instabilities of the Pomeranchuk type. These states include the isotropic β -phase and the anisotropic α -phase, which are is the particle-hole channel analogy to the superfluid $^3\text{He-B}$ and A phases, respectively. Different from the relativistic spin-orbit coupling, these dynamically generated spin-orbit couplings possess collective excitations of Goldstone modes, whose dynamics couples to spin moment and induces resonances in the inelastic neutron scattering spectroscopy. Possible realizations of “unconventional magnetism” in hidden order systems and experimental detections are discussed.

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References

1. T. D. Lee and C. N. Yang. Question of parity conservation in weak interactions. *Phys. Rev.*, 104:254–258, Oct 1956.
2. C. N. Yang and R. L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.*, 96:191–195, Oct 1954.
3. C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.*, 19:1312–1315, Dec 1967.
4. Tai Tsun Wu and Chen Ning Yang. Concept of nonintegrable phase factors and global formulation of gauge fields. *Phys. Rev. D*, 12:3845–3857, Dec 1975.
5. Tai Tsun Wu and Chen Ning Yang. Dirac monopole without strings:monopole harmonics. *Nucl. Phys. B*, 107:365, 1976.
6. H.Weyl. *Symmetry*. Princeton University Press; Reprint edition, 2016.
7. David J. Gross. The role of symmetry in fundamental physics. *Proceedings of the National Academy of Sciences*, 93(25):14256–14259, 1996.

8. C. N. Yang. Symmetry and physics. *Proceedings of the American Philosophical Society*, 140:267, 1996.
9. M. Lax. *Symmetry Principles in Solid State and Molecular Physics*. Dover Publications, 2012.
10. H. Georgi. *Lie Algebras In Particle Physics: from Isospin To Unified Theories*. CRC Press; 1st edition, 1999.
11. E. Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1918:235–257, 1918.
12. E. P. Wigner. *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*. Academic Press, New York, 1959.
13. H. Weyl. *The Theory of Groups and Quantum Mechanics*. Dover, New York, 1950.
14. V. Fock. On the theory of the hydrogen atom. *Z. Phys.*, 98:145, 1935.
15. J. J. Sakurai and J. J. Napolitano. *Modern Quantum Mechanics*. Pearson, 2010.
16. S Glashow. The renormalizability of vector meson interactions. *Nucl. Phys. B*, 10:107, 1959.
17. A Salam and J. C. Ward. Weak and electromagnetic interactions. *Nuovo Cimento.*, 11:568–577, 1959.
18. Steven Weinberg. A model of leptons. *Phys. Rev. Lett.*, 19:1264–1266, Nov 1967.
19. M.E. Peskin and D.V. Schroeder. *An Introduction To Quantum Field Theory*. Frontiers in Physics. Avalon Publishing, 1995.
20. Lev D. Landau. On the theory of phase transition. *Zh. Eksp. Teor. Fiz.*, 7:19–32, 1937.
21. V. L. Ginzburg and L. D. Landau. *Zh. Eksp. Teor. Fiz.*, 20:1064, 1950.
22. L D Landau and E.M. Lifshitz. *Statistical Physics*. Butterworth-Heinemann; 3rd edition, 1980.
23. Jeffrey Goldstone, Abdus Salam, and Steven Weinberg. Broken symmetries. *Phys. Rev.*, 127:965–970, Aug 1962.
24. P. W. Anderson. Plasmons, gauge invariance, and mass. *Phys. Rev.*, 130:439–442, Apr 1963.
25. Peter W. Higgs. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.*, 13:508–509, Oct 1964.
26. C. N. Yang. Eta-pairing and off-diagonal long-range order in a hubbard-model. *Phys. Rev. Lett.*, 63(19):2144–2147, 1989.
27. CHEN NING YANG and S.C. ZHANG. So4 symmetry in a hubbard model. *Modern Physics Letters B*, 04(11):759–766, 1990.
28. S. C. Zhang. So-4 symmetry of the hubbard-model and its experimental consequences. *International Journal of Modern Physics B*, 5(1-2):153–167, 1991.
29. S. C. Zhang. A unified theory based on so(5) symmetry of superconductivity and antiferromagnetism. *Science*, 275(5303):1089–1096, 1997.
30. Eugene Demler, Werner Hanke, and Shou-Cheng Zhang. SO(5) theory of antiferromagnetism and superconductivity. *Rev. Mod. Phys.*, 76:909–974, Nov 2004.
31. E. Demler and S. C. Zhang. Theory of the resonant neutron-scattering of high-t-c superconductors. *Phys. Rev. Lett.*, 75(22):4126–4129, 1995.
32. C. Kittel. *Quantum theory of Solids*. Wiley, New York, 1987.
33. Y. H. Wang, H. Steinberg, P. Jarillo-Herrero, and N. Gedik. Observation of floquet-bloch states on the surface of a topological insulator. *Science*, 342:453, 2013.
