量子场论　自由量子场，粒子与反粒子

本章建立和描述 量子场论 中的量子“自由场”

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单粒子态：

$$U(\Lambda,a)\Psi_{p,\sigma,n}=e^{ia_\mu(\Lambda p)^\mu}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}D_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\Lambda p,\sigma',n}$$

- 自旋为 $j,\sigma=-j,\ldots,j$ 的有质量态: $D_{\sigma'\sigma}(W)=(e^{\frac{i}{2}\Theta_{ik}(\Lambda,p)J^{(j)}_{ik}})_{\sigma'\sigma}$ p转成$\Lambda(p-k/\alpha)$的转动
- 自旋、 螺旋度为 σ 的无质量态: $D_{\sigma'\sigma}(W)=\delta_{\sigma'\sigma}e^{i\theta\sigma}$ 和p垂直矢量在Λ空间转动后与Λp垂直矢量的夹角

无相互作用的多粒子态: 单粒子态的直乘 能否从本征值鉴别是单粒子态还是多粒子态？

- 讨论不同种类的粒子，引入分立的指标n来代表粒子所属的种类
- 每个粒子用其能动量和参考动量系角动量第三分量及粒子种类来标记

一个一般的无相互作用的多粒子态可以写为 $\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$
它的能动量、自旋的第三分量(只对基本参考动量)和可能的$U(1)$本征值为:

$$P_0^\mu\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=(p_1^\mu+p_2^\mu+\cdots)\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$
$$J_0^3\Phi_{k_1,\sigma_1,n_1;k_2,\sigma_2,n_2;\cdots}=(\sigma_1+\sigma_2+\cdots)\Phi_{k_1,\sigma_1,n_1;k_2,\sigma_2,n_2;\cdots}$$
$$(Q_0)_a\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=(q_{a1}+q_{a2}+\cdots)\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$

对称性生成元标记下标0: 没有相互作用的多粒子体系

单粒子态：

$$U(\Lambda,a)\Psi_{p,\sigma,n}=e^{ia_\mu(\Lambda p)^\mu}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}D_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\Lambda p,\sigma',n}$$

- 自旋为j的有质量态：$D_{\sigma'\sigma}(W)=D^{(j)}_{\sigma'\sigma}(W)$ p转成$\Lambda(p-k/\alpha)$的转动
- 自旋为j,螺旋度为σ的无质量态：$D_{\sigma'\sigma}(W)=\delta_{\sigma'\sigma}e^{i\theta\sigma}$ 和p垂直矢量在Λ空间转动后与Λp垂直矢量的夹角

无相互作用的多粒子态：单粒子态的直乘 $\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$

$$P_0^\mu\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=(p_1^\mu+p_2^\mu+\cdots)\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$

$$J_0^3\Phi_{k_1,\sigma_1,n_1;k_2,\sigma_2,n_2;\cdots}=(\sigma_1+\sigma_2+\cdots)\Phi_{k_1,\sigma_1,n_1;k_2,\sigma_2,n_2;\cdots}$$

为什么？整体运动

$$(Q_0)_a\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=(q_{a1}+q_{a2}+\cdots)\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$

在时空平移和转动洛伦兹变换$U_0(\Lambda,a)$下：

$$U_0(\Lambda,a)\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=e^{ia_\mu((\Lambda p_1)^\mu+(\Lambda p_2)^\mu+\cdots)}\sqrt{\frac{(\Lambda p_1)^0(\Lambda p_2)^0\cdots}{p_1^0p_2^0\cdots}}\sum_{\sigma_1'\sigma_2'\cdots}D_{\sigma_1'\sigma_1}(W(\Lambda,p_1))$$

$$\times D_{\sigma_2'\sigma_2}(W(\Lambda,p_2))\cdots\Phi_{\Lambda p_1,\sigma_1',n_1;\Lambda p_2,\sigma_2',n_2;\cdots}$$

在相互独立的内部$U_0(1)$对称性变换下：

$$U_0(T(\theta))\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=e^{i(q_{a1}+q_{a2}+\cdots)\theta^a}\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$

排序问题：

如果所有的粒子都各属于不同的种类，它们之间可以区分，原则上可以规定一种标准的排序方式，例如第一种粒子排在第一位，第二种粒子排在第二位，$\cdots$,等等.

但如果在一个多粒子态中有某两个粒子的种类相同,$\Phi_{\cdots;p,\sigma,n;\cdots;p',\sigma',n;\cdots}$，交换此两个粒子的态$\Phi_{\cdots;p',\sigma',n;\cdots;p,\sigma,n;\cdots}$与原来的态是无法区分的 这似乎只在微观世界才会发生，因而限制了结果的应用范围，需要对同类粒子的排序进行详细的讨论。

交换两个同类粒子无法区分，交换前后的态之间只能相差一个相角:

$$\Phi_{\cdots;p,\sigma,n;\cdots;p',\sigma',n;\cdots} = \alpha_n(p,\sigma;p',\sigma')\,\Phi_{\cdots;p',\sigma',n;\cdots;p,\sigma,n;\cdots}$$

相角$\alpha_n(p,\sigma;p',\sigma')$与其它粒子的$p,\sigma,n$无关. 我们讨论的是自由粒子的多粒子态，如果交换其中的某两个粒子还要受到其它粒子的影响，意味着其它粒子对这两个参与交换的粒子有相互作用，就不是自由粒子态了。

$$\Phi_{\cdots;p,\sigma,n;\cdots;p',\sigma',n;\cdots} = \alpha_n(p,\sigma;p',\sigma')\ \Phi_{\cdots;p',\sigma',n;\cdots;p,\sigma,n;\cdots}$$

两边实施洛伦兹变换

$$\sum_{\bar\sigma\bar\sigma'\cdots} D_{\bar\sigma\sigma}(W(\Lambda,p))D_{\bar\sigma'\sigma'}(W(\Lambda,p'))\Phi_{\cdots,\Lambda p,\bar\sigma,n;\cdots;\Lambda p',\bar\sigma',n;\cdots}$$

$$= \alpha_n(p,\sigma;p',\sigma')\sum_{\bar\sigma'\bar\sigma} D_{\bar\sigma'\sigma'}(W(\Lambda,p'))D_{\bar\sigma\sigma}(W(\Lambda,p))\Phi_{\cdots,\Lambda p',\bar\sigma',n;\cdots;\Lambda p,\bar\sigma,n;\cdots}$$

$$\Phi_{\cdots,\Lambda p,\bar\sigma,n;\cdots;\Lambda p',\bar\sigma',n;\cdots} = \alpha_n(p,\sigma;p',\sigma')\ \Phi_{\cdots,\Lambda p',\bar\sigma',n;\cdots;\Lambda p,\bar\sigma,n;\cdots}$$

$$\Rightarrow\quad \alpha_n(p,\sigma;p',\sigma') = \alpha_n(\Lambda p,\bar\sigma;\Lambda p',\bar\sigma')$$

$\alpha_n(p,\sigma;p',\sigma')$与$\sigma,\sigma'$无关, 可略去$\alpha_n$中的$\sigma$指标,并且在时空转动下是不变的,

$$\alpha_n(p,\sigma;p',\sigma') = \alpha_n(p,p') \qquad\qquad \alpha_n(p,p') = \alpha_n(\Lambda p,\Lambda p')$$

在1+3维时空由p和p'构造的时空转动不变量只可能为p^2,p'^2和$p^\mu p'_\mu$,
考虑到$p^2 = M^2 = p'^2$并且$p^\mu p'_\mu$对交换p和p'是对称的

1+2维时空中同类粒子交换可出现任意相角任意子统计

- 交换粒子可等价为绕两粒子中点的转角为 π 的转动，产生相角 $e^{i2\pi\sigma}$
- 1+2维时空只一个非三个J $\Rightarrow$不量子化！ $\Rightarrow$ σ可取连续值 拓扑非平凡可转一圈不回归

$\Phi_{\cdots;p,\sigma,n;\cdots;p',\sigma',n;\cdots} = \alpha_n(p,p')\ \Phi_{\cdots;p',\sigma',n;\cdots;p,\sigma,n;\cdots}$

$\alpha_n(p,p') = \alpha_n(\Lambda p, \Lambda p')$ 在四维时空$\alpha_n(p,p')$只能依赖p^2,p'^2和$p^\mu p'_\mu$
结合$p^2{=}M^2{=}p'^2$及$p^\mu p'_\mu$对交换p和p'是对称的$\alpha_n(p,p') = \alpha_n(p',p)$

$$\Phi_{\cdots;p,\sigma,n;\cdots;p',\sigma',n;\cdots} = \alpha_n(p,p')\Phi_{\cdots;p',\sigma',n;\cdots;p,\sigma,n;\cdots}$$
$$\Phi_{\cdots;p',\sigma',n;\cdots;p,\sigma,n;\cdots} = \alpha_n(p',p)\Phi_{\cdots;p,\sigma,n;\cdots;p',\sigma',n;\cdots}$$
$$\Rightarrow \alpha_n(p,p')\alpha_n(p',p) = 1 \quad \Rightarrow \quad \alpha_n(p,p') = \pm 1$$

1+3维时空中同类粒子交换只可能出现两种情况：变号或不变号

- $\alpha_n(p,p') = +1$的粒子叫玻色子。 对玻色子同类粒子交换不变号
- $\alpha_n(p,p') = -1$的粒子叫费米子。 对费米子同类粒子交换一次出一负号

进一步对不同类粒子之间的排序在标准的排序方式基础上作如下的安排：

- 最近邻的费米子与费米子之间交换一次出一负号
- 最近邻的玻色子与玻色子，玻色子与费米子之间交换不变号

关于对全同粒子对称性的评述

- ♣ 我们的量子场论是建筑在粒子具有全同性的要求基础上的
- ♠ 它是假设吗？还是与生俱来，是我们用单粒子态定义量子场论所导致的呢！
- ♡ 核心是要建立粒子态的完备描述！
- ♢ 当粒子态的完备描述不再被需要时，全同性就丧失了！
- ✠ 对全同性丧失的临界尺度的研究还很少！
- ¶ 我们目前涉猎的物理理论居然要跨越全同性有无的边界，十分匪夷所思！

多粒子态的归一化：记$\Phi_{p_1,\sigma_1,n;p_2,\sigma_2,n_2;\cdots} = \Phi_{p_1,p_2,\cdots}$

真空态Φ_0和单粒子态Φ_q

$$(\Phi_0,\Phi_0) = 1 \qquad (\Phi_{q'},\Phi_q) = \delta(q'-q) \equiv \delta^3(\vec{q}'-\vec{q})\delta_{\sigma'\sigma}\delta_{n'n}$$ 非洛伦兹不变！

对两粒子态$\Phi_{q_1q_2}$ $(\Phi_{q_1'q_2'},\Phi_{q_1q_2}) = \frac{1}{2!}[\delta(q_1'-q_1)\delta(q_2'-q_2) \pm \delta(q_2'-q_1)\delta(q_1'-q_2)]$

负号对两个粒子都是费米子的情形,正号对其它情形(两个粒子都是玻色子或一个是玻色子一个是费米子).

一般情况：$$(\Phi_{q_1'q_2'\cdots q_M'},\Phi_{q_1q_2\cdots q_N}) = \frac{\delta_{MN}}{N!}\sum_{\mathcal{P}}\delta_{\mathcal{P}}\prod_i \delta(q_i - q'_{\mathcal{P}i})$$ N只针对同种粒子

求和对所有可能的对指标$1,2,\cdots,N$的交换排序$\mathcal{P}$实行.对交换排序中涉及奇数次费米子交换时,$\delta_{\mathcal{P}}=-1$,其它情况的交换排序$\delta_{\mathcal{P}}=1$.

约定：$$(\Phi_{\alpha'},\Phi_\alpha) = \delta(\alpha'-\alpha) \qquad \int d\alpha\cdots \equiv \sum_{n_1\sigma_1n_2\sigma_2\cdots}\int d\vec{p}_1\int d\vec{p}_2\cdots$$

$$\Phi = \int d\alpha\, \Phi_\alpha(\Phi_\alpha,\Phi) \qquad 1 = \int d\alpha\, \Phi_\alpha(\Phi_\alpha,\quad)$$ 多粒子态构成完备集！ 只要H厄米、有下界无上界

散射问题:

一组在宏观上相距很远的相互之间没有相互作用的粒子逐渐相互接近到微观上很小的区域发生相互作用，再逐渐互相分离到宏观上相距很远相互之间不再有相互作用的区域。

相互作用发生在粒子逐渐相互接近到微观上很小的区域。记一个有相互作用体系的总体时间平移生成元算符为H,被称为体系的哈密顿量，是体系总四动量的零分量。把这个体系的相互作用撤除得到的无相互作用的自由粒子体系的哈密顿量记为H_0.将两个哈密顿量的差定义为体系的相互作用V：

$$H = H_0 + V$$

将体系中粒子相距很远，相互之间没有相互作用的初始和末了状态分别称为进态和出态，记为：Ψ_α^+和Ψ_α^-

- 进态和出态分别构成完备集
- 相互之间无相互作用⇒进态和出态分别可被看成一组自由多粒子态
- 进态和出态满足与自由多粒子态同样的洛仑兹和内部对称性变换关系
- 过程的持续：无相互作用的无穷将来⇐发生相互作用的现在⇐无相互作用的无穷过去

进态和出态满足的洛仑兹变换: 满足与自由多粒子态同样的有相互作用系统的洛仑兹变换关系

$$U(\Lambda,a)\Psi^{\pm}_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=e^{ia_\mu((\Lambda p_1)^\mu+(\Lambda p_2)^\mu+\cdots)}\sqrt{\frac{(\Lambda p_1)^0(\Lambda p_2)^0\cdots}{p_1^0p_2^0\cdots}}\sum_{\sigma_1'\sigma_2'\cdots}D_{\sigma_1'\sigma_1}(W(\Lambda,p_1))$$

$$\times D_{\sigma_2'\sigma_2}(W(\Lambda,p_2))\cdots\Psi^{\pm}_{\Lambda p_1,\sigma_1',n_1;\Lambda p_2,\sigma_2',n_2;\cdots}\equiv\int d\alpha'\,(U_0)_{\alpha'\alpha}\Psi^{\pm}_{\alpha'}$$

$$U(T(\theta))\Psi^{\pm}_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=e^{i(q_{a1}+q_{a2}+\cdots)\theta^a}\Psi^{\pm}_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$

$$\Lambda=1,\ a=(\epsilon,\vec{0})$$

$$e^{i\epsilon H}\Psi^{\pm}_\alpha=e^{i\epsilon E_\alpha}\Psi^{\pm}_\alpha\qquad H\Psi^{\pm}_\alpha=E_\alpha\Psi^{\pm}_\alpha\qquad E_\alpha=p_1^0+p_2^0+\cdots$$

将H_0选择的使其具有与H完全一样的本征值谱,即

$$H_0\Phi_\alpha=E_\alpha\Phi_\alpha\qquad(\Phi_{\alpha'},\Phi_\alpha)=\delta(\alpha'-\alpha)$$

Φ_α是H_0的本征态。

进态和出态满足的洛仑兹变换: 满足与自由多粒子态同样的有相互作用系统的洛仑兹变换关系

$$U(\Lambda,a)\Psi^{\pm}_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=e^{ia_\mu((\Lambda p_1)^\mu+(\Lambda p_2)^\mu+\cdots)}\sqrt{\frac{(\Lambda p_1)^0(\Lambda p_2)^0\cdots}{p_1^0p_2^0\cdots}}\sum_{\sigma_1'\sigma_2'\cdots}D_{\sigma_1'\sigma_1}(W(\Lambda,p_1))$$

$$\times D_{\sigma_2'\sigma_2}(W(\Lambda,p_2))\cdots\Psi^{\pm}_{\Lambda p_1,\sigma_1',n_1;\Lambda p_2,\sigma_2',n_2;\cdots}\equiv\int d\alpha'\ (U_0)_{\alpha'\alpha}\Psi^{\pm}_{\alpha'}$$

$$U(T(\theta))\Psi^{\pm}_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=e^{i(q_{a1}+q_{a2}+\cdots)\theta^a}\Psi^{\pm}_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$

$$H\Psi^{\pm}_\alpha=E_\alpha\Psi^{\pm}_\alpha \qquad H_0\Phi_\alpha=E_\alpha\Phi_\alpha \qquad E_\alpha=p_1^0+p_2^0+\cdots$$

过程的持续 $\Rightarrow$ 量子态随时间的演化 $\xRightarrow{\text{理论中没定义}}$ 需定义态随时间的演化!

时间的演化

关于时间的附注：

绝对的、真实的和数学的时间，由其特性决定，自身均匀地流逝，与一切外在事务无关，又名延续；

相对的、表象的和普通的时间是可感知和外在的（不论是精确的或是不均匀的）对运动之延续的量度，它常被用以代替真实的时间，如一小时、一天、一个月、一年。

《自然哲学之数学原理–宇宙体系》 伊萨克.牛顿 1686年5月8日

时间的演化是： 均匀流逝

E.Wigner, Ann.Math.40,149(1939) On Unitary Representations of the Inhomogeneous Lorentz Group

If we knew, e.g., the operator K corresponding to the measurement of a physical quantity at the time $t = 0$, we could follow up the change of this quantity throughout time. In order to obtain its value for the time $t = t_1$, we could transform the original wave function ϕ_l by $D(l', l)$ to a coordinate system l' the time scale of which begins a time t_1 later.

$\underbrace{\text{begins a time } t_1 \text{ later}}_{t'=t-t_1,\ t\geq t_1}$

The measurement of the quantity in question in this coordinate system for the time 0 ($\underbrace{\text{the time } 0}_{t'=0}$) is given–as in original one–by the operator K. This measurement is identical, however, with the measurement of the quantity at time $t_1 = t$ in the original system. · · · · · ·

量子态随时间的演化

将态的演化时间翻译为两观测者的观测时间差 就像一个人自己O_τ和他手上戴的手表O

- 观测者O的时钟标记t，观测到的为理论原始假设中的态Ψ
- 观测者O_τ的时钟标记$t' = t - \tau$，感受 态的演化，观测到的态为$\Psi(\tau)$

观测者O_τ自我时间为“现在” $t' = 0$时，观测者O的时间纪录为$t = \tau$. $\Psi(\tau)$对τ的依赖就像一个人在看自己的手表。

$$\Psi(\tau) = U(1, -\tau)\Psi = e^{-iH\tau}\Psi \qquad i\frac{\partial}{\partial\tau}\Psi(\tau) = H\,\Psi(\tau)$$ 薛定谔方程！

suppose that a standard observer $\mathcal{O}$ sets his or her clock so that $t = 0$ is at some time during the collision process, while some other observer $\mathcal{O}'$ at rest with respect to the first uses a clock set so that $t' = 0$ is at a time $t = \tau$; that is, the two observers' time coordinates are related by $t' = t - \tau$. Then if $\mathcal{O}$ sees the system to be in a state Ψ, $\mathcal{O}'$ will see the system in a state $U(1, -\tau)\Psi = \exp(-iH\tau)\Psi$. Thus the appearance of the state long before or long after the collision (in whatever basis is used by $\mathcal{O}$) is found by applying a time-translation operator $\exp(-iH\tau)$ with $\tau \to -\infty$ or $\tau \to +\infty$, respectively. Of course, if the state is really

时间的均匀流逝体现为一系列观测者之间的 均匀的时间观测 间隔！

进态和出态与H_0的本征态的关系：

进态：

要求在无穷过去$O_{-\infty}$观测的进态与自由粒子态完全相同：

$$\Psi^+(-\infty) = \Phi(-\infty) \qquad U(1,-\tau)|_{\tau=-\infty}\Psi^+ = U_0(1,-\tau)|_{\tau=-\infty}\Phi$$

公式应在波包（不同本征态的叠加）意义下理解，否则带入本征值将导致$\Psi^+_\alpha = \Phi_\alpha$

$\Rightarrow e^{-iH\tau}\Psi^+_\alpha|_{\tau\to-\infty} = e^{-iH_0\tau}\Phi_\alpha|_{\tau\to-\infty}$ 或 $\Psi^+_\alpha = \Omega(-\infty)\Phi_\alpha$ 先将态自由地演化到无穷将来再相互作用地演化回现在

$$\Omega(\tau) \equiv U^\dagger(1,-\tau)U_0(1,-\tau) = U(1,\tau)U_0^\dagger(1,\tau) = e^{iH\tau}e^{-iH_0\tau}$$

出态：

要求在无穷将来$O_{+\infty}$观测的出态与自由粒子态完全相同：

$$\Psi^-(+\infty) = \Phi(+\infty) \qquad U(1,-\tau)|_{\tau=+\infty}\Psi^- = U_0(1,-\tau)|_{\tau=+\infty}\Phi$$

公式应在波包（不同本征态的叠加）意义下理解，否则带入本征值将导致$\Psi^-_\alpha = \Phi_\alpha$

$\Rightarrow e^{-iH\tau}\Psi^-_\alpha|_{\tau\to+\infty} = e^{-iH_0\tau}\Phi_\alpha|_{\tau\to+\infty}$ 或 $\Psi^-_\alpha = \Omega(+\infty)\Phi_\alpha$ 先将态自由地演化回无穷过去再相互作用地演化到现在

作业1,2,3

关于进态和出态:

♣ 引入系统的演化: 时间演化≡时间平移!

♢ 自由粒子态与相互作用的混合体?

♡ 相互作用暧昧地引入? 含糊不清 和撤除? 绝热近似

♠ 与自由粒子同样的质量谱?

¶ $e^{-iH\tau}\Psi_\alpha^\pm|_{\tau\to\mp\infty} = e^{-iH_0\tau}\Phi_\alpha|_{\tau\to\mp\infty}$?

♡ 不含时间的 $\Psi_\alpha^\pm$ 囊括了全部的系统演化!

进态和出态的内积定义为S矩阵元： $S_{\beta\alpha} = (\Psi_\beta^-, \Psi_\alpha^+)$ $\alpha \to \beta$ 几率幅

S矩阵元的性质： 作业4,5 $S_{\beta\alpha} = \delta(\beta-\alpha) - 2\pi i \delta(E_\alpha - E_\beta) T_{\beta\alpha}^+$

利用进出态的完备性 是厄米、有下界无上界的H的本征态所张开的态空间的完备性

$$\Psi_\alpha^+ = \int d\beta\ \Psi_\beta^- (\Psi_\beta^-, \Psi_\alpha^+) = \int d\beta\ S_{\beta\alpha} \Psi_\beta^-$$

$$\Psi_\alpha^- = \int d\beta\ \Psi_\beta^+ (\Psi_\beta^+, \Psi_\alpha^-) = \int d\beta\ S_{\alpha\beta}^* \Psi_\beta^+$$

幺正性质

$$\int d\beta\ S_{\beta\gamma}^* S_{\beta\alpha} = \int d\beta\ (\Psi_\gamma^+, \Psi_\beta^-)(\Psi_\beta^-, \Psi_\alpha^+) = (\Psi_\gamma^+, \Psi_\alpha^+) = \delta(\gamma - \alpha)$$

将S矩阵元建立在自由粒子基上，引入S矩阵算符: $\Psi_\alpha^\pm = \Omega(\mp\infty)\Phi_\alpha$

$$S_{\beta\alpha} \equiv (\Phi_\beta, S\Phi_\alpha) \ \rightarrow \ S = \Omega^\dagger(\infty)\Omega(-\infty) = U(+\infty, -\infty)$$

$$U(\tau, \tau_0) = \Omega^\dagger(\tau)\Omega(\tau_0) = e^{iH_0\tau} e^{-iH(\tau-\tau_0)} e^{-iH_0\tau_0} = U_0(1,\tau) U^\dagger(1, \tau - \tau_0) U_0^\dagger(1, \tau_0)$$

$$\Omega(\tau) = e^{iH\tau} e^{-iH_0\tau} = U(1,\tau) U_0^\dagger(1,\tau)$$

注意 $U(\tau, \tau_0)$ 和 $U(\Lambda, a)$ 是不同的量！

定义为进态和出态内积的S矩阵元 $S_{\beta\alpha}=(\Psi^-_\beta,\Psi^+_\alpha)$

利用洛仑兹变换和内部对称性变换算符的幺正性质

$\Rightarrow$ **S矩阵元在洛仑兹变换和内部对称性变换下是不变的!**

S矩阵元的洛仑兹变换不变性: $S_{\beta\alpha}=(\Psi^-_\beta,\Psi^+_\alpha)=(U(\Lambda,a)\Psi^-_\beta,U(\Lambda,a)\Psi^+_\alpha)$

$$S_{p_1',\sigma_1',n_1';p_2',\sigma_2',n_2';\cdots p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}=\int d\bar\beta d\bar\alpha\ (U_0^*)_{\bar\beta\beta}S_{\bar\beta\bar\alpha}(U_0)_{\bar\alpha\alpha}$$ 见后

$$=e^{ia_\mu((\Lambda p_1)^\mu+(\Lambda p_2)^\mu+\cdots-(\Lambda p_1)^{\mu\prime}-(\Lambda p_2)^{\mu\prime}-\cdots)}\sqrt{\frac{(\Lambda p_1)^0(\Lambda p_2)^0\cdots(\Lambda p_1')^0(\Lambda p_2')^0\cdots}{p_1^0p_2^0\cdots p_1^{0\prime}p_2^{0\prime}\cdots}}$$

$$\times\sum_{\bar\sigma_1,\bar\sigma_2,\cdots}D^{(j_1)}_{\bar\sigma_1\sigma_1}(W(\Lambda,p_1))D^{(j_2)}_{\bar\sigma_2\sigma_2}(W(\Lambda,p_2))\cdots\sum_{\bar\sigma_1',\bar\sigma_2',\cdots}D^{(j_1')*}_{\bar\sigma_1'\sigma_1'}(W(\Lambda,p_1'))D^{(j_2')*}_{\bar\sigma_2'\sigma_2'}(W(\Lambda,p_2'))\cdots$$

$$\times S_{\Lambda p_1',\bar\sigma_1',n_1';\Lambda p_2',\bar\sigma_2',n_2';\cdots\Lambda p_1,\bar\sigma_1,n_1;\Lambda p_2,\bar\sigma_2,n_2;\cdots}$$

左边与平移参量a无关 $\Rightarrow$ 要求等式右边a无关 $\Rightarrow$ $p_1^\mu+p_2^\mu+\cdots-p_1^{\mu\prime}-p_2^{\mu\prime}-\cdots=0$

记$p_\alpha=p_\beta$，四动量是连续变量，动量守恒意味S矩阵元中含因子$\delta(\vec p_\beta-\vec p_\alpha)$

$$S_{\beta\alpha}-\delta(\beta-\alpha)\overset{\text{作业4}}{=\!=\!=}-2\pi i\delta(E_\alpha-E_\beta)T^+_{\beta\alpha}=-2\pi iM_{\beta\alpha}\delta^4(p_\beta-p_\alpha)$$

定义为进态和出态内积的S矩阵元 $S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\alpha)$

利用洛仑兹变换和内部对称性变换算符的幺正性质

$\Rightarrow$ **S矩阵元在洛仑兹变换和内部对称性变换下是不变的!**

S矩阵元的洛仑兹变换不变性: $S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\alpha) = (U(\Lambda, a)\Psi^-_\beta, U(\Lambda, a)\Psi^+_\alpha)$

$$p_\alpha = p_\beta \qquad S_{\beta\alpha} - \delta(\beta - \alpha) = -2\pi i M_{\beta\alpha}\delta^4(p_\beta - p_\alpha)$$

S矩阵元的内部对称性变换不变性: $S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\alpha) = (U(T(\theta))\Psi^-_\beta, U(T(\theta))\Psi^+_\alpha)$

$$S_{p'_1,\sigma'_1,n'_1;p'_2,\sigma'_2,n'_2;\cdots p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$
$$= e^{i(q_{a1}+q_{a2}+\cdots-q'_{a1}-q'_{a2}-\cdots)\theta^a} S_{p'_1,\sigma'_1,n'_1;p'_2,\sigma'_2,n'_2;\cdots p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$

左边是与内部转动参量θ^a无关 $\Rightarrow$ 要求等式右边θ^a无关 $q_{a1} + q_{a2} + \cdots - q'_{a1} - q'_{a2} - \cdots = 0$

记 $q_\alpha = q_\beta$

S矩阵

定义为进态和出态内积的S矩阵元 $S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\alpha)$

利用洛仑兹变换和内部对称性变换算符的幺正性质

$\Rightarrow$ **S矩阵元在洛仑兹变换和内部对称性变换下是不变的!**

S矩阵元的洛仑兹变换不变性: $S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\alpha) = (U(\Lambda,a)\Psi^-_\beta, U(\Lambda,a)\Psi^+_\alpha)$

$p_\alpha = p_\beta \qquad S_{\beta\alpha} - \delta(\beta-\alpha) = -2\pi i M_{\beta\alpha}\delta^4(p_\beta - p_\alpha)$

用自由粒子的洛仑兹变换和内部对称性变换: $U_0(\Lambda,a) \quad U_0(T(\theta))$

$$U_0(\Lambda,a)\Phi_\alpha = \int d\bar\alpha (U_0)_{\bar\alpha\alpha}\Phi_{\bar\alpha} \qquad S_{\beta\alpha} \equiv (\Phi_\beta, S\Phi_\alpha) \Rightarrow S = \int d\beta d\alpha\, \Phi_\beta S_{\beta\alpha}\Phi_\alpha$$

$$S_{\beta\alpha} = (U\Psi^-_\beta, U\Psi^+_\alpha) \overset{\text{见}U_0\text{矩阵元最早定义}}{=\!=\!=\!=\!=} \int d\bar\beta d\bar\alpha ((U_0)_{\bar\beta\beta}\Psi^-_{\bar\beta}, (U_0)_{\bar\alpha\alpha}\Psi^+_{\bar\alpha}) = \int d\bar\beta d\bar\alpha (U_0^*)_{\bar\beta\beta} S_{\bar\beta\bar\alpha}(U_0)_{\bar\alpha\alpha}$$

$$\Rightarrow S = \int d\beta d\alpha\, \Phi_\beta S_{\beta\alpha}\Phi_\alpha = \int d\bar\beta d\bar\alpha d\beta d\alpha\, \Phi_\beta (U_0^*)_{\bar\beta\beta} S_{\bar\beta\bar\alpha}(U_0)_{\bar\alpha\alpha}\Phi_\alpha$$

$$= \int d\bar\beta d\bar\alpha d\beta d\alpha \Phi_\beta (U_0^*)_{\bar\beta\beta}(\Phi_{\bar\beta}, S\Phi_{\bar\alpha})(U_0)_{\bar\alpha\alpha}\Phi_\alpha = \int d\beta d\alpha \Phi_\beta (U_0\Phi_\beta, SU_0\Phi_\alpha)\Phi_\alpha = U_0^{-1}(\Lambda,a) S U_0(\Lambda,a)$$

U_0可以是任意一个幺正算符 **S矩阵算符是洛仑兹变换和内部对称性变换下不变的!**

S矩阵的微扰展开

$$S = U(+\infty, -\infty) \qquad U(\tau, \tau_0) = \Omega^\dagger(\tau)\Omega(\tau_0) = e^{iH_0\tau} e^{-iH(\tau-\tau_0)} e^{-iH_0\tau_0}$$

$$i\frac{d}{d\tau}U(\tau,\tau_0) = e^{iH_0\tau}(-H_0+H)e^{-iH(\tau-\tau_0)}e^{-iH_0\tau_0} = V(\tau)U(\tau,\tau_0) \qquad V(\tau) = e^{iH_0\tau}Ve^{-iH_0\tau}$$

$$U(\tau,\tau_0) = 1 - i\int_{\tau_0}^{\tau} dt_1 V(t_1) + (-i)^2\int_{\tau_0}^{\tau} dt_1 \int_{\tau_0}^{t_1} dt_2 V(t_1)V(t_2)$$

$$+(-i)^3\int_{\tau_0}^{\tau} dt_1 \int_{\tau_0}^{t_1} dt_2 \int_{\tau_0}^{t_2} dt_3 V(t_1)V(t_2)V(t_3) + \cdots$$

$$= \mathbf{T}\, e^{-i\int_{\tau_0}^{\tau} dt\ V(t)}$$

$\mathbf{T}$ 是编时乘积，它将时间早的的算符排在右边　　作业6,7

$$V(t) = \int d\vec{x}\ \tilde{\mathcal{H}}(\vec{r},t) \qquad \tilde{\mathcal{H}}(x) = \tilde{\mathcal{H}}(\vec{r},t)$$

是局域的相互作用哈密顿量密度　生成元开始直接和时空坐标发生关系！

局域性也可由 Cluster Decomposition Principle 得到.

