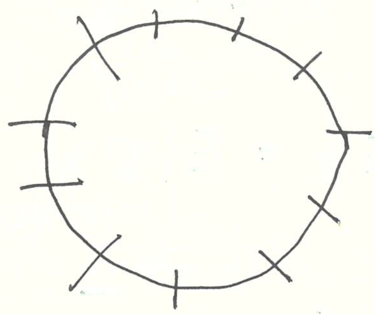
Heisenberg spin 1/2 chain

$$H = \frac{J}{2} \sum_{x=1}^{N} \left(S_x^+ S_{x+1}^- + S_x^- S_{x+1}^+ + 2\Delta \left(S_x^3 S_x^{3+1} - \frac{1}{4} \right) \right)$$

- Boundary:



symmetry: $\left[\sum_x S_x^3 , H \right] = 0$ $\quad$ $U(1)$ symmetry

if: $\Delta = 1$, $\quad SU(2)$ symmetry

Def reference state: $|\uparrow, \cdots\cdots \uparrow\rangle$: Ferromagnetic state

$|\Psi\rangle$: M particle state

$$\boxed{\sum_x S_x^3 = \frac{N}{2} - M}$$

$\phi(x_1 \cdots x_m)$

Expansion: $|\Psi\rangle = \sum_{x_1 < \cdots x_m} \phi(x_1 \cdots x_m)\, S_{x_1}^- \cdots\cdots S_{x_m}^- |\uparrow \cdots \uparrow\rangle$

$\Phi(x_1 \cdots x_m)$ 交换不变性 (Boson)

$$\Phi(x_1 \cdots x_m) = e^{i k_{p(1)} x_1 + \cdots k_{p(N)} x_N}$$

plane wave + scattering amplitude

$$H|\Psi\rangle = E|\Psi\rangle$$

① $\sum_x S_x^+ S_x^- |\Psi\rangle = \sum_{x_1 \cdots x_m} \sum_{l=1}^{M} \Psi(x_1 \cdots\cdots x_{l-1}, x_l - 1, \cdots x_m) | x_1 \cdots\cdots x_m \rangle$

if: x_1 和 x_2 相邻

$\Phi(x_1 \cdots\cdots x_l, x_{l+1}, \cdots\cdots) \Rightarrow$

if $x_l + 1 = x_{l+1}$, $\Phi(x_1 \cdots\cdots x_m) = 0$

② $\sum_x S_x^- S_x^+ |\Psi\rangle = \sum_{x_1 \cdots x_m} \sum_{l=1}^{M} \Psi(x_1 \cdots\cdots x_{l-1}, x_l, \cdots\cdots x_m)$

③

· Diagonal term:

$$2\Delta \sum_{\Delta} \left(S_x \cdot S_{x+1} - \frac{1}{4} \right) |\Psi\rangle = -\Delta \sum_{x_1 \cdots x_m}$$