34. Hanyu Zhu, Jun Yi, Ming-Yang Li, Jun Xiao, Lifa Zhang, Chih-Wen Yang, Robert A Kaindl, Lain-Jong Li, Yuan Wang, and Xiang Zhang. Observation of chiral phonons. *Science*, 359:579–582, 2018.

35. Colin V. Parker, Li-Chung Ha, and Cheng Chin. Direct observation of effective ferromagnetic domains of cold atoms in a shaken optical lattice. *Nature Physics*, 9(12):769C774, Oct 2013.
36. Brandon M. Anderson, Logan W. Clark, Jennifer Crawford, Andreas Glatz, Igor S. Aranson, Peter Scherpelz, Lei Feng, Cheng Chin, and K. Levin. Direct lattice shaking of bose condensates: Finite momentum superfluids. *Physical Review Letters*, 118(22), May 2017.
37. Mikael C. Rechtsman, Julia M. Zeuner, Yonatan Plotnik, Yaakov Lumer, Daniel Podolsky, Felix Dreisow, Stefan Nolte, Mordechai Segev, and Alexander Szameit. Photonic Floquet topological insulators. *Nature*, 496(7444):196–200, apr 2013.
38. Mark S. Rudner, Netanel H. Lindner, Erez Berg, and Michael Levin. Anomalous Edge States and the Bulk-Edge Correspondence for Periodically Driven Two-Dimensional Systems. *Phys. Rev. X*, 3(3):031005, jul 2013.
39. Manisha Thakurathi, Aavishkar A. Patel, Diptiman Sen, and Amit Dutta. Floquet generation of Majorana end modes and topological invariants. *Phys. Rev. B - Condens. Matter Mater. Phys.*, 88(15):155133, oct 2013.
40. C. W. von Keyserlingk and S. L. Sondhi. Phase structure of one-dimensional interacting Floquet systems. II. Symmetry-broken phases. *Phys. Rev. B*, 93(24):245146, jun 2016.
41. Zhenghao Gu, H. A. Fertig, Daniel P. Arovas, and Assa Auerbach. Floquet spectrum and transport through an irradiated graphene ribbon. *Phys. Rev. Lett.*, 107(21):216601, nov 2011.
42. Netanel H. Lindner, Gil Refael, and Victor Galitski. Floquet topological insulator in semiconductor quantum wells. *Nat. Phys.*, 7(6):490–495, jun 2011.
43. Dominic V. Else, Bela Bauer, and Chetan Nayak. Floquet time crystals. *Phys. Rev. Lett.*, 117:090402, Aug 2016.
44. C. W. Von Keyserlingk and S. L. Sondhi. Phase structure of one-dimensional interacting Floquet systems. I. Abelian symmetry-protected topological phases. *Phys. Rev. B - Condens. Matter Mater. Phys.*, 93(24):245145, jun 2016.
45. Andrew C. Potter, Takahiro Morimoto, and Ashvin Vishwanath. Classification of Interacting Topological Floquet Phases in One Dimension. *Phys. Rev. X*, 6(4):041001, oct 2016.
46. Rahul Roy and Fenner Harper. Periodic Table for Floquet Topological Insulators. *arXiv:1603.06944*, mar 2016.
47. Frederik Nathan and Mark S Rudner. Topological singularities and the general classification of Floquet-Bloch systems. *New J. Phys.*, 17(12):125014, dec 2015.
48. N. Fläschner, Benno S. Rem, M. Tarnowski, D. Vogel, D.-S. Lühmann, Klaus Sengstock, and C. Weitenberg. Experimental reconstruction of the Berry curvature in a topological Bloch band. *1509.05763*, 05882(2015):1–8, 2015.
49. Shenglong Xu and Congjun Wu. Space-time crystal and space-time group. *Phys. Rev. Lett.*, 120:096401, Feb 2018.
50. Takahiro Morimoto, Hoi Chun Po, and Ashvin Vishwanath. Floquet topological phases protected by time glide symmetry. *Phys. Rev. B*, 95:195155, May 2017.
51. Ian Affleck. Large- n limit of $SU(n)$ quantum "spin" chains. *Phys. Rev. Lett.*, 54:966–969, Mar 1985.
52. Daniel P. Arovas and Assa Auerbach. Functional integral theories of low-dimensional quantum heisenberg models. *Phys. Rev. B*, 38:316–332, Jul 1988.
53. Ian Affleck and J. Brad Marston. Large- n limit of the heisenberg-hubbard model: Implications for high- T_c superconductors. *Phys. Rev. B*, 37:3774–3777, Mar 1988.
54. N. Read and Subir Sachdev. Spin-peierls, valence-bond solid, and néel ground states

- of low-dimensional quantum antiferromagnets. *Phys. Rev. B*, 42:4568–4589, Sep 1990.
55. Congjun Wu, Jiang-ping Hu, and Shou-cheng Zhang. Exact $so(5)$ symmetry in the spin-3/2 fermionic system. *Phys. Rev. Lett.*, 91:186402, Oct 2003.