由于V是H中的相互作用部分,不是洛仑兹变换的协变量, 如此引入相互作用哈密顿量密度可以确保它是洛仑兹变换的标量。

S矩阵的洛仑兹和内部对称性不变性要求$\tilde{\mathcal{H}}(x)$是洛仑兹和内部对称性不变量

猜测：$U_0(\Lambda,a)\tilde{\mathcal{H}}(x)U_0^{-1}(\Lambda,a) = \tilde{\mathcal{H}}(\Lambda x+a)$ 　　$U_0(T(\theta))\tilde{\mathcal{H}}(x)U_0^{-1}(T(\theta)) = \tilde{\mathcal{H}}(x)$

$[\tilde{\mathcal{H}}(x), \tilde{\mathcal{H}}(x')] = 0$ 　　$x - x'$ 类空间隔　　类时、类光间隔保时序，类空不保，因而要可对易！

反应率

将系统限制在有限的空间中：周期性边条件 $\Rightarrow e^{i\vec{p}\cdot\vec{L}} = 1$

$$\vec{p} = \frac{2\pi}{L}(n_1, n_2, n_3) \quad \delta_V^3(\vec{p}' - \vec{p}) = \frac{1}{(2\pi)^3}\int_V d^3x\, e^{i(\vec{p}-\vec{p}')\cdot\vec{x}} = \frac{V}{(2\pi)^3}\delta_{\vec{p}',\vec{p}}$$

$$\Psi_\alpha^{\rm Box} \equiv [\frac{(2\pi)^3}{V}]^{N_\alpha/2}\Psi_\alpha \rightarrow (\Psi_\beta^{\rm Box}, \Psi_\alpha^{\rm Box}) = \delta_{\beta\alpha}$$ **Kronecker** $\rightarrow S_{\beta\alpha} = [\frac{V}{(2\pi)^3}]^{\frac{N_\beta+N_\alpha}{2}} S_{\beta\alpha}^{\rm Box}$

$$\delta_T(E_\alpha - E_\beta) = \frac{1}{2\pi}\int_{-T/2}^{T/2} dt\, e^{i(E_\alpha - E_\beta)t} \qquad d\beta = d\vec{p}_1 d\vec{p}_2\cdots$$

$$P(\alpha\rightarrow\beta) \equiv |S_{\beta\alpha}^{\rm Box}|^2 = [\frac{(2\pi)^3}{V}]^{N_\alpha+N_\beta}|S_{\beta\alpha}|^2$$ $d\beta$区间态的数目：$d\mathcal{N}_\beta = [\frac{V}{(2\pi)^3}]^{N_\beta} d\beta$ 按第二行的积分，它是归一的

$$dP(\alpha\rightarrow\beta) = P(\alpha\rightarrow\beta)d\mathcal{N}_\beta = [\frac{(2\pi)^3}{V}]^{N_\alpha}|S_{\beta\alpha}|^2 d\beta$$

对有偏转的部分：

$$S_{\beta\alpha} \equiv -2i\pi\delta_V^3(\vec{p}_\beta - \vec{p}_\alpha)\delta_T(E_\beta - E_\alpha)M_{\beta\alpha} \xrightarrow{VT\ \text{large}} -2i\pi\delta^4(p_\beta - p_\alpha)M_{\beta\alpha}$$

$$dP(\alpha\rightarrow\beta) = (2\pi)^2[\frac{(2\pi)^3}{V}]^{N_\alpha-1}\frac{T}{2\pi}|M_{\beta\alpha}|^2\delta_V^3(\vec{p}_\beta - \vec{p}_\alpha)\delta_T(E_\beta - E_\alpha)d\beta$$

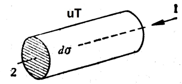
$$d\Gamma(\alpha\rightarrow\beta) \equiv \frac{dP(\alpha\rightarrow\beta)}{T} = (2\pi)^{3N_\alpha-2}V^{1-N_\alpha}|M_{\beta\alpha}|^2\delta^4(p_\beta - p_\alpha)d\beta$$

碰撞截面

$$S_{\beta\alpha} \overset{\text{connect part}}{- - - - - - - \rightarrow} -2i\pi\delta^4(p_\beta - p_\alpha)M_{\beta\alpha}$$

$$d\Gamma(\alpha\to\beta) \equiv \frac{dP(\alpha\to\beta)}{T} = (2\pi)^{3N_\alpha-2}V^{1-N_\alpha}|M_{\beta\alpha}|^2\delta^4(p_\beta-p_\alpha)d\beta$$

$N_\alpha = 1:$ $\quad d\Gamma(\alpha\to\beta) = 2\pi|M_{\beta\alpha}|^2\delta^4(p_\beta-p_\alpha)d\beta$



在体积边界面上各处向相对运动方向随机发射一个粒子1，此粒子在全空间V各处等几率出现

$N_\alpha = 2:$ $\quad \Phi_\alpha = \frac{u_\alpha}{V}$

$$d\sigma(\alpha\to\beta) \equiv \frac{d\Gamma(\alpha\to\beta)}{\Phi_\alpha} = (2\pi)^4 u_\alpha^{-1}|M_{\beta\alpha}|^2\delta^4(p_\beta-p_\alpha)d\beta$$

$$dP(1+2\to\beta) = \frac{d\sigma(1+2\to\beta)u_\alpha T}{V}$$

洛伦兹变换性质：

$$S_{p_1',\sigma_1',n_1';p_2',\sigma_2',n_2';\cdots p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\cdots}$$

$$= e^{ia_\mu((\Lambda p_1)^\mu+(\Lambda p_2)^\mu+\cdots-(\Lambda p_1)^{\mu\prime}-(\Lambda p_2)^{\mu\prime}-\cdots)}\sqrt{\frac{(\Lambda p_1)^0(\Lambda p_2)^0\cdots(\Lambda p_1')^0(\Lambda p_2')^0\cdots}{p_1^0 p_2^0\cdots p_1^{0\prime}p_2^{0\prime}\cdots}}$$

$$\times\sum_{\bar\sigma_1,\bar\sigma_2,\cdots} D^{(j_1)}_{\bar\sigma_1\sigma_1}(W(\Lambda,p_1))D^{(j_2)}_{\bar\sigma_2\sigma_2}(W(\Lambda,p_2))\cdots\sum_{\bar\sigma_1',\bar\sigma_2',\cdots} D^{(j_1')*}_{\bar\sigma_1'\sigma_1'}(W(\Lambda,p_1'))D^{(j_2')*}_{\bar\sigma_2'\sigma_2'}(W(\Lambda,p_2'))\cdots$$

$$\times S_{\Lambda p_1',\bar\sigma_1',n_1';\Lambda p_2',\bar\sigma_2',n_2';\cdots\Lambda p_1,\bar\sigma_1,n_1;\Lambda p_2,\bar\sigma_2,n_2;\cdots}$$

$$S_{\beta\alpha}-\delta(\beta-\alpha)=-2\pi i M_{\beta\alpha}\delta^4(p_\beta-p_\alpha)$$

$$R_{\beta\alpha}\equiv\sum_{\text{spins}}|M_{\beta\alpha}|^2\prod_\beta E\prod_\alpha E$$

需对初态σ末态σ'求和以分别消去D和D^*矩阵

is invariant

$$N_\alpha=1:\quad\sum_{\text{spins}} d\Gamma(\alpha\to\beta)=2\pi E_\alpha^{-1}R_{\beta\alpha}\delta^4(p_\beta-p_\alpha)\frac{d\beta}{\prod_\beta E}$$

$$N_\alpha=2:\quad\sum_{\text{spins}} d\sigma(\alpha\to\beta)=(2\pi)^4u_\alpha^{-1}E_1^{-1}E_2^{-1}R_{\beta\alpha}\delta^4(p_\beta-p_\alpha)\frac{d\beta}{\prod_\beta E}$$

反应率与碰撞截面

洛伦兹变换性质：

$$R_{\beta\alpha} \equiv \sum_{\text{spins}} |M_{\beta\alpha}|^2 \prod_{\beta} E \prod_{\alpha} E \qquad \textbf{is invariant}$$

$$N_\alpha = 1: \quad \sum_{\text{spins}} d\Gamma(\alpha \to \beta) = 2\pi E_\alpha^{-1} R_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \frac{d\beta}{\prod_\beta E}$$

按$1/E_\alpha$变换

$$N_\alpha = 2: \quad \sum_{\text{spins}} d\sigma(\alpha \to \beta) = (2\pi)^4 u_\alpha^{-1} E_1^{-1} E_2^{-1} R_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \frac{d\beta}{\prod_\beta E}$$

不变！

$$u_\alpha \equiv \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}$$

$$\vec{p}_1 = 0 \to E_1 = m_1 \to p_1 \cdot p_2 = m_1 E_2 \to u_\alpha = \frac{|\vec{p}_2|}{E_2} \qquad \vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - v^2}} \quad E = \frac{m_0}{\sqrt{1 - v^2}}$$

$$p_1 = (\vec{p}, E_1) \;\; p_2 = (-\vec{p}, E_2) \;\; E = E_1 + E_2 \to u_\alpha = \frac{|\vec{p}|E}{E_1 E_2} = \left|\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right|$$

$$(|\vec{p}|^2 + E_1 E_2)^2 - (E_1^2 - |\vec{p}|^2)(E_2^2 - |\vec{p}|^2) = |\vec{p}|^2 (E_1 + E_2)^2$$

反应率与碰撞截面

相空间：

$$N_\alpha = 1: \quad \sum_{\text{spins}} d\Gamma(\alpha\to\beta) = 2\pi E_\alpha^{-1} R_{\beta\alpha}\delta^4(p_\beta - p_\alpha)\frac{d\beta}{\prod_\beta E}$$

按$1/E_\alpha$变换

$$\vec{p}_\alpha = 0 \qquad \vec{p}_1' = -\vec{p}_2' - \vec{p}_3' - \cdots$$

$$\delta^4(p_\beta - p_\alpha)d\beta = \delta^3(\vec{p}_1' + \vec{p}_2' + \cdots)\delta(E_1' + E_2' + \cdots - E)d\vec{p}_1' d\vec{p}_2'\cdots$$

$$N_\alpha = 2: \quad \sum_{\text{spins}} d\sigma(\alpha\to\beta) = (2\pi)^4 u_\alpha^{-1} E_1^{-1} E_2^{-1} R_{\beta\alpha}\delta^4(p_\beta - p_\alpha)\frac{d\beta}{\prod_\beta E} \qquad \vec{p}_1 + \vec{p}_2 = 0$$

$$N_\beta = 2: \quad \delta^4(p_\beta - p_\alpha)d\beta = \delta(E_1' + E_2' - E)d\vec{p}_1' \qquad 1\to 1' + 2' \qquad 1 + 2 \to 1' + 2'$$

$$= \delta\left(\sqrt{|\vec{p}_1'|^2 + m_1'^2} + \sqrt{|\vec{p}_1'|^2 + m_2'^2} - E\right)|\vec{p}_1'|^2 d|\vec{p}_1'|d\Omega = \frac{|\vec{p}_1'|E_1'E_2'}{E}d\Omega$$

$$(E - E_1')^2 = E_2'^2 + m_2'^2 - m_1'^2 \qquad (E - E_2')^2 = E_2'^2 + m_1'^2 - m_2'^2$$

$$E_1' = \sqrt{|\vec{p}_1'|^2 + m_1'^2} = \frac{E^2 - m_2'^2 + m_1'^2}{2E} \qquad E_2' = \sqrt{|\vec{p}_1'|^2 + m_2'^2} = \frac{E^2 - m_1'^2 + m_2'^2}{2E}$$

$$|\vec{p}_1'| = \frac{\sqrt{(E^2 - m_1'^2 - m_2'^2)^2 - 4m_1'^2 m_2'^2}}{2E} \qquad \left.\frac{d\Gamma(\alpha\to\beta)}{d\Omega}\right|_{N_\alpha=1} \xlongequal{d\Gamma = 2\pi|M_{\beta\alpha}|^2\delta^4(p_\beta - p_\alpha)d\beta} \frac{2\pi|\vec{p}_1'|E_1'E_2'}{E}|M_{\beta\alpha}|^2$$

相空间：

$$N_\alpha = 2: \quad \sum_{\text{spins}} d\sigma(\alpha\to\beta) = (2\pi)^4 u_\alpha^{-1} E_1^{-1} E_2^{-1} R_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \frac{d\beta}{\prod_\beta E}$$

不变！

$$N_\beta = 2: \qquad 1+2\to 1'+2' \qquad \vec{p}_1+\vec{p}_2=0$$

$$\frac{d\sigma(\alpha\to\beta)}{d\Omega} = \frac{(2\pi)^4|\vec{p}_1'|E_1'E_2'}{Eu_\alpha}|M_{\beta\alpha}|^2 = \frac{(2\pi)^4|\vec{p}_1'|E_1'E_2'E_1E_2}{E^2|\vec{p}_1|}|M_{\beta\alpha}|^2$$

$$N_\beta = 3:$$

$$\delta^4(p_\beta - p_\alpha)d\beta = d\vec{p}_2' d\vec{p}_3' \delta\left(\sqrt{(\vec{p}_2'+\vec{p}_3')^2+m_1'^2} + \sqrt{\vec{p}_2'^2+m_2'^2} + \sqrt{\vec{p}_3'^2+m_3'^2} - E\right)$$

$$d\vec{p}_2' d\vec{p}_3' = |\vec{p}_2'|^2 d|\vec{p}_2'| |\vec{p}_3'|^2 d|\vec{p}_3'| d\Omega_3 d\phi_{23} d\cos\theta_{23} \qquad \frac{\partial E_1'}{\partial\cos\theta_{23}} = \frac{|\vec{p}_2'||\vec{p}_3'|}{E_1'}$$

$$\delta^4(p_\beta - p_\alpha)d\beta = |\vec{p}_2'|d|\vec{p}_2'|\ |\vec{p}_3'|d|\vec{p}_3'|\ E_1' d\Omega_3 d\phi_{23} = E_1'E_2'E_3' dE_2' dE_3' d\Omega_3 d\phi_{23}$$

integrate out $\cos\theta_{23}$

关于量子场理论，目前我们已经：

- 建立了自由粒子态
- 引入时空平移和转动及内部对称性的生成元算符
- 用H建立了S矩阵理论直接描述散射实验 相互作用通过e^{-iHt}演化引入

只需知道$V(t)$对自由粒子态的作用：$S_{\alpha\beta}=(\Phi_\beta,S\Phi_\alpha)=\delta(\beta-\alpha)-2\pi iM_{\beta\alpha}\delta^4(p_\beta-p_\alpha)$

$$S=\mathbf{T}e^{-i\int_{-\infty}^{\infty}dt\ V(t)}\qquad V(t)=e^{iH_0t}Ve^{-iH_0t}=\int d\vec{x}\mathcal{H}(x)\qquad[\mathcal{H}(\mathbf{x}),\mathcal{H}(\mathbf{x}')]_{\text{类空}}=\mathbf{0}\qquad H=H_0+V$$

$$N_\alpha=1:\quad d\Gamma(\alpha\to\beta)=2\pi|M_{\beta\alpha}|^2\delta^4(p_\beta-p_\alpha)d\beta$$ 按 $1/E_\alpha$ 变换

$$N_\alpha=2:\quad d\sigma(\alpha\to\beta)\equiv\frac{d\Gamma(\alpha\to\beta)}{\Phi_\alpha}=(2\pi)^4u_\alpha^{-1}|M_{\beta\alpha}|^2\delta^4(p_\beta-p_\alpha)d\beta$$ 洛伦兹变换不变量

建立局域的$\mathcal{H}(x)$对自由粒子态的作用? 本章

- 引入联系不同粒子态的算符：产生与湮灭算符
- 将H_0和V及其它生成元算符用产生与湮灭算符局域地表达出来

建立完整的计算体系： 下章 Wick定理； 约化公式； 路径积分

产生算符作用在某个多粒子态上定义为在此态上多加入一个粒子

$$a^\dagger(q)\Phi_{q_1q_2\cdots q_N} \equiv \Phi_{qq_1q_2\cdots q_N} \qquad \Phi_{q_1q_2\cdots q_N} = a^\dagger(q_1)a^\dagger(q_2)\cdots a^\dagger(q_N)\Phi_0$$

产生算符的厄米共轭算符: $(\Phi_{q_1'q_2'\cdots q_M'}, \Phi_{q_1q_2\cdots q_N}) = \frac{\delta_{MN}}{N!}\sum_{\mathcal{P}}\delta_{\mathcal{P}}\prod_i \delta(q_i - q'_{\mathcal{P}i})$

$$(\Phi_{q_1'\cdots q_M'}, a(q)\Phi_{q_1\cdots q_N}) = (a^\dagger(q)\Phi_{q_1'\cdots q_M'}, \Phi_{q_1\cdots q_N}) = (\Phi_{qq_1'\cdots q_M'}, \Phi_{q_1\cdots q_N})$$

对指标$1,2,\cdots,N$交换$\mathcal{P}$的求和可写成对一特殊的要被置换到首位$(\mathcal{P}r=1)$的指标r的求和,再加上对剩余的指标$1,\cdots,r-1,r+1,\cdots,N$的所有置换$\bar{\mathcal{P}}$求和$\sum_{\mathcal{P}} = \sum_{r=1}^N \sum_{\bar{\mathcal{P}}}$. 利用$\delta_{\mathcal{P}} = \delta_{r1}\delta_{\bar{\mathcal{P}}}$($\delta_{r1}$是将换到首位所贡献的$\delta_{\mathcal{P}}$),

$$(\Phi_{q_1'\cdots q_M'}, a(q)\Phi_{q_1\cdots q_N}) = \frac{\delta_{N,M+1}}{N!}\sum_{r=1}^N\sum_{\bar{\mathcal{P}}}\delta_{r1}\delta_{\bar{\mathcal{P}}}\delta(q-q_r)\prod_{i=1}^M\delta(q_i' - q_{\bar{\mathcal{P}}i})$$

$$= \begin{cases} \sum_{r=1}^N \delta_{r1}\delta(q-q_r)(\Phi_{q_1'\cdots q_M'}, \Phi_{q_1\cdots q_{r-1}q_{r+1}\cdots q_N}) & N\geq 1 \\ 0 & N=0 \end{cases}$$

湮灭算符: $a(q)\Phi_{q_1\cdots q_N} = \sum_{r=1}^N \delta_{r1}\delta(q-q_r)\Phi_{q_1\cdots q_{r-1}q_{r+1}\cdots q_N} \qquad N\geq 1 \qquad a(q)\Phi_0 = 0$

产生算符： $a^\dagger(q)\Phi_{q_1q_2\cdots q_N} \equiv \Phi_{qq_1q_2\cdots q_N}$ $\quad \Phi_{q_1q_2\cdots q_N} = a^\dagger(q_1)a^\dagger(q_2)\cdots a^\dagger(q_N)\Phi_0$

湮灭算符： $a(q)\Phi_{q_1\cdots q_N} = \sum_{r=1}^{N}\delta_{r1}\delta(q-q_r)\Phi_{q_1\cdots q_{r-1}q_{r+1}\cdots q_N}$ $\quad N\geq 1 \quad a(q)\Phi_0 = 0$

产生与湮灭算符的性质：作业8

$$a(q')a^\dagger(q) \mp a^\dagger(q)a(q') = \delta(q'-q)$$

$$a(q')a(q) \mp a(q)a(q') = 0 \qquad a^\dagger(q')a^\dagger(q) \mp a^\dagger(q)a^\dagger(q') = 0$$

- 负号对应两个粒子都是玻色子或一个是玻色子一个是费米子的情况
- 正号对应两个粒子都是费米子的情况

粒子数算符：作业9

$$N \equiv \int d\vec{q}\; a^\dagger(q)a(q) \qquad [N, a^\dagger(q)] = a^\dagger(q) \qquad [N, a(q)] = -a(q)$$

产生湮灭算符在洛伦兹变换下的行为： $a^\dagger(p)\Phi_{p_1p_2\cdots p_N} \equiv \Phi_{pp_1p_2\cdots p_N}$

$$U_0(\Lambda,b)\Phi_{p,\sigma,n;p_1,\sigma_1,n_1;\cdots} = e^{ib_\mu((\Lambda p)^\mu+(\Lambda p_1)^\mu+\cdots)}\sqrt{\frac{(\Lambda p)^0(\Lambda p_1)^0\cdots}{p^0p_1^0\cdots}}\sum_{\bar\sigma\bar\sigma_1\cdots}D_{\bar\sigma\sigma}(W(\Lambda,p))$$

$$= U_0(\Lambda,b)a^\dagger(\vec p,\sigma,n)U_0^{-1}(\Lambda,b)U_0(\Lambda,b)\Phi_{p_1,\sigma_1,n_1;\cdots} \qquad \times D_{\bar\sigma_1\sigma_1}(W(\Lambda,p_1))\cdots\Phi_{\Lambda p,\bar\sigma,n;\Lambda p_1,\sigma_1',n_1;\cdots}$$

$$U_0(\Lambda,b)a^\dagger(\vec p,\sigma,n)U_0^{-1}(\Lambda,b) = e^{i(\Lambda p)\cdot b}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar\sigma}D_{\bar\sigma\sigma}(W(\Lambda,p))a^\dagger(\vec p_\Lambda,\bar\sigma,n)$$

$$= e^{i(\Lambda p)\cdot b}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar\sigma}D^*_{\sigma\bar\sigma}(W^{-1}(\Lambda,p))a^\dagger(\vec p_\Lambda,\bar\sigma,n)$$

$$U_0(\Lambda,b)a(\vec p,\sigma,n)U_0^{-1}(\Lambda,b) = e^{-i(\Lambda p)\cdot b}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar\sigma}D^\dagger_{\bar\sigma\sigma}(W(\Lambda,p))a(\vec p_\Lambda,\bar\sigma,n)$$

$$= e^{-i(\Lambda p)\cdot b}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar\sigma}D_{\sigma\bar\sigma}(W^{-1}(\Lambda,p))a(\vec p_\Lambda,\bar\sigma,n)$$

产生湮灭算符在空间反射变换下的行为： $a^\dagger(p)\Phi_{p_1p_2\cdots p_N} \equiv \Phi_{pp_1p_2\cdots p_N}$

有质量正能: $P\Psi_{p,\sigma} = \eta\Psi_{\mathcal{P}p,\sigma}$

无质量正能: $P\Psi_{p,\sigma} = \eta_\sigma e^{\mp i\pi\sigma}\Psi_{\mathcal{P}p,-\sigma}$ 负号: $0 \le \phi < \pi$，正号: $\pi \le \phi < 2\pi$ ϕ是$\vec{p}$在xy平面上投影与x轴的夹角

产生湮灭有质量的粒子算符在空间反射变换下的行为：

$$Pa^\dagger(\vec{p},\sigma,n)P^{-1} = \eta a^\dagger(\mathcal{P}\vec{p},\sigma,n) \qquad Pa(\vec{p},\sigma,n)P^{-1} = \eta^* a(\mathcal{P}\vec{p},\sigma,n)$$

产生湮灭无质量的粒子算符在空间反射变换下的行为：

$$Pa^\dagger(\vec{p},\sigma,n)P^{-1} = \eta_\sigma e^{\mp i\pi\sigma} a^\dagger(\mathcal{P}\vec{p},-\sigma,n) \qquad Pa(\vec{p},\sigma,n)P^{-1} = \eta_\sigma^* e^{\pm i\pi\sigma} a(\mathcal{P}\vec{p},-\sigma,n)$$

负号: $0 \le \phi < \pi$，正号: $\pi \le \phi < 2\pi$ （ϕ是$\vec{p}$在xy平面上的投影与x轴的夹角）

产生湮灭算符在时间反演变换下的行为： $a^\dagger(p)\Phi_{p_1p_2\cdots p_N} \equiv \Phi_{pp_1p_2\cdots p_N}$

有质量正能： $T\Psi_{p,\sigma} = (-1)^{j-\sigma}\Psi_{\mathcal{P}p,-\sigma}$

无质量正能： $T\Psi_{p,\sigma} = \zeta_\sigma e^{\pm i\pi\sigma}\Psi_{\mathcal{P}p,\sigma}$

$\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ 中：正号$0 \le \phi < \pi$,负号对应$\pi \le \phi < 2\pi$

产生湮灭有质量的粒子算符在时间反演变换下的行为：

$$Ta^\dagger(\vec{p},\sigma,n)T^{-1} = (-1)^{j-\sigma}a^\dagger(\mathcal{P}\vec{p},-\sigma,n) \qquad Ta(\vec{p},\sigma,n)T^{-1} = (-1)^{j-\sigma}a(\mathcal{P}\vec{p},-\sigma,n)$$

产生湮灭无质量的粒子算符在时间反演变换下的行为：

$$Ta^\dagger(\vec{p},\sigma,n)T^{-1} = \zeta_\sigma e^{\pm i\pi\sigma}a^\dagger(\mathcal{P}\vec{p},\sigma,n) \qquad Ta(\vec{p},\sigma,n)T^{-1} = \zeta_\sigma^* e^{\mp i\pi\sigma}a(\mathcal{P}\vec{p},\sigma,n)$$

$\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ 中正号$0 \le \phi < \pi$,负号对应$\pi \le \phi < 2\pi$

产生湮灭算符在内部对称性变换下的行为： $a^\dagger(p)\Phi_{p_1p_2\cdots p_N} \equiv \Phi_{pp_1p_2\cdots p_N}$

$$U_0(T(\theta))\Psi_{q,\sigma} = e^{iq_a\theta^a}\Psi_{q,\sigma} \qquad Q_a\ \Psi_{q,\sigma} = q_a\ \Psi_{q,\sigma}$$

产生湮灭粒子算符在内部对称性$U(1)$生成元Q_a变换下的行为：

$$[Q_a,\ a^\dagger(\vec{q})]\Phi_{q_1q_2\cdots q_n} = Q_a\Phi_{qq_1q_2\cdots q_n} - a^\dagger(\vec{q})(q_{a1}+q_{a2}+\cdots)\Phi_{q_1q_2\cdots q_n}$$

$$= q_a\Phi_{qq_1q_2\cdots q_n} = q_a a^\dagger(\vec{q})\Phi_{q_1q_2\cdots q_n}$$

$$[Q_a,\ a^\dagger(\vec{q})] = q_a a^\dagger(\vec{q}) \qquad [Q_a,\ a(\vec{q})] = -q_a a(\vec{q})$$

产生和湮灭算符在坐标空间的表达:

$$\tilde{\mathcal{H}}(x) = \sum_{NM}\int dp_1\cdots dp_N dp_1'\cdots dp_M'\ \tilde{c}(x,p_1,\cdots,p_N,p_1',\cdots,p_M')\ a(p_1)\cdots a(p_N)a^\dagger(p_1')\cdots a^\dagger(p_M')$$

$$= \sum_{NM} c_{NM}\ \psi_l^{+,N}(x)\psi_l^{-,M}(x)$$

$$\psi_l^+(x) = \sum_{\sigma,n}\int d^3p\ u_l(x;\vec{p},\sigma,n)a(\vec{p},\sigma,n) \qquad \psi_l^-(x) = \sum_{\sigma,n}\int d^3p\ v_l(x;\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

♣ 局域相互作用哈密顿量的表达需要

$$S = \mathbf{T}\ e^{-i\int_{\tau_0}^{\tau} dt\ V(t)} \qquad V(t) = e^{iH_0t}Ve^{-iH_0t} = \int d\vec{x}\ \tilde{\mathcal{H}}(\vec{r},t)$$

$$U_0(\Lambda,a)\tilde{\mathcal{H}}(x)U_0^{-1}(\Lambda,a) = \tilde{\mathcal{H}}(\Lambda x+a)$$

✠ 态空间的产生与湮灭直接与坐标空间联系 量子场: 产生湮灭算符的集合！

♡ 局域性! 它导致的结果可满足**cluster decomposition**原理，但反过来呢 ？

♠ 因果性! $[\tilde{\mathcal{H}}(x),\tilde{\mathcal{H}}(x')] = 0 \quad x - x'$ 类空间隔

产生和湮灭算符在坐标空间的表达:

$$\psi_l^+(x)=\sum_{\sigma,n}\int d^3p\ u_l(x;\vec{p},\sigma,n)a(\vec{p},\sigma,n) \quad \psi_l^-(x)=\sum_{\sigma,n}\int d^3p\ v_l(x;\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$[Q_a,\ a^\dagger(\vec{q})] = q_a a^\dagger(\vec{q}) \quad [Q_a,\ a(\vec{q})] = -q_a a(\vec{q})$$

$$[Q_a,\ \psi_l^\pm(x)] = \mp q_a \psi_l^\pm(x)$$

要求$u_l(x;\vec{p},\sigma,n)$和$v_l(x;\vec{p},\sigma,n)$满足: $U_0(\Lambda,a)\tilde{\mathcal{H}}(x)U_0^{-1}(\Lambda,a) = \tilde{\mathcal{H}}(\Lambda x+a)$

$$U_0(\Lambda,a)\psi_l^+(x)U_0^{-1}(\Lambda,a) = \sum_{\bar{l}} D_{l\bar{l}}^+(\Lambda^{-1})\psi_{\bar{l}}^+(\Lambda x+a)$$

$$U_0(\Lambda,a)\psi_l^-(x)U_0^{-1}(\Lambda,a) = \sum_{\bar{l}} D_{l\bar{l}}^-(\Lambda^{-1})\psi_{\bar{l}}^-(\Lambda x+a)$$

$$D^\pm(\Lambda^{-1})D^\pm(\bar{\Lambda}^{-1}) = D^\pm((\bar{\Lambda}\Lambda)^{-1}) \qquad D^\pm(\Lambda_1)D^\pm(\Lambda_2) = D^\pm(\Lambda_1\Lambda_2)$$

产生和湮灭算符在坐标空间的表达:

$$\psi_l^+(x)=\sum_{\sigma,n}\int d^3p\, u_l(x;\vec{p},\sigma,n)a(\vec{p},\sigma,n) \quad \psi_l^-(x)=\sum_{\sigma,n}\int d^3p\, v_l(x;\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$[Q_a,\ \psi_l^\pm(x)]=\mp q_a\psi_l^\pm(x) \qquad U_0(\Lambda,a)\psi_l^\pm(x)U_0^{-1}(\Lambda,a)=\sum_{\bar{l}}D^\pm_{l\bar{l}}(\Lambda^{-1})\psi_{\bar{l}}^\pm(\Lambda x+a)$$

$$U_0(e^\omega,\epsilon)=e^{\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}+i\epsilon_\rho P^\rho} \qquad D^\pm(e^\omega)=e^{\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}}$$

算$D^\pm$等价于算$\mathcal{J}^{\sigma\rho}$的矩阵元！场表示没么正性要求

$$i[\mathcal{J}^{\mu\nu},\mathcal{J}^{\rho\sigma}]=g^{\nu\rho}\mathcal{J}^{\mu\sigma}-g^{\mu\rho}\mathcal{J}^{\nu\sigma}-g^{\sigma\mu}\mathcal{J}^{\rho\nu}+g^{\sigma\nu}\mathcal{J}^{\rho\mu}$$

$$[\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}+i\epsilon_\rho P^\rho,\psi_l^\pm(x)]=-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}_{l\bar{l}}\psi_{\bar{l}}^\pm(x)+[\omega_{\rho\sigma}x^\sigma+\epsilon_\rho]\partial^\rho\psi_l^\pm(x)$$

$$[P^\rho,\psi_l^\pm(x)]=-i\partial^\rho\psi_l^\pm(x) \qquad [x^\sigma,-i\partial^\rho]=ig^{\sigma\rho}$$