$$\eta(x_1, \cdots x_m)\, \Psi(x_1 \cdots x_m)\, |x \cdots x_m\rangle$$

$\Downarrow$ bond 的数量

· Wavefunction condition

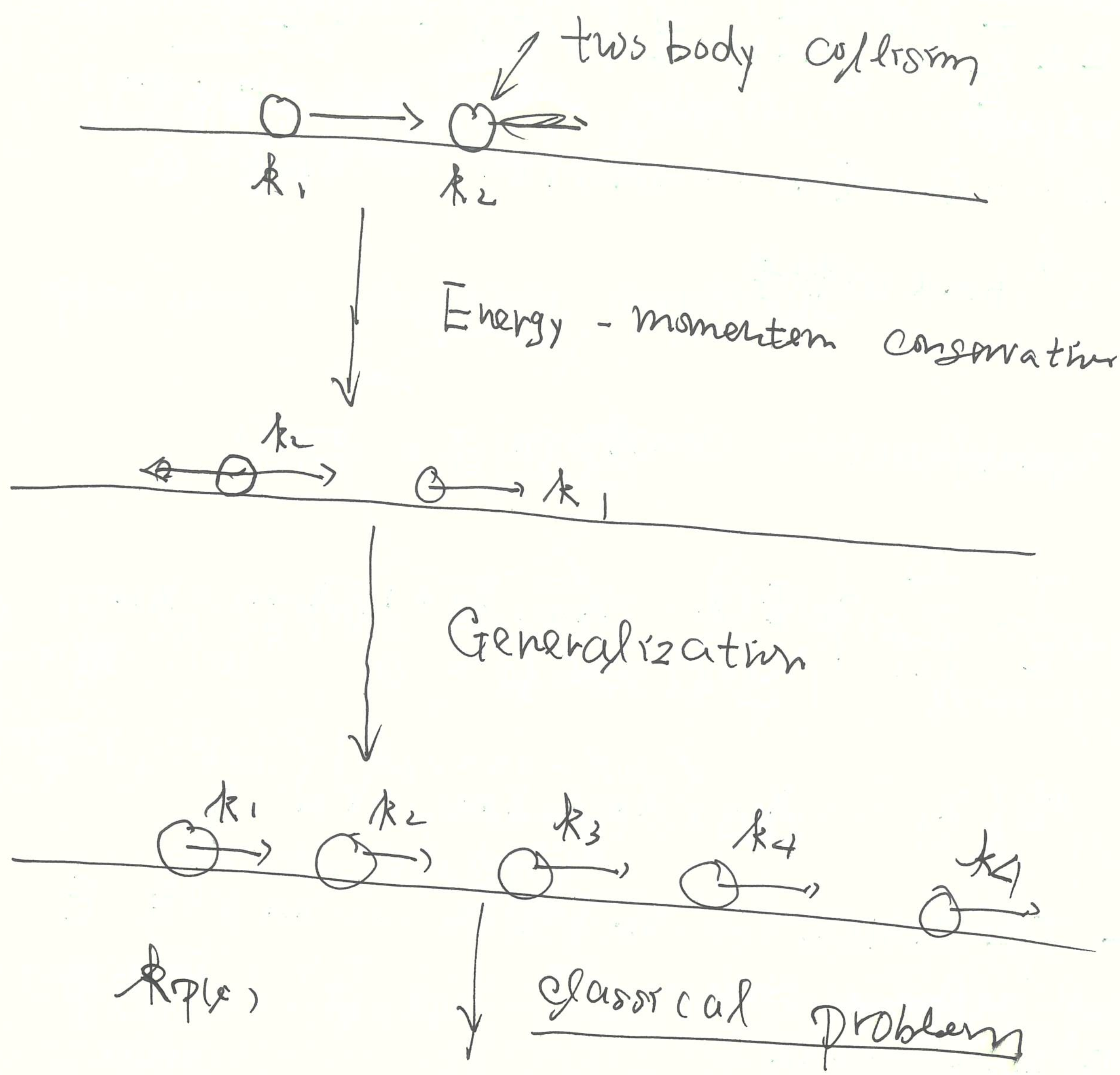
$$\frac{J}{2} \sum_{l=1}^{M} \Big\{ \Psi(x_1 \cdots x_{l-1}, x_l + 1, x_{l+1}, x_m)$$

$$+ \Psi(x_1 \cdots x_{l-1}, x_l - 1, x_{l+1}, x_m) \Big\}$$

$$- J/2\, \Delta\, \eta(x_1 \cdots x_m)\, \Psi(x_1 \cdots x_m) = E$$

$$\Psi(x_1, x_2 \cdots x_m)$$

Fixed our view on $l + n$



如果 Quantum mechanics，允许态叠加，

Bethe ansatz :

$$\phi(x_1, \cdots x_n) = e^{i\, k_{P(1)} x_1 + \cdots k_{P(n)} x_n}$$

P: ~~a~~ permutation group

$$\phi(x_1 \ldots x_m) = \sum_{P} A_P\, e^{i(k_{P(1)} x_1 + \cdots + k_{P(m)} x_m)}$$

$x_m \downarrow$ magnon

Whether this solution is complete. (Yang, Zhen, Ping)

Non hermitian Bethe ansatz

interaction 会改变 A_P，不会改变 kinetic energy

- ~~step 1~~: relation: $A_P, - A_{P'}$ scattering amplitude

periodic condition:

two condition $\longrightarrow$ determin k

Bethe ansatz solution $\longrightarrow$ k

$$E = J\sum_{l=1}^{m}(\cos k_l - \Delta) \qquad k_l\,?$$

- single body: $M = 1$

$$|\Psi\rangle = \sum e^{ik_N \cdot x_N} |\uparrow, \cdots\cdots \uparrow\rangle$$

it's equivalen to spin wave

$$k = \frac{2\pi}{L} \cdot N \quad : \quad k = 0 : \text{ same weight}$$

gap less

- two body :