 56. Shu Chen, Congjun Wu, Shou-Cheng Zhang, and Yupeng Wang. Exact spontaneous plaquette ground states for high-spin ladder models. *Phys. Rev. B*, 72:214428, Dec 2005.
 57. Congjun Wu. Competing orders in one-dimensional spin-3/2 fermionic systems. *Phys. Rev. Lett.*, 95:266404, Dec 2005.
 58. C. Wu. Hidden Symmetry and Quantum Phases in SPIN-3/2 Cold Atomic Systems. *Modern Physics Letters B*, 20:1707–1738, 2006.
 59. Cenke Xu and Congjun Wu. Resonating plaquette phases in $su(4)$ heisenberg anti-ferromagnet. *Phys. Rev. B*, 77:134449, Apr 2008.
 60. CONGJUN WU, JIANGPING HU, and SHOU-CHENG ZHANG. Quintet pairing and non-abelian vortex string in spin-3/2 cold atomic systems. *International Journal of Modern Physics B*, 24(03):311–322, 2010.
 61. Hsiang-Hsuan Hung, Yupeng Wang, and Congjun Wu. Quantum magnetism in ultracold alkali and alkaline-earth fermion systems with symplectic symmetry. *Phys. Rev. B*, 84:054406, Aug 2011.
 62. Zhi-Qiang Gao and Congjun Wu. Exceptional symmetry of g_2 in spin- $\frac{3}{2}$ fermion systems, 2020.
 63. A. V. Gorshkov, M. Hermele, V. Gurarie, C. Xu, P. S. Julienne, J. Ye, P. Zoller, E. Demler, M. D. Lukin, and A. M. Rey. Two-orbital $su(n)$ magnetism with ultracold alkaline-earth atoms. *Nat Phys*, 6(4):289–295, April 2010.
 64. Michael Hermele, Victor Gurarie, and Ana Maria Rey. Mott insulators of ultracold fermionic alkaline Earth atoms: underconstrained magnetism and chiral spin liquid. *Phys. Rev. Lett.*, 103(13):135301, sep 2009.
 65. Cenke Xu. Liquids in multiorbital $SU(n)$ magnets made up of ultracold alkaline-earth atoms. *Phys. Rev. B*, 81:144431, Apr 2010.
 66. M A Cazalilla, A F Ho, and M Ueda. Ultracold gases of ytterbium: ferromagnetism and mott states in an $su(6)$ fermi system. *New Journal of Physics*, 11(10):103033, 2009.
 67. D. Controzzi and A. M. Tsvelik. Exactly solvable model for isospin $s = 3/2$ fermionic atoms on an optical lattice. *Phys. Rev. Lett.*, 96:097205, Mar 2006.
 68. P. Lecheminant, E. Boulat, and P. Azaria. Confinement and superfluidity in one-dimensional degenerate fermionic cold atoms. *Phys. Rev. Lett.*, 95:240402, Dec 2005.
 69. Kazumasa Hattori. Critical nature of non-fermi liquid in spin 3/2 multipolar kondo model. *Journal of the Physical Society of Japan*, 74(12):3135–3138, 2005.
 70. Hong-Hao Tu, Guang-Ming Zhang, and Lu Yu. Spin-quadrupole ordering of spin- $\frac{3}{2}$ ultracold fermionic atoms in optical lattices in the one-band hubbard model. *Phys. Rev. B*, 74:174404, Nov 2006.
 71. Hong-Hao Tu, Guang-Ming Zhang, and Lu Yu. Mott insulating phases and quantum phase transitions of interacting spin- $\frac{3}{2}$ fermionic cold atoms in optical lattices at half filling. *Phys. Rev. B*, 76:014438, Jul 2007.
 72. Stellan Östlund, T. H. Hansson, and A. Karlhede. Properties of the doped spin- $\frac{3}{2}$ mott insulator near half filling. *Phys. Rev. B*, 71:165121, Apr 2005.
 73. M. Bartenstein, A. Altmeyer, S. Riedl, R. Geursen, S. Jochim, C. Chin, J. Hecker Denschlag, R. Grimm, A. Simoni, E. Tiesinga, C. J. Williams, and P. S. Julienne. Precise determination of ^6Li cold collision parameters by radio-frequency spectroscopy on weakly bound molecules. *Phys. Rev. Lett.*, 94:103201, Mar 2005.
 74. Ákos Rapp, Gergely Zaránd, Carsten Honerkamp, and Walter Hofstetter. Color su-

- perfluidity and “baryon” formation in ultracold fermions. *Phys. Rev. Lett.*, 98:160405, Apr 2007.
75. Á. Rapp, W. Hofstetter, and G. Zaránd. Trionic phase of ultracold fermions in an optical lattice: A variational study. *Phys. Rev. B*, 77(14):144520, April 2008.