$H\sim -i\partial_t,\ \vec{P}\sim i\nabla$ 作用算符上与通常作用态上相差一个负号！

$$[J^{\rho\sigma},\psi_l^\pm(x)]=-\mathcal{J}^{\rho\sigma}_{l\bar{l}}\psi_{\bar{l}}^\pm(x)+i(x^\rho\partial^\sigma-x^\sigma\partial^\rho)\psi_l^\pm(x) \qquad L^{\rho\sigma}\equiv -i(x^\rho\partial^\sigma-x^\sigma\partial^\rho)$$

$$i[L^{\mu\nu},L^{\rho\sigma}]=g^{\nu\rho}L^{\mu\sigma}-g^{\mu\rho}L^{\nu\sigma}-g^{\sigma\mu}L^{\rho\nu}+g^{\sigma\nu}L^{\rho\mu} \qquad [\mathcal{J}^{\mu\nu},L^{\rho\sigma}]=0$$

$$i[i\partial^\mu,L^{\rho\sigma}]=g^{\mu\rho}i\partial^\sigma-g^{\mu\sigma}i\partial^\rho \qquad [i\partial^\mu,\mathcal{J}^{\rho\sigma}]=0 \qquad [i\partial^\mu,i\partial^\rho]=0$$

场空间的Pauli-Lubanski算符: $W^\mu\equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}i\partial_\nu(L_{\rho\sigma}+\mathcal{J}_{\rho\sigma})=-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}i\partial_\nu\mathcal{J}_{\rho\sigma}$

产生和湮灭算符在坐标空间的表达:

$$\psi_l^+(x) = \sum_{\sigma,n} \int d^3p\, u_l(x;\vec{p},\sigma,n) a(\vec{p},\sigma,n) \quad \psi_l^-(x) = \sum_{\sigma,n} \int d^3p\, v_l(x;\vec{p},\sigma,n) a^\dagger(\vec{p},\sigma,n)$$

$$U_0(\Lambda,a)\psi_l^\pm(x)U_0^{-1}(\Lambda,a) = \sum_{\bar{l}} D_{l\bar{l}}^\pm(\Lambda^{-1})\psi_{\bar{l}}^\pm(\Lambda x + a)$$

$$U_0(\Lambda,b)a(\vec{p},\sigma,n)U_0^{-1}(\Lambda,b) = e^{-i(\Lambda p)\cdot b}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar{\sigma}} D_{\sigma\bar{\sigma}}(W^{-1}(\Lambda,p))a(\vec{p}_\Lambda,\bar{\sigma},n)$$

$$U_0(\Lambda,b)a^\dagger(\vec{p},\sigma,n)U_0^{-1}(\Lambda,b) = e^{i(\Lambda p)\cdot b}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar{\sigma}} D_{\sigma\bar{\sigma}}^*(W^{-1}(\Lambda,p))a^\dagger(\vec{p}_\Lambda,\bar{\sigma},n)$$

$$\sum_{\bar{l}} D_{l\bar{l}}^+(\Lambda^{-1})u_{\bar{l}}(\Lambda x + b;\vec{p}_\Lambda,\bar{\sigma},n) = \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{\sigma} D_{\sigma\bar{\sigma}}(W^{-1}(\Lambda,p))e^{-i(\Lambda p)\cdot b}u_l(x;\vec{p},\sigma,n)$$

$$\sum_{\bar{l}} D_{l\bar{l}}^-(\Lambda^{-1})v_{\bar{l}}(\Lambda x + b;\vec{p}_\Lambda,\bar{\sigma},n) = \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{\sigma} D_{\sigma\bar{\sigma}}^*(W^{-1}(\Lambda,p))e^{i(\Lambda p)\cdot b}v_l(x;\vec{p},\sigma,n)$$

$$d^3p\sqrt{\frac{(\Lambda p)^0}{p^0}} = \frac{d^3p}{p^0}\sqrt{p^0(\Lambda p)^0} = \frac{d^3(\Lambda p)}{(\Lambda p)^0}\sqrt{p^0(\Lambda p)^0} = d^3(\Lambda p)\sqrt{\frac{p^0}{(\Lambda p)^0}}$$

产生和湮灭算符在坐标空间的表达:

$$\psi_l^+(x)=\sum_{\sigma,n}\int d^3p\ u_l(x;\vec{p},\sigma,n)a(\vec{p},\sigma,n)\quad \psi_l^-(x)=\sum_{\sigma,n}\int d^3p\ v_l(x;\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$\sum_{\bar{l}}D_{l\bar{l}}^+(\Lambda^{-1})u_{\bar{l}}(\Lambda x+b;\vec{p}_\Lambda,\bar{\sigma},n)=\sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{\sigma}D_{\sigma\bar{\sigma}}(W^{-1}(\Lambda,p))e^{-i(\Lambda p)\cdot b}u_l(x;\vec{p},\sigma,n)$$

$$\sum_{\bar{l}}D_{l\bar{l}}^-(\Lambda^{-1})v_{\bar{l}}(\Lambda x+b;\vec{p}_\Lambda,\bar{\sigma},n)=\sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{\sigma}D_{\sigma\bar{\sigma}}^*(W^{-1}(\Lambda,p))e^{i(\Lambda p)\cdot b}v_l(x;\vec{p},\sigma,n)$$

$$\sum_{\bar{\sigma}}u_{\bar{l}}(\Lambda x+b;\vec{p}_\Lambda,\bar{\sigma},n)D_{\bar{\sigma}\sigma}(W(\Lambda,p))=\sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{l}D_{\bar{l}l}^+(\Lambda)e^{-i(\Lambda p)\cdot b}u_l(x;\vec{p},\sigma,n)$$

$$\sum_{\bar{\sigma}}v_{\bar{l}}(\Lambda x+b;\vec{p}_\Lambda,\bar{\sigma},n)D_{\bar{\sigma}\sigma}^*(W(\Lambda,p))=\sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{l}D_{\bar{l}l}^-(\Lambda)e^{i(\Lambda p)\cdot b}v_l(x;\vec{p},\sigma,n)$$

平移

$$\psi_l^+(x)=\sum_{\sigma,n}\int d^3p\, u_l(x;\vec{p},\sigma,n)a(\vec{p},\sigma,n) \quad \psi_l^-(x)=\sum_{\sigma,n}\int d^3p\, v_l(x;\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$\sum_{\bar\sigma}u_{\bar l}(\Lambda x+b;\vec{p}_\Lambda,\bar\sigma,n)D_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^+_{\bar l l}(\Lambda)e^{-i(\Lambda p)\cdot b}u_l(x;\vec{p},\sigma,n)$$

$$\sum_{\bar\sigma}v_{\bar l}(\Lambda x+b;\vec{p}_\Lambda,\bar\sigma,n)D^*_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^-_{\bar l l}(\Lambda)e^{i(\Lambda p)\cdot b}v_l(x;\vec{p},\sigma,n)$$

$\Lambda=1, b$任意$,\Rightarrow D^+(\Lambda)=D^-(\Lambda)=1$ $\Rightarrow$ $u_l(x+b;\vec{p},\sigma,n)=e^{-ip\cdot b}u_l(x;\vec{p},\sigma,n) \quad v_l(x+b;\vec{p},\sigma,n)=e^{ip\cdot b}v_l(x;\vec{p},\sigma,n)$

$\Rightarrow u_l(x;\vec{p},\sigma,n)=\dfrac{e^{-ip\cdot x}}{(2\pi)^{3/2}}u_l(\vec{p},\sigma,n)$ 注意指数上的正负号！ $v_l(x;\vec{p},\sigma,n)=\dfrac{e^{ip\cdot x}}{(2\pi)^{3/2}}v_l(\vec{p},\sigma,n)$

$$\psi_l^+(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ip\cdot x}u_l(\vec{p},\sigma,n)a(\vec{p},\sigma,n) \quad \underbrace{\psi_l^-(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{ip\cdot x}v_l(\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)}$$

$$\langle 0|\psi_l^-(x)p^\rho|\Psi\rangle\sim -i\partial^\rho\langle 0|\psi_l^-(x)|\Psi\rangle=\langle 0|[P^\rho,\psi_l^-(x)]|\Psi\rangle=-\langle 0|\psi_l^-(x)P^\rho|\Psi\rangle$$

$$\sum_{\bar\sigma}u_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^+_{\bar l l}(\Lambda)u_l(\vec{p},\sigma,n)$$

$$\sum_{\bar\sigma}v_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D^*_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^-_{\bar l l}(\Lambda)v_l(\vec{p},\sigma,n)$$

有质量情况

$$\psi_l^+(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ip\cdot x}u_l(\vec{p},\sigma,n)a(\vec{p},\sigma,n)\quad \psi_l^-(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{ip\cdot x}v_l(\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$\sum_{\bar\sigma}u_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^+_{\bar l l}(\Lambda)u_l(\vec{p},\sigma,n)$$

$$\sum_{\bar\sigma}v_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D^*_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^-_{\bar l l}(\Lambda)v_l(\vec{p},\sigma,n)$$

推进： 定义$q=\Lambda p$， 取$p=k=(M,0,0,0)$

$\Lambda=L(q)$, $L(q)$由第一章给出是沿$\vec{q}$方向的"推进"： $q=L(q)(M,0,0,0)=\Lambda p$

则$L(p)=1$， 导致$W(\Lambda,p)\equiv L^{-1}(\Lambda p)\Lambda L(p)=L^{-1}(q)L(q)=1$

$$u_{\bar l}(\vec{q},\sigma,n)=\sqrt{\frac{M}{q^0}}\sum_l D^+_{\bar l l}(L(q))u_l(0,\sigma,n)$$

$$v_{\bar l}(\vec{q},\sigma,n)=\sqrt{\frac{M}{q^0}}\sum_l D^-_{\bar l l}(L(q))v_l(0,\sigma,n)$$

有质量情况

$$\psi_l^+(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ip\cdot x}u_l(\vec{p},\sigma,n)a(\vec{p},\sigma,n)\quad \psi_l^-(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{ip\cdot x}v_l(\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$\sum_{\bar\sigma}u_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^+_{\bar l l}(\Lambda)u_l(\vec{p},\sigma,n)$$

$$\sum_{\bar\sigma}v_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D^*_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^-_{\bar l l}(\Lambda)v_l(\vec{p},\sigma,n)$$

转动: $D^+=D^-$ 为了适应未来反粒子的引入!

取$p=k=(M,0,0,0)$,$\Lambda=R$, 导致$\vec{p}_\Lambda=0$: $q=\Lambda p=p$

则上章给出的$L(p)=1$，导致$W(\Lambda,p)\equiv L^{-1}(\Lambda p)\Lambda L(p)=L^{-1}(p)R=R$

$$\sum_{\bar\sigma}u_{\bar l}(0,\bar\sigma,n)D_{\bar\sigma\sigma}(R)=\sum_l D^+_{\bar l l}(R)u_l(0,\sigma,n)\xrightarrow{\text{非平庸 (平庸是恒等式)}}\sum_{\bar\sigma}u_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)}_{\bar\sigma\sigma}=\sum_l\vec{\mathcal{J}}_{\bar l l}u_l(0,\sigma,n)$$

$\Updownarrow$ 它们是可以构造出来的! 对固定 j_n' 的 $\vec{\mathcal{J}}_{\bar l l}$ 选择 $j_n\le j_n'$ 的 $\vec{J}^{(j_n)}$ 进行求解

$$\sum_{\bar\sigma}v_{\bar l}(0,\bar\sigma,n)D^*_{\bar\sigma\sigma}(R)=\sum_l D^-_{\bar l l}(R)v_l(0,\sigma,n)\xrightarrow{\text{非平庸 (平庸是恒等式)}}\sum_{\bar\sigma}v_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)*}_{\bar\sigma\sigma}=-\sum_l\vec{\mathcal{J}}_{\bar l l}v_l(0,\sigma,n)$$

无质量情况

$$\psi_l^+(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ip\cdot x}u_l(\vec p,\sigma,n)a(\vec p,\sigma,n)\quad \psi_l^-(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{ip\cdot x}v_l(\vec p,\sigma,n)a^\dagger(\vec p,\sigma,n)$$

$$\sum_{\bar\sigma}u_{\bar l}(\vec p_\Lambda,\bar\sigma,n)D_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^+_{\bar l l}(\Lambda)u_l(\vec p,\sigma,n)\quad D_{\bar\sigma\sigma}(W(\Lambda,p))=e^{i\theta(\Lambda,p)\sigma}\delta_{\bar\sigma\sigma}$$

$$\sum_{\bar\sigma}v_{\bar l}(\vec p_\Lambda,\bar\sigma,n)D^*_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^-_{\bar l l}(\Lambda)v_l(\vec p,\sigma,n)\quad D^*_{\bar\sigma\sigma}(W(\Lambda,p))=e^{-i\theta(\Lambda,p)\sigma}\delta_{\bar\sigma\sigma}$$

取$p=k=(\kappa,0,0,\kappa)$,$\Lambda=L(q)$,: $q=L(q)(\kappa,0,0,\kappa)=\Lambda p$

则$L(p)=1$，导致$W(\Lambda,p)\equiv L^{-1}(\Lambda p)\Lambda L(p)=L^{-1}(q)L(q)=1$

$$u_{\bar l}(\vec q,\sigma,n)=\sqrt{\frac{k^0}{q^0}}\sum_l D^+_{\bar l l}(L(q))u_l(\vec k,\sigma,n)\quad v_{\bar l}(\vec q,\sigma,n)=\sqrt{\frac{k^0}{q^0}}\sum_l D^-_{\bar l l}(L(q))v_l(\vec k,\sigma,n)$$

取：$p=k=(\kappa,0,0,\kappa)$，$\Lambda=W\quad q=\Lambda p=k$

$$u_{\bar l}(\vec k,\sigma,n)e^{i\theta(W,k)\sigma}=\sum_l D^+_{\bar l l}(W)u_l(\vec k,\sigma,n)\quad v_{\bar l}(\vec k,\sigma,n)e^{-i\theta(W,k)\sigma}=\sum_l D^-_{\bar l l}(W)v_l(\vec p,\sigma,n)$$

无质量情况: $W(\theta,\alpha,\beta)=S(\alpha,\beta)R(\theta)$

$$\psi_l^+(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ip\cdot x}u_l(\vec{p},\sigma,n)a(\vec{p},\sigma,n)\quad \psi_l^-(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{ip\cdot x}v_l(\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$\sum_{\bar\sigma}u_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^+_{\bar l l}(\Lambda)u_l(\vec{p},\sigma,n)\quad D_{\bar\sigma\sigma}(W(\Lambda,p))=e^{i\theta(\Lambda,p)\sigma}\delta_{\bar\sigma\sigma}$$

$$\sum_{\bar\sigma}v_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D^*_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^-_{\bar l l}(\Lambda)v_l(\vec{p},\sigma,n)\quad D^*_{\bar\sigma\sigma}(W(\Lambda,p))=e^{-i\theta(\Lambda,p)\sigma}\delta_{\bar\sigma\sigma}$$

$$u_{\bar l}(\vec{k},\sigma,n)e^{i\theta(W,k)\sigma}=\sum_l D^+_{\bar l l}(W)u_l(\vec{k},\sigma,n)\quad v_{\bar l}(\vec{k},\sigma,n)e^{-i\theta(W,k)\sigma}=\sum_l D^-_{\bar l l}(W)v_l(\vec{p},\sigma,n)$$

$$u_{\bar l}(\vec{k},\sigma,n)e^{i\theta\sigma}=\sum_l D^+_{\bar l l}(R(\theta))u_l(\vec{k},\sigma,n)\quad v_{\bar l}(\vec{k},\sigma,n)e^{-i\theta\sigma}=\sum_l D^-_{\bar l l}(R(\theta))v_l(\vec{k},\sigma,n)$$

⇕ 它们是可以构造出来的! ⇕

$$u_{\bar l}(\vec{k},\sigma,n)=\sum_l D^+_{\bar l l}(S(\alpha,\beta))u_l(\vec{k},\sigma,n)\quad v_{\bar l}(\vec{k},\sigma,n)=\sum_l D^-_{\bar l l}(S(\alpha,\beta))v_l(\vec{k},\sigma,n)$$

推进与转动

$$\psi_l^+(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ip\cdot x}u_l(\vec{p},\sigma,n)a(\vec{p},\sigma,n)\quad \psi_l^-(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{ip\cdot x}v_l(\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

有质量： $u_{\bar{l}}(\vec{q},\sigma,n)=\sqrt{\frac{M}{q^0}}\sum_l D^+_{\bar{l}l}(L(q))u_l(0,\sigma,n)\quad v_{\bar{l}}(\vec{q},\sigma,n)=\sqrt{\frac{M}{q^0}}\sum_l D^-_{\bar{l}l}(L(q))v_l(0,\sigma,n)$

$$\sum_{\bar{\sigma}}u_{\bar{l}}(0,\bar{\sigma},n)D_{\bar{\sigma}\sigma}(R)=\sum_l D^+_{\bar{l}l}(R)u_l(0,\sigma,n)\xRightarrow{\text{非平庸 (平庸是恒等式)}}\sum_{\bar{\sigma}}u_{\bar{l}}(0,\bar{\sigma},n)\vec{J}^{(j_n)}_{\bar{\sigma}\sigma}=\sum_l\vec{\mathcal{J}}_{\bar{l}l}u_l(0,\sigma,n)$$

$$\sum_{\bar{\sigma}}v_{\bar{l}}(0,\bar{\sigma},n)D^*_{\bar{\sigma}\sigma}(R)=\sum_l D^-_{\bar{l}l}(R)v_l(0,\sigma,n)\xRightarrow{\text{非平庸 (平庸是恒等式)}}\sum_{\bar{\sigma}}v_{\bar{l}}(0,\bar{\sigma},n)\vec{J}^{(j_n)*}_{\bar{\sigma}\sigma}=-\sum_l\vec{\mathcal{J}}_{\bar{l}l}v_l(0,\sigma,n)$$

无质量： $u_{\bar{l}}(\vec{q},\sigma,n)=\sqrt{\frac{k^0}{q^0}}\sum_l D^+_{\bar{l}l}(L(q))u_l(\vec{k},\sigma,n)\quad v_{\bar{l}}(\vec{q},\sigma,n)=\sqrt{\frac{k^0}{q^0}}\sum_l D^-_{\bar{l}l}(L(q))v_l(\vec{k},\sigma,n)$

$$u_{\bar{l}}(\vec{k},\sigma,n)e^{i\theta\sigma}=\sum_l D^+_{\bar{l}l}(R(\theta))u_l(\vec{k},\sigma,n)\quad v_{\bar{l}}(\vec{k},\sigma,n)e^{-i\theta\sigma}=\sum_l D^-_{\bar{l}l}(R(\theta))v_l(\vec{k},\sigma,n)$$

$D^+=D^-=$实矩阵

$$u_{\bar{l}}(\vec{k},\sigma,n)=\sum_l D^+_{\bar{l}l}(S(\alpha,\beta))u_l(\vec{k},\sigma,n)\quad v_{\bar{l}}(\vec{k},\sigma,n)=\sum_l D^-_{\bar{l}l}(S(\alpha,\beta))v_l(\vec{k},\sigma,n)$$

$v_l=u_l^*$

后面对自旋0, 1/2的场只讨论有质量的情形（零质量取有质量的零质量极限，且无奇异），对自旋1及以上的单独讨论零质量情形。

$$\psi_l^+(x) = \sum_{\sigma,n}\int \frac{d^3p}{(2\pi)^{3/2}} e^{-ip\cdot x} u_l(\vec{p},\sigma,n)a(\vec{p},\sigma,n) \quad \psi_l^-(x) = \sum_{\sigma,n}\int \frac{d^3p}{(2\pi)^{3/2}} e^{ip\cdot x} v_l(\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$\sum_{\bar{\sigma}} u_{\bar{l}}(\vec{p}_\Lambda,\bar{\sigma},n) D_{\bar{\sigma}\sigma}(W(\Lambda,p)) = \sqrt{p^0/(\Lambda p)^0} \sum_{l} D^+_{\bar{l}l}(\Lambda) u_l(\vec{p},\sigma,n)$$

$$\sum_{\bar{\sigma}} v_{\bar{l}}(\vec{p}_\Lambda,\bar{\sigma},n) D^*_{\bar{\sigma}\sigma}(W(\Lambda,p)) = \sqrt{p^0/(\Lambda p)^0} \sum_{l} D^-_{\bar{l}l}(\Lambda) v_l(\vec{p},\sigma,n)$$

标量量子场: $D^\pm(\Lambda) = 1 \quad D(W) = 1,\ u_l(\vec{p}_\Lambda)\sqrt{(\Lambda p)^0} = u_l(\vec{p})\sqrt{p^0} \quad v_l(\vec{p}_\Lambda)\sqrt{(\Lambda p)^0} = v_l(\vec{p})\sqrt{p^0}$

只考虑有质量的标量场: $u(\vec{p}) = v(\vec{p}) = 1/\sqrt{2p^0}$ 无质量可看作质量趋于0的极限

$$\phi^+(x) = \int \frac{d\vec{p}}{(2\pi)^{3/2}\sqrt{2p^0}} e^{-ip\cdot x} a(\vec{p}) \quad \phi^-(x) = \int \frac{d\vec{p}}{(2\pi)^{3/2}\sqrt{2p^0}} e^{ip\cdot x} a^\dagger(\vec{p}) = (\phi^+(x))^\dagger$$

$$[\phi^+(x),\phi^+(y)]_\mp = 0 \qquad [\phi^-(x),\phi^-(y)]_\mp = 0$$

对易子: 玻色子; 反对易子: 费米子

$$\Delta_+(M,x-y) \equiv [\phi^+(x),\phi^-(y)]_\mp = \int \frac{d\vec{p}\, d\vec{p}'}{(2\pi)^3(2p^0 2p'^0)^{1/2}} e^{-ipx} e^{ip'y} \delta(\vec{p}-\vec{p}') = \int \frac{d\vec{p}}{(2\pi)^3 2p^0} e^{-ip\cdot(x-y)}$$

作业10

♣ 动量空间的产生与湮灭算符:

$$[a(q), a^\dagger(q')]_\mp = \delta(\vec{q}-\vec{q}') \qquad [a(q), a(q')]_\mp = [a^\dagger(q), a^\dagger(q')]_\mp = 0 \quad a(q)\Phi_0 = 0$$

$$N \equiv \int d\vec{p}\, N(p) \qquad [N(p), a^\dagger(q)] = \delta(\vec{p}-\vec{q})a^\dagger(q) \qquad [N, a^\dagger(q)] = a^\dagger(q)$$

$$N(p) \equiv a^\dagger(p)a(p) \qquad [N(p), a(q)] = -\delta(\vec{p}-\vec{q})a(q) \qquad [N, a(q)] = -a(q)$$

♠ 坐标空间的产生与湮灭算符:

$$[\phi^+(x), \phi^-(x')]_\mp = \Delta_+(M, x-x') \quad [\phi^+(x), \phi^+(x')]_\mp = [\phi^-(x), \phi^-(x')]_\mp = 0 \quad \phi^+(x)\Phi_0 = 0$$

$$\Delta_+(M, x) = \int \frac{d\vec{p}}{(2\pi)^3 2p^0} e^{-ip\cdot x} \overset{x类空}{=\!=\!=} \frac{M}{4\pi^2\sqrt{-x^2}} K_1(M\sqrt{-x^2}) \overset{M\to\infty, t=0}{=\!=\!=\!=\!=\!=} \frac{1}{2M}\delta(\vec{x})$$

$$\tilde{N}(t) \equiv \int d\vec{x}\, \phi^-(x)\phi^+(x) \qquad [\tilde{N}(t), \phi^-(\vec{x}, t)] = \int d\vec{x}'\phi^-(\vec{x}', t)\Delta_+(M, \vec{x}'-\vec{x}, 0)$$

$$[\tilde{N}(t), \phi^+(\vec{x}, t)] = -\int d\vec{x}'\phi^+(\vec{x}', t)\Delta_+(M, \vec{x}-\vec{x}', 0)$$

关于坐标空间的产生与湮灭算符

$$[\phi^+(x),\phi^-(x')]_\mp = \Delta_+(M,x-x') \quad [\phi^+(x),\phi^+(x')]_\mp = [\phi^-(x),\phi^-(x')]_\mp = 0 \quad \phi^+(x)\Phi_0 = 0$$

$$\Delta_+(M,x) = \int \frac{d\vec{p}}{(2\pi)^3 2p^0} e^{-ip\cdot x} \overset{x\text{类空}}{=\!=\!=} \frac{M}{4\pi^2\sqrt{-x^2}} K_1(M\sqrt{-x^2}) \overset{M\to\infty,t=0}{=\!=\!=\!=\!=\!=} \frac{1}{2M}\delta(\vec{x})$$

$$\tilde{N}(t) \equiv \int d\vec{x}\,\phi^-(x)\phi^+(x) \qquad [\tilde{N}(t),\phi^-(\vec{x},t)] = \int d\vec{x}'\phi^-(\vec{x}',t)\Delta_+(M,\vec{x}'-\vec{x},0)$$

♣ 波包：

$$[\tilde{N}(t),\phi^+(\vec{x},t)] = -\int d\vec{x}'\phi^+(\vec{x}',t)\Delta_+(M,\vec{x}-\vec{x}',0)$$

若存在：$\rho(\vec{x}) = 2M\int d\vec{x}'\rho(\vec{x}')\Delta_+(M,\vec{x}'-\vec{x},0)$ $\qquad \tilde{\phi}^-(t) \equiv \int d\vec{x}\,\rho(\vec{x})\phi^-(\vec{x},t)$

$$[\tilde{N}(t),\tilde{\phi}^-(t)] = \int d\vec{x}d\vec{x}'\rho(\vec{x}')\Delta_+(M,\vec{x}'-\vec{x},0)\phi^-(\vec{x},t) = \tilde{\phi}^-(t)$$

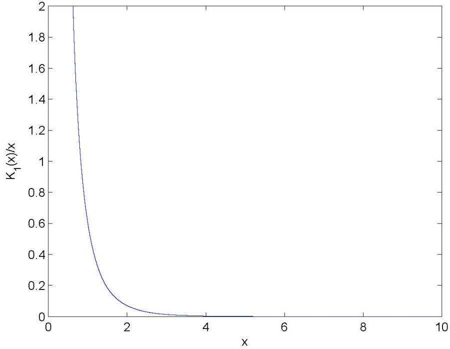
$$[\tilde{N}(t),\tilde{\phi}^+(t)] = -\int d\vec{x}d\vec{x}'\rho(\vec{x}')\Delta_+(M,\vec{x}'-\vec{x},0)\phi^+(\vec{x},t) = -\tilde{\phi}^+(t)$$

坐标空间的产生与湮灭算符: $\Delta_+(M,x) = \frac{M}{4\pi^2\sqrt{-x^2}}K_1(M\sqrt{-x^2})$

$$\tilde{N}(t) \equiv \int d\vec{x}\, \phi^-(x)\phi^+(x)$$

$$[\tilde{N}(t), \phi^-(\vec{x},t)] = \int d\vec{x}'\phi^-(\vec{x}',t)\Delta_+(M,\vec{x}'-\vec{x},0)$$

$$[\tilde{N}(t), \phi^+(\vec{x},t)] = -\int d\vec{x}'\phi^+(\vec{x}',t)\Delta_+(M,\vec{x}-\vec{x}',0)$$



$$K_1(x) = \int_0^\infty dt \frac{t\sin xt}{\sqrt{1+t^2}}$$

ϕ^+和ϕ^-场之间的不对易（或不反对易）将对构造$\tilde{\mathcal{H}}(x)$产生很大的困难。如果$\tilde{\mathcal{H}}(x)$是由ϕ^+和ϕ^-场构造的，$\tilde{\mathcal{H}}(x)$在类空区对易要求ϕ^+和ϕ^-场之间必须是对易或反对易的，否则ϕ^+场和ϕ^-场之间必须达成某种平衡，以使ϕ^+场和ϕ^-场之间的不对易（或不反对易）的影响能够以某种形式被消掉。

为了寻找这种能够消除这种不对易性对$\tilde{\mathcal{H}}(x)$在类空区的对易性的影响的规则，将ϕ^+场和ϕ^-场进行一个线性组合$\phi(x) \equiv \kappa\phi^+(x) + \lambda\phi^-(x)$,看是否能够实现这个线性组合场在类空区间的对易或反对易性质

$$[\phi(x),\phi^\dagger(y)]_\mp = |\kappa|^2\Delta_+(M,x-y) \mp |\lambda|^2\Delta_+(M,y-x) \quad [\phi(x),\phi(y)]_\mp = \kappa\lambda\{\Delta_+(M,x-y) \mp \Delta_+(M,y-x)\}$$

$$\Delta_+(M,x-y)\text{是洛伦兹不变量}\xRightarrow{\text{类空间隔可实现}x-y\to y-x}\Delta_+(M,x-y) = \Delta_+(M,y-x)$$

$$\Rightarrow \text{选对易子和}|\kappa| = |\lambda|$$

产生和湮灭算符还有一个相角的任意性，ϕ前的一个整体常数是无关紧要的.可将κ和λ的值选择为1,即：$\phi(x) = \phi^+(x) + \phi^-(x)$. 以这样一种组合的标量玻色子场$\phi(x)$和其共轭$\phi^\dagger(x)$来构造$\mathcal{H}(x)$可以保证其在类控区间相互对易。

ϕ^+和ϕ^-场之间的不对易（或不反对易）将对构造$\tilde{\mathcal{H}}(x)$产生很大的困难。
改进办法是将ϕ^+场和ϕ^-场进行一个线性组合$\phi(x) \equiv \phi^+(x) + \phi^-(x)$,它使得在类空区间 $[\phi(x), \phi^\dagger(y)] = [\phi(x), \phi(y)] = 0$

引发问题: 如ϕ场带某种守恒内部荷, $\tilde{\mathcal{H}}(x)$与生成此对称性的生成元对易,

$$[Q^a, \tilde{\mathcal{H}}(x)] = 0 \quad \Rightarrow \quad [Q_a, \phi^+(x)]_- = -q_a\phi^+(x) \qquad [Q_a, \phi^-(x)]_- = q_a\phi^-(x)$$

当荷$q_a \neq 0$时，若以ϕ^+和ϕ^-场为基本元素构造$\tilde{\mathcal{H}}(x)$,要保证$[Q^a, \tilde{\mathcal{H}}(x)] = 0$,每一项中含$\phi^+$的数目和含$\phi^-$的数目相等即可.

但若改用以ϕ和$\phi^\dagger$场作为基本元素构造$\tilde{\mathcal{H}}(x)$,不管怎样构造,都不可能保证每一项中含ϕ^+的数目和含ϕ^-的数目相等.无法实现要求$[Q^a, \tilde{\mathcal{H}}(x)] = 0$.

ϕ^+和ϕ^-场之间的不对易（或不反对易）将对构造$\tilde{\mathcal{H}}(x)$产生很大的困难。
改进办法是将ϕ^+场和ϕ^-场进行一个线性组合$\phi(x) \equiv \phi^+(x) + \phi^-(x)$,它使得在类空区间 $[\phi(x), \phi^\dagger(y)] = [\phi(x), \phi(y)] = 0$ 但无法实现$[Q^a, \tilde{\mathcal{H}}(x)] = 0$.

以ϕ和$\phi^\dagger$场作为基本元素构造$\tilde{\mathcal{H}}(x)$造成无法实现要求$[Q^a, \tilde{\mathcal{H}}(x)] = 0$的困难的根本原因是$\phi$场不像$\phi^+$那样有确定的荷,因为它是由带$-q_a$荷的$\phi^+$部分和带$+q_a$荷的$\phi^-$部分叠加而成的.