$k \neq 0$ (magnon, $S=1$)

$$\phi \quad \underset{k_1}{\uparrow} \quad \underset{k_2}{\uparrow} \quad - \quad \underset{k_2}{\uparrow} \quad \underset{k_1}{\uparrow}$$

$$\Psi = A_1 e^{i k_1 x_1 + k_2 x_2} + A_2 e^{i k_2 x_1 + k_1 x_2}$$

$$\frac{J}{2}\left(\Psi(x_1\pm1,x_2)+\Psi(x_1,x_2\pm1)\right)=(E+2J\Delta)$$

$\Psi(x_1,x_2)$ ①

$\big\downarrow$ 体现 interaction

$$\frac{J}{2}\left(\Psi(x_1-1,x_2)+\Psi(x_1,x_2+1)\right)=(E+J\Delta)$$

$\Psi(x_1,x_2)$ ②

①、② $\Rightarrow$ $\frac{J}{2}\left(\Psi(x_2,x_2)+\Psi(x_1,x_1)\right)$

$=J\Delta\Psi(x_1,x_2)$ ③

①、②方程匹配的充分必要条件为③

$$\cancel{\frac{1}{2}\left(Ae^{i(k_1+k_2)x_1}+A'e^{i(k_1+k_2)x_2}\right)}$$

$$\frac{1}{2}(A+A')\left(e^{i(k_1+k_2)x_2}+e^{i(k_1+k_2)x_2}\right)$$

$$=\Delta\left(Ae^{ik_1x_1+k_2x_2}+A'e^{ik_2x_1+k_1x_2}\right)$$

⑧

$$\frac{A'}{A} = -\frac{e^{i(k_1+k_2)} - 2\Delta e^{ik_2} - 1}{e^{ik_1+ik_2} - 2\Delta e^{ik_2} - 1} = e^{i\Theta(k_1,k_2)}$$

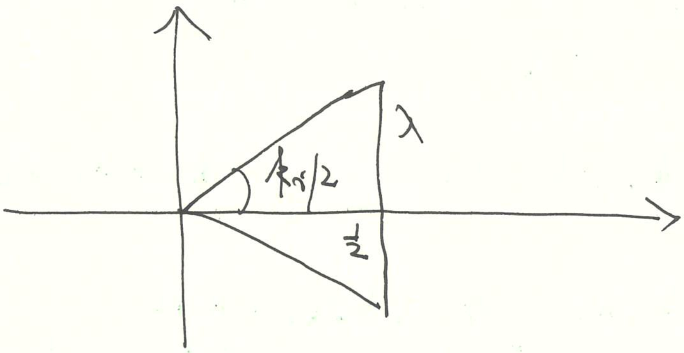
$$\Theta(k_1, k_2) = -\Theta(k_1, k_2)$$

$$e^{i\frac{k_1+k_2}{2}} - 2\Delta e^{i\frac{k_2-k_1}{2}} + e^{-i\frac{k_1+k_2}{2}}$$

$$= 2\cos\frac{k_1+k_2}{2} - 2\Delta\left(\cos\frac{k_2-k_1}{2} + i\sin\frac{k_2-k_1}{2}\right)$$

$$\cong -2\sin\frac{k_1}{2}\cdot\sin\frac{k_2}{2} - 2i\sin\frac{k_2-k_1}{2}\ )$$

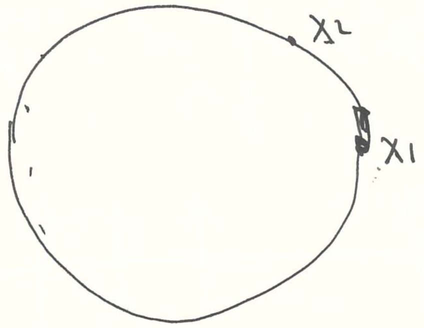
$$e^{ik_i} = \frac{\lambda_i + i/2}{\lambda_i - i/2} \Rightarrow \boxed{\lambda_j = \frac{1}{2}\cot\frac{k_i}{2}}$$

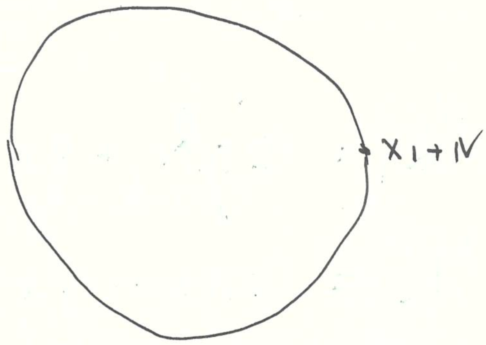


$$e^{i\Theta(k_1,k_2)} = -\frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i}$$