 76. Shintaro Taie, Yosuke Takasu, Seiji Sugawa, Rekishu Yamazaki, Takuya Tsujimoto, Ryo Murakami, and Yoshiro Takahashi. Realization of a $SU(2) \times SU(6)$ system of fermions in a cold atomic gas. *Phys. Rev. Lett.*, 105:190401, Nov 2010.
 77. Shintaro Taie, Rekishu Yamazaki, Seiji Sugawa, and Yoshiro Takahashi. An $su(6)$ mott insulator of an atomic fermi gas realized by large-spin pomeranchuk cooling. *Nat Phys*, 8(11):825–830, November 2012.
 78. S. Sugawa, K. Inaba, S. Taie, R. Yamazaki, M. Yamashita, and Y. Takahashi. Interaction and filling-induced quantum phases of dual Mott insulators of bosons and fermions. *Nature Physics*, 7:642–648, August 2011.
 79. Hideaki Hara, Yosuke Takasu, Yoshifumi Yamaoka, John M. Doyle, and Yoshiro Takahashi. Quantum degenerate mixtures of alkali and alkaline-earth-like atoms. *Phys. Rev. Lett.*, 106:205304, May 2011.
 80. Guido Pagano, Marco Mancini, Giacomo Cappellini, Pietro Lombardi, Florian Sch?fer, Hui Hu, Xia-Ji Liu, Jacopo Catani, Carlo Sias, Massimo Inguscio, and Leonardo Fallani. A one-dimensional liquid of fermions with tunable spin. *Nature Physics*, 10(3):198C201, Feb 2014.
 81. B. J. DeSalvo, M. Yan, P. G. Mickelson, Y. N. Martinez de Escobar, and T. C. Killian. Degenerate fermi gas of ^{87}Sr . *Phys. Rev. Lett.*, 105:030402, Jul 2010.
 82. P. G. Mickelson, Y. N. Martinez de Escobar, M. Yan, B. J. Desalvo, and T. C. Killian. Bose-Einstein condensation of Sr88 through sympathetic cooling with Sr87. *Phys. Rev. A*, 81(5):051601, May 2010.
 83. J. Heinze, J. S. Krauser, N. Fläschner, K. Sengstock, C. Becker, U. Ebling, A. Eckardt, and M. Lewenstein. Engineering spin waves in a high-spin ultracold Fermi gas. *ArXiv e-prints*, February 2013.
 84. Jasper S. Krauser, Jannes Heinze, Nick Flaschner, Soren Gotze, Ole Jurgensen, Dirk-Soren Luhmann, Christoph Becker, and Klaus Sengstock. Coherent multi-flavour spin dynamics in a fermionic quantum gas. *Nat Phys*, 8(11):813–818, November 2012.
 85. M. Bishof, M. J. Martin, M. D. Swallows, C. Benko, Y. Lin, G. Quémener, A. M. Rey, and J. Ye. Inelastic collisions and density-dependent excitation suppression in a ^{87}Sr optical lattice clock. *Phys. Rev. A*, 84(5):052716, November 2011.
 86. M. Bishof, Y. Lin, M. D. Swallows, A. V. Gorshkov, J. Ye, and A. M. Rey. Resolved Atomic Interaction Sidebands in an Optical Clock Transition. *Physical Review Letters*, 106(25):250801, June 2011.
 87. M. J. Martin, M. Bishof, M. D. Swallows, X. Zhang, C. Benko, J. von-Stecher, A. V. Gorshkov, A. M. Rey, and J. Ye. A quantum many-body spin system in an optical lattice clock. *ArXiv e-prints*, December 2012.
 88. Congjun Wu. Exotic many-body physics with large-spin fermi gases. *Physics*, 3:92, Nov 2010.
 89. Congjun Wu. Quantum gases: Mott made easy. *Nat Phys*, 8(11):784–785, November 2012.
 90. P Schlottmann. Ground-state and elemental excitations of the one-dimensional multi-component fermi gas with delta -function interaction. *Journal of Physics: Condensed Matter*, 6(7):1359, 1994.
 91. A. S. Stepanenko and J. M. F. Gunn. ‘Baryonic’ bound-state instability in trapped fermionic atoms. *eprint arXiv:cond-mat/9901317*, January 1999.
 92. Da Wang, Yi Li, Zi Cai, Zhichao Zhou, Yu Wang, and Congjun Wu.

- Competing Orders in the 2D Half-Filled $SU(2) \times N$ Hubbard Model through the Pinning-Field Quantum Monte Carlo Simulations. *Phys. Rev. Lett.*, 112(15):156403, apr 2014.
93. Da Wang, Lei Wang, and Congjun Wu. Slater and mott insulating states in the $su(6)$ hubbard model. *Phys. Rev. B*, 100:115155, Sep 2019.
 94. Zhichao Zhou, Da Wang, Zi Yang Meng, Yu Wang, and Congjun Wu. Mott insulating states and quantum phase transitions of correlated $SU(2n)$ dirac fermions. *Phys. Rev. B*, 93:245157, Jun 2016.