为了使ϕ场有确定的荷,将ϕ中的ϕ^-部分的粒子换为另外一种能使其带有$-q_a$荷但具有与原来粒子相同质量的标量玻色粒子,$\phi(x) = \phi^+(x) + \phi^{+c\dagger}(x)$

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} e^{-ip\cdot x} a(\vec{p}) \qquad \phi^{+c\dagger}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} e^{ip\cdot x} a^{c\dagger}(\vec{p}) = (\phi^{+c}(x))^\dagger$$

$$[Q_a, \phi^+(x)]_- = -q_a\phi^+(x) \qquad [Q_a, (\phi^{+c}(x))^\dagger]_- = -q_a(\phi^{+c}(x))^\dagger$$

$$[Q_a,\ a(\vec{p})]_- = -q_a a(\vec{p}) \qquad [Q_a,\ a^c(\vec{p})]_- = +q_a a^c(\vec{p})$$

$a(\vec{p})$和$a^c(\vec{p})$是各带相反荷但具有相同质量的两种不同标量玻色子湮灭算符.
上标^c 用于指示电荷共轭,反映它是另一种具有同样质量但带相反荷的粒子.

$$[\phi(x),\phi^\dagger(y)] = [\phi^+(x),\phi^{+\dagger}(y)] - [\phi^{+c}(x),\phi^{+c\dagger}(y)] = \Delta(M,x-y) \overset{\text{类空区间}}{=\!=\!=} 0 \qquad [Q_a,\phi(x)] = -q_a\phi(x)$$

$$\Delta(M,x) = \Delta_+(M,x) - \Delta_+(M,-x) = \int \frac{d\vec{p}}{2p^0(2\pi)^3}[e^{-ip\cdot x} - e^{ip\cdot x}]$$

作业11

$\phi(x)=\phi^+(x)+\phi^{+c\dagger}(x)$ $[Q_a,\phi^+(x)]_-=-q_a\phi^+(x)$ $[Q_a,(\phi^{+c}(x))^\dagger]_-=-q_a(\phi^{+c}(x))^\dagger$

$$\phi^+(x)=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}e^{-ip\cdot x}a(\vec{p}) \qquad \phi^{+c\dagger}(x)=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}e^{ip\cdot x}a^{c\dagger}(\vec{p})=(\phi^{+c}(x))^\dagger$$

$$[Q_a,\ a(\vec{p})]_-=-q_a a(\vec{p}) \qquad [Q_a,\ a^c(\vec{p})]_-=+q_a a^c(\vec{p})$$

$a(\vec{p})$和$a^c(\vec{p})$是各带相反荷但具有相同质量的两种不同标量玻色子湮灭算符.
上标^c 用于指示电荷共轭,反映它是另一种具有同样质量但带相反荷的粒子.

$$[\phi(x),\phi^\dagger(y)]=\Delta(M,x-y)\overset{\text{类空区间}}{=\!=\!=\!=}0 \qquad [Q_a,\phi(x)]=-q_a\phi(x)$$

第一式保证用以ϕ和$\phi^\dagger$场构造$\tilde{\mathcal{H}}(x)$可以使其在类空区间对易因而保证因果性。这只有两种粒子的质量相同才能达到!

第二式保证用以ϕ和$\phi^\dagger$场来构造$\tilde{\mathcal{H}}(x)$可以实现要求$[Q^a,\tilde{\mathcal{H}}(x)]=0$.

如果$q_a\neq 0$,则$a^c(\vec{p})\neq a(\vec{p})$. 这时体系中具有两种带有相反荷但质量相同的玻色标量粒子. 我们将这种质量相同但荷相反的粒子叫反粒子.

如果$q_a=0$,则可以选择$a^c(\vec{p})=a(\vec{p})$, 此时,粒子本身就是它自己的反粒子, 这样的粒子不带荷,相应的场是自共轭场$\phi(x)=\phi^\dagger(x)$.

复数域为反粒子留出空间

反粒子的引入

$$\phi(x) = \phi^{+}(x) + \phi^{+c\dagger}(x) \quad [Q_a, \phi^{+}(x)]_{-} = -q_a \phi^{+}(x) \quad [Q_a, (\phi^{+c}(x))^{\dagger}]_{-} = -q_a (\phi^{+c}(x))^{\dagger}$$

$$\phi^{+}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} e^{-ip\cdot x} a(\vec{p}) \qquad \phi^{+c\dagger}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} e^{ip\cdot x} a^{c\dagger}(\vec{p}) = (\phi^{+c}(x))^{\dagger}$$

$$[Q_a,\ a(\vec{p})]_{-} = -q_a a(\vec{p}) \qquad [Q_a,\ a^{c}(\vec{p})]_{-} = +q_a a^{c}(\vec{p})$$

$a(\vec{p})$和$a^c(\vec{p})$是各带相反荷但具有相同质量的两种不同标量玻色子湮灭算符.
上标^c 用于指示电荷共轭,反映它是另一种具有同样质量但带相反荷的粒子.

$$[\phi(x), \phi^{\dagger}(y)] = \Delta(M, x-y) \overset{\text{类空区间}}{=\!=\!=\!=} 0 \qquad [Q_a, \phi(x)] = -q_a \phi(x)$$

S矩阵的相对论不变性和内部对称性不变性要求在体系中存在反粒子!

理论将粒子湮灭和反粒子产生等权重地分配在场的定义中, 意味物理上把它们看成是"等价"的从荷的角度, 它意味着:

- 产生粒子"等价"于消灭反粒子，或产生反粒子"等价"于消灭粒子
- 粒子反粒子碰到一起将有发生 湮灭反应 的可能性!

这正是狄拉克的空穴理论,空穴现在被反粒子所取代.

关于反粒子的评注：

♣ 我们生存在一个粒子的世界，所有反粒子都是不稳定的!

♢ 可理解为所有反粒子都碰上粒子而湮灭掉了！ 需要相互作用

♡ 但这要求现在世界中 粒子的数目远大于反粒子的数目!

♠ 为什么会有这种 物质反物质的不对称性?

¶ 它要求基本相互作用中有产生物质反物质不对称效应的项!

✠ C或CP破坏可望达到这个目的!

空间反射变换: $Pa^\dagger(\vec{p},\sigma,n)P^{-1}=\eta a^\dagger(\mathcal{P}\vec{p},\sigma,n)$ $Pa(\vec{p},\sigma,n)P^{-1}=\eta^* a(\mathcal{P}\vec{p},\sigma,n)$

$$Pa(\vec{p})P^{-1}=\eta^* a(-\vec{p}) \qquad Pa^c(\vec{p})P^{-1}=\eta^{c*}a^c(-\vec{p})$$

$$P\phi^+(x)P^{-1}=\eta^*\phi^+(\mathcal{P}x) \qquad P\phi^{+c}(x)P^{-1}=\eta^{c*}\phi^{+c}(\mathcal{P}x)$$

空间反射变换后的场$\phi_P=\eta^*\phi^+ +\eta^c\phi^{+c\dagger}$及其共轭$\phi_P^\dagger$来构造$\tilde{\mathcal{H}}(x)$已能够使其在类空区间相互对易,并可以实现$[Q^a,\tilde{\mathcal{H}}(x)]=0$! 我们进一步可以通过选择$a$与$a^\dagger$及$a^c$与$a^{c\dagger}$之间的相对相角，使得在保证$\phi=\phi^+ +\phi^{+c\dagger}$的同时，还对称地有 作业12

$$\eta^c=\eta^* \qquad \Rightarrow \qquad \eta\eta^c=1$$

无自旋的粒子和反粒子的联合宇称位相为偶. 我们现在拥有统一的对ϕ场的空间反射变换:

$$P\phi(x)P^{-1}=\eta^*\phi(\mathcal{P}x)$$

$\eta^*=1$标量 $\eta^*=-1$赝标量

标量场的分立对称性变换性质

时间反演变换: $Ta^\dagger(\vec{p},\sigma,n)T^{-1}=(-1)^{j-\sigma}a^\dagger(\mathcal{P}\vec{p},-\sigma,n)$ $Ta(\vec{p},\sigma,n)T^{-1}=(-1)^{j-\sigma}a(\mathcal{P}\vec{p},-\sigma,n)$

$$Ta(\vec{p})T^{-1}=(-1)^{j-\sigma}a(-\vec{p}) \qquad Ta^c(\vec{p})T^{-1}=(-1)^{j-\sigma}a^c(-\vec{p})$$

$$T\phi^+(x)T^{-1}=(-1)^{j-\sigma}\phi^+(-\mathcal{P}x) \qquad T\phi^{+c}(x)T^{-1}=(-1)^{j-\sigma}\phi^{+c}(-\mathcal{P}x)$$

无自旋的粒子和反粒子的联合时间宇称位相为偶. 我们现在拥有统一的对ϕ场的时间反演变换:

$$T\phi(x)T^{-1}=(-1)^{j-\sigma}\phi(-\mathcal{P}x)$$

电荷共轭变换C:

$$Ca(\vec{p})C^{-1}=\xi^* a^c(\vec{p}) \Rightarrow C\phi^+(x)C^{-1}=\xi^*\phi^{+c}(\pm x) \quad Ca^c(\vec{p})C^{-1}=\xi^{c*}a(\vec{p}) \Rightarrow C\phi^{+c}(x)C^{-1}=\xi^{c*}\phi^+(\pm x)$$

不可以连续变形到单位变换的变换,需要判断它是幺正算符,还是反幺正算符

$$CU_0(\Lambda,a)C^{-1}\phi^{+,c}(x)CU_0^{-1}(\Lambda,a) = CU_0(\Lambda,a)C^{-1}\xi C\phi^+(\pm x)C^{-1}CU_0^{-1}(\Lambda,a)$$ +幺正; −反幺正

$$= \xi CU_0(\Lambda,a)\phi^+(\pm x)U_0^{-1}(\Lambda,a) = \xi C\phi^+(\pm(\Lambda x+a)) = \phi^{+,c}(\Lambda x+a)C$$

$$= U_0(\Lambda,a)\phi^{+,c}(x)U_0^{-1}(\Lambda,a)C \quad \Rightarrow \quad CU_0(\Lambda,a)C^{-1}=U_0(\Lambda,a) \quad U_0(\Lambda,a)C=CU_0(\Lambda,a)$$

取$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$ 和$a^\mu = \epsilon^\mu$,准到ω和ϵ一阶 $U(1+\omega,\epsilon)=1+\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}+i\epsilon_\rho P^\rho+\cdots$

$$CiJ^{\rho\sigma}C^{-1} = iJ^{\rho\sigma} \qquad CiP^\rho C^{-1} = iP^\rho$$

由于还不能确定C是幺正算符,还是反幺正算符,暂把虚数i保留在了C和C^{-1}算符的中间. 在上式中对四动量的零分量有 $CiHC^{-1} = iH$

如果C是反幺正算符将导致$CHC^{-1} = -H$.它意味如果假设物理体系具有电荷共轭对称性, 则对应能量为E每一个正能态都应有相应的负能态$-E$在物理谱中出现, 实验上并没有发现负能态, 因此要求C是反幺正算符是不对的.

应取C为幺正算符.

$$C\vec{J}C^{-1} = \vec{J} \qquad C\vec{K}C^{-1} = -\vec{K} \qquad C\vec{P}C^{-1} = -\vec{P} \qquad CHC^{-1} = H$$

电荷共轭变换C及CPT联合变换:

$$Ca(\vec{p})C^{-1} = \xi^* a^c(\vec{p}) \qquad Ca^c(\vec{p})C^{-1} = \xi^{c*} a(\vec{p})$$

C 幺正算符:

$$C\phi^+(x)C^{-1} = \xi^* \phi^{+c}(x) \qquad C\phi^{+c}(x)C^{-1} = \xi^{c*} \phi^+(x)$$

电荷共轭变换后的场$\phi_C = \xi^* \phi^{c+} + \xi^c \phi^{+\dagger}$ 及其共轭$\phi_C^\dagger$来构造$\tilde{\mathcal{H}}(x)$已能够使其在类空区间相互对易,并可以实现$[Q^a, \tilde{\mathcal{H}}(x)] = 0$! 我们进一步可以通过选择$a$与$a^\dagger$,$a^c$与$a^{c\dagger}$及$a$与$a^c$之间的相对相角，相角可以与$\vec{p}$、c相关，使得在保证$\phi = \phi^+ + \phi^{+c\dagger}$和 $\phi_C = \xi^*(\phi^+ + \phi^{+c\dagger})$的同时，还对称地有

$$\xi^c = \xi^* \qquad \Rightarrow \qquad \xi\xi^c = 1$$

无自旋的粒子和反粒子的联合电荷共轭宇称位相为偶. 我们现在拥有统一的对ϕ场的电荷共轭变换：

$$C\phi(x)C^{-1} = \xi^* \phi^\dagger(x) \qquad T\phi(x)T^{-1} = (-1)^{j-\sigma}\phi(-\mathcal{P}x) \qquad P\phi(x)P^{-1} = \eta^* \phi(\mathcal{P}x)$$

对CPT联合变换: $CPT\ \phi(x)\ [CPT]^{-1} = \xi^*(-1)^{j-\sigma}\eta^*\phi^\dagger(-x)$

自由标量场的场方程、荷流矢量、哈密顿量和作用量、正则对易关系

玻色统计；反粒子 $\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}a^{c\dagger}(\vec{p})]$ $U_0(\Lambda,a)\phi(x)U_0^{-1}(\Lambda,a)=\phi(\Lambda x+a)$

$C\phi(x)C^{-1}=\xi^*\phi^\dagger(x)$ $T\phi(x)T^{-1}=(-1)^{j-\sigma}\phi(-\mathcal{P}x)$ $P\phi(x)P^{-1}=\eta^*\phi(\mathcal{P}x)$

$$p^\mu p_\mu = M^2 \quad\Rightarrow\quad (\partial^2+M^2)\phi(x)=0$$ 自由粒子场的Klein-Gordon方程！

$$\phi^*(x)(\partial^2+M^2)\phi(x)-\phi(x)(\partial^2+M^2)\phi^*(x)=0$$

流矢量： $j_\mu(x)\equiv -iq\{[\partial_\mu\phi^*(x)]\phi(x)-\phi^*(x)[\partial_\mu\phi(x)]\}$ $\partial^\mu j_\mu(x)=0$

荷： $$\int d^3x\, j^0(x,t) = -iq\int d^3x\,\{[\partial^0\phi^*(x)]\phi(x)-\phi^*(x)\partial^0\phi(x)\} = q\underbrace{\int d^3p[a^\dagger(\vec{p})a(\vec{p})-a^c(\vec{p})a^{c\dagger}(\vec{p})]}_{\text{对中性粒子为零}}$$

$$[\int d^3x\, j^0(x,t), a(\vec{p})] = -qa(\vec{p}) \qquad [\int d^3x\, j^0(x,t), a^\dagger(\vec{p})] = qa^\dagger(\vec{p})$$

$$[\int d^3x\, j^0(x,t), a^c(\vec{p})] = qa^c(\vec{p}) \qquad [\int d^3x\, j^0(x,t), a^{c\dagger}(\vec{p})] = -qa^{c\dagger}(\vec{p})$$

$$Q(t)\equiv\int d^3x\, j^0(x,t) = Q^\dagger(t) \qquad [Q,\phi(x)]=-q\phi(x) \qquad [Q,\phi^\dagger(x)]=q\phi^\dagger(x)$$

$$\dot{Q}(t) = -\int d^3x\,\nabla\cdot\vec{j}(x,t) = 0 \qquad CQ(t)C^{-1} = q\int d^3p[a^{c\dagger}(\vec{p})a^c(\vec{p})-a(\vec{p})a^\dagger(\vec{p})] = -Q(t)$$

自由标量场的场方程、荷流矢量、哈密顿量和作用量、正则对易关系

$$\phi(x)=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p})+e^{ip\cdot x}a^{c\dagger}(\vec{p})]$$

$$p^\mu p_\mu=M^2 \quad\Rightarrow\quad (\partial^2+M^2)\phi(x)=0$$ 自由粒子场的**Klein-Gordon**方程！

自共轭自由标量场$\phi=\phi^++\phi^{+\dagger}$的哈密顿量就是体系的总能量算符

$$H_0'=\int d\vec{p}\; a^\dagger(\vec{p})a(\vec{p})\sqrt{\vec{p}^2+M^2}=\int d\vec{p}\;\frac{1}{2}\sqrt{\vec{p}^2+M^2}:\{a^\dagger(\vec{p})a(\vec{p})+a(\vec{p})a^\dagger(\vec{p})\}:$$

$:O:$代表对算符O的正规乘积，即将算符O中的湮灭算符排在右边.

如果去掉正规乘积，将会导致真空的零点能。

验证哈密顿量算符确实是体系平移变换的生成元：

$$a(q)\Phi_{q_1\cdots q_N}=\sum_{r=1}^N\delta(q-q_r)\Phi_{q_1\cdots q_{r-1}q_{r+1}\cdots q_N} \qquad H_0\Phi_{q_1\cdots q_N}=(\sqrt{\vec{q}_1^2+M^2}+\cdots+\sqrt{\vec{q}_N^2+M^2})\Phi_{q_1\cdots q_N}$$

$$H_0'\Phi_{q_1\cdots q_N}=\int d\vec{p}\; a^\dagger(\vec{p})a(\vec{p})\sqrt{\vec{p}^2+M^2}\Phi_{q_1\cdots q_N}=\int d\vec{p}\; a^\dagger(\vec{p})\sum_{r=1}^N\delta(p-q_r)\Phi_{q_1\cdots q_{r-1}q_{r+1}\cdots q_N}\sqrt{\vec{p}^2+M^2}$$

$$=\sum_{r=1}^N\Phi_{q_1\cdots q_{r-1}q_rq_{r+1}\cdots q_N}\sqrt{\vec{q}_r^2+M^2}=H_0\Phi_{q_1\cdots q_N}\quad\Rightarrow\quad H_0'=H_0$$ 可类似地讨论其它洛伦兹群生成元

$$\phi(x)=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p})+e^{ip\cdot x}a^{c\dagger}(\vec{p})]$$

$p^\mu p_\mu = M^2 \quad\Rightarrow\quad (\partial^2+M^2)\phi(x)=0$ 自由粒子场的Klein-Gordon方程！

自共轭自由标量场$\phi=\phi^+ + \phi^{+\dagger}$ 的哈密顿量就是体系的总能量算符

$$H_0=\int d\vec{p}\, a^\dagger(\vec{p})a(\vec{p})\sqrt{\vec{p}^2+M^2}=\int d\vec{p}\,\frac{1}{2}\sqrt{\vec{p}^2+M^2}:\{a^\dagger(\vec{p})a(\vec{p})+a(\vec{p})a^\dagger(\vec{p})\}:$$

$:O:$代表对算符O的正规乘积，即将算符O中的湮灭算符排在右边.

如果去掉正规乘积，将会导致真空的零点能。

哈密顿量的坐标空间表达：$\phi(x)=\int\frac{d\vec{p}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p})+e^{ip\cdot x}a^\dagger(\vec{p})]$

$$\dot{\phi}(x)=i\int\frac{d\vec{p}}{(2\pi)^{3/2}}\frac{p^0}{\sqrt{2p^0}}[-e^{-ip\cdot x}a(\vec{p})+e^{ip\cdot x}a^\dagger(\vec{p})] \qquad \nabla\phi(x)=i\int\frac{d\vec{p}}{(2\pi)^{3/2}}\frac{\vec{p}}{\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p})-e^{ip\cdot x}a^\dagger(\vec{p})]$$

$\dot{O}$是对O的时间依赖求导数。自共轭自由标量场作业14哈密顿量还可表达为：

$$H_0=\int d\vec{x}\,\frac{1}{2}:[\dot{\phi}^2(x)+(\nabla\phi(x))^2+M^2\phi^2(x)]:$$

自由标量场的场方程、荷流矢量、哈密顿量和作用量、正则对易关系

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}a^{c\dagger}(\vec{p})] \qquad (\partial^2 + M^2)\phi(x) = 0$$

哈密顿量的坐标空间表达：$\phi(x)=\int \frac{d\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p})+e^{ip\cdot x}a^\dagger(\vec{p})]$

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$\dot{O}$是对O的时间依赖求导数。自共轭自由标量场作业14哈密顿量还可表达为：

$$H_0 = \int d\vec{x}\, \frac{1}{2} : [\dot{\phi}^2(x) + (\nabla\phi(x))^2 + M^2\phi^2(x)] :$$

将$\phi(x)$取为体系的广义坐标，H_0在广义动量固定情形下对$\phi(x)$的泛函微商定义了体系的广义动量对时间的导数的负值$-\dot{\pi}(x)$

$$-\dot{\pi}(x) \equiv \left.\frac{\delta H_0}{\delta\phi(x)}\right|_{\pi固定} = (-\nabla^2 + M^2)\phi(x) = -\ddot{\phi}(x)$$ 略去边界积分，在微商过程中将$\dot{\phi}$项固定

$$\pi(x) = \dot{\phi}(x) \qquad S_0=\int d^4x\ :\dot{\phi}^2(x):-\int dt\ H_0=\int d^4x\ \frac{1}{2} : [(\partial_\mu\phi(x))^2 - M^2\phi^2(x)] :$$

将作用量S_0取极值就得到场方程！

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}a^{c\dagger}(\vec{p})] \qquad (\partial^2 + M^2)\phi(x) = 0$$

哈密顿量的坐标空间表达: $\phi(x)=\int \frac{d\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p})+e^{ip\cdot x}a^\dagger(\vec{p})]$

$$\dot{\phi}(x)=i\int \frac{d\vec{p}}{(2\pi)^{3/2}} \frac{p^0}{\sqrt{2p^0}}[-e^{-ip\cdot x}a(\vec{p})+e^{ip\cdot x}a^\dagger(\vec{p})] \quad \nabla\phi(x)=i\int \frac{d\vec{p}}{(2\pi)^{3/2}} \frac{\vec{p}}{\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p})-e^{ip\cdot x}a^\dagger(\vec{p})]$$

$\dot{O}$是对O的时间依赖求导数。自共轭自由标量场作业14哈密顿量还可表达为:

$$H_0=\int d\vec{x}\, \frac{1}{2} :[\dot{\phi}^2(x) + (\nabla\phi(x))^2 + M^2\phi^2(x)]: \quad S_0=\int d^4x\, \frac{1}{2} :[(\partial_\mu\phi(x))^2 - M^2\phi^2(x)]:$$

等时对易关系:

$$[\phi(\vec{x},t), \pi(\vec{y},t)]_- = i\delta(\vec{x}-\vec{y}) \qquad \pi(x) = \dot{\phi}(x)$$

$$[\phi(\vec{x},t), \phi(\vec{y},t)]_- = [\pi(\vec{x},t), \pi(\vec{y},t)]_- = 0$$ 作业13

$$\dot{\phi}(\vec{x},t) = i[H_0, \phi(\vec{x},t)] = \frac{\delta H_0}{\delta\pi(\vec{x},t)} \qquad \dot{\pi}(\vec{x},t) = i[H_0, \pi(\vec{x},t)] = -\frac{\delta H_0}{\delta\phi(\vec{x},t)}$$ 作业15

扣除作用量中的四度时空体积积分,我们就得到体系的拉格朗日量密度:

$$S_0=\int d^4x\, \mathcal{L}_0 \qquad \mathcal{L}_0=\frac{1}{2} :[(\partial_\mu\phi(x))^2 - M^2\phi^2(x)]: \qquad \pi(x)=\frac{\partial\mathcal{L}_0}{\partial\dot{\phi}(x)}=\dot{\phi}(x)$$

标量场结果集锦

- 场及场方程：

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}[e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}a^{c\dagger}(\vec{p})] \qquad (\partial^2 + M^2)\phi(x) = 0$$

- 哈密顿量、作用量与广义动量：

$$H_0 = \int d\vec{x}\,\frac{1}{2} : [\dot{\phi}^2(x) + (\nabla\phi(x))^2 + M^2\phi^2(x)] :$$

$$S_0 = \int d^4x\,\mathcal{L}_0 \qquad \mathcal{L}_0 = \frac{1}{2} : [(\partial_\mu\phi(x))^2 - M^2\phi^2(x)] : \qquad \pi(x) = \frac{\partial\mathcal{L}_0}{\partial\dot{\phi}(x)} = \dot{\phi}(x)$$

- 等时对易关系：

$$[\phi(\vec{x},t), \pi(\vec{y},t)]_- = i\delta(\vec{x}-\vec{y}) \qquad \pi(x) = \dot{\phi}(x)$$

$$[\phi(\vec{x},t), \phi(\vec{y},t)]_- = [\pi(\vec{x},t), \pi(\vec{y},t)]_- = 0$$

- 正则场方程：

$$\dot{\phi}(\vec{x},t) = i[H_0, \phi(\vec{x},t)] = \frac{\delta H_0}{\delta\pi(\vec{x},t)} \qquad \dot{\pi}(\vec{x},t) = i[H_0, \pi(\vec{x},t)] = -\frac{\delta H_0}{\delta\phi(\vec{x},t)}$$

零质量标量场的结果可在有质量的标量场结果中将质量趋于零得到！

标量场的地位:

♣ 是各种量子场中最简单、平庸的场!

♢ 总被拿来作为例子或玩具模型首先研究!

♡ 是公理化场论、构造性场论的研究对象！

♠ 但在现实世界 刚刚发现 其对应的基本粒子! Higgs ?

¶ 是否它太简单了?

✠ 发现了很多它不好的性质！ 见第四章关于平庸性和不自然性的讨论

γ矩阵

$$\sum_{\bar\sigma} u_{\bar l}(0,\bar\sigma,n) D_{\bar\sigma\sigma}(R) = \sum_l D^{+}_{\bar l l}(R) u_l(0,\sigma,n) \xRightarrow{\text{非平庸 (平庸是恒等式)}} \sum_{\bar\sigma} u_{\bar l}(0,\bar\sigma,n) \vec{J}^{(j_n)}_{\bar\sigma\sigma} = \sum_l \vec{\mathcal{J}}_{\bar l l} u_l(0,\sigma,n)$$

Clifford 代数与洛伦兹变换生成元

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \qquad \mathcal{J}^{\mu\nu} \equiv -\frac{i}{4}[\gamma^\mu,\gamma^\nu] = -\mathcal{J}^{\nu\mu}$$

$$\begin{aligned}
i[\mathcal{J}^{\mu\nu},\mathcal{J}^{\rho\sigma}] &= -\frac{i}{16}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu - \mu\Leftrightarrow\nu - \rho\Leftrightarrow\sigma] \\
&= -\frac{i}{16}[2g^{\nu\rho}\gamma^\mu\gamma^\sigma - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu - \mu\Leftrightarrow\nu - \rho\Leftrightarrow\sigma] \\
&= -\frac{i}{16}[2g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + \gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu - \mu\Leftrightarrow\nu - \rho\Leftrightarrow\sigma] \\
&= -\frac{i}{16}[2g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2\gamma^\rho\gamma^\mu g^{\nu\sigma} - \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu - \mu\Leftrightarrow\nu - \rho\Leftrightarrow\sigma] \\
&= -\frac{i}{16}[2g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2\gamma^\rho\gamma^\mu g^{\nu\sigma} - 2g^{\mu\sigma}\gamma^\rho\gamma^\nu - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma \\
&\quad +2g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\gamma^\rho\gamma^\nu g^{\mu\sigma} + 2g^{\nu\sigma}\gamma^\rho\gamma^\mu - 2g^{\nu\sigma}\gamma^\mu\gamma^\rho + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho - 2\gamma^\sigma\gamma^\mu g^{\nu\rho} \\
&\quad +2g^{\mu\rho}\gamma^\sigma\gamma^\nu + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho - 2g^{\nu\sigma}\gamma^\mu\gamma^\rho + 2\gamma^\sigma\gamma^\nu g^{\mu\rho} - 2g^{\nu\rho}\gamma^\sigma\gamma^\mu \\
&= g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\sigma\mu}\mathcal{J}^{\rho\nu} + g^{\sigma\nu}\mathcal{J}^{\rho\mu}
\end{aligned}$$

Clifford 代数的洛伦兹变换

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu} \qquad \mathcal{J}^{\mu\nu}\equiv-\frac{i}{4}[\gamma^\mu,\gamma^\nu]=-\mathcal{J}^{\nu\mu}$$

$$i[\mathcal{J}^{\mu\nu},\mathcal{J}^{\rho\sigma}]=g^{\nu\rho}\mathcal{J}^{\mu\sigma}-g^{\mu\rho}\mathcal{J}^{\nu\sigma}-g^{\sigma\mu}\mathcal{J}^{\rho\nu}+g^{\sigma\nu}\mathcal{J}^{\rho\mu}$$

$$\Lambda=1+\omega \qquad D(\Lambda)=1+\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}$$

$$[\mathcal{J}^{\mu\nu},\gamma^\rho]=-\frac{i}{4}[\gamma^\mu\gamma^\nu\gamma^\rho-\gamma^\rho\gamma^\mu\gamma^\nu-\mu\Leftrightarrow\nu]=-\frac{i}{4}[2\gamma^\mu g^{\nu\rho}-\gamma^\mu\gamma^\rho\gamma^\nu-\gamma^\rho\gamma^\mu\gamma^\nu-\mu\Leftrightarrow\nu]$$

$$=-\frac{i}{4}[2\gamma^\mu g^{\nu\rho}-2g^{\mu\rho}\gamma^\nu-\mu\Leftrightarrow\nu]=-i(\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho})$$

$$D(\Lambda)\gamma^\rho D^{-1}(\Lambda)=\gamma^\rho+\frac{i}{2}\omega_{\mu\nu}[\mathcal{J}^{\mu\nu},\gamma^\rho]=\gamma^\rho+\frac{1}{2}\omega_{\mu\nu}(\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho})=\gamma^\rho+\omega_\mu{}^\rho\gamma^\mu$$

$$=(g_\mu{}^\rho+\omega_\mu{}^\rho)\gamma^\mu=\Lambda_\mu{}^\rho\gamma^\mu$$

$$D(\Lambda)1D^{-1}(\Lambda)=1 \qquad D(\Lambda)\mathcal{J}^{\rho\sigma}D^{-1}(\Lambda)=\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\mathcal{J}^{\mu\nu}$$

Clifford 代数的完备性

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \mathcal{J}^{\mu\nu} \equiv -\frac{i}{4}[\gamma^\mu, \gamma^\nu] = -\mathcal{J}^{\nu\mu}$$

$$i[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\sigma\mu}\mathcal{J}^{\rho\nu} + g^{\sigma\nu}\mathcal{J}^{\rho\mu}$$

任何多于4个的γ^μ的乘积一定可用4个和4个以下的γ^μ的乘积表达

- 多于4个的γ^μ的乘积至少有一对指标重复
- 将重复指标的这对γ^μ可以去掉，相差± 1
- 剩下的γ^μ的乘积如仍多于4个，重复上面过程
- 直到剩下的γ^μ的乘积等于或少于4个

16个独立的γ^μ

$1 \qquad \gamma^\mu \qquad \mathcal{J}^{\mu\nu} \qquad \gamma^\mu\gamma_5 \qquad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \gamma_5^2 = 1 \qquad \{\gamma_5, \gamma^\mu\} = 0$

若用矩阵构造γ^μ,16个独立自由度意味至少需要在$\sqrt{16} = 4$维空间才能实现！

Clifford 代数的在Minkovski空间的非厄米性

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \qquad \mathcal{J}^{\mu\nu} \equiv -\frac{i}{4}[\gamma^\mu,\gamma^\nu] = -\mathcal{J}^{\nu\mu}$$

$$i[\mathcal{J}^{\mu\nu},\mathcal{J}^{\rho\sigma}] = g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\sigma\mu}\mathcal{J}^{\rho\nu} + g^{\sigma\nu}\mathcal{J}^{\rho\mu}$$

$$D(\Lambda)\gamma^\rho D^{-1}(\Lambda) = \Lambda_\mu^{\ \rho}\gamma^\mu \qquad D(\Lambda)1D^{-1}(\Lambda) = 1 \qquad D(\Lambda)\mathcal{J}^{\rho\sigma}D^{-1}(\Lambda) = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma}\mathcal{J}^{\mu\nu}$$