- periodic boundary condition

$$\Psi(x_1, x_2+N) = \Psi(x_1, x_2)$$





$$\cos k_1 + \cos k_2 = \frac{\sin k_1\cdot\cos k_2 + \sin k_2\cos k_1}{\sin k_1\cdot\sin k_2}$$

$$= \frac{\sin(k_1+k_2)}{\sin k_1\cdot\sin k_2}$$

$$x_1,\ k_1 \longrightarrow x_1+N$$

$$\Psi(x_1,x_2) = \Psi(x_1, x_2+N)$$

$$\Rightarrow e^{ik_1N}\frac{A'}{A} = 1$$

由相互作用贡献

$$1 = \frac{A'}{A} e^{ik_1 N} \qquad 1 = \frac{A}{A'} e^{ik_2 N}$$

$$\Rightarrow \quad 1 = e^{i(k_1+k_2) N}$$

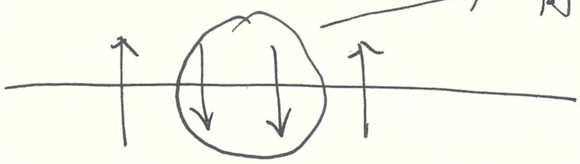
$$e^{i\delta_\ell} = e^{i(k_2-k_1) N}$$

$$\begin{cases} N k_1 = (2\pi N + \delta_\ell)/2 \\ N k_2 = (2\pi N - \delta_\ell)/2 \end{cases}$$

$$\Rightarrow \begin{cases} \left(\dfrac{\lambda_1 + i/2}{\lambda_1 - i/2}\right)^N = \dfrac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 + i} \\[2ex] \left(\dfrac{\lambda_2 + i/2}{\lambda_1 - i/2}\right)^N = \left(\dfrac{\lambda_2 - \lambda_1 + i}{\lambda_2 - \lambda_1 - i}\right) \end{cases}$$

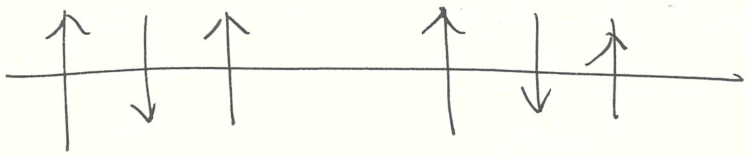
Prove the existence of bound state

$M=2, \quad N \to \infty$



$\to$ 有一个吸引势，不管有多弱 都有bound state

$\downarrow$ 拆开 需要能量



$$\left(\frac{\lambda_1 + i/2}{\lambda_2 - i/2}\right)^N = \left(\frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i}\right)^{\cancel{N}} \longrightarrow 0$$

$$\left(\frac{\lambda_2 + i/2}{\lambda_1 - i/2}\right)^N = \left(\frac{\lambda_2 - \lambda_1 - i}{\lambda_1 - \lambda_2 - i}\right) \longrightarrow \infty$$

$\lambda_1 = x_1 + \frac{i}{2}$

$\lambda_2 = x - i/2$

$$E = -\frac{J}{2}\left(\frac{1}{\lambda_1^2 + 1/4} + \frac{1}{\lambda_2^2 + 1/4}\right)$$

$$= -\frac{J}{2}\,\frac{1}{x^2+1}$$

string state: rapidity 是 complex 的.

不存在 bound state:

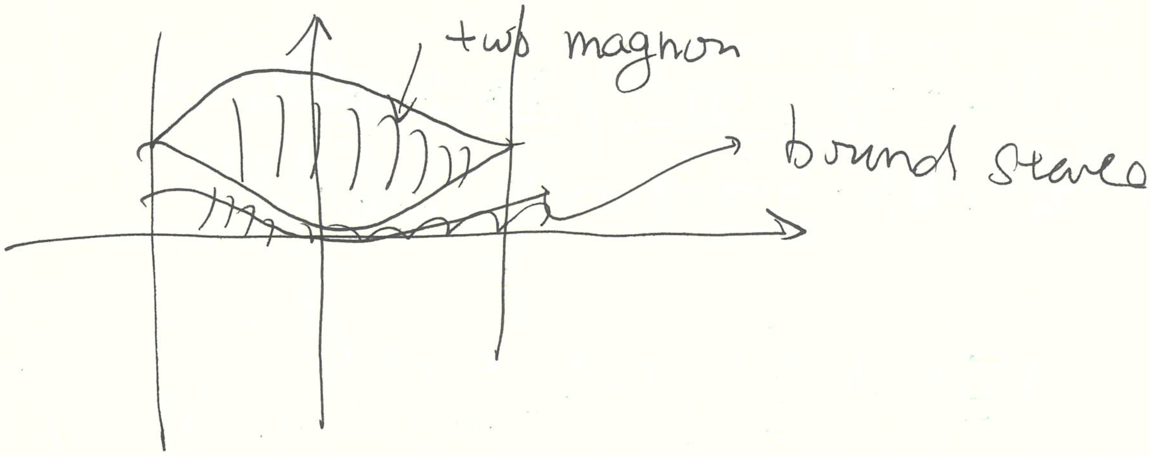
if $\lambda_1 = \lambda_2^*$

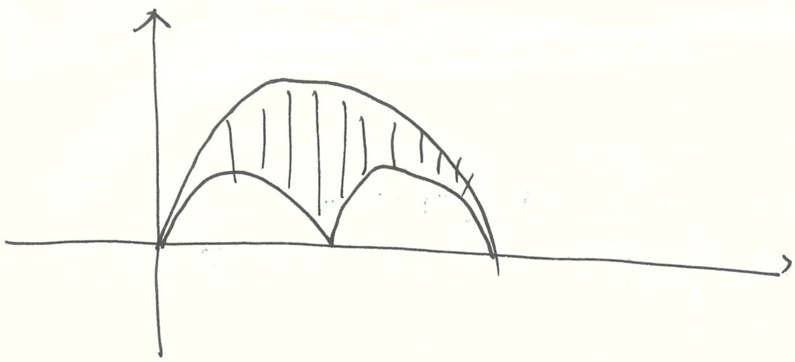
$$e^{i(k_1+k_2)} = \frac{x+i}{x-i}$$

$$\Rightarrow \quad \cos(k_1+k_2) = 1 - \frac{2}{x^2+1}$$

$$\Rightarrow \quad E = \frac{J}{2}\left(\cos(k_1+k_2) - 1\right)$$

$$E_{\pm}(k) = 2|J|\left(1 \pm \cos\frac{k}{2}\right)$$





$P \qquad k_{P1} \cdots k_{P(j)}\, k_{P(j+1)} \cdots k_{P(N)}$

$P' \qquad k_{P(1)} \cdots k_{P(j+1)}\, k_{P(j)} \cdots k_{P(N)}$

$$\Rightarrow \frac{k_{P'}}{k_P} = - \frac{e^{i(k+k')} - 2\Delta e^{ik'} + 1}{e^{i(k+k')} - 2\Delta e^{ik} + 1}$$

+ periodic condition

$$A_{1 \cdots N} = e^{i(k_1 x_1 + \cdots k_N x_N)}$$

↓ 巡る一次

$$A_{2 \cdots N} = e^{ik_2 x_2 + k_1 x_2 + \cdots k_N x_N}$$

↓

$$e^{ik_N} \prod_{l=1} \frac{A_{23 \cdots N1}}{A_{12 \cdots N}} = 1$$

particle 1:

$$e^{ik_1N}(-1)^{M-1}e^{i\sum_{l=1}^{M-1}\Theta(k_l,k_{l+1})}=1$$

⋮

particle N:

Ground state: $\{1,3,5,\cdots 9\}=m_j$

$$2\lambda_j=\cot\frac{k_j}{2}$$

$$\boxed{\Rightarrow\ \frac{k_j}{2}=\frac{\pi}{2}-\tan^{-1}2\lambda_j}$$

$$\cot\frac{1}{2}\phi_{j,l}=\lambda_j-\lambda_l$$

$$\Rightarrow\ \boxed{\phi_{j,l}=\begin{cases}\pi-2\tan^{-1}(\lambda_j-\lambda_l)\\-\pi-2\tan^{-1}(\lambda_j-\lambda_l)\end{cases}}$$