 95. Han Xu, Yu Wang, Zhichao Zhou, and Congjun Wu. Mott insulating states of the anisotropic $su(4)$ dirac fermions, 2019.
 96. Shenglong Xu, Julio T. Barreiro, Yu Wang, and Congjun Wu. Interaction effects with varying n in $SU(n)$ symmetric fermion lattice systems. *Phys. Rev. Lett.*, 121:167205, Oct 2018.
 97. I. I. Pomeranchuk. On the stability of a fermi liquid. *Soviet Physics Jetp-Ussr*, 8(2):361–362, 1959.
 98. C. Wu and S. C. Zhang. Dynamic generation of spin orbit coupling. *Physical Review Letters*, 93:36403, 2004.
 99. Congjun Wu, Kai Sun, Eduardo Fradkin, and Shou-Cheng Zhang. Fermi liquid instabilities in the spin channel. *Phys. Rev. B*, 75:115103, Mar 2007.
 100. T. Xiang and C. Wu. *D-Wave Superconductivity*. Peking University Press and Cambridge University Press.
 101. A. J. Leggett. Theoretical description of new phases of liquid-he-3. *Rev. Mod. Phys.*, 47(2):331–414, 1975.
 102. Manfred Sigris and Kazuo Ueda. Phenomenological theory of unconventional superconductivity. *Rev. Mod. Phys.*, 63:239–311, Apr 1991.
 103. Eduardo Fradkin, Steven A. Kivelson, Michael J. Lawler, James P. Eisenstein, and Andrew P. Mackenzie. Nematic fermi fluids in condensed matter physics. *Annual Review of Condensed Matter Physics*, 1(1):153C178, Aug 2010.
 104. Liang Fu. Topological Crystalline Insulators. *Phys. Rev. Lett.*, 106(10):106802, mar 2011.
 105. Siddharth A Parameswaran, Ari M Turner, Daniel P Arovas, and Ashvin Vishwanath. Topological order and absence of band insulators at integer filling in non-symmorphic crystals. *Nat. Phys.*, 9(5):299–303, apr 2013.
 106. Steve M. Young and Charles L. Kane. Dirac Semimetals in Two Dimensions. *Phys. Rev. Lett.*, 115(12):126803, sep 2015.
 107. Zhijun Wang, A Alexandradinata, R J Cava, and B Andrei Bernevig. Hourglass fermions. *Nature*, 532:189, 2016.
 108. Haruki Watanabe, Hoi Chun Po, Michael P. Zaletel, and Ashvin Vishwanath. Filling-Enforced Gaplessness in Band Structures of the 230 Space Groups. *Phys. Rev. Lett.*, 117(9):096404, aug 2016.
 109. G.W. Kadic, M. and Milton, M. van Hecke, and M. Wegener. 3d metamaterials. *Nature Reviews Physics*, 1:198, 2019.
 110. M. S. Rudner and N. H. Linder. Band structure engineering and non-equilibrium dynamics in floquet topological insulators. *Nature Reviews Physics*, 2:229, 2020.
 111. Fenner Harper, Rahul Roy, Mark S. Rudner, and S.L. Sondhi. Topology and broken symmetry in floquet systems. *Annual Review of Condensed Matter Physics*, 11(1):345–368, 2020.
 112. L. Guo and P. Liang. Condensed matter physics in time crystals. *New. J. Phys.*,

- 22:075003, 2021.
113. J Yu, R. X. Zhang, and Z. D. Song. Dynamical symmetry indicators for floquet crystals. *Nat. Comm.*, 12:5985, 2021.
 114. K. Giergiell, M. Dauphin, A. Lewenstein, J. Zakrzewski, and K. Sacha. Topological time crystals. *New. J. Phys.*, 21:052003, 2019.
 115. Yang Peng and Gil Refael. Floquet second-order topological insulators from nonsymmetric space-time symmetries. *Phys. Rev. Lett.*, 123:016806, Jul 2019.
 116. Reinhold Kleiner, Xianjing Zhou, Eric Dorsch, Xufeng Zhang, Dieter Koelle, and Dafei Jin. Space-time crystalline order of a high-critical-temperature superconductor with intrinsic Josephson junctions. *NATURE COMMUNICATIONS*, 12(1), OCT 15 2021.
 117. Qiang Gao and Qian Niu. Floquet-bloch oscillations and intraband zener tunneling in an oblique spacetime crystal. *Phys. Rev. Lett.*, 127:036401, Jul 2021.
 118. Alexander Muenchinger, Vincent Hahn, Dominik Beutel, Simon Woska, Joel Monti, Carsten Rockstuhl, Eva Blasco, and Martin Wegener. Multi-Photon 4D Printing of Complex Liquid Crystalline Microstructures by In Situ Alignment Using Electric Fields. *ADVANCED MATERIALS TECHNOLOGIES*, 7(1), JAN 2022.
 119. Lifa Zhang and Qian Niu. Chiral phonons at high-symmetry points in monolayer hexagonal lattices. *Phys. Rev. Lett.*, 115:115502, Sep 2015.