16个基: 1 $\quad\gamma^\mu$ $\quad\mathcal{J}^{\mu\nu}$ $\quad\gamma^\mu\gamma_5$ $\quad\gamma_5$

非厄米性: $\gamma^\mu|a^\mu\rangle = a^\mu|a^\mu\rangle$ 不求和 $\Rightarrow 2g^{\mu\mu}|a^\mu\rangle = \{\gamma^\mu,\gamma^\mu\}|a^\mu\rangle = 2a^\mu a^\mu|a^\mu\rangle$

注: γ^μ无共同本征态 $\qquad (a^0)^2 = 1 \qquad (a^1)^2 = (a^2)^2 = (a^3)^2 = -1$

- 若取$\gamma^\mu = \gamma^{\mu\dagger} \Rightarrow \gamma^\mu$的本征值是实的$\Rightarrow (a^0)^2 = (a^1)^2 = (a^2)^2 = (a^3)^2 = 1$
- 若取$\gamma^\mu = -\gamma^{\mu\dagger} \Rightarrow \gamma^\mu$的本征值是虚的$\Rightarrow (a^0)^2 = (a^1)^2 = (a^2)^2 = (a^3)^2 = -1$
- 只有取$\gamma^0 = \gamma^{0\dagger} \quad \gamma^i = -\gamma^{i\dagger}$
- 它导致$\mathcal{J}^{\sigma\rho}$不全厄米，在场空间生成的 表示不幺正！ 但不是在态空间
- 但$\mathcal{J}^{ij}$厄米，在场空间由 纯转动 生成的表示仍幺正！

Clifford 代数与时空转动的表示矩阵

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \mathcal{J}^{\mu\nu} \equiv -\frac{i}{4}[\gamma^\mu, \gamma^\nu] = -\mathcal{J}^{\nu\mu}$$

$$i[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\sigma\mu}\mathcal{J}^{\rho\nu} + g^{\sigma\nu}\mathcal{J}^{\rho\mu}$$

$$D(\Lambda)\gamma^\rho D^{-1}(\Lambda) = \Lambda_\mu^{\ \rho}\gamma^\mu \qquad D(\Lambda)1D^{-1}(\Lambda) = 1 \qquad D(\Lambda)\mathcal{J}^{\rho\sigma}D^{-1}(\Lambda) = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma}\mathcal{J}^{\mu\nu}$$

16个基： $1 \qquad \gamma^\mu \qquad \mathcal{J}^{\mu\nu} \qquad \gamma^\mu\gamma_5 \qquad \gamma_5$

厄米性的自洽取法： $\gamma^0 = \gamma^{0\dagger} \quad \gamma^i = -\gamma^{i\dagger}$

角动量的平方： $j = \frac{1}{2} \qquad \mathcal{J}^i = -\frac{1}{2}\epsilon_{ijk}\mathcal{J}^{jk} = \frac{i}{8}\epsilon_{ijk}[\gamma^j, \gamma^k]$

为什么要用**clifford**代数构造的生成元来描述自旋1/2的场？存不存在其它的方式？

$$\mathcal{J}^i\mathcal{J}^i = \frac{-1}{64}\epsilon_{ijk}\epsilon_{ij'k'}[\gamma^j, \gamma^k][\gamma^{j'}, \gamma^{k'}] = \frac{-1}{32}[\gamma^j, \gamma^k][\gamma^j, \gamma^k]$$

$$= \frac{-1}{32}[\gamma^j\gamma^k\gamma^j\gamma^k - \gamma^j\gamma^k\gamma^k\gamma^j - \gamma^k\gamma^j\gamma^j\gamma^k + \gamma^k\gamma^j\gamma^k\gamma^j] = \frac{-1}{16}[\gamma^j\gamma^k\gamma^j\gamma^k - \gamma^j\gamma^k\gamma^k\gamma^j]$$

$$= \frac{-1}{16}[\gamma^j(2g^{kj} - \gamma^j\gamma^k)\gamma^k - 9] = \frac{-1}{16}[6 - 9 - 9] = \frac{3}{4} = \frac{1}{2}(\frac{1}{2} + 1)$$

γ矩阵

$$\psi_l^+(x) = \sum_{\sigma,n}\int \frac{d^3p}{(2\pi)^{3/2}} e^{-ip\cdot x} u_l(\vec{p},\sigma,n) a(\vec{p},\sigma,n) \qquad \psi_l^-(x) = \sum_{\sigma,n}\int \frac{d^3p}{(2\pi)^{3/2}} e^{ip\cdot x} v_l(\vec{p},\sigma,n) a^\dagger(\vec{p},\sigma,n)$$

$$u(\vec{p},\sigma) = \sqrt{M/p^0} D(L(p)) u(0,\sigma) \qquad v(\vec{p},\sigma) = \sqrt{M/p^0} D(L(p)) v(0,\sigma)$$

$$\sum_{\bar\sigma} u_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)}_{\bar\sigma\sigma} = \sum_{l} \vec{\mathcal{J}}_{\bar l l} u_l(0,\sigma,n) \qquad [J^i, J^j] = i\epsilon_{ijk}J^k \quad J^i = -\frac{1}{2}\epsilon_{ijk}J^{jk}$$

$$\sum_{\bar\sigma} v_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)*}_{\bar\sigma\sigma} = -\sum_{l} \vec{\mathcal{J}}_{\bar l l} v_l(0,\sigma,n) \qquad [\mathcal{J}^i, \mathcal{J}^j] = i\epsilon_{ijk}\mathcal{J}^k \quad \mathcal{J}^i = -\frac{1}{2}\epsilon_{ijk}\mathcal{J}^{jk}$$

旋量量子场: $\vec{\mathcal{J}} = \frac{1}{2}\vec{\sigma}I \qquad [\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k \qquad \mathcal{J}^2 = \frac{1}{4}\vec{\sigma}\cdot\vec{\sigma}I = \frac{3}{4}I = \frac{1}{2}(\frac{1}{2}+1)I$

只考虑有质量的旋量场： 无质量可以看作质量趋于零的极限

- 非平庸的最低表示是$\vec{J}^{(1/2)} = \frac{1}{2}\vec{\sigma} \Rightarrow$ Weyl旋量表示
- $[\sigma_2\sigma^i\sigma_2, \sigma_2\sigma^j\sigma_2] = 2i\epsilon_{ijk}\sigma_2\sigma^k\sigma_2$,
  $\sigma_2\sigma^i\sigma_2 = -\sigma^{i*} \quad \vec{J}^{(1/2)*} = \frac{1}{2}\vec{\sigma}^* = -\frac{1}{2}\sigma_2\vec{\sigma}\sigma_2$
  $\Rightarrow$有两个互为共轭的Weyl旋量表示
- γ矩阵最低4维，考虑两个旋量表示的直和dim$I = 2$： Dirac旋量表示

γ矩阵

$$\psi_l^+(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ip\cdot x}u_l(\vec{p},\sigma,n)a(\vec{p},\sigma,n)\quad \psi_l^-(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{ip\cdot x}v_l(\vec{p},\sigma,n)a^\dagger(\vec{p},\sigma,n)$$

$$u(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma)\qquad v(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma)$$

$$\sum_{\bar\sigma}u_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)}_{\bar\sigma\sigma}=\sum_l\vec{\mathcal{J}}_{\bar l l}u_l(0,\sigma,n)\qquad [J^i,J^j]=i\epsilon_{ijk}J^k$$

$$\sum_{\bar\sigma}v_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)*}_{\bar\sigma\sigma}=-\sum_l\vec{\mathcal{J}}_{\bar l l}v_l(0,\sigma,n)\qquad [\mathcal{J}^i,\mathcal{J}^j]=i\epsilon_{ijk}\mathcal{J}^k$$

旋量量子场: $\vec{\mathcal{J}}=\frac{1}{2}\vec\sigma I\quad \vec{J}^{(1/2)}=\frac{1}{2}\vec\sigma\quad \vec{J}^{(1/2)*}=-\frac{1}{2}\sigma_2\vec\sigma\sigma_2$

$$\psi_l^+(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma)\quad \psi_l^{-c}(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

$$\sum_{\bar\sigma}u_{\bar m\bar a}(0,\bar\sigma)\frac{1}{2}\vec\sigma_{\bar\sigma\sigma}=\sum_{m,a}\frac{1}{2}\underbrace{\vec\sigma_{\bar m m}I_{\bar a a}}_{\mathcal{J}_{\bar l l}}u_{ma}(0,\sigma)\quad \sum_{\bar\sigma,\bar a}v_{\bar m\bar a}(0,\bar\sigma)\frac{1}{2}(\sigma_2\vec\sigma\sigma_2)_{\bar\sigma\sigma}=\sum_{m,a}\frac{1}{2}\underbrace{\vec\sigma_{\bar m m}I_{\bar a a}}_{\mathcal{J}_{\bar l l}}v_{ma}(0,\sigma)$$

$$\sum_{\bar\sigma}u_{\bar m\pm}(0,\bar\sigma)\frac{1}{2}\vec\sigma_{\bar\sigma\sigma}=\sum_m\frac{1}{2}\vec\sigma_{\bar m m}u_{m\pm}(0,\sigma)\quad \sum_{\bar\sigma}v_{\bar m\pm}(0,\bar\sigma)\frac{1}{2}(\sigma_2\vec\sigma\sigma_2)_{\bar\sigma\sigma}=\sum_m\frac{1}{2}\vec\sigma_{\bar m m}v_{m\pm}(0,\sigma)$$

$u_{m\pm}(0,\sigma)=c_\pm\delta_{m\sigma}\quad v_{m\pm}(0,\sigma)=-id_\pm(\sigma_2)_{m\sigma}\qquad \pm$ 代表I空间的分量 $\quad -i$ 消去σ_2中的i

γ矩阵

旋量量子场: $\vec{J}^{(1/2)} = \frac{1}{2}\vec{\sigma}$ $\quad \vec{J}^{(1/2)*} = -\frac{1}{2}\sigma_2\vec{\sigma}\sigma_2$ $\quad \vec{\mathcal{J}} = \frac{1}{2}\vec{\sigma}I$

$$\psi_l^+(x) = (2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\, u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) \qquad \psi_l^{-c}(x) = (2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\, v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

$$u(\vec{p},\sigma) = \sqrt{M/p^0}D(L(p))u(0,\sigma) \qquad v(\vec{p},\sigma) = \sqrt{M/p^0}D(L(p))v(0,\sigma)$$

$$\sum_{\bar\sigma} u_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)}_{\bar\sigma\sigma} = \sum_l \vec{\mathcal{J}}_{\bar l l}u_l(0,\sigma,n) \qquad \sum_{\bar\sigma} v_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)*}_{\bar\sigma\sigma} = -\sum_l \vec{\mathcal{J}}_{\bar l l}v_l(0,\sigma,n)$$

$$\sum_{\bar\sigma} u_{\bar m\pm}(0,\bar\sigma)\frac{1}{2}\vec{\sigma}_{\bar\sigma\sigma} = \sum_m \frac{1}{2}\vec{\sigma}_{\bar m m}u_{m\pm}(0,\sigma) \qquad \sum_{\bar\sigma} v_{\bar m\pm}(0,\bar\sigma)\frac{1}{2}(\sigma_2\vec{\sigma}\sigma_2)_{\bar\sigma\sigma} = \sum_m \frac{1}{2}\vec{\sigma}_{\bar m m}v_{m\pm}(0,\sigma)$$

$$u_{m\pm}(0,\sigma) = c_\pm\delta_{m\sigma} \qquad v_{m\pm}(0,\sigma) = -id_\pm(\sigma_2)_{m\sigma} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathcal{J}^i = -\frac{1}{2}\epsilon_{ijk}\mathcal{J}^{jk} = \begin{pmatrix} \frac{\sigma^i}{2} & 0 \\ 0 & \frac{\sigma^i}{2} \end{pmatrix}$$

原角标 l 现用$m = \pm 1/2, \pm$;四个分量代表: 选c_+, d_+或c_-, d_-为零⇒Weyl旋量

$$\begin{bmatrix} m=\frac{1}{2},+ \\ m=-\frac{1}{2},+ \\ m=\frac{1}{2},- \\ m=-\frac{1}{2},- \end{bmatrix} \qquad u(0,\frac{1}{2}) = \begin{bmatrix} c_+ \\ 0 \\ c_- \\ 0 \end{bmatrix} \quad u(0,-\frac{1}{2}) = \begin{bmatrix} 0 \\ c_+ \\ 0 \\ c_- \end{bmatrix} \quad v(0,\frac{1}{2}) = \begin{bmatrix} 0 \\ d_+ \\ 0 \\ d_- \end{bmatrix} \quad v(0,-\frac{1}{2}) = -\begin{bmatrix} d_+ \\ 0 \\ d_- \\ 0 \end{bmatrix}$$

γ矩阵

旋量量子场: $\mathcal{J}^i = -\frac{1}{2}\epsilon_{ijk}\mathcal{J}^{jk} = \frac{i}{8}\epsilon_{ijk}[\gamma^j,\gamma^k] = \begin{pmatrix} \frac{\sigma^i}{2} & 0 \\ 0 & \frac{\sigma^i}{2} \end{pmatrix}$

$$\psi_l^+(x) = (2\pi)^{-3/2}\sum_\sigma \int d\vec{p}\; u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) \qquad \psi_l^{-c}(x) = (2\pi)^{-3/2}\sum_\sigma \int d\vec{p}\; v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

$$u(\vec{p},\sigma) = \sqrt{M/p^0}D(L(p))u(0,\sigma) \qquad v(\vec{p},\sigma) = \sqrt{M/p^0}D(L(p))v(0,\sigma)$$

$$\sum_{\bar\sigma} u_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)}_{\bar\sigma\sigma} = \sum_l \vec{\mathcal{J}}_{\bar l l}u_l(0,\sigma,n) \qquad \sum_{\bar\sigma} v_{\bar l}(0,\bar\sigma,n)\vec{J}^{(j_n)*}_{\bar\sigma\sigma} = -\sum_l \vec{\mathcal{J}}_{\bar l l}v_l(0,\sigma,n)$$

$$\begin{bmatrix} m=\frac{1}{2},+ \\ m=-\frac{1}{2},+ \\ m=\frac{1}{2},- \\ m=-\frac{1}{2},- \end{bmatrix} \quad u(0,\frac{1}{2}) = \begin{bmatrix} c_+ \\ 0 \\ c_- \\ 0 \end{bmatrix} \quad u(0,-\frac{1}{2}) = \begin{bmatrix} 0 \\ c_+ \\ 0 \\ c_- \end{bmatrix} \quad v(0,\frac{1}{2}) = \begin{bmatrix} 0 \\ d_+ \\ 0 \\ d_- \end{bmatrix} \quad v(0,-\frac{1}{2}) = -\begin{bmatrix} d_+ \\ 0 \\ d_- \\ 0 \end{bmatrix}$$

$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ Itzykson书中的手征表象

$$[\gamma^j,\gamma^k] = -2i\epsilon_{ijk}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \quad \gamma^j\gamma^k = \begin{pmatrix} -\sigma^j\sigma^k & 0 \\ 0 & -\sigma^j\sigma^k \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$$

旋量量子场: $D(\Lambda)=e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}$, $\Lambda=e^{\omega}$, $\mathcal{J}^{\mu\nu}=-\frac{i}{4}[\gamma^\mu,\gamma^\nu]$, $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$

$$\gamma^j=\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix}\quad\gamma^0=\begin{pmatrix}0&-I\\-I&0\end{pmatrix}\quad u(0,\tfrac{1}{2})=\begin{bmatrix}c_+\\0\\c_-\\0\end{bmatrix}\quad u(0,-\tfrac{1}{2})=\begin{bmatrix}0\\c_+\\0\\c_-\end{bmatrix}\quad v(0,\tfrac{1}{2})=\begin{bmatrix}0\\d_+\\0\\d_-\end{bmatrix}\quad v(0,-\tfrac{1}{2})=-\begin{bmatrix}d_+\\0\\d_-\\0\end{bmatrix}$$

$$\psi_l^+(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma)\quad\psi_l^{-c}(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

$$u(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma)\qquad v(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma)$$

$\beta=\beta^{-1}=\beta^\dagger=\gamma^0$ γ^0厄米; γ^i反厄米 $\beta\gamma^i\beta^{-1}=-\gamma^i$ $\beta\gamma^0\beta^{-1}=\gamma^0$ $\beta\gamma^{\mu\dagger}\beta=\gamma^\mu$

$$\beta\mathcal{J}^{ij}\beta^{-1}=\mathcal{J}^{ij}\qquad\beta\mathcal{J}^{i0}\beta^{-1}=-\mathcal{J}^{i0}\qquad\Rightarrow\quad\beta\mathcal{J}^{\rho\sigma\dagger}\beta=\mathcal{J}^{\rho\sigma}$$

$$P\vec{J}P^{-1}=\vec{J}\quad P\vec{K}P^{-1}=-\vec{K}\quad L(p)\equiv e^{\omega}\quad D(L(-\vec{p}))=D(\mathcal{P}L(\vec{p})\mathcal{P}^{-1})=D(e^{\mathcal{P}\omega\mathcal{P}^{-1}})=\beta D(L(\vec{p}))\beta$$

$$u(-\vec{p},\sigma)=\sqrt{m/p^0}\beta D(L(p))\beta u(0,\sigma)\qquad v(-\vec{p},\sigma)=\sqrt{m/p^0}\beta D(L(p))\beta v(0,\sigma)$$

$$P\psi^\pm(x)P^{-1}\propto\psi^\pm(\mathcal{P}x)\ \Rightarrow\ \beta u(0,\sigma)=b_u u(0,\sigma)\qquad\beta v(0,\sigma)=b_v v(0,\sigma)\Rightarrow b_u^2=b_v^2=1$$

空间反射对称性要求Dirac旋量: $c_-=-b_u c_+,\quad d_-=-b_v d_+$

γ矩阵

$$\psi_l^+(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\; u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) \qquad \psi_l^{-c}(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\; v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

$$u(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma) \qquad v(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma)$$

$$Pa(\vec{p},\sigma,n)P^{-1}=\eta^*a(\mathcal{P}\vec{p},\sigma,n) \qquad Pa^{c\dagger}(\vec{p},\sigma,n)P^{-1}=\eta^c a^\dagger(\mathcal{P}\vec{p},\sigma,n)$$

$$P\psi_l^+(x)P^{-1}=\eta^*(2\pi)^{-\frac{3}{2}}\sum_\sigma\int d\vec{p}\; u_l(\vec{p},\sigma)e^{-ip\cdot x}a(-\vec{p},\sigma)=\eta^*(2\pi)^{-\frac{3}{2}}\sum_\sigma\int d\vec{p}\; u_l(-\vec{p},\sigma)e^{-ip\cdot\mathcal{P}x}a(\vec{p},\sigma)$$

$$\psi_l^+(\mathcal{P}x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\; u_l(\vec{p},\sigma)e^{-ip\cdot\mathcal{P}x}a(\vec{p},\sigma)$$

$$P\psi_l^{-c}(x)P^{-1}=\eta^c(2\pi)^{-\frac{3}{2}}\sum_\sigma\int d\vec{p}\; v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(-\vec{p},\sigma)=\eta^c(2\pi)^{-\frac{3}{2}}\sum_\sigma\int d\vec{p}\; v_l(-\vec{p},\sigma)e^{ip\cdot\mathcal{P}x}a^{c\dagger}(\vec{p},\sigma)$$

$$\psi_l^{-c}(\mathcal{P}x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\; v_l(\vec{p},\sigma)e^{ip\cdot\mathcal{P}x}a^{c\dagger}(\vec{p},\sigma)$$

$$u(-\vec{p},\sigma)=\sqrt{m/p^0}\beta D(L(p))\beta u(0,\sigma) \qquad v(-\vec{p},\sigma)=\sqrt{m/p^0}\beta D(L(p))\beta v(0,\sigma)$$

$$P\psi^\pm(x)P^{-1}\propto\psi^\pm(\mathcal{P}x)\;\Rightarrow\;\beta u(0,\sigma)=b_u u(0,\sigma) \qquad \beta v(0,\sigma)=b_v v(0,\sigma)\Rightarrow b_u^2=b_v^2=1$$

旋量量子场: $D(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}$, $\Lambda = e^{\omega}$, $\mathcal{J}^{\mu\nu} = -\frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]$, $\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}$

$$\gamma^j=\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \quad \gamma^0=\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \quad u(0,\tfrac{1}{2})=\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-b_u}{\sqrt{2}} \\ 0 \end{bmatrix} \quad u(0,-\tfrac{1}{2})=\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-b_u}{\sqrt{2}} \end{bmatrix} \quad v(0,\tfrac{1}{2})=\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-b_v}{\sqrt{2}} \end{bmatrix} \quad v(0,-\tfrac{1}{2})=\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{b_v}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\psi_l^{+}(x)=(2\pi)^{-3/2}\sum_{\sigma}\int d\vec{p}\; u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) \qquad \psi_l^{-c}(x)=(2\pi)^{-3/2}\sum_{\sigma}\int d\vec{p}\; v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

$$u(\vec{p},\sigma) = \sqrt{M/p^0}D(L(p))u(0,\sigma) \qquad v(\vec{p},\sigma) = \sqrt{M/p^0}D(L(p))v(0,\sigma)$$

$\beta = \beta^{-1} = \beta^{\dagger} = \gamma^0$ γ^0厄米; γ^i反厄米 $\beta\gamma^i\beta^{-1} = -\gamma^i$ $\beta\gamma^0\beta^{-1} = \gamma^0$ $\beta\gamma^{\mu\dagger}\beta = \gamma^{\mu}$

$$\beta\mathcal{J}^{ij}\beta^{-1} = \mathcal{J}^{ij} \qquad \beta\mathcal{J}^{i0}\beta^{-1} = -\mathcal{J}^{i0} \qquad \Rightarrow \quad \beta\mathcal{J}^{\rho\sigma\dagger}\beta = \mathcal{J}^{\rho\sigma}$$

$P\vec{J}P^{-1}=\vec{J}$ $P\vec{K}P^{-1}=-\vec{K}$ $L(p)\equiv e^{\omega}$ $D(L(-\vec{p}))=D(\mathcal{P}L(\vec{p})\mathcal{P}^{-1})=D(e^{\mathcal{P}\omega\mathcal{P}^{-1}})=\beta D(L(\vec{p}))\beta$

$$u(-\vec{p},\sigma) = \sqrt{m/p^0}\beta D(L(p))\beta u(0,\sigma) \qquad v(-\vec{p},\sigma) = \sqrt{m/p^0}\beta D(L(p))\beta v(0,\sigma)$$

$$P\psi^{\pm}(x)P^{-1} \propto \psi^{\pm}(\mathcal{P}x) \Rightarrow \beta u(0,\sigma) = b_u u(0,\sigma) \qquad \beta v(0,\sigma) = b_v v(0,\sigma) \Rightarrow b_u^2 = b_v^2 = 1$$

<u>空间反射对称性要求Dirac旋量</u>: $c_- = -b_u c_+, \quad d_- = -b_v d_+$

$$\psi_l^+(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\, u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) \quad \psi_l^{-c}(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\, v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

旋量量子场: $D(\Lambda)=e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}$, $\Lambda=e^\omega$, $\mathcal{J}^{\mu\nu}=-\frac{i}{4}[\gamma^\mu,\gamma^\nu]$, $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$

$$\gamma^j=\begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix} \quad \gamma^0=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix} \quad u(0,\frac{1}{2})=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{-b_u}{\sqrt{2}}\\ 0\end{bmatrix} \quad u(0,-\frac{1}{2})=\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\ 0\\ \frac{-b_u}{\sqrt{2}}\end{bmatrix} \quad v(0,\frac{1}{2})=\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\ 0\\ \frac{-b_v}{\sqrt{2}}\end{bmatrix} \quad v(0,-\frac{1}{2})=\begin{bmatrix}\frac{-1}{\sqrt{2}}\\ 0\\ \frac{b_v}{\sqrt{2}}\\ 0\end{bmatrix}$$

$$u(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma) \quad v(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma) \quad \beta\gamma^{\mu\dagger}\beta=\gamma^\mu \quad \beta\mathcal{J}^{\rho\sigma\dagger}\beta=\mathcal{J}^{\rho\sigma}$$

将$\psi_l^+(x)$和$\psi_l^{-c}(x)$进行线性组合来构造$\tilde{\mathcal{H}}(x)$以保证其在类空区间的对易性

$$\psi(x)=\kappa\psi^+(x)+\lambda\psi^{-c}(x) \quad [\psi_l(x),\psi_{\bar{l}}^\dagger(y)]_\mp=\int d\vec{p}\,[|\kappa|^2N_{l\bar{l}}(\vec{p})e^{-ip\cdot(x-y)}\mp|\lambda|^2M_{l\bar{l}}(\vec{p})e^{ip\cdot(x-y)}]$$

$$N_{l\bar{l}}(\vec{p})\equiv\sum_\sigma u_l(\vec{p},\sigma)u_{\bar{l}}^*(\vec{p},\sigma) \quad M_{l\bar{l}}(\vec{p})\equiv\sum_\sigma v_l(\vec{p},\sigma)v_{\bar{l}}^*(\vec{p},\sigma) \quad N(0)=\frac{1}{2}(1+b_u\beta) \quad M(0)=\frac{1}{2}(1+b_v\beta)$$

$$N(\vec{p})=M/(2p^0)D(L(p))[1+b_u\beta]D^\dagger(L(p)) \qquad M(\vec{p})=M/(2p^0)D(L(p))[1+b_v\beta]D^\dagger(L(p))$$

费米统计

$$\psi_l^+(x)=(2\pi)^{-3/2}\sum\int d\vec{p}\; u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) \quad \psi_l^{-c}(x)=(2\pi)^{-3/2}\sum\int d\vec{p}\; v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

将$\psi_l^+(x)$和$\psi_l^{\sigma c}(x)$进行线性组合来构造$\tilde{\mathcal{H}}(x)$以保证其在类空区间的对易性

$$\psi(x)=\kappa\psi^+(x)+\lambda\psi^{-c}(x) \qquad [\psi_l(x),\psi_{\bar{l}}^\dagger(y)]_\mp=\int d\vec{p}\;[|\kappa|^2N_{l\bar{l}}(\vec{p})e^{-ip\cdot(x-y)}\mp|\lambda|^2M_{l\bar{l}}(\vec{p})e^{ip\cdot(x-y)}]$$

$$N_{l\bar{l}}(\vec{p})\equiv\sum_\sigma u_l(\vec{p},\sigma)u_{\bar{l}}^*(\vec{p},\sigma) \quad M_{l\bar{l}}(\vec{p})\equiv\sum_\sigma v_l(\vec{p},\sigma)v_{\bar{l}}^*(\vec{p},\sigma) \quad N(0)=\frac{1}{2}(1+b_u\beta) \quad M(0)=\frac{1}{2}(1+b_v\beta)$$

$$N(\vec{p})=M/(2p^0)D(L(p))[1+b_u\beta]D^\dagger(L(p)) \qquad M(\vec{p})=M/(2p^0)D(L(p))[1+b_v\beta]D^\dagger(L(p))$$

$$D(\Lambda)\gamma^\rho D^{-1}(\Lambda)=\Lambda_\sigma^{\ \rho}\gamma^\sigma \qquad [e^{\frac{i}{2}\mathcal{J}^{\mu\nu\dagger}\omega_{\mu\nu}}]=[e^{\frac{i}{2}\mathcal{J}^{\mu\nu}\omega_{\mu\nu}}]^{-1,\dagger} \;\Rightarrow\; \beta D(L(p))\beta=D^{\dagger-1}(L(p))$$

$$D(L(p))\beta D^{-1}(L(p))=L_\mu^{\ 0}(p)\gamma^\mu=p_\mu\gamma^\mu/M \quad D(L(p))D^\dagger(L(p))=D(L(p))\beta D^{-1}(L(p))\beta=p_\mu\gamma^\mu\beta/M$$

$$N(\vec{p})=\frac{1}{2p^0}[p^\mu\gamma_\mu+b_uM]\beta \qquad M(\vec{p})=\frac{1}{2p^0}[p^\mu\gamma_\mu+b_vM]\beta$$

$$[\psi_l(x),\psi_{\bar{l}}^\dagger(y)]_\mp=[|\kappa|^2(i\gamma^\mu\partial_{x,\mu}+b_uM)\beta\Delta_+(x-y)\mp|\lambda|^2(-i\gamma^\mu\partial_{x,\mu}+b_\nu M)\beta\Delta_+(y-x)]_{l\bar{l}}$$

$|\kappa|^2=\mp|\lambda|^2 \quad b_u|\kappa|^2=\pm b_v|\lambda|^2 \quad M\neq 0 \;\Rightarrow\; \kappa=\lambda=1 \quad b_u=-b_v=1$ **-1选择等价乘γ_5** 与Weinberg书符号相反

$$N(\vec{p})=1/(2p^0)[p^\mu\gamma_\mu+M]\beta \qquad M(\vec{p})=1/(2p^0)[p^\mu\gamma_\mu-M]\beta$$

旋量量子场: $D(\Lambda)=e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}$, $\Lambda=e^{\omega}$, $\mathcal{J}^{\mu\nu}=-\frac{i}{4}[\gamma^\mu,\gamma^\nu]$, $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$

$$\gamma^j=\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix}\quad \gamma^0=\begin{pmatrix}0&-I\\-I&0\end{pmatrix}\quad u(0,\tfrac{1}{2})=\begin{bmatrix}\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\\0\end{bmatrix}\quad u(0,-\tfrac{1}{2})=\begin{bmatrix}0\\\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\end{bmatrix}\quad v(0,\tfrac{1}{2})=\begin{bmatrix}0\\\frac{1}{\sqrt2}\\0\\\frac{1}{\sqrt2}\end{bmatrix}\quad v(0,-\tfrac{1}{2})=\begin{bmatrix}\frac{-1}{\sqrt2}\\0\\\frac{-1}{\sqrt2}\\0\end{bmatrix}$$

$$\psi_l^+(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec p\,u_l(\vec p,\sigma)e^{-ip\cdot x}a(\vec p,\sigma)\qquad \psi_l^{-c}(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec p\,v_l(\vec p,\sigma)e^{ip\cdot x}a^{c\dagger}(\vec p,\sigma)$$

$$u(\vec p,\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma)\quad v(\vec p,\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma)\quad \beta\gamma^{\mu\dagger}\beta=\gamma^\mu\quad \beta\mathcal{J}^{\rho\sigma\dagger}\beta=\mathcal{J}^{\rho\sigma}$$

$$\gamma_\mu^T=-\mathcal{C}\gamma_\mu\mathcal{C}^{-1}\quad \mathcal{C}\equiv i\gamma^2\beta=-\mathcal{C}^{-1}=\begin{pmatrix}-i\sigma^2&0\\0&i\sigma^2\end{pmatrix}\quad \gamma_\mu^*=-\beta\mathcal{C}\gamma_\mu\mathcal{C}^{-1}\beta\quad \mathcal{J}_{\mu\nu}^T=-\mathcal{C}\mathcal{J}_{\mu\nu}\mathcal{C}^{-1}$$

$$\gamma^{\substack{3\\1}}=\mathcal{C}\gamma^{\substack{3\\1}}\mathcal{C}^{-1}\quad \gamma^{\substack{2\\0}}=-\mathcal{C}\gamma^{\substack{2\\0}}\mathcal{C}^{-1}\quad [e^{\frac{i}{2}\mathcal{J}^{\mu\nu}\omega_{\mu\nu}}]^*=\beta\mathcal{C}[e^{\frac{i}{2}\mathcal{J}^{\mu\nu}\omega_{\mu\nu}}]\mathcal{C}^{-1}\beta\quad D(L(p))^*=\beta\mathcal{C}D(L(p))\mathcal{C}^{-1}\beta$$

$$\mathcal{C}^{-1}\beta u(0,\sigma)=v(0,\sigma)\quad \mathcal{C}^{-1}\beta v(0,\sigma)=u(0,\sigma)\quad \underbrace{u_l^*(\vec p,\sigma)=\beta\mathcal{C}v_l(\vec p,\sigma)\quad v_l^*(\vec p,\sigma)=\beta\mathcal{C}u_l(\vec p,\sigma)}$$