Bethe eq:

$$N(\pi - 2\tan^{-1} 2\lambda_j) = 2\pi m_j + \sum_{l \neq j} \pi \, \mathrm{sgn}(\lambda_j - \lambda_l)$$

$$- 2\tan^{-1}(\lambda_j - \lambda_l)$$

$$\Rightarrow \boxed{\tan^{-1} 2\lambda_j = \frac{\pi}{N} I_j + \frac{1}{N} \sum_{l \neq j} \tan^{-1}(\lambda_j - \lambda_l)}$$

$$\boxed{I_j = \frac{N}{2} - m_j - \frac{1}{2} \sum_{l \neq j} \mathrm{sgn}(\lambda_j - \lambda_l)}$$

$\lambda_1 > \lambda_2 \cdots \lambda_3$

$$\frac{1}{2} \sum_{l \neq j} \mathrm{sgn}(\lambda_j - \lambda_l) = -(j-1) + \frac{N}{2} - j$$

$$= \frac{N}{2} + 1 - 2j$$ 代入 I_j

$$\boxed{I_j = \frac{N}{4} - j + \frac{1}{2}}$$

$$I_j = \left\{ -\frac{N}{4} + \frac{1}{2}, \; -\frac{N}{4} + \frac{3}{2}, \; \cdots, \; \frac{N}{4} - \frac{1}{2} \right\}$$

利用一个 integral Eq:

$$x = \frac{I_j}{N} \in \left[-\frac{1}{4}, \frac{1}{4}\right]$$

$$\frac{1}{N}\sum_j = \int_{-1/4}^{1/4} dx$$

如何解

$$\boxed{\tan^{-1} 2\lambda(x) = \pi\lambda + \int_{-1/4}^{1/4} dy \tan^{-1}(\lambda(x) - \lambda(y))}$$

- Spectral function

$$\rho(\lambda) = \frac{dx}{d\lambda}$$

$$\frac{2}{1+4\lambda^2} = \rho(\lambda) + \int_{-1/4}^{1/4} \frac{1}{1+(\lambda-\lambda(y))^2} dy$$

$$= \rho(\lambda) + \int_{-1/4}^{1/4} d(\lambda(y)) \cdot \frac{dy}{d\lambda(y)} \frac{1}{1+(\lambda-\lambda(y))^2}$$

$$= \rho(\lambda) + \int_{\cancel{-1/4}\ -\infty}^{\cancel{1/4}\ +\infty} du\ \rho(u) \frac{2}{1+(\lambda-u)^2}$$

convolution

$\longrightarrow$ Fourier transformation

$$\int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \frac{1}{1+4\lambda^2} e^{i\omega\lambda} = \pi \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \rho(\lambda) e^{i\omega\lambda} + 2\pi \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \frac{e^{\omega(\lambda-\mu)} e^{i\omega\mu} \rho(\lambda)}{1+(\lambda-\mu)^2}$$

$$\Rightarrow \quad \frac{1}{2} e^{-1/2|\omega|} = \pi \widetilde{\rho(\omega)} + 2\pi \widetilde{\rho(\omega)} \frac{e^{-|\omega|}}{2}$$

$$\Rightarrow \quad \widetilde{\rho(\omega)} = \frac{e^{-1/2|\omega|}}{2\pi(1+e^{-|\omega|})}$$

$$= \frac{1}{4\pi \cosh \frac{|\omega|}{2}} \quad \text{(Remarkable)}$$

$$\Rightarrow \quad \rho(\lambda) = \frac{dx}{d\lambda} \int_{-\infty}^{+\infty} e^{-i\omega\lambda} \frac{1}{\cosh \frac{\omega}{2}} = \frac{1}{2\cosh \pi\lambda}$$

$$\frac{E}{NJ} = -\frac{1}{2} \sum_{j=1}^{N/2} \frac{1}{\lambda_j^2 + 1/4} = -\frac{J}{2} \int_{-1/4}^{1/4} dx \frac{1}{\lambda^2(x) + 1/4} = -\frac{J}{2} \int_{-\infty}^{+\infty} d\lambda \, \rho(\lambda) \frac{1}{\lambda^2 + 1/4}$$

$$= -\frac{J}{2} \int_{-\infty}^{+\infty} d\lambda \frac{1}{2\cosh \pi\lambda} \cdot \frac{1}{\lambda^2 + 1/4}$$

$$\Rightarrow -\int_{-\infty}^{+\infty} dy \, \frac{\operatorname{sech}\frac{\pi}{2}y}{y^2+1} = -\operatorname{Log} 2$$