 120. Di Xiao, Ming-Che Chang, and Qian Niu. Berry phase effects on electronic properties. *Rev. Mod. Phys.*, 82:1959–2007, Jul 2010.
 121. Ching Kai Chiu, Jeffrey C Y Teo, Andreas P. Schnyder, and Shinsei Ryu. Classification of topological quantum matter with symmetries. *Rev. Mod. Phys.*, 88(3):1–63, 2016.
 122. A. Kyprianidis, F. Machado, W. Morong, P. Becker, K. S. Collins, D. V. Else, L. Feng, P. W. Hess, C. Nayak, G. Pagano, N. Y. Yao, and C. Monroe. Observation of a prethermal discrete time crystal. *Science*, 372(6547):1192C1196, Jun 2021.
 123. Alfred Shapere and Frank Wilczek. Classical time crystals. *Phys. Rev. Lett.*, 109:160402, Oct 2012.
 124. Frank Wilczek. Quantum time crystals. *Phys. Rev. Lett.*, 109:160401, Oct 2012.
 125. Patrick Bruno. Impossibility of spontaneously rotating time crystals: A no-go theorem. *Phys. Rev. Lett.*, 111:070402, Aug 2013.
 126. Haruki Watanabe and Masaki Oshikawa. Absence of quantum time crystals. *Phys. Rev. Lett.*, 114:251603, Jun 2015.
 127. Krzysztof Sacha. Modeling spontaneous breaking of time-translation symmetry. *Phys. Rev. A*, 91:033617, Mar 2015.
 128. N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath. Discrete time crystals: Rigidity, criticality, and realizations. *Phys. Rev. Lett.*, 118:030401, Jan 2017.
 129. Vedika Khemani, Achilleas Lazarides, Roderich Moessner, and S. L. Sondhi. Phase structure of driven quantum systems. *Phys. Rev. Lett.*, 116:250401, Jun 2016.
 130. Google Quantum AI and collaborations. Observation of time-crystalline eigenstate order on a quantum processor, 2021.
 131. J. Randall, C. E. Bradley, F. V. van der Gronden, A. Galicia, M. H. Abobeih, M. Markham, D. J. Twitchen, F. Machado, N. Y. Yao, and T. H. Taminiau. Many-body-localized discrete time crystal with a programmable spin-based quantum simulator. *Science*, 374(6574):1474C1478, Dec 2021.
 132. Tin-Lun Ho and Sungkit Yip. Pairing of fermions with arbitrary spin. *Phys. Rev. Lett.*, 82:247–250, Jan 1999.
 133. S.-K. Yip and Tin-Lun Ho. Zero sound modes of dilute fermi gases with arbitrary spin. *Phys. Rev. A*, 59:4653–4656, Jun 1999.

134. Congjun Wu, B. Andrei Bernevig, and Shou-Cheng Zhang. Helical liquid and the edge of quantum spin hall systems. *Phys. Rev. Lett.*, 96:106401, Mar 2006.
135. Hsiu-Hau Lin, Leon Balents, and Matthew P. A. Fisher. Exact $so(8)$ symmetry in the weakly-interacting two-leg ladder. *Phys. Rev. B*, 58:1794–1825, Jul 1998.
136. Erez Berg, Eduardo Fradkin, and Steven A. Kivelson. Charge- $4e$ superconductivity from pair-density-wave order in certain high-temperature superconductors. *Nature Physics*, 5(11):830C833, Sep 2009.
137. Daniel F. Agterberg, J.C. Samus Davis, Stephen D. Edkins, Eduardo Fradkin, Dale J. Van Harlingen, Steven A. Kivelson, Patrick A. Lee, Leo Radzihovsky, John M. Tranquada, and Yuxuan Wang. The physics of pair-density waves: Cuprate superconductors and beyond. *Annual Review of Condensed Matter Physics*, 11(1):231C270, Mar 2020.
138. Shao-Kai Jian, Yingyi Huang, and Hong Yao. Charge- $4e$ superconductivity from nematic superconductors in two and three dimensions. *Phys. Rev. Lett.*, 127:227001, Nov 2021.
139. Rafael M. Fernandes and Liang Fu. Charge- $4e$ superconductivity from multicomponent nematic pairing: Application to twisted bilayer graphene. *Phys. Rev. Lett.*, 127:047001, Jul 2021.
140. Meng Zeng, Lun-Hui Hu, Hong-Ye Hu, Yi-Zhuang You, and Congjun Wu. Phase-fluctuation induced time-reversal symmetry breaking normal state, arXiv:2102.06158.
141. Jun Ge, Pinyuan Wang, Ying Xing, Qiangwei Yin, Hechang Lei, Ziqiang Wang, and Jian Wang. Discovery of charge- $4e$ and charge- $6e$ superconductivity in kagome superconductor csv3sb5, arXiv:2201.10352.