$\beta\mathcal{C}$的效果：u,v互换加共轭

$$\beta\gamma_5\beta^{-1}=-\gamma_5\qquad \mathcal{C}\gamma_5\mathcal{C}^{-1}=\gamma_5^T\qquad \beta\mathcal{C}\gamma_5\mathcal{C}^{-1}\beta=-\gamma_5^*\qquad [\mathcal{J}^{\mu\nu},\gamma_5]=0$$

$$\gamma_5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3\qquad \gamma_5^2=1\qquad \{\gamma_5,\gamma^\mu\}=0$$

费米统计

旋量量子场: $D(\Lambda)=e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}$, $\Lambda=e^{\omega}$, $\mathcal{J}^{\mu\nu}=-\frac{i}{4}[\gamma^\mu,\gamma^\nu]$, $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$

$$\gamma^j=\begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix}\quad \gamma^0=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix}\quad u(0,\tfrac{1}{2})=\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\ \frac{-1}{\sqrt{2}}\\0\end{bmatrix}\quad u(0,-\tfrac{1}{2})=\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\0\\ \frac{-1}{\sqrt{2}}\end{bmatrix}\quad v(0,\tfrac{1}{2})=\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\0\\ \frac{1}{\sqrt{2}}\end{bmatrix}\quad v(0,-\tfrac{1}{2})=\begin{bmatrix}\frac{-1}{\sqrt{2}}\\0\\ \frac{-1}{\sqrt{2}}\\0\end{bmatrix}$$

$$\psi_l^+(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\; u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma)\qquad \psi_l^{-c}(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\; v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)$$

$$u(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma)\quad v(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma)\quad \{\gamma_5,\gamma^\mu\}=0\quad [\mathcal{J}^{\mu\nu},\gamma_5]=0$$

$$\gamma_5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\quad P_L\equiv\frac{1-\gamma_5}{2}=\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}\quad P_R\equiv\frac{1+\gamma_5}{2}=\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}$$

$$P_L+P_R=1\quad u_{\substack{L\\R}}(\vec{p},\sigma)\equiv P_{\substack{L\\R}}u(\vec{p},\sigma)\quad v_{\substack{L\\R}}(\vec{p},\sigma)\equiv P_{\substack{L\\R}}v(\vec{p},\sigma)\quad [D(\Lambda),P_{\substack{L\\R}}]=0$$

$$u_{\substack{L\\R}}(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u_{\substack{L\\R}}(0,\sigma)\qquad v_{\substack{L\\R}}(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v_{\substack{L\\R}}(0,\sigma)$$

最后的γ矩阵结果: Itzykson-Zuber书中手征表象 可以相差一个实正交变换

$$\gamma^0=\beta=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix}\qquad \gamma^i=\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\qquad \gamma_5=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$$

$$\sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\quad \sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\quad \sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad \mathcal{C}=\begin{pmatrix}-i\sigma^2 & 0\\ 0 & i\sigma^2\end{pmatrix}$$

$$u(0,\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\quad u(0,-\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\quad v(0,\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\quad v(0,-\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\-1\\0\end{bmatrix}$$

$$u(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma)\quad v(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma)$$

注：$\gamma_5 u(\vec{p},\sigma)$, $\gamma_5 v(\vec{p},\sigma)$ 对应$b_u=-b_v=-1$的选择！

$$\gamma_5\mathcal{C}^{-1}u(0,-\sigma)=(-1)^{\frac{1}{2}-\sigma}u(0,\sigma)\qquad \gamma_5\mathcal{C}^{-1}v(0,-\sigma)=(-1)^{\frac{1}{2}-\sigma}v(0,\sigma)$$

$$\beta=\beta^*\quad\Rightarrow\quad D^*(L(-\vec{p}))=\beta D^*(L(\vec{p}))\beta=\gamma_5\beta D^*(L(\vec{p}))\beta\gamma_5=\gamma_5\mathcal{C}D(L(\vec{p}))\gamma_5\mathcal{C}^{-1}$$

$$(-1)^{\frac{1}{2}+\sigma}u^*(-\vec{p},-\sigma)=-\gamma_5\mathcal{C}u(\vec{p},\sigma)\qquad (-1)^{\frac{1}{2}+\sigma}v^*(-\vec{p},-\sigma)=-\gamma_5\mathcal{C}v(\vec{p},\sigma)$$

计算$D(L(p))$–证明$B(|\vec{p}|)$是推进变换: $L(p) = R(\hat{p})B(|\vec{p}|)R^{-1}(\hat{p})$ $\quad \gamma \equiv \frac{\sqrt{p^2+M^2}}{M}$

$$B(|\vec{p}|) = \begin{pmatrix} \gamma & 0 & 0 & \sqrt{\gamma^2-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\gamma^2-1} & 0 & 0 & \gamma \end{pmatrix} = \exp\begin{pmatrix} 0 & 0 & 0 & \operatorname{arc\,cosh}(\gamma) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \operatorname{arc\,cosh}(\gamma) & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & \operatorname{arc\,cosh}(\gamma) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \operatorname{arc\,cosh}(\gamma) & 0 & 0 & 0 \end{pmatrix}^{2n} = \begin{pmatrix} \operatorname{arc\,cosh}^{2n}(\gamma) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{arc\,cosh}^{2n}(\gamma) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & \operatorname{arc\,cosh}(\gamma) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \operatorname{arc\,cosh}(\gamma) & 0 & 0 & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & 0 & 0 & \operatorname{arc\,cosh}^{2n+1}(\gamma) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \operatorname{arc\,cosh}^{2n+1}(\gamma) & 0 & 0 & 0 \end{pmatrix}$$

$$\sum_{n=0}\frac{1}{(2n)!}\operatorname{arc\,cosh}^{2n}(\gamma) = \gamma \qquad \sum_{n=1}\frac{1}{(2n+1)!}\operatorname{arc\,cosh}^{2n+1}(\gamma) = \sinh(\operatorname{arc\,cosh}^{2n+1}(\gamma)) = \sqrt{\gamma^2-1}$$

纯推进变换: $\omega_3^{\ 0} = \omega_{03} = -\omega_{30} = \omega_0^{\ 3} = \operatorname{arc\,cosh}(\gamma)$ $\qquad \omega_\rho^{\ \sigma} \overset{\text{其它}}{=\!=\!=\!=} 0$

费米统计

计算$D(L(p))$: $L(p) = R(\hat{p})B(|\vec{p}|)R^{-1}(\hat{p})$ $\qquad D(L(\vec{p})) = D(R(\hat{p}))D(B(|\vec{p}|))D(R^{-1}(\hat{p}))$

$B(|\vec{p}|) = e^{\omega}$ 纯推进变换: $\omega_3^{\ 0} = \omega_{03} = -\omega_{30} = \omega_0^{\ 3} = \text{arc}\cosh(\gamma)$ $\qquad \omega_\rho^{\ \sigma} \overset{\text{其它}}{=\!=\!=} 0$

$$D(B(|\vec{p}|)) = e^{i\omega_{03}\mathcal{J}^{03}} = \begin{pmatrix} e^{\frac{1}{2}\omega_{03}\sigma^3} & 0 \\ 0 & e^{-\frac{1}{2}\omega_{03}\sigma^3} \end{pmatrix} = \cosh(\frac{1}{2}\omega_{03}) + \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \sinh(\frac{1}{2}\omega_{03})$$

$$\mathcal{J}^{03} = -\frac{i}{4}[\gamma^0, \gamma^3] = -\frac{i}{2}\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$$

$\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ $\qquad \mathcal{J}^i = -\frac{1}{2}\epsilon_{ijk}\mathcal{J}^{jk} = \frac{i}{8}\epsilon_{ijk}[\gamma^j, \gamma^k] = \begin{pmatrix} \frac{\sigma^i}{2} & 0 \\ 0 & \frac{\sigma^i}{2} \end{pmatrix}$

$$D(R(\hat{p})) = e^{-i\phi\mathcal{J}^3}e^{-i\theta\mathcal{J}^2} = [\cos\frac{\phi}{2} - i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}][\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\sin\frac{\theta}{2}]$$

$$= \cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} - i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R^{-1}(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

计算$D(L(p))$: $D(B(|\vec{p}|)) = \cosh(\frac{1}{2}\text{arc}\cosh(\gamma)) + \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \sinh(\frac{1}{2}\text{arc}\cosh(\gamma))$

$$D(R(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} - i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R^{-1}(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R(\hat{p}))\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} D(R^{-1}(\hat{p}))$$

$$= \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\left[\cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} - i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}\right]$$

$$\times\left[\cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}\right]$$

$$= \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\Big[[\cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}]^2 + \sin^2\frac{\phi}{2}\cos^2\frac{\theta}{2}$$

$$-2i[\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}]\sin\frac{\phi}{2}\cos\frac{\theta}{2}\Big]$$

费米统计

计算$D(L(p))$:

$$D(B(|\vec{p}|)) = \cosh(\frac{1}{2}\text{arc}\cosh(\gamma)) + \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \sinh(\frac{1}{2}\text{arc}\cosh(\gamma))$$

$$D(R(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} - i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R^{-1}(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R(\hat{p}))\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} D(R^{-1}(\hat{p}))$$

$$= \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\Big[[\cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}]^2 + \sin^2\frac{\phi}{2}\cos^2\frac{\theta}{2}$$
$$-2i[\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}]\sin\frac{\phi}{2}\cos\frac{\theta}{2}\Big]$$

$$= \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\Big[\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} - 2i[-\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}]\cos\frac{\phi}{2}\cos\frac{\theta}{2}$$
$$-2i[\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}]\sin\frac{\phi}{2}\cos\frac{\theta}{2}\Big]$$

计算$D(L(p))$: $$D(B(|\vec{p}|)) = \cosh(\frac{1}{2}\text{arc}\cosh(\gamma)) + \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \sinh(\frac{1}{2}\text{arc}\cosh(\gamma))$$

$$D(R(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} - i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R^{-1}(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R(\hat{p}))\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} D(R^{-1}(\hat{p}))$$

$$= \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\Big[\cos\theta - i[-\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos^2\frac{\phi}{2}\sin\theta + \frac{1}{2}\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\phi\sin\theta] - \frac{i}{2}\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\phi\sin\theta$$
$$-i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\sin^2\frac{\phi}{2}\sin\theta\Big]$$

$$= \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\Big[\cos\theta + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\phi\sin\theta - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\phi\sin\theta\Big]$$

$$= \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\cos\theta + \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}\cos\phi\sin\theta + \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}\sin\phi\sin\theta$$

计算$D(L(p))$:

$$L(p) = R(\hat{p})B(|\vec{p}|)R^{-1}(\hat{p}) \qquad D(L(\vec{p})) = D(R(\hat{p}))D(B(|\vec{p}|))D(R^{-1}(\hat{p}))$$

$$D(B(|\vec{p}|)) = \cosh(\frac{1}{2}\text{arc}\cosh(\gamma)) + \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \sinh(\frac{1}{2}\text{arc}\cosh(\gamma))$$

$$D(R(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} - i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R^{-1}(\hat{p})) = \cos\frac{\phi}{2}\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\cos\frac{\phi}{2}\sin\frac{\theta}{2} + i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\phi}{2}\cos\frac{\theta}{2} - i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\sin\frac{\phi}{2}\sin\frac{\theta}{2}$$

$$D(R(\hat{p}))\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}D(R^{-1}(\hat{p})) = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\cos\theta + \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}\cos\phi\sin\theta + \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}\sin\phi\sin\theta$$

$$D(L(\vec{p})) = \cosh(\frac{1}{2}\text{arc}\cosh(\gamma)) + \begin{pmatrix} \Theta(\theta,\phi) & 0 \\ 0 & -\Theta(\theta,\phi) \end{pmatrix} \sinh(\frac{1}{2}\text{arc}\cosh(\gamma))$$

$$\Theta(\theta,\phi) = \sigma^3\cos\theta + \sigma^1\cos\phi\sin\theta + \sigma^2\sin\phi\sin\theta = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

费米统计

计算$u(\vec{p},\sigma), v(\vec{p},\sigma)$: $u(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma)$ $v(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma)$

$$u(0,\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\quad u(0,-\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\quad v(0,\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\quad v(0,-\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\-1\\0\end{bmatrix}$$

$$D(L(\vec{p}))=\cosh(\frac{1}{2}\text{arc}\cosh(\gamma))+\begin{pmatrix}\Theta(\theta,\phi) & 0\\ 0 & -\Theta(\theta,\phi)\end{pmatrix}\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))$$

$$\Theta(\theta,\phi)=\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}\quad \Theta(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\ e^{i\phi}\sin\theta\end{pmatrix}\quad \Theta(\theta,\phi)\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}e^{-i\phi}\sin\theta\\ -\cos\theta\end{pmatrix}$$

$$\begin{matrix}u(\vec{p},\frac{1}{2})\\ v(\vec{p},-\frac{1}{2})\end{matrix}=\frac{1}{\sqrt{2\gamma}}\begin{bmatrix}\pm\cosh(\frac{1}{2}\text{arc}\cosh(\gamma))\pm\cos\theta\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))\\ \pm e^{i\phi}\sin\theta\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))\\ -\cosh(\frac{1}{2}\text{arc}\cosh(\gamma))+\cos\theta\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))\\ e^{i\phi}\sin\theta\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))\end{bmatrix}$$

$\hat{p}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$

$$\begin{matrix}u(\vec{p},-\frac{1}{2})\\ v(\vec{p},\frac{1}{2})\end{matrix}=\frac{1}{\sqrt{2\gamma}}\begin{bmatrix}e^{-i\phi}\sin\theta\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))\\ \cosh(\frac{1}{2}\text{arc}\cosh(\gamma))-\cos\theta\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))\\ \pm e^{-i\phi}\sin\theta\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))\\ \mp\cosh(\frac{1}{2}\text{arc}\cosh(\gamma))\mp\cos\theta\sinh(\frac{1}{2}\text{arc}\cosh(\gamma))\end{bmatrix}$$

$$\gamma=\frac{\sqrt{\vec{p}^2+M^2}}{M}$$

$u(\vec{p},\sigma), v(\vec{p},\sigma)$的显式表达:

$$u(\vec{p},\frac{1}{2})=\frac{1}{2\sqrt{\gamma}}\begin{bmatrix}\sqrt{\gamma+1}+\sqrt{\gamma-1}\cos\theta\\ \sqrt{\gamma-1}e^{i\phi}\sin\theta\\ -\sqrt{\gamma+1}+\sqrt{\gamma-1}\cos\theta\\ \sqrt{\gamma-1}e^{i\phi}\sin\theta\end{bmatrix}$$

$$\hat{p}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$$

$$u(\vec{p},-\frac{1}{2})=\frac{1}{2\sqrt{\gamma}}\begin{bmatrix}\sqrt{\gamma-1}e^{-i\phi}\sin\theta\\ \sqrt{\gamma+1}-\sqrt{\gamma-1}\cos\theta\\ \sqrt{\gamma-1}e^{-i\phi}\sin\theta\\ -\sqrt{\gamma+1}-\sqrt{\gamma-1}\cos\theta\end{bmatrix}$$

$$\gamma=\frac{\sqrt{\vec{p}^2+M^2}}{M}$$

$$v(\vec{p},\frac{1}{2})=\frac{1}{2\sqrt{\gamma}}\begin{bmatrix}\sqrt{\gamma-1}e^{-i\phi}\sin\theta\\ \sqrt{\gamma+1}-\sqrt{\gamma-1}\cos\theta\\ -\sqrt{\gamma-1}e^{-i\phi}\sin\theta\\ \sqrt{\gamma+1}+\sqrt{\gamma-1}\cos\theta\end{bmatrix}$$

$$\cosh(\frac{1}{2}\mathrm{arc}\cosh(\gamma))=\frac{\sqrt{\gamma+1}}{\sqrt{2}}$$

$$v(\vec{p},-\frac{1}{2})=\frac{1}{2\sqrt{\gamma}}\begin{bmatrix}-\sqrt{\gamma+1}-\sqrt{\gamma-1}\cos\theta\\ -\sqrt{\gamma-1}e^{i\phi}\sin\theta\\ -\sqrt{\gamma+1}+\sqrt{\gamma-1}\cos\theta\\ \sqrt{\gamma-1}e^{i\phi}\sin\theta\end{bmatrix}$$

$$\sinh(\frac{1}{2}\mathrm{arc}\cosh(\gamma))=\frac{\sqrt{\gamma-1}}{\sqrt{2}}$$

$u(\vec{p},\sigma), v(\vec{p},\sigma)$的显式表达的零质量极限：$\hat{p}=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$u(\vec{p},\frac{1}{2}) \overset{M=0}{=\!=\!=} \frac{1}{2}\begin{bmatrix} 1+\cos\theta \\ e^{i\phi}\sin\theta \\ -1+\cos\theta \\ e^{i\phi}\sin\theta \end{bmatrix} = \frac{1}{2p^0}\begin{bmatrix} p^0+p^3 \\ p^1+ip^2 \\ -p^0+p^3 \\ p^1+ip^2 \end{bmatrix}$$

$$= -\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} v(\vec{p},-\frac{1}{2})$$

$$v(\vec{p},\frac{1}{2}) \overset{M=0}{=\!=\!=} \frac{1}{2}\begin{bmatrix} e^{-i\phi}\sin\theta \\ 1-\cos\theta \\ -e^{-i\phi}\sin\theta \\ 1+\cos\theta \end{bmatrix} = \frac{1}{2p^0}\begin{bmatrix} p^1-ip^2 \\ p^0-p^3 \\ -p^1+ip^2 \\ p^0+p^3 \end{bmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} u(\vec{p},-\frac{1}{2})$$

$$u(\vec{p},-\frac{1}{2}) \overset{M=0}{=\!=\!=} \frac{1}{2}\begin{bmatrix} e^{-i\phi}\sin\theta \\ 1-\cos\theta \\ e^{-i\phi}\sin\theta \\ -1-\cos\theta \end{bmatrix} = \frac{1}{2p^0}\begin{bmatrix} p^1-ip^2 \\ p^0-p^3 \\ p^1-ip^2 \\ -p^0-p^3 \end{bmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} v(\vec{p},\frac{1}{2})$$

$$v(\vec{p},-\frac{1}{2}) \overset{M=0}{=\!=\!=} \frac{1}{2}\begin{bmatrix} -1-\cos\theta \\ -e^{i\phi}\sin\theta \\ -1+\cos\theta \\ e^{i\phi}\sin\theta \end{bmatrix} = \frac{1}{2p^0}\begin{bmatrix} -p^0-p^3 \\ -p^1-ip^2 \\ -p^0+p^3 \\ p^1+ip^2 \end{bmatrix}$$

$$= -\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} u(\vec{p},\frac{1}{2})$$

- **u和v线性相关，且都不是螺旋度算符的本征态**
- 因为此时不再存在静止系，$\sigma=\pm\frac{1}{2}$已经没有意义
- 需要用$\sigma=\pm\frac{1}{2}$的**u(或v)**态叠加出螺旋度的本征态

用零质量$u(\vec{p},\sigma), v(\vec{p},\sigma)$构造的螺旋度本征态: $\hat{p}=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\mathcal{J}^i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} \qquad \Sigma \equiv \frac{\vec{\mathcal{J}}\cdot\vec{p}}{p^0} = \frac{1}{2p^0}\begin{pmatrix}p^i\sigma^i & 0\\ 0 & p^i\sigma^i\end{pmatrix} \qquad p^i\sigma^i = \begin{pmatrix}p^3 & p^1-ip^2\\ p^1+ip^2 & -p^3\end{pmatrix}$$

$$v_+(p) \equiv \frac{\sqrt{1-\cos\theta}}{\sqrt{2}}[u(\vec{p},-\frac{1}{2}) + \frac{(1+\cos\theta)e^{-i\phi}}{\sin\theta}u(\vec{p},\frac{1}{2})] = \frac{1}{\sqrt{2p^0}}\begin{pmatrix}|p]_a\\ 0\end{pmatrix} \quad |p]_a \equiv \begin{pmatrix}\frac{p^1-ip^2}{\sqrt{p^0-p^3}}\\ \frac{p^0-p^3}{\sqrt{p^0-p^3}}\end{pmatrix}$$

$$= \frac{\sqrt{1-\cos\theta}}{\sqrt{2}}[v(\vec{p},\frac{1}{2}) - \frac{(1+\cos\theta)e^{-i\phi}}{\sin\theta}v(\vec{p},-\frac{1}{2})] \qquad \frac{p^i\sigma^i}{p^0}|p]_a = |p]_a \quad \Sigma v_+(p) = \frac{1}{2}v_+(p)$$

$$v_-(p) \equiv \frac{\sqrt{1+\cos\theta}}{\sqrt{2}}[-u(\vec{p},-\frac{1}{2}) + \frac{(1-\cos\theta)e^{-i\phi}}{\sin\theta}u(\vec{p},\frac{1}{2})] = \frac{1}{\sqrt{2p^0}}\begin{pmatrix}0\\ |p\rangle^{\dot{a}}\end{pmatrix} \quad |p\rangle^{\dot{a}} \equiv \begin{pmatrix}\frac{-p^1+ip^2}{\sqrt{p^0+p^3}}\\ \frac{p^0+p^3}{\sqrt{p^0+p^3}}\end{pmatrix}$$

$$= \frac{\sqrt{1+\cos\theta}}{\sqrt{2}}[v(\vec{p},\frac{1}{2}) + \frac{(1-\cos\theta)e^{-i\phi}}{\sin\theta}v(\vec{p},-\frac{1}{2})] \quad \frac{p^i\sigma^i}{p^0}|p\rangle^{\dot{a}} = -|p\rangle^{\dot{a}} \quad \Sigma v_-(p) = -\frac{1}{2}v_-(p)$$

$\bar{u}_+(p) \equiv ([p|^a, 0)$ $\quad [p|^a \equiv (|p\rangle^{\dot{a}})^\dagger$ $\quad \bar{u}_-(p) \equiv (0, \langle p|_{\dot{a}})$ $\quad \langle p|_{\dot{a}} \equiv (|p]_a)^\dagger$ $\quad \langle pq\rangle \equiv \langle p|_{\dot{a}}|q\rangle^{\dot{a}}$ $\quad [pq] \equiv [p|^a|q]_a$

$$p_{a\dot{b}} \equiv -|p]_a\langle p|_{\dot{b}} = -\begin{pmatrix}p^0+p^3 & p^1-ip^2\\ p^1+ip^2 & p^0-p^3\end{pmatrix} \qquad p^{\dot{a}b} \equiv -|p\rangle^{\dot{a}}[p|^b = -\begin{pmatrix}p^0-p^3 & -p^1+ip^2\\ -p^1-ip^2 & p^0+p^3\end{pmatrix}$$

空间反射变换: $Pa(\vec{p},\sigma)P^{-1} = \eta^* a(-\vec{p},\sigma) \quad Pa^c(\vec{p},\sigma)P^{-1} = \eta^{c*} a^c(-\vec{p},\sigma)$

$u_l(-\vec{p},\sigma) = b_u \beta u_l(\vec{p},\sigma) = \beta u_l(\vec{p},\sigma) \quad v_l(-\vec{p},\sigma) = b_v \beta v_l(\vec{p},\sigma) = -\beta v_l(\vec{p},\sigma)$

$$\psi_l(x) = (2\pi)^{-3/2} \sum_\sigma \int d\vec{p}\, [u_l(\vec{p},\sigma) e^{-ip\cdot x} a(\vec{p},\sigma) + v_l(\vec{p},\sigma) e^{ip\cdot x} a^{c\dagger}(\vec{p},\sigma)]$$

$$P\psi_l(x)P^{-1} = (2\pi)^{-3/2} \sum_\sigma \int d\vec{p}\, [\eta^* u_l(-\vec{p},\sigma) e^{-ip\cdot\mathcal{P}x} a(\vec{p},\sigma) + \eta^c v_l(-\vec{p},\sigma) e^{ip\cdot\mathcal{P}x} a^{c\dagger}(\vec{p},\sigma)]$$

$$= (2\pi)^{-3/2} \sum_\sigma \int d\vec{p}\, [\eta^* \beta u_l(\vec{p},\sigma) e^{-ip\cdot\mathcal{P}x} a(\vec{p},\sigma) - \eta^c \beta v_l(\vec{p},\sigma) e^{ip\cdot\mathcal{P}x} a^{c\dagger}(\vec{p},\sigma)]$$

为保证用空间反射态来构造$\tilde{\mathcal{H}}(x)$同样保证能够使其在类空区间相互对易，只能取

$\eta^c = -\eta^* \qquad \Rightarrow \qquad P\psi(x)P^{-1} = \eta^* \beta \psi(\mathcal{P}x)$ 作业21

$P\psi_L(x)P^{-1} = \eta^* \beta \psi_R(\mathcal{P}x) \qquad P\psi_R(x)P^{-1} = \eta^* \beta \psi_L(\mathcal{P}x)$

旋量场的分立对称性变换性质

时间反演变换: $Ta(\vec{p},\sigma)T^{-1}=(-1)^{\frac{1}{2}-\sigma}a(-\vec{p},-\sigma)$ $\quad Ta^c(\vec{p},\sigma)T^{-1}=(-1)^{\frac{1}{2}-\sigma}a^c(-\vec{p},-\sigma)$

$(-1)^{\frac{1}{2}+\sigma}u^*(-\vec{p},-\sigma)=-\gamma_5\mathcal{C}u(\vec{p},\sigma)$ $\quad (-1)^{\frac{1}{2}+\sigma}v^*(-\vec{p},-\sigma)=-\gamma_5\mathcal{C}v(\vec{p},\sigma)$

$$\psi_l(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,[u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma)+v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)]$$

$$T\psi_l(x)T^{-1}=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,(-1)^{\frac{1}{2}-\sigma}[u_l^*(\vec{p},\sigma)e^{ip\cdot x}a(-\vec{p},-\sigma)+v_l^*(\vec{p},\sigma)e^{-ip\cdot x}a^{c\dagger}(-\vec{p},-\sigma)]$$

$$=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,(-1)^{\frac{1}{2}+\sigma}[u_l^*(-\vec{p},-\sigma)e^{ip\cdot\mathcal{P}x}a(\vec{p},\sigma)+v_l^*(-\vec{p},-\sigma)e^{-ip\cdot\mathcal{P}x}a^{c\dagger}(\vec{p},\sigma)]$$

$$=-\gamma_5\mathcal{C}(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,[u_l(\vec{p},\sigma)e^{ip\cdot\mathcal{P}x}a(\vec{p},\sigma)+v_l(\vec{p},\sigma)e^{-ip\cdot\mathcal{P}x}a^{c\dagger}(\vec{p},\sigma)]$$

$$T\psi(x)T^{-1}=-\gamma_5\mathcal{C}\psi(-\mathcal{P}x)$$ 作业22

旋量场的分立对称性变换性质

电荷共轭与联合CPT变换： $Ca(\vec{p},\sigma)C^{-1}=\xi^* a^c(\vec{p},\sigma)$ $\quad Ca^c(\vec{p},\sigma)C^{-1}=\xi^{c*}a(\vec{p},\sigma)$

$$\beta\mathcal{C}u_l^*(\vec{p},\sigma)=v_l(\vec{p},\sigma) \quad \beta\mathcal{C}v_l^*(\vec{p},\sigma)=u_l(\vec{p},\sigma)$$

$$\psi_l(x)=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,[u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma)+v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)]$$

$$C\psi_l(x)C^{-1}=(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,[\xi^* u_l(\vec{p},\sigma)e^{-ip\cdot x}a^c(\vec{p},\sigma)+\xi^c v_l(\vec{p},\sigma)e^{ip\cdot x}a^\dagger(\vec{p},\sigma)]$$

$$=\beta\mathcal{C}(2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,[\xi^* v_l^*(\vec{p},\sigma)e^{-ip\cdot x}a^c(\vec{p},\sigma)+\xi^c u_l^*(\vec{p},\sigma)e^{ip\cdot x}a^\dagger(\vec{p},\sigma)]$$

为保证用电荷共轭态来构造$\tilde{\mathcal{H}}(x)$同样保证能够使其在类空区间相互对易，只能取

$$\xi^c=\xi^* \qquad \Rightarrow \qquad C\psi(x)C^{-1}=\xi^*\beta\mathcal{C}\psi^*(x)$$

如果$a^c(\vec{p},\sigma)=a(\vec{p},\sigma)\Rightarrow C\psi(x)C^{-1}=\xi^*\psi(x)$,此种费米子叫Majorana费米子

Majorana费米子： $\psi(x)=\beta\mathcal{C}\psi^*(x)$ 作业23

对CPT联合变换： $CPT\ \psi(x)\ [CPT]^{-1}=\xi^*\eta^*\gamma_5\psi^*(-x)$

自由旋量场的场方程、哈密顿量和作用量、正则对易关系

$$\psi_l(x) = (2\pi)^{-3/2}\sum_\sigma \int d\vec{p}\,[u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) + v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)]$$

$$D(L(p))\beta D^{-1}(L(p)) = L_\mu^{\ 0}(p)\gamma^\mu = p_\mu\gamma^\mu/M$$

$$\frac{1}{M}p_\mu\gamma^\mu\ u(\vec{p},\sigma) = \frac{1}{M}p_\mu\gamma^\mu\sqrt{\frac{M}{p^0}}D(L(p))\ u(0,\sigma) = \sqrt{\frac{M}{p^0}}D(L(p))\beta u(0,\sigma) = u(\vec{p},\sigma)$$

$$\frac{1}{M}p_\mu\gamma^\mu\ v(\vec{p},\sigma) = \frac{1}{M}p_\mu\gamma^\mu\sqrt{\frac{M}{p^0}}D(L(p))\ v(0,\sigma) = \sqrt{\frac{M}{p^0}}D(L(p))\beta v(0,\sigma) = -v(\vec{p},\sigma)$$

$$(p^\mu\gamma_\mu - M)u(\vec{p},\sigma) = 0 \qquad (p^\mu\gamma_\mu + M)v(\vec{p},\sigma) = 0$$

注：$\gamma_5 u(\vec{p},\sigma)$，$\gamma_5 v(\vec{p},\sigma)$ 对应$b_u = -b_v = -1$的选择！

$$\Rightarrow \quad (i\gamma^\mu\partial_\mu - M)\psi(x) = 0$$

自由粒子场的Dirac方程！

M前的符号依赖$b_u = -b_v$的约定！

$$i\gamma^\mu\partial_\mu\psi_{\substack{L\\R}}(x) - M\psi_{\substack{R\\L}}(x) = 0$$

自由旋量场的哈密顿量是体系的总能量算符

$$H_0' = \sum_\sigma \int d\vec{p}\ [a^\dagger(\vec{p},\sigma)a(\vec{p},\sigma) + a^{c\dagger}(\vec{p},\sigma)a^c(\vec{p},\sigma)]\sqrt{\vec{p}^2 + M^2} = H_0$$

作业31

检验Dirac方程: $(p^\mu\gamma_\mu - M)u(\vec{p},\sigma) = 0$ $\quad (p^\mu\gamma_\mu + M)v(\vec{p},\sigma) = 0$

$$\begin{matrix} u(\vec{p},\frac{1}{2}) \\ v(\vec{p},-\frac{1}{2}) \end{matrix} = \frac{1}{2\sqrt{\gamma}}\begin{bmatrix} \pm\sqrt{\gamma+1}\pm\sqrt{\gamma-1}\cos\theta \\ \pm\sqrt{\gamma-1}e^{i\phi}\sin\theta \\ -\sqrt{\gamma+1}+\sqrt{\gamma-1}\cos\theta \\ \sqrt{\gamma-1}e^{i\phi}\sin\theta \end{bmatrix} \qquad \hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\begin{matrix} u(\vec{p},-\frac{1}{2}) \\ v(\vec{p},\frac{1}{2}) \end{matrix} = \frac{1}{2\sqrt{\gamma}}\begin{bmatrix} \sqrt{\gamma-1}e^{-i\phi}\sin\theta \\ \sqrt{\gamma+1}-\sqrt{\gamma-1}\cos\theta \\ \pm\sqrt{\gamma-1}e^{-i\phi}\sin\theta \\ \mp\sqrt{\gamma+1}\mp\sqrt{\gamma-1}\cos\theta \end{bmatrix} \qquad \gamma = \frac{\sqrt{\vec{p}^2+M^2}}{M} \qquad M\sqrt{\gamma^2-1} = \sqrt{\vec{p}^2}$$

$$\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \qquad p_\mu\gamma^\mu = M[\gamma\gamma^0 - \sqrt{\gamma^2-1}(\sin\theta\cos\phi\gamma^1 + \sin\theta\sin\phi\gamma^2 + \cos\theta\gamma^3)]$$

$$p_\mu\gamma^\mu = M\begin{pmatrix} 0 & -\gamma-\sqrt{\gamma^2-1}(\sin\theta\cos\phi\sigma^1+\sin\theta\sin\phi\sigma^2+\cos\theta\sigma^3) \\ -\gamma+\sqrt{\gamma^2-1}(\sin\theta\cos\phi\sigma^1+\sin\theta\sin\phi\sigma^2+\cos\theta\sigma^3) & 0 \end{pmatrix}$$

$$= M\begin{pmatrix} 0 & 0 & -\gamma-\sqrt{\gamma^2-1}\cos\theta & -\sqrt{\gamma^2-1}e^{-i\phi}\sin\theta \\ 0 & 0 & -\sqrt{\gamma^2-1}e^{i\phi}\sin\theta & -\gamma+\sqrt{\gamma^2-1}\cos\theta \\ -\gamma+\sqrt{\gamma^2-1}\cos\theta & \sqrt{\gamma^2-1}e^{-i\phi}\sin\theta & 0 & 0 \\ \sqrt{\gamma^2-1}e^{i\phi}\sin\theta & -\gamma-\sqrt{\gamma^2-1}\cos\theta & 0 & 0 \end{pmatrix}$$

$$\psi_l(x) = (2\pi)^{-3/2}\sum\int d\vec{p}\,[u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma)+v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)] \quad (i\gamma^\mu\partial_\mu - M)\psi(x)=0$$

自由旋量场的哈密顿量是体系的总能量算符

$$H_0 = \sum_\sigma \int d\vec{p}\ \ [a^\dagger(\vec{p},\sigma)a(\vec{p},\sigma)+a^{c\dagger}(\vec{p},\sigma)a^c(\vec{p},\sigma)]\sqrt{\vec{p}^2+M^2}$$

哈密顿量的坐标空间表达:

$$\psi^\dagger = (2\pi)^{-3/2}\sum_\sigma\int d\vec{p}\,[u^\dagger(\vec{p},\sigma)e^{ip\cdot x}a^\dagger(\vec{p},\sigma)+v^\dagger(\vec{p},\sigma)e^{-ip\cdot x}a^c(\vec{p},\sigma)]$$

$$\nabla\psi(x) = (2\pi)^{-3/2}i\sum_\sigma\int d\vec{p}\,\vec{p}[u(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma)-v(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)]$$

$$u^\dagger(\vec{p},\sigma)u(\vec{p},\sigma')\overset{\text{作业16}}{=\!=\!=\!=}v^\dagger(\vec{p},\sigma)v(\vec{p},\sigma')=\delta_{\sigma\sigma'} \qquad u^\dagger(\vec{p},\sigma)v(-\vec{p},\sigma')\overset{\text{作业18}}{=\!=\!=\!=}v^\dagger(\vec{p},\sigma)u(-\vec{p},\sigma')=0$$

$$\overset{\text{作业17}}{=\!=\!=\!\Rightarrow}\quad H_0 = \int d\vec{x}:\ [\psi^\dagger(x)\beta(-i\vec{\gamma}\cdot\nabla+M)\psi(x)]:$$

$$\int d^4x\ [\overline{\psi}i\vec{\gamma}\cdot\nabla\psi(x)]^* = -i\int d^4x[\partial^i\psi^\dagger(x)]\gamma^{i\dagger}\beta\psi(x) = -i\int d^4x[\partial^i\overline{\psi}(x)]\gamma^i\psi(x) = \int d^4x\ \overline{\psi}(x)i\vec{\gamma}\cdot\nabla\psi(x)$$

自由旋量场的场方程、哈密顿量和作用量、正则对易关系

$$\psi_l(x) = (2\pi)^{-3/2}\sum_\sigma \int d\vec{p}\,[u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) + v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)] \quad (i\gamma^\mu\partial_\mu - M)\psi(x) = 0$$

$$(i\partial_t + i\beta\vec{\gamma}\cdot\nabla - \beta M)\psi(x) = 0 \qquad H_0 = \int d\vec{x}\; :[\psi^\dagger(x)\beta(-i\vec{\gamma}\cdot\nabla + M)\psi(x)]:$$

$\psi(x)$为广义坐标，H_0对$\psi(x)$的泛函微商定义了广义动量对时间导数的负值$-\dot{\pi}(x)$

$$-\dot{\pi}^T(x) \equiv \left.\frac{\delta H_0}{\delta\psi^T(x)}\right|_{\pi\text{固定}} = \beta^*(i\vec{\gamma}^*\cdot\nabla + M)\psi^* = -i\dot{\psi}^*(x)$$

左泛函微商

$$\pi(x) = i\psi^*(x) = i[\overline{\psi}(x)\beta]^T \qquad \overline{\psi}(x) \equiv \psi^\dagger(x)\beta$$

$$S_0 = \int d^4x\; :\pi^T(x)\cdot\dot{\psi}(x): - \int dt\, H_0 = \int d^4x\; :\overline{\psi}(x)(i\gamma^\mu\partial_\mu - M)\psi(x):$$

$$\gamma^\mu\partial_\mu = \beta\partial_t + \vec{\gamma}\cdot\nabla$$

略去边界积分，将S_0取极值得到Dirac方程!

$$\int d^4x\,[\overline{\psi}i\gamma^\mu\partial_\mu\psi(x)]^* = -i\int d^4x[\partial_\mu\psi^\dagger(x)]\gamma^{\mu\dagger}\beta\psi(x) = -i\int d^4x[\partial_\mu\overline{\psi}(x)]\gamma^\mu\psi(x) = \int d^4x\,\overline{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)$$

$$\psi_l(x) = (2\pi)^{-3/2}\sum_\sigma \int d\vec{p}\, [u_l(\vec{p},\sigma)e^{-ip\cdot x}a(\vec{p},\sigma) + v_l(\vec{p},\sigma)e^{ip\cdot x}a^{c\dagger}(\vec{p},\sigma)] \quad (i\gamma^\mu\partial_\mu - M)\psi(x) = 0$$

$$H_0 = \mathrm{Re}\int d\vec{x}\ :[\psi^\dagger(x)\beta(-i\vec{\gamma}\cdot\nabla + M)\psi(x)]: \qquad -\dot{\vec{\pi}}(x) \equiv \left.\frac{\delta H_0}{\delta\psi(x)}\right|_{\pi\text{固定}} = -i\dot{\psi}^*(x)$$

$$S_0 = \int d^4x\ :\overline{\psi}(x)(i\gamma^\mu\partial_\mu - M)\psi(x): \qquad \pi(x) = i\psi^*(x) \quad \overline{\psi}(x) \equiv \psi^\dagger(x)\beta$$

$$\{\psi_l(\vec{x},t), \pi_{\bar{l}}(\vec{y},t)\} = i\delta_{l\bar{l}}\delta(\vec{x}-\vec{y}) \qquad \{\psi_l(\vec{x},t), \psi_{\bar{l}}(\vec{y},t)\} = \{\pi_l(\vec{x},t), \pi_{\bar{l}}(\vec{y},t)\} = 0$$ 作业19

$$\dot{\psi}(\vec{x},t) = i[H_0, \psi(\vec{x},t)] = \frac{\delta H_0}{\delta\pi(\vec{x},t)} \qquad \dot{\pi}(\vec{x},t) = i[H_0, \pi(\vec{x},t)] = -\frac{\delta H_0}{\delta\psi(\vec{x},t)}$$ 作业20

$$S_0 = \int d^4x\, \mathcal{L}_0 \qquad \mathcal{L}_0 =: \overline{\psi}(x)(i\gamma^\mu\partial_\mu - M)\psi(x): \quad \pi^i(x) = \frac{\partial\mathcal{L}_0}{\partial\dot{\psi}(x)} = i\psi^*(x)$$

零质量旋量场的结果可在有质量的旋量场结果中将质量趋于零得到！

费米子质量: $\psi_R = P_R\psi \quad \psi_L = P_L\psi \quad \psi^c \equiv C\psi C^{-1} = \xi^*\beta\mathcal{C}\psi^* \quad (\psi_L)^c = \xi^*\beta\mathcal{C}\psi_L^* = P_R\xi^*\beta\mathcal{C}\psi^* = (\psi^c)_R$

$$S_0 = \int d^4x : \overline{\psi}(x)(i\gamma^\mu\partial_\mu - M)\psi(x) :$$

$$= \int d^4x : [\overline{\psi_L}(x) i\gamma^\mu\partial_\mu\psi_L(x) + \overline{\psi_R}(x) i\gamma^\mu\partial_\mu\psi_R(x) - M\overline{\psi_R}(x)\psi_L(x) - M\overline{\psi_L}(x)\psi_R(x)] :$$

♣ 如果费米子质量为零，左手场与右手场不发生作用!

♢ 这时左手场和右手场可以有相互独立的对称性! 手征对称性

♡ 质量负责联系左手场和右手场!

♠ 这时左、右手场不能再有相互独立的对称性! $\psi_R \to R\psi_R \quad \psi_L \to L\psi_L \quad R \neq L$

¶ 若费米子存在手征对称性将禁戒质量项的出现!

✠ 费米子有质量意味手征对称性必须发生破坏!

自由旋量场的场方程、哈密顿量和作用量、正则对易关系

Marjorana中微子? 质量: $\psi^c \equiv C\psi C^{-1} = \xi^*\beta\mathcal{C}\psi^* \overset{\text{Marjorana}}{\Longrightarrow} \xi^*\psi$ 纯中性

$$(\psi_L)^c = \xi^*\beta\mathcal{C}\psi_L^* = P_R\xi^*\beta\mathcal{C}\psi^* = (\psi^c)_R \overset{\text{Marjorana}}{\Longrightarrow} \xi^*\psi_R \qquad (\psi_R)^c = (\psi^c)_L \overset{\text{Marjorana}}{\Longrightarrow} \xi^*\psi_L$$

$$\overline{(\psi_L)^c} = \overline{(\psi^c)_R} \overset{\text{Marjorana}}{\Longrightarrow} \xi\overline{\psi_R} \qquad \overline{(\psi_R)^c} = \overline{(\psi^c)_L} \overset{\text{Marjorana}}{\Longrightarrow} \xi\overline{\psi_L}$$

$$\overline{\psi}\gamma^\mu\psi = \overline{\psi^c}\gamma^\mu\psi^c = [\beta\mathcal{C}\psi^*]^\dagger\beta\gamma^\mu\beta\mathcal{C}\psi^* = \psi^T\mathcal{C}\gamma^\mu\mathcal{C}\beta\psi^* = \psi^T\gamma^{\mu T}\beta\psi^* = -\overline{\psi}\gamma^\mu\psi = 0$$

Dirac mass term $= \int d^4x : [-D\overline{\psi_R}(x)\psi_L(x) - D\overline{\psi_L}(x)\psi_R(x)] :$

Marjorana mass term $= \int d^4x : [-\frac{A}{2}\overline{(\psi_L)^c}(x)\psi_L(x) - \frac{A}{2}\overline{\psi_L}(x)(\psi_L)^c(x)] :$ **Fermion#violation**

Marjorana mass term $= \int d^4x : [-\frac{B}{2}\overline{(\psi_R)^c}(x)\psi_R(x) - \frac{B}{2}\overline{\psi_R}(x)(\psi_R)^c(x)] :$ **Fermion#violation**

对左右手场都存在的情形：

$$\sqrt{2}\chi = \psi_L + (\psi_L)^c \qquad \chi^c = \chi \qquad \sqrt{2}\omega = \psi_R + (\psi_R)^c \qquad \omega^c = \omega$$

$$\psi_L = \sqrt{2}P_L\chi \qquad (\psi_L)^c = \sqrt{2}P_R\chi \qquad \psi_R = \sqrt{2}P_R\omega \qquad (\psi_R)^c = \sqrt{2}P_L\omega$$

Marjorana质量: $\psi^c \equiv C\psi C^{-1} = \xi^* \beta \mathcal{C} \psi^* \xrightarrow{\text{Marjorana}} \psi$ 纯中性

$$\sqrt{2}\chi = \psi_L + (\psi_L)^c \qquad \chi^c = \chi \qquad \sqrt{2}\omega = \psi_R + (\psi_R)^c \qquad \omega^c = \omega$$

$$\psi_L = \sqrt{2}P_L\chi \qquad (\psi_L)^c = \sqrt{2}P_R\chi \qquad \psi_R = \sqrt{2}P_R\omega \qquad (\psi_R)^c = \sqrt{2}P_L\omega$$

$$\overline{\psi_L} i\gamma^\mu \partial_\mu \psi_L + \overline{\psi_R} i\gamma^\mu \partial_\mu \psi_R = \overline{\chi} i\gamma^\mu \partial_\mu \chi + \overline{\omega} i\gamma^\mu \partial_\mu \omega$$

$$D\overline{\psi_R}\psi_L + D\overline{\psi_L}\psi_R + \frac{A}{2}\overline{(\psi_L)^c}\psi_L + \frac{A}{2}\overline{\psi_L}(\psi_L)^c + \frac{B}{2}\overline{(\psi_R)^c}\psi_R + \frac{B}{2}\overline{\psi_R}(\psi_R)^c$$

$$= D(\overline{\chi}\omega + \overline{\omega}\chi) + A\overline{\chi}\chi + B\overline{\omega}\omega = (\overline{\chi}, \overline{\omega}) \begin{pmatrix} A & D \\ D & B \end{pmatrix} \begin{pmatrix} \chi \\ \omega \end{pmatrix}$$

$$M_{1,2} = \frac{A}{2} + \frac{B}{2} \pm \sqrt{(\frac{A}{2} - \frac{B}{2})^2 + D^2}$$

Eigenstates: $\eta_1 = \chi\cos\theta - \omega\sin\theta \qquad \eta_2 = \chi\sin\theta + \omega\cos\theta \qquad \tan 2\theta = \frac{2D}{A-B}$

$$D = \frac{1}{2}(M_1 - M_2)\sin 2\theta \qquad A = M_1\cos^2\theta + M_2\sin^2\theta \qquad B = M_1\sin^2\theta + M_2\cos^2\theta$$

自由旋量场的场方程、哈密顿量和作用量、正则对易关系

Marjorana中微子？质量： $\psi^c \equiv C\psi C^{-1} = \xi^*\beta\mathcal{C}\psi^* \xRightarrow{\text{Marjorana}} \xi^*\psi$ 纯中性

$$\text{Marjorana mass term} = \int d^4x : [-\frac{A}{2}\overline{(\psi_L)^c}(x)\psi_L(x) - \frac{A}{2}\overline{\psi_L}(x)(\psi_L)^c(x)] : \text{Fermion\#violation}$$

$$\text{Marjorana mass term} = \int d^4x : [-\frac{B}{2}\overline{(\psi_R)^c}(x)\psi_R(x) - \frac{B}{2}\overline{\psi_R}(x)(\psi_R)^c(x)] : \text{Fermion\#violation}$$

对左右手场分别单独存在的情形，不能有Dirac质量：

$$\sqrt{2}\chi = \psi_L + (\psi_L)^c \qquad \chi^c = \chi \qquad \sqrt{2}\omega = \psi_R + (\psi_R)^c \qquad \omega^c = \omega$$

$$\psi_L = \sqrt{2}P_L\chi \qquad (\psi_L)^c = \sqrt{2}P_R\chi \qquad \psi_R = \sqrt{2}P_R\omega \qquad (\psi_R)^c = \sqrt{2}P_L\omega$$

$$S_L = \int d^4x : [\overline{\psi_L}i\gamma^\mu\partial_\mu\psi_L - \frac{A}{2}\overline{(\psi_L)^c}\psi_L - \frac{A}{2}\overline{\psi_L}(\psi_L)^c] : \quad = \int d^4x : \overline{\chi}[\, i\gamma^\mu\partial_\mu - A\,]\chi :$$

$$S_R = \int d^4x : [\overline{\psi_R}i\gamma^\mu\partial_\mu\psi_R - \frac{B}{2}\overline{(\psi_R)^c}\psi_R - \frac{B}{2}\overline{\psi_R}(\psi_R)^c] : \quad = \int d^4x : \overline{\omega}[\, i\gamma^\mu\partial_\mu - B\,]\omega :$$

$$[i\gamma^\mu\partial_\mu - A]\chi = 0 \Rightarrow i\gamma^\mu\partial_\mu P_L\chi - AP_R\chi = 0 \xRightarrow{\text{两分量方程}} i\gamma^\mu\partial_\mu\psi_L - A(\psi_L)^c = 0$$

$$[i\gamma^\mu\partial_\mu - B]\omega = 0 \Rightarrow i\gamma^\mu\partial_\mu P_R\omega - BP_L\omega = 0 \xRightarrow{\text{两分量方程}} i\gamma^\mu\partial_\mu\psi_R - B(\psi_R)^c = 0$$

旋量场的自由度与两种质量:

♣ 具有空间反射对称的旋量场有4个自由度!

♢ 若粒子中性或无质量， 可只有2个自由度 Weyl！ 但无空间反射对称性

♠ 中性粒子每两自由度左或右 可单独具有自己的Majorana质量但破坏费米子数!

♡ 也可看成是中性左(右)手Dirac场分别和自己的电荷共轭场的关联

¶ 左右各俩中性自由度两个粒子之间可以通过Dirac质量相互关联!

✠ 对中性粒子，两种质量可能共存，导致跷跷板机制中微子属那种？

自然界中的中微子的几种存在可能性:

♣ 至少有三种，每种都有左手态，至少两种有很小质量！每种可能

(1) 是纯 Dirac粒子 $A = B = 0$。有宇称；有右手（四分量）。无法解释小质量

(2) 是一组两分量纯 Majorana粒子 $D = B = 0$。无宇称；无右手。无法解释小质量

(3) 是两组通过Dirac质量耦合的两分量 Majorana粒子 $D = 0$对应退耦合

♠ 两组耦合Majorana粒子可产生跷跷板机制解释小质量！产生重中微子

✠ 三种中微子的每种都可以是上面三个可能中的某一种！

旋量场的地位:

♣ 是各种非平庸量子场中最简单的场!

♢ 十分复杂! 且没有经典对应!

♠ 现实世界已发现的所有物质型基本粒子都由旋量场描述!

♡ Dirac、Weyl、Majorana粒子正被凝聚态、量子信息密切关注!

¶ 存在两种的基本旋量场!

✠ 左右手场的不同对称性手征对称性 在现实世界起关键作用

按自旋分类

$$\psi_l^+(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{-ip\cdot x}u_l(\vec p,\sigma,n)a(\vec p,\sigma,n)\quad \psi_l^-(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^{3/2}}e^{ip\cdot x}v_l(\vec p,\sigma,n)a^\dagger(\vec p,\sigma,n)$$

$$\sum_{\bar\sigma}u_{\bar l}(\vec p_\Lambda,\bar\sigma,n)D_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^+_{\bar l l}(\Lambda)u_l(\vec p,\sigma,n)$$

$$\sum_{\bar\sigma}v_{\bar l}(\vec p_\Lambda,\bar\sigma,n)D^*_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_l D^-_{\bar l l}(\Lambda)v_l(\vec p,\sigma,n)$$

矢量量子场: $D(\Lambda)^\mu{}_\nu{}^\pm\equiv\Lambda^\mu{}_\nu\quad (J^1\pm iJ^2)_{\sigma'\sigma}=\delta_{\sigma',\sigma\pm1}\sqrt{(j\mp\sigma)(j\pm\sigma+1)}\ j=0,1\quad J^3_{\sigma'\sigma}=\sigma\delta_{\sigma'\sigma}$

$$D(1+\omega)^\mu{}_\nu=g^\mu{}_\nu+\omega^\mu{}_\nu=g^\mu{}_\nu+\frac{i}{2}\omega_{\sigma\rho}(\mathcal{J}^{\sigma\rho})^\mu{}_\nu\ \overset{e^\omega\ \text{不幺正!}}{=\!=\!=\!=\!=\!\Rightarrow}\ (\mathcal{J}^{\sigma\rho})^\mu{}_\nu=i(g^\sigma{}_\nu g^{\rho\mu}-g^\rho{}_\nu g^{\sigma\mu})$$

$$\phi^{+\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec p\ u^\mu(\vec p,\sigma)a(\vec p,\sigma)e^{-ip\cdot x}\quad \phi^{-\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec p\ v^\mu(\vec p,\sigma)a^\dagger(\vec p,\sigma)e^{ip\cdot x}$$

$$u^\mu(\vec p,\sigma)=(M/p^0)^{1/2}L(p)^\mu{}_\nu u^\nu(0,\sigma)\qquad v^\mu(\vec p,\sigma)=(M/p^0)^{1/2}L(p)^\mu{}_\nu v^\nu(0,\sigma)$$

$$\sum_{\bar\sigma}u^\mu(0,\bar\sigma)\vec J^{(j)}_{\bar\sigma\sigma}=\vec{\mathcal{J}}^\mu{}_\nu u^\nu(0,\sigma)\qquad \sum_{\bar\sigma}v^\mu(0,\bar\sigma)\vec J^{(j)*}_{\bar\sigma\sigma}=-\vec{\mathcal{J}}^\mu{}_\nu v^\nu(0,\sigma)$$

转动生成元 $\vec{\mathcal{J}}^\mu{}_\nu$是$\Lambda^\mu{}_\nu$的纯转动部分的四矢量表达: $(\mathcal{J}_k)^\mu{}_\nu=-\frac{1}{2}\epsilon_{ijk}(\mathcal{J}^{ij})^\mu{}_\nu=i\epsilon_{ijk}g^{i\mu}g^j{}_\nu$

$(\mathcal{J}_k)^0{}_0=(\mathcal{J}_k)^0{}_i=(\mathcal{J}_k)^i{}_0=0\quad (\mathcal{J}_k)^i{}_j=i\epsilon_{ijk}\quad (\vec{\mathcal{J}}^2)^0{}_0=(\vec{\mathcal{J}}^2)^0{}_i=(\vec{\mathcal{J}}^2)^i{}_0=0\quad (\vec{\mathcal{J}}^2)^i{}_j=2\delta^i{}_j$ 场自旋为1

按自旋分类

$$\phi^{+\mu}(x)=\sum_{\sigma}(2\pi)^{-3/2}\int d\vec{p}\; u^{\mu}(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} \quad \phi^{-\mu}(x)=\sum_{\sigma}(2\pi)^{-3/2}\int d\vec{p}\; v^{\mu}(\vec{p},\sigma)a^{\dagger}(\vec{p},\sigma)e^{ip\cdot x}$$

$$\sum_{\bar\sigma}u_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_{l}D^{+}_{\bar l l}(\Lambda)u_l(\vec{p},\sigma,n) \quad u^{\mu}(\vec{p},\sigma)=(M/p^0)^{1/2}L(p)^{\mu}{}_{\nu}u^{\nu}(0,\sigma)$$

$$\sum_{\bar\sigma}v_{\bar l}(\vec{p}_\Lambda,\bar\sigma,n)D^{*}_{\bar\sigma\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_{l}D^{-}_{\bar l l}(\Lambda)v_l(\vec{p},\sigma,n) \quad v^{\mu}(\vec{p},\sigma)=(M/p^0)^{1/2}L(p)^{\mu}{}_{\nu}v^{\nu}(0,\sigma)$$

$$(\mathcal{J}_k)^0{}_0=(\mathcal{J}_k)^0{}_i=(\mathcal{J}_k)^i{}_0=0 \qquad (\mathcal{J}_k)^i{}_j=i\epsilon_{ijk} \qquad (\vec{\mathcal{J}}^2)^0{}_0=(\vec{\mathcal{J}}^2)^0{}_i=(\vec{\mathcal{J}}^2)^i{}_0=0 \qquad (\vec{\mathcal{J}}^2)^i{}_j=2\delta^i_j$$

$$\sum_{\bar\sigma}u^{\mu}(0,\bar\sigma)\vec{J}^{(j)}_{\bar\sigma\sigma}=\vec{\mathcal{J}}^{\mu}{}_{\nu}u^{\nu}(0,\sigma) \quad \sum_{\bar\sigma}u^{0}(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=\sum_{\sigma'}\vec{\mathcal{J}}^{0}{}_{\nu}u^{\nu}(0,\sigma')\cdot\vec{J}^{(j)}_{\sigma'\sigma}=(\vec{\mathcal{J}}^2)^0{}_{\nu}u^{\nu}(0,\sigma)=0$$

$$\sum_{\bar\sigma}u^{i}(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=\sum_{\sigma'}\vec{\mathcal{J}}^{i}{}_{\nu}u^{\nu}(0,\sigma')\cdot\vec{J}^{(j)}_{\sigma'\sigma}=(\vec{\mathcal{J}}^2)^i{}_{\nu}u^{\nu}(0,\sigma)=2u^i(0,\sigma)$$

$$\sum_{\bar\sigma}v^{\mu}(0,\bar\sigma)\vec{J}^{(j)*}_{\bar\sigma\sigma}=-\vec{\mathcal{J}}^{\mu}{}_{\nu}v^{\nu}(0,\sigma) \quad \sum_{\bar\sigma}v^{0}(0,\bar\sigma)(\vec{J}^2)^{(j)*}_{\bar\sigma\sigma}=-\sum_{\sigma'}\vec{\mathcal{J}}^{0}{}_{\nu}v^{\nu}(0,\sigma')\cdot\vec{J}^{(j)*}_{\sigma'\sigma}=(\vec{\mathcal{J}}^2)^0{}_{\nu}v^{\nu}(0,\sigma)=0$$

$$\sum_{\bar\sigma}v^{i}(0,\bar\sigma)(\vec{J}^2)^{(j)*}_{\bar\sigma\sigma}=-\sum_{\sigma'}\vec{\mathcal{J}}^{i}{}_{\nu}v^{\nu}(0,\sigma')\cdot\vec{J}^{(j)*}_{\sigma'\sigma}=(\vec{\mathcal{J}}^2)^i{}_{\nu}v^{\nu}(0,\sigma)=2v^i(0,\sigma)$$

按自旋分类

$$\phi^{+\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\; u^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} \qquad \phi^{-\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\; v^\mu(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}$$

$$(\mathcal{J}_k)^0{}_0=(\mathcal{J}_k)^0{}_i=(\mathcal{J}_k)^i{}_0=0 \qquad (\mathcal{J}_k)^i{}_j=i\epsilon_{ijk} \qquad (\vec{\mathcal{J}}^2)^0{}_0=(\vec{\mathcal{J}}^2)^0{}_i=(\vec{\mathcal{J}}^2)^i{}_0=0 \qquad (\vec{\mathcal{J}}^2)^i{}_j=2\delta^i{}_j$$

$$u^\mu(\vec{p},\sigma)=(M/p^0)^{1/2}L(p)^\mu{}_\nu u^\nu(0,\sigma) \qquad \sum_{\bar\sigma}u^0(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=0 \qquad \sum_{\bar\sigma}u^i(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=2u^i(0,\sigma)$$

$$v^\mu(\vec{p},\sigma)=(M/p^0)^{1/2}L(p)^\mu{}_\nu v^\nu(0,\sigma) \qquad \sum_{\bar\sigma}v^0(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=0 \qquad \sum_{\bar\sigma}v^i(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=2v^i(0,\sigma)$$

$$\sum_{\bar\sigma}u^\mu(0,\bar\sigma)\vec{J}^{(j)}_{\bar\sigma\sigma}=\vec{\mathcal{J}}^\mu{}_\nu u^\nu(0,\sigma) \qquad \sum_{\bar\sigma}v^\mu(0,\bar\sigma)\vec{J}^{(j)*}_{\bar\sigma\sigma}=-\vec{\mathcal{J}}^\mu{}_\nu v^\nu(0,\sigma)$$

自旋为0的情况: $j=0 \qquad L^\mu{}_0(p)=p^\mu/M$

$$u^i(0)=v^i(0)=0 \qquad u^0(0)=-v^0(0)=i(M/2)^{1/2} \qquad u^\mu(\vec{p})=-v^\mu(\vec{p})=ip^\mu(2p^0)^{-1/2}$$

$$\phi^{+\mu}(x)=-\partial^\mu\phi^+(x) \qquad \phi^{-\mu}(x)=-\partial^\mu\phi^-(x)$$

标量场的基础上加上一个微商即可得到现在的矢量场

按自旋分类

$$\phi^{+\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\, u^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} \qquad \phi^{-\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\, v^\mu(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}$$

$$u^\mu(\vec{p},\sigma)=(M/p^0)^{1/2}L(p)^\mu{}_\nu u^\nu(0,\sigma) \qquad \sum_{\bar\sigma} u^0(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=0 \qquad \sum_{\bar\sigma} u^i(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=2u^i(0,\sigma)$$

$$v^\mu(\vec{p},\sigma)=(M/p^0)^{1/2}L(p)^\mu{}_\nu v^\nu(0,\sigma) \qquad \sum_{\bar\sigma} v^0(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=0 \qquad \sum_{\bar\sigma} v^i(0,\bar\sigma)(\vec{J}^2)^{(j)}_{\bar\sigma\sigma}=2v^i(0,\sigma)$$

自旋为1有质量的情况: $j=1$ ⇑态自旋为1/2和大于1上式只有零解⇑

$$\sum_{\bar\sigma} u^\mu(0,\bar\sigma)\vec{J}_{\bar\sigma\sigma}=\vec{\mathcal{J}}^\mu{}_\nu u^\nu(0,\sigma) \qquad \sum_{\bar\sigma} v^\mu(0,\bar\sigma)\vec{J}^*_{\bar\sigma\sigma}=-\vec{\mathcal{J}}^\mu{}_\nu v^\nu(0,\sigma)$$

$$J^\pm_{\sigma'\sigma}=(J^1\pm iJ^2)_{\sigma'\sigma}=\delta_{\sigma',\sigma\pm1}\sqrt{(1\mp\sigma)(1\pm\sigma+1)} \qquad J^3_{\sigma'\sigma}=\sigma\delta_{\sigma'\sigma}$$

$$(\mathcal{J}^{\sigma\rho})^\mu{}_\nu=i(g^\sigma{}_\nu g^{\rho\mu}-g^\rho{}_\nu g^{\sigma\mu}) \qquad (\mathcal{J}_k)^0{}_0=(\mathcal{J}_k)^0{}_i=(\mathcal{J}_k)^i{}_0=0 \qquad (\mathcal{J}_k)^i{}_j=i\epsilon_{ijk}$$

$$u^0(0,\sigma)=v^0(0,\sigma)=0 \qquad u^i(0,\sigma)\sigma=i\epsilon_{ij3}u^j(0,\sigma) \qquad v^i(0,\sigma)\sigma=-i\epsilon_{ij3}v^j(0,\sigma)$$

$$u^1(0,0)=u^2(0,0)=v^1(0,0)=v^2(0,0)=0 \qquad u^3(0,\pm1)=v^3(0,\pm1)=0$$

$$u^1(0,\sigma)\sigma=iu^2(0,\sigma) \quad u^2(0,\sigma)\sigma=-iu^1(0,\sigma) \quad v^1(0,\sigma)\sigma=-iv^2(0,\sigma) \quad v^2(0,\sigma)\sigma=iv^1(0,\sigma)$$

按自旋分类

$$\phi^{+\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\, u^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} \qquad \phi^{-\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\, v^\mu(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}$$

$$\sum_{\bar\sigma} u^\mu(0,\bar\sigma)\vec{J}_{\bar\sigma\sigma}=\vec{\mathcal{J}}^\mu{}_\nu u^\nu(0,\sigma) \qquad \sum_{\bar\sigma} v^\mu(0,\bar\sigma)\vec{J}^*_{\bar\sigma\sigma}=-\vec{\mathcal{J}}^\mu{}_\nu v^\nu(0,\sigma)$$

$$u^0(0,\sigma)=v^0(0,\sigma)=0 \qquad u^1(0,\sigma)\sigma=iu^2(0,\sigma) \qquad v^1(0,\sigma)\sigma=-iv^2(0,\sigma)$$

$$u^1(0,0)=u^2(0,0)=v^1(0,0)=v^2(0,0)=0 \qquad u^3(0,\pm1)=v^3(0,\pm1)=0$$

自旋为1有质量的情况: $j=1$

$$u^\mu(\vec{p},\sigma)=v^{\mu*}(\vec{p},\sigma)=(2p^0)^{-1/2}e^\mu(\vec{p},\sigma) \qquad u^\mu(0,0)=v^\mu(0,0)=(2M)^{-1/2}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

作业24

$$u^\mu(0,1)=-v^\mu(0,-1)=-\frac{(2M)^{-1/2}}{\sqrt{2}}\begin{bmatrix}0\\1\\i\\0\end{bmatrix} \qquad u^\mu(0,-1)=-v^\mu(0,1)=\frac{(2M)^{-1/2}}{\sqrt{2}}\begin{bmatrix}0\\1\\-i\\0\end{bmatrix}$$

自旋为1有质量的情况: $u^\mu(\vec{p},\sigma) = v^{\mu *}(\vec{p},\sigma) = (2p^0)^{-1/2} e^\mu(\vec{p},\sigma)$

$$e^\mu(\vec{p},\sigma) \equiv L^\mu{}_\nu(\vec{p}) e^\nu(0,\sigma) \qquad e^\mu(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad e^\mu(0,1) = \frac{-1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ i \\ 0 \end{bmatrix}, \quad e^\mu(0,-1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -i \\ 0 \end{bmatrix}$$

$$L(\vec{p}) = \begin{pmatrix} \gamma & \sqrt{\gamma^2-1}\sin\theta\cos\phi & \sqrt{\gamma^2-1}\sin\theta\sin\phi & \sqrt{\gamma^2-1}\cos\theta \\ \sqrt{\gamma^2-1}\sin\theta\cos\phi & 1+(\gamma-1)\sin^2\theta\cos^2\phi & (\gamma-1)\sin^2\theta\sin\phi\cos\phi & (\gamma-1)\sin\theta\cos\theta\cos\phi \\ \sqrt{\gamma^2-1}\sin\theta\sin\phi & (\gamma-1)\sin^2\theta\sin\phi\cos\phi & 1+(\gamma-1)\sin^2\theta\sin^2\phi & (\gamma-1)\sin\theta\cos\theta\sin\phi \\ \sqrt{\gamma^2-1}\cos\theta & (\gamma-1)\sin\theta\cos\theta\cos\phi & (\gamma-1)\sin\theta\cos\theta\sin\phi & 1+(\gamma-1)\cos^2\theta \end{pmatrix}$$

$$e^\mu(\vec{p},0) = \begin{bmatrix} \sqrt{\gamma^2-1}\cos\theta \\ (\gamma-1)\sin\theta\cos\theta\cos\phi \\ (\gamma-1)\sin\theta\cos\theta\sin\phi \\ 1+(\gamma-1)\cos^2\theta \end{bmatrix} \qquad e^\mu(\vec{p},\pm 1) = \mp\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\gamma^2-1}e^{\pm i\phi}\sin\theta \\ 1+(\gamma-1)e^{\pm i\phi}\cos\phi\sin^2\theta \\ \pm i+(\gamma-1)e^{\pm i\phi}\sin\phi\sin^2\theta \\ (\gamma-1)e^{\pm i\phi}\sin\theta\cos\theta \end{bmatrix}$$

按自旋分类

自旋为1有质量的情况: $u^\mu(\vec{p},\sigma) = v^{\mu *}(\vec{p},\sigma) = (2p^0)^{-1/2} e^\mu(\vec{p},\sigma)$

$$e^\mu(\vec{p},\sigma) \equiv L^\mu{}_\nu(\vec{p}) e^\nu(0,\sigma) \qquad e^\mu(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad e^\mu(0,1) = \frac{-1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ i \\ 0 \end{bmatrix}, \quad e^\mu(0,-1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -i \\ 0 \end{bmatrix}$$

$$\phi^{+\mu}(x) = \phi^{-\mu\dagger}(x) = (2\pi)^{-3/2} \sum_\sigma \int \frac{d\vec{p}}{\sqrt{2p^0}} e^\mu(\vec{p},\sigma) a(\vec{p},\sigma) e^{-ip\cdot x}$$

$$[\phi^{+\mu}(x), \phi^{-\nu}(y)]_\mp = \int \frac{d\vec{p}}{(2\pi)^3 2p^0} e^{-ip\cdot(x-y)} \Pi^{\mu\nu}(\vec{p}) = [-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{M^2}] \Delta_+(M, x-y)$$ 作业25

$$\Pi^{\mu\nu}(\vec{p}) \equiv \sum_\sigma e^\mu(\vec{p},\sigma) e^{\nu *}(\vec{p},\sigma) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}$$

为保证因果性，须把矢量粒子看成是玻色子，必须以组合的v^μ 构造$\tilde{\mathcal{H}}(x)$

- 对不带荷的矢量粒子 $v^\mu(x) \equiv \phi^{+\mu}(x) + \phi^{+\mu\dagger}(x)$
- 对带荷的矢量粒子须引入带相反荷的反粒子 $v^\mu(x) \equiv \phi^{+\mu}(x) + \phi^{+c\mu\dagger}(x)$

自旋为1有质量的情况: $u^\mu(\vec{p},\sigma) = v^{\mu *}(\vec{p},\sigma) = (2p^0)^{-1/2} e^\mu(\vec{p},\sigma)$

$$e^\mu(\vec{p},\sigma) \equiv L^\mu{}_\nu(\vec{p}) e^\nu(0,\sigma) \qquad e^\mu(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad e^\mu(0,1) = \frac{-1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ i \\ 0 \end{bmatrix}, \quad e^\mu(0,-1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -i \\ 0 \end{bmatrix}$$

$$\phi^{+\mu}(x) = \phi^{-\mu\dagger}(x) = (2\pi)^{-3/2} \sum_\sigma \int \frac{d\vec{p}}{\sqrt{2p^0}} e^\mu(\vec{p},\sigma) a(\vec{p},\sigma) e^{-ip\cdot x}$$

为保证因果性，须把矢量粒子看成是玻色子，必须以组合的v^μ构造$\tilde{\mathcal{H}}(x)$

$$v^\mu(x) = (2\pi)^{-3/2} \sum_\sigma \int \frac{d\vec{p}}{\sqrt{2p^0}} [e^\mu(\vec{p},\sigma) a(\vec{p},\sigma) e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma) a^{c\dagger}(\vec{p},\sigma) e^{ip\cdot x}]$$

不带荷的情形相当于$a^c(\vec{p},\sigma) = a(\vec{p},\sigma)$

$$[v^\mu(x), v^\nu(y)]_- = [v^\mu(x), v^{\nu\dagger}(y)]_- = -[g^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{M^2}] \Delta(M, x-y)$$ 作业26

按自旋分类

$$\phi^{+\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\; u^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} \qquad \phi^{-\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\; v^\mu(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}$$

$$u^\mu(\vec{p},\sigma)=(k^0/p^0)^{1/2}L(p)^\mu{}_\nu u^\nu(0,\sigma) \qquad u^\mu(\vec{k},\sigma)e^{i\theta\sigma}=R^\mu{}_\nu(\theta)u^\nu(\vec{k},\sigma) \qquad u^\mu(\vec{k},\sigma)=S^\mu{}_\nu u^\nu(\vec{k},\sigma)$$

$$v^\mu(\vec{p},\sigma)=(k^0/p^0)^{1/2}L(p)^\mu{}_\nu v^\nu(0,\sigma) \qquad v^\mu(\vec{k},\sigma)e^{-i\theta\sigma}=R^\mu{}_\nu(\theta)v^\nu(\vec{k},\sigma) \qquad v^\mu(\vec{k},\sigma)=S^\mu{}_\nu v^\nu(\vec{k},\sigma)$$

自旋为1无质量的情况： $j=1$ 有质量的无质量极限产生发散！

$$u^\mu(\vec{p},\sigma)=v^{\mu *}(\vec{p},\sigma)=(2p^0)^{-1/2}e^\mu(\vec{p},\sigma) \qquad e^\mu(\vec{p},\sigma)=L(p)^\mu{}_\nu e^\nu(\vec{k},\sigma)\text{ 的螺旋度指标取 } \pm 1 \qquad p^\mu=L(p)^\mu{}_\nu k^\nu$$

$$e^\mu(\vec{k},\sigma)e^{i\sigma\theta}=R^\mu{}_\nu(\theta)e^\nu(\vec{k},\sigma) \qquad e^\mu(\vec{k},\sigma)=S^\mu{}_\nu(\alpha,\beta)e^\nu(\vec{k},\sigma) \qquad e^\mu(\vec{k},\pm1)=\frac{1}{\sqrt{2}}(0,1,\pm i,0)$$

$$S^\mu{}_\nu(\alpha,\beta)=\begin{bmatrix}1+\zeta & \alpha & \beta & -\zeta\\ \alpha & 1 & 0 & -\alpha\\ \beta & 0 & 1 & -\beta\\ \zeta & \alpha & \beta & 1-\zeta\end{bmatrix} \qquad R^\mu{}_\nu(\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

自旋为1无质量的情况: $j=1$有质量的无质极限产生发散!

$$\phi^{+\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\; u^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} \quad \phi^{-\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\; v^\mu(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}$$

$u^\mu(\vec{p},\sigma)=v^{\mu*}(\vec{p},\sigma)=(2p^0)^{-1/2}e^\mu(\vec{p},\sigma)$ $\quad e^\mu(\vec{p},\sigma)=L(p)^\mu{}_\nu e^\nu(\vec{k},\sigma)$的螺旋度指标取 ±1 $\quad p^\mu=L(p)^\mu{}_\nu k^\nu$

$$e^\mu(\vec{k},\sigma)e^{i\sigma\theta}=R^\mu{}_\nu(\theta)e^\nu(\vec{k},\sigma) \qquad e^\mu(\vec{k},\sigma)=S^\mu{}_\nu(\alpha,\beta)e^\nu(\vec{k},\sigma) \qquad e^\mu(\vec{k},\pm1)=\frac{1}{\sqrt{2}}(0,1,\pm i,0)$$

$$S^\mu{}_\nu(\alpha,\beta)e^\nu(\vec{k},\pm1)=\frac{1}{\sqrt{2}}\begin{bmatrix}1+\zeta & \alpha & \beta & -\zeta\\ \alpha & 1 & 0 & -\alpha\\ \beta & 0 & 1 & -\beta\\ \zeta & \alpha & \beta & 1-\zeta\end{bmatrix}\begin{bmatrix}0\\1\\\pm i\\0\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\alpha\pm i\beta\\1\\\pm i\\\alpha\pm i\beta\end{bmatrix}$$

$$S^\mu{}_\nu(\alpha,\beta)e^\nu(\vec{k},\sigma)=e^\mu(\vec{k},\sigma)+(\alpha+i\sigma\beta)\frac{k^\mu}{\sqrt{2}|\vec{k}|} \qquad W(\theta,\alpha,\beta)=S(\alpha,\beta)R(\theta)$$

$$D^\mu{}_\nu(W(\theta,\alpha,\beta))e^\nu(\vec{k},\sigma)=S^\mu{}_\lambda(\alpha,\beta)R^\lambda{}_\nu(\theta)e^\nu(\vec{k},\sigma)=e^{i\sigma\theta}[e^\mu(\vec{k},\sigma)+\frac{\alpha+i\sigma\beta}{\sqrt{2}|\vec{k}|}k^\mu]$$

自旋为1无质量的情况:

$$\phi^{+\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\; u^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} \qquad \phi^{-\mu}(x)=\sum_\sigma(2\pi)^{-3/2}\int d\vec{p}\; v^\mu(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}$$

$$u^\mu(\vec{p},\sigma)=v^{\mu*}(\vec{p},\sigma)=(2p^0)^{-1/2}e^\mu(\vec{p},\sigma)$$

$e^\mu(\vec{p},\sigma)=L(p)^\mu{}_\nu e^\nu(\vec{k},\sigma)$ 的螺旋度指标取 ± 1 $\quad p^\mu=L(p)^\mu{}_\nu k^\nu$

$$e^\mu(\vec{k},\sigma)e^{i\sigma\theta}=R^\mu{}_\nu(\theta)e^\nu(\vec{k},\sigma) \qquad e^\mu(\vec{k},\sigma)=S^\mu{}_\nu(\alpha,\beta)e^\nu(\vec{k},\sigma) \qquad e^\mu(\vec{k},\pm1)=\frac{1}{\sqrt{2}}(0,1,\pm i,0)$$

$$S^\mu{}_\nu(\alpha,\beta)e^\nu(\vec{k},\sigma)=e^\mu(\vec{k},\sigma)+(\alpha+i\sigma\beta)k^\mu/(\sqrt{2}|\vec{k}|) \qquad W(\theta,\alpha,\beta)=S(\alpha,\beta)R(\theta)$$

$$D^\mu{}_\nu(W(\theta,\alpha,\beta))e^\nu(\vec{k},\sigma)=S^\mu{}_\lambda(\alpha,\beta)R^\lambda{}_\nu(\theta)e^\nu(\vec{k},\sigma)=e^{i\sigma\theta}[e^\mu(\vec{k},\sigma)+k^\mu(\alpha+i\sigma\beta)/(\sqrt{2}|\vec{k}|)]$$

$$\begin{aligned}D^\mu{}_\nu(\Lambda)e^\nu(\vec{p},\sigma)&=D^\mu{}_\nu(\Lambda L(p))e^\nu(\vec{k},\sigma)=D^\mu{}_\nu(L(\Lambda p)W)e^\nu(\vec{k},\sigma) \qquad W\equiv L^{-1}(\Lambda p)\Lambda L(p)\\&=D^\mu{}_\nu(L(\Lambda p))e^{i\sigma\theta(\Lambda,p)}[e^\nu(\vec{k},\sigma)+\{\alpha(\Lambda,p)+i\sigma\beta(\Lambda,p)\}k^\nu/(\sqrt{2}|\vec{k}|)]\\&=e^{i\sigma\theta(\Lambda,p)}[e^\mu(\vec{p}_\Lambda,\sigma)+p^\mu_\Lambda\{\alpha(\Lambda,p)+i\sigma\beta(\Lambda,p)\}/(\sqrt{2}|\vec{k}|)] \qquad p^\mu_\Lambda=\Lambda^\mu{}_\nu p^\nu\end{aligned}$$

$$e^{-i\sigma\theta(\Lambda,p)}e^\mu(\vec{p},\sigma)=D^\mu{}_\nu(\Lambda^{-1})e^\nu(\vec{p}_\Lambda,\sigma)+p^\mu\{\alpha(\Lambda,p)+i\sigma\beta(\Lambda,p)\}/(\sqrt{2}|\vec{k}|)$$

自旋为1无质量的情况:

$$v^\mu(x) = (2\pi)^{-3/2}\sum_{\sigma=\pm 1}\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

$$U(\Lambda)a(\vec{p},\sigma)U^{-1}(\Lambda)=\sqrt{\frac{(\Lambda p)^0}{p^0}}e^{-i\sigma\theta(\Lambda p)}a(\vec{p}_\Lambda,\sigma) \qquad U(\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(\Lambda)=\sqrt{\frac{(\Lambda p)^0}{p^0}}e^{i\sigma\theta(\Lambda p)}a^\dagger(\vec{p}_\Lambda,\sigma)$$

$$e^{-i\sigma\theta(\Lambda,p)}e^\mu(\vec{p},\sigma) = D^\mu{}_\nu(\Lambda^{-1})e^\nu(\vec{p}_\Lambda,\sigma) + p^\mu\{\alpha(\Lambda,p)+i\sigma\beta(\Lambda,p)\}/(\sqrt{2}|\vec{k}|)$$

$$U(\Lambda)v^\mu(x)U^{-1}(\Lambda)=(2\pi)^{-3/2}\sum_{\sigma=\pm 1}\int\frac{d\vec{p}}{\sqrt{2p^0}}\sqrt{\frac{(\Lambda p)^0}{p^0}}[e^\mu(\vec{p},\sigma)e^{-i\sigma\theta(\Lambda,p)}a(\vec{p}_\Lambda,\sigma)e^{-ip\cdot x}+\cdots]$$

$$=(2\pi)^{-3/2}\sum_{\sigma=\pm 1}\int\frac{d\vec{p}_\Lambda}{\sqrt{2p^0_\Lambda}}\{[D^\mu{}_\nu(\Lambda^{-1})e^\nu(\vec{p}_\Lambda,\sigma)+p^\mu\frac{\alpha(\Lambda,p)+i\sigma\beta(\Lambda,p)}{\sqrt{2}|\vec{k}|}]a(\vec{p}_\Lambda,\sigma)e^{-ip_\Lambda(\Lambda x)}+\cdots\}$$

$$= D^\mu{}_\nu(\Lambda^{-1})v^\nu(\Lambda x) + \partial^\mu\Omega(x,\Lambda) \qquad D^\mu{}_\nu(\Lambda^{-1}) = [D^\mu{}_\nu(\Lambda^{-1})]^* = (\Lambda^{-1})^\mu{}_\nu$$

$$\Omega(x,\Lambda)\equiv(2\pi)^{-\frac{3}{2}}i\sum_{\sigma=\pm 1}\int\frac{d\vec{p}}{\sqrt{2p^0}}\sqrt{\frac{(\Lambda p)^0}{p^0}}[\frac{\alpha(\Lambda,p)+i\sigma\beta(\Lambda,p)}{\sqrt{2}|\vec{k}|}a(\vec{p}_\Lambda,\sigma)e^{-ipx}-\frac{\alpha(\Lambda,p)-i\sigma\beta(\Lambda,p)}{\sqrt{2}|\vec{k}|}a^{c\dagger}(\vec{p}_\Lambda,\sigma)e^{ipx}]$$

自旋为1无质量的情况:

$$v^\mu(x) = (2\pi)^{-3/2}\sum_{\sigma=\pm 1}\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

$$U(\Lambda)a(\vec{p},\sigma)U^{-1}(\Lambda)=\sqrt{\frac{(\Lambda p)^0}{p^0}}e^{-i\sigma\theta(\Lambda p)}a(\vec{p}_\Lambda,\sigma) \quad U(\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(\Lambda)=\sqrt{\frac{(\Lambda p)^0}{p^0}}e^{i\sigma\theta(\Lambda p)}a^\dagger(\vec{p}_\Lambda,\sigma)$$

$$U(\Lambda)v^\mu(x)U^{-1}(\Lambda) = D^\mu{}_\nu(\Lambda^{-1})v^\nu(\Lambda x) + \partial^\mu\Omega(x,\Lambda) \qquad D^\mu{}_\nu(\Lambda^{-1}) = [D^\mu{}_\nu(\Lambda^{-1})]^* = (\Lambda^{-1})^\mu{}_\nu$$

$$\Omega(x,\Lambda)\equiv(2\pi)^{-\frac{3}{2}}i\sum_{\sigma=\pm 1}\int\frac{d\vec{p}}{\sqrt{2p^0}}\sqrt{\frac{(\Lambda p)^0}{p^0}}\left[\frac{\alpha(\Lambda,p)+i\sigma\beta(\Lambda,p)}{\sqrt{2}|\vec{k}|}a(\vec{p}_\Lambda,\sigma)e^{-ipx} - \frac{\alpha(\Lambda,p)-i\sigma\beta(\Lambda,p)}{\sqrt{2}|\vec{k}|}a^{c\dagger}(\vec{p}_\Lambda,\sigma)e^{ipx}\right]$$

洛仑兹转动不只给出时空转动结果,还多出全散度项.虽避免了单纯在有质量矢量场理论中取零质量极限导致的发散,但矢量场洛仑兹变换性质无法实现!

只有矢量场多一个全散度项不产生任何影响的理论能够保证实现理论本该具有的洛仑兹变换的性质, 它要求理论在如下变换下是不变的:

$$v^\mu(x) \to v^{\mu'}(x) = v^\mu(x) + \partial^\mu\Omega(x)$$ 冗余的自由度!

这个变换叫规范变换. 在规范变换下保持不变的理论叫规范理论.

规范对称性的引入回顾： $v^{\mu+}(x)=v^{\mu-\dagger}(x)=\sum_{\sigma,n}\int\frac{d^3p}{(2\pi)^3}e^{-ip\cdot x}u^{\mu}(\vec{p},\sigma)a(\vec{p},\sigma)$

$$\sum_{\bar{\sigma}}u^{\mu}(\vec{p}_{\Lambda},\bar{\sigma})D_{\bar{\sigma}\sigma}(W(\Lambda,p))=\sqrt{p^0/(\Lambda p)^0}\sum_{l}\Lambda^{\mu}{}_{\nu}u^{\nu}(\vec{p},\sigma)\quad D_{\bar{\sigma}\sigma}(W(\Lambda,p))=e^{i\theta(\Lambda,p)\sigma}\delta_{\bar{\sigma}\sigma}$$

$$u^{\mu}(\vec{p},\sigma)=v^{\mu*}(\vec{p},\sigma)=(2p^0)^{-1/2}e^{\mu}(\vec{p},\sigma)\quad\sum_{\bar{\sigma}}e^{\mu}(\vec{p}_{\Lambda},\bar{\sigma})D_{\bar{\sigma}\sigma}(W(\Lambda,p))=\sum_{l}\Lambda^{\mu}{}_{\nu}e^{\nu}(\vec{p},\sigma)$$

取$p=k,\quad\Lambda=L(q),\quad q=\Lambda p\ \Rightarrow\ L(p)=1\ \Rightarrow\ W(\Lambda,p)\equiv L^{-1}(\Lambda p)\Lambda L(p)=1$

$$e^{\mu}(\vec{q},\sigma)=\sum_{l}L^{\mu}{}_{\nu}(q)e^{\nu}(\vec{k},\sigma)$$

取：$p=k,\quad\Lambda=S(\alpha,\beta),\quad q=\Lambda p=k\ \Rightarrow\ e^{\mu}(\vec{k},\sigma)=\sum_{l}S^{\mu}{}_{\nu}(\alpha,\beta)e^{\nu}(\vec{k},\sigma)$

取：$p=k,\quad\Lambda=R(\theta),\quad q=\Lambda p=k\ \Rightarrow\ e^{\mu}(\vec{k},\sigma)e^{i\theta\sigma}=\sum_{l}R^{\mu}{}_{\nu}(\theta)e^{\nu}(\vec{k},\sigma)$

$$S^{\mu}{}_{\nu}(\alpha,\beta)e^{\nu}(\vec{k},\sigma)=e^{\mu}(\vec{k},\sigma)+(\alpha+i\sigma\beta)\frac{k^{\mu}}{\sqrt{2}|\vec{k}|}\ \Rightarrow\ U(\Lambda)v^{\mu}(x)U^{-1}(\Lambda)=(\Lambda^{-1})^{\mu}{}_{\nu}v^{\nu}(\Lambda x)+\partial^{\mu}\Omega(x,\Lambda)$$

空间反射变换:

$$v^{\mu}(x) = (2\pi)^{-3/2}\sum_{\sigma}\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^{\mu}(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

有质量情形: $L^{\mu}{}_{\nu}(-\vec{p}) = \mathcal{P}^{\mu}{}_{\rho}L^{\rho}{}_{\sigma}(\vec{p})\mathcal{P}^{\sigma}{}_{\nu}$ $\quad$ $e^{\mu}(\vec{p},\sigma) = L^{\mu}{}_{\nu}(\vec{p})e^{\nu}(0,\sigma)$

$$e^{\mu}(0,0) = \begin{bmatrix}0\\0\\0\\1\end{bmatrix},\quad e^{\mu}(0,1) = \frac{-1}{\sqrt{2}}\begin{bmatrix}0\\1\\i\\0\end{bmatrix},\quad e^{\mu}(0,-1) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-i\\0\end{bmatrix}$$

$$e^{\mu}(-\vec{p},\sigma) = -\mathcal{P}^{\mu}{}_{\nu}e^{\nu}(\vec{p},\sigma) \quad Pa(\vec{p},\sigma)P^{-1} = \eta^{*}a(-\vec{p},\sigma) \quad Pa^{c}(\vec{p},\sigma)P^{-1} = \eta^{c*}a^{c}(-\vec{p},\sigma)$$

为保证用空间反射态来构造$\tilde{\mathcal{H}}(x)$同样保证能够使其在类空区间相互对易,只能取

$$\eta^{c} = \eta^{*} \qquad \Rightarrow \qquad Pv^{\mu}(x)P^{-1} = -\eta^{*}\mathcal{P}^{\mu}{}_{\nu}v^{\nu}(\mathcal{P}x)$$

空间反射变换（续）：

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu*}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

零质量情形: $L^\mu{}_\nu(-\vec{p}) = \mathcal{P}^\mu{}_\rho L^\rho{}_\sigma(\vec{p})\mathcal{P}^\sigma{}_\nu$ $\qquad e^\mu(\vec{p},\sigma) = L^\mu{}_\nu(\vec{p})e^\nu(0,\sigma)$

$$e^\mu(\vec{k},1) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\i\\0\end{bmatrix} \qquad e^\mu(\vec{k},-1) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-i\\0\end{bmatrix}$$

$e^\mu(-\vec{p},-\sigma) = -\mathcal{P}^\mu{}_\nu e^{\nu*}(\vec{p},\sigma)$ $\quad Pa(\vec{p},\sigma)P^{-1} = \eta^*_\sigma e^{\pm i\pi\sigma}a(-\vec{p},-\sigma)$ $\quad Pa^c(\vec{p},\sigma)P^{-1} = \eta^c_\sigma e^{\mp i\pi\sigma}a^c(-\vec{p},-\sigma)$

为保证用空间反射态来构造$\tilde{\mathcal{H}}(x)$同样保证能够使其在类空区间相互对易，只能取

$$\eta^c = \eta^* \qquad \Rightarrow \qquad Pv^\mu(x)P^{-1} = -\eta^*\mathcal{P}^\mu{}_\nu v^\nu(\mathcal{P}x)$$

时间反演变换:

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu*}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

有质量的情形: $L^\mu{}_\nu(-\vec{p}) = \mathcal{P}^\mu{}_\rho L^\rho{}_\sigma(\vec{p})\mathcal{P}^\sigma{}_\nu \qquad e^\mu(\vec{p},\sigma) = L^\mu{}_\nu(\vec{p})e^\nu(0,\sigma)$

$$e^\mu(0,0) = \begin{bmatrix}0\\0\\0\\1\end{bmatrix},\quad e^\mu(0,1) = \frac{-1}{\sqrt{2}}\begin{bmatrix}0\\1\\i\\0\end{bmatrix},\quad e^\mu(0,-1) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-i\\0\end{bmatrix}$$

$$e^{\mu*}(0,\sigma) = (-1)^\sigma e^\mu(0,-\sigma) \qquad (-1)^{1+\sigma}e^{\mu*}(-\vec{p},-\sigma) = \mathcal{P}^\mu{}_\nu e^\nu(\vec{p},\sigma)$$

$$Ta(\vec{p},\sigma)T^{-1} = (-1)^{1-\sigma}a(-\vec{p},-\sigma) \qquad Ta^c(\vec{p},\sigma)T^{-1} = (-1)^{1-\sigma}a^c(-\vec{p},-\sigma)$$

为保证用时间反演态来构造$\tilde{\mathcal{H}}(x)$同样保证能够使其在类空区间相互对易,只能取

$$Tv^\mu(x)T^{-1} = \mathcal{P}^\mu{}_\nu v^\nu(-\mathcal{P}x)$$

时间反演变换（续）：

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

零质量的情形： $L^\mu{}_\nu(-\vec{p})=\mathcal{P}^\mu{}_\rho L^\rho{}_\sigma(\vec{p})\mathcal{P}^\sigma{}_\nu$ $\quad e^\mu(\vec{p},\sigma)=L^\mu{}_\nu(\vec{p})e^\nu(0,\sigma)$

$$e^\mu(\vec{k},1) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\i\\0\end{bmatrix}, \qquad e^\mu(\vec{k},-1) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-i\\0\end{bmatrix}$$

$$e^\mu(-\vec{p},-\sigma)=-\mathcal{P}^\mu{}_\nu e^{\nu *}(\vec{p},\sigma) \quad Ta(\vec{p},\sigma)T^{-1}=\zeta^*_\sigma e^{\mp i\pi\sigma}a(-\vec{p},\sigma) \quad Ta^c(\vec{p},\sigma)T^{-1}=\zeta^c_\sigma e^{\pm i\pi\sigma}a^{c\dagger}(-\vec{p},\sigma)$$

为保证用时间反演态来构造$\tilde{\mathcal{H}}(x)$同样保证能够使其在类空区间相互对易，只能取

$$\zeta^c = \zeta^* \qquad \Rightarrow \qquad Tv^\mu(x)T^{-1} = \zeta^*\mathcal{P}^\mu{}_\nu v^\nu(-\mathcal{P}x)$$

电荷共轭变换与CPT联合变换:

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma \int \frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

$$L^\mu{}_\nu(-\vec{p}) = \mathcal{P}^\mu{}_\rho L^\rho{}_\sigma(\vec{p})\mathcal{P}^\sigma{}_\nu \qquad e^\mu(\vec{p},\sigma) = L^\mu{}_\nu(\vec{p})e^\nu(0,\sigma)$$

$$e^\mu(0,0) = \begin{bmatrix} 0\\0\\0\\1\end{bmatrix}, \quad e^\mu(0,1) = -e^\mu(\vec{k},1) = \frac{-1}{\sqrt{2}}\begin{bmatrix} 0\\1\\i\\0\end{bmatrix}, \quad e^\mu(0,-1) = e^\mu(\vec{k},-1) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0\\1\\-i\\0\end{bmatrix}$$

$$Ca(\vec{p},\sigma)C^{-1} = \xi^* a^c(\vec{p},\sigma) \qquad Ca^c(\vec{p},\sigma)C^{-1} = \xi^{c*}a(\vec{p},\sigma)$$

为保证用电荷共轭态来构造$\tilde{\mathcal{H}}(x)$同样保证能够使其在类空区间相互对易,只能取

$$\xi^c = \xi^* \qquad \Rightarrow \qquad Cv^\mu(x)(x)C^{-1} = \xi^* v^{\mu\dagger}(x)$$

$$Pv^\mu(x)(x)P^{-1} = -\eta^*\mathcal{P}^\mu{}_\nu v^\nu(\mathcal{P}x) \qquad Tv^\mu(x)(x)T^{-1} = \zeta^*\mathcal{P}^\mu{}_\nu v^\nu(-\mathcal{P}x)$$

$$CPTv^\mu(x)[CPT]^{-1} = -\xi^*\eta^*\zeta^* v^{\mu\dagger}(-x)$$

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma \int \frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

$$p^\mu p_\mu = M^2 \qquad (\partial^2 + M^2)v^\mu(x) = 0$$

有质量的矢量场:

$$e^\mu(\vec{p},\sigma) = L^