142. M. van den Bossche, F.-C. Zhang, and F. Mila. Plaquette ground state in the two-dimensional $su(4)$ spin-orbital model. *The European Physical Journal B - Condensed Matter and Complex Systems*, 17(3):367–370, 2000.
143. S. Pankov, R. Moessner, and S. L. Sondhi. Resonating singlet valence plaquettes. *Phys. Rev. B*, 76:104436, Sep 2007.
144. Daniel S. Rokhsar and Steven A. Kivelson. Superconductivity and the quantum hardcore dimer gas. *Phys. Rev. Lett.*, 61:2376–2379, Nov 1988.
145. Rahul M. Nandkishore and Michael Hermele. Fractons. *Annual Review of Condensed Matter Physics*, 10(1):295–313, 2019.
146. J. E. Hirsch. Discrete hubbard-stratonovich transformation for fermion lattice models. *Phys. Rev. B*, 28:4059–4061, Oct 1983.
147. J. E. Hirsch. Two-dimensional hubbard model: Numerical simulation study. *Phys. Rev. B*, 31:4403–4419, Apr 1985.
148. J. E. Hirsch and S. Tang. Antiferromagnetism in the two-dimensional hubbard model. *Phys. Rev. Lett.*, 62:591–594, Jan 1989.
149. N. Read and Subir Sachdev. Large- N expansion for frustrated quantum antiferromagnets. *Phys. Rev. Lett.*, 66:1773–1776, Apr 1991.
150. R. Jackiw and C. Rebbi. Spin from isospin in a gauge theory. *Phys. Rev. Lett.*, 36:1116–1119, May 1976.
151. Congjun Wu, Han-dong Chen, Jiang-piang Hu, and Shou-Cheng Zhang. Vortex configurations of bosons in an optical lattice. *Phys. Rev. A*, 69:043609, Apr 2004.
152. G. Baym and C. Pethick. *Landau Fermi-liquid theory*. John Wiley & Sons, Inc., 1984.
153. J. E. Hirsch. Spin-split states in metals. *Phys. Rev. B*, 41:6820–6827, Apr 1990.
154. J. E. Hirsch. Chromium: A possible spin-split metal. *Phys. Rev. B*, 41:6828–6835, Apr 1990.
155. Daniel G. Barci and Luis E. Oxman. Strongly correlated fermions with nonlinear energy dispersion and spontaneous generation of anisotropic phases. *Phys. Rev. B*,

- 67:205108, May 2003.
156. M. J. Lawler, D. G. Barci, V. Fernandez, E. Fradkin, and L. Oxman. Nonperturbative behavior of the quantum phase transition to a nematic fermi fluid. *Phys. Rev. B*, 73:085101, 2006.
 157. Christoph J. Halboth and Walter Metzner. *d*-wave superconductivity and pomeranchuk instability in the two-dimensional hubbard model. *Phys. Rev. Lett.*, 85:5162–5165, Dec 2000.
 158. L. Dell’Anna and W. Metzner. Fermi surface fluctuations and single electron excitations near pomeranchuk instability in two dimensions. *Phys. Rev. B*, 73:045127, 2006.
 159. Hae-Young Kee. Sound propagation in a nematic fermi liquid. *Phys. Rev. B*, 67:073105, Feb 2003.
 160. S. A. Kivelson, I. P. Bindloss, E. Fradkin, V. Oganesyan, J. M. Tranquada, A. Kapitulnik, and C. Howald. How to detect fluctuating stripes in the high-temperature superconductors. *Rev. Mod. Phys.*, 75:1201–1241, Oct 2003.
 161. L. P. Gorkov and A. Sokol. Fermi surface fluctuations and single electron excitations near pomeranchuk instability in two dimensions. *Phys. Rev. Lett.*, 69:2586, 1999.
 162. CM Varma. Cure to the landau–pomeranchuk and associated long-wavelength fermi-surface instabilities on the lattice. *Philosophical Magazine*, 85(15):1657–1666, 2005.
 163. C. M. Varma and L. J. Zhu. Helicity order: Hidden order parameter in uru2si2. *Phys. Rev. Lett.*, 96:036405, 2006.
 164. H. Y. Kee and E. H. Kim. Itinerant metamagnetism induced by electronic nematic order. *Phys. Rev. B*, 71:184402, 2005.
 165. Carsten Honerkamp. Charge instabilities at the metamagnetic transition. *Physical Review B*, 72:115103, 2005.
 166. Vadim Oganesyan, Steven A. Kivelson, and Eduardo Fradkin. Quantum theory of a nematic fermi fluid. *Phys. Rev. B*, 64:195109, Oct 2001.
 167. S. A. Kivelson, E. Fradkin, and V. J. Emery. Electronic liquid crystal phases of a doped mott insulator. *Nature*, 393:550, 1998.
 168. M. P. Lilly, K. B. Cooper, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West. Evidence for an anisotropic state of two-dimensional electrons in high landau levels. *Phy. Rev. Lett.*, 82:394, 1999.
 169. R. R. Du, D. C. Tsui, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, and K. W. West. *Sol. Stat. Comm.*, 109:389, 1999.
 170. S. A. Grigera, R. S. Perry, A. J. Schofield, M. Chiao, S. R. Julian, G. G. Lonzarich, S. I. Ikeda, Y. Maeno, A. J. Millis, and A. P. Mackenzie. Magnetic field-tuned quantum criticality in the metallic ruthenate sr3ru2o7. *Science*, 294:329, 2001.
 171. S. A. Grigera, P. Gegenwart, R. A. Borzi, F. Weickert, A. J. Schofield, R. S. Perry, T. Tayama, T. Sakakibara, Y. Maeno, A. G. Green, and A. P. Mackenzie. Disorder-sensitive phase formation linked to metamagnetic quantum criticality. *Science*, 306:1154, 2004.
 172. R. A. Borzi, S. A. Grigera, J. Farrell, R. S. Perry, S. J. S. Lister, S. L. Lee, D. A. Tennant, Y. Maeno, and A. P. Mackenzie. Formation of a nematic fluid at high fields in sr3ru2o7. *Science*, 315:214, 2007.
 173. Andrey V. Chubukov and Dmitrii L. Maslov. Spin conservation and fermi liquid near a ferromagnetic quantum critical point. *Phys. Rev. Lett.*, 103:216401, Nov 2009.
 174. J. A. Hertz. Quantum critical phenomena. *Phys. Rev. B*, 14(3):1165–1184, Aug 1976.
 175. A. J. Millis. Effect of a nonzero temperature on quantum critical points in itinerant fermion systems. *Phys. Rev. B*, 48(10):7183–7196, Sep 1993.
 176. D Vollhardt and P Wolfle. *The superfluid phases of helium 3*. Taylor & Francis,

- London, 1990.
177. P. G. de Gennes and J. Prost. *The Physics of Liquid Crystals*. Clarendon Press; 2nd edition, 1995.
 178. Yi Li and Congjun Wu. Spin-orbit coupled fermi liquid theory of ultracold magnetic dipolar fermions. *Phys. Rev. B*, 85:205126, May 2012.
 179. Liang Fu. Parity-breaking phases of spin-orbit-coupled metals with gyrotropic, ferroelectric, and multipolar orders. *Phys. Rev. Lett.*, 115:026401, Jul 2015.
 180. Wei-Cheng Lee and Congjun Wu. Theory of unconventional metamagnetic electron states in orbital band systems. *Phys. Rev. B*, 80:104438, Sep 2009.
 181. A J Leggett. Spin diffusion and spin echoes in liquid ^3He at low temperature. *Journal of Physics C: Solid State Physics*, 3(2):448, 1970.
 182. L. R. Corruccini, D. D. Osheroff, D. M. Lee, and R. C. Richardson. Spin diffusion in liquid ^3He : The effect of leggett and rice. *Phys. Rev. Lett.*, 27:650–653, Sep 1971.
 183. D.D. Osheroff. Textural spin waves in ^3He . *Physica B+C*, 90(1):20 – 34, 1977.
 184. Dennis S. Greywall. Specific heat of normal liquid ^3He . *Phys. Rev. B*, 27:2747–2766, Mar 1983.
 185. J A Mydosh, P M Oppeneer, and P S Riseborough. Hidden order and beyond: an experimentaltheoretical overview of the multifaceted behavior of Uru_2Si_2 . *Journal of Physics: Condensed Matter*, 32(14):143002, Jan 2020.
 186. Christian T. Wolowiec, Noravee Kanchanavatee, Kevin Huang, Sheng Ran, Alexander J. Breindel, Naveen Pouse, Kalyan Sasmal, Ryan E. Baumbach, Greta Chappell, Peter S. Riseborough, and M. Brian Maple. Isoelectronic perturbations to f-d-electron hybridization and the enhancement of hidden order in Uru_2Si_2 . *Proceedings of the National Academy of Sciences*, 118(20), 2021.
 187. J. W. Harter, Z. Y. Zhao, J.-Q. Yan, D. G. Mandrus, and D. Hsieh. A parity-breaking electronic nematic phase transition in the spin-orbit coupled metal $\text{Cd}_2\text{Re}_2\text{O}_7$. *Science*, 356(6335):295C299, Apr 2017.
 188. J. W. Harter, D. M. Kennes, H. Chu, A. de la Torre, Z. Y. Zhao, J.-Q. Yan, D. G. Mandrus, A. J. Millis, and D. Hsieh. Evidence of an improper displacive phase transition in $\text{Cd}_2\text{Re}_2\text{O}_7$ via time-resolved coherent phonon spectroscopy. *Phys. Rev. Lett.*, 120:047601, Jan 2018.
 189. M. R. Norman. Crystal structure of the inversion-breaking metal $\text{Cd}_2\text{Re}_2\text{O}_7$. *Phys. Rev. B*, 101:045117, Jan 2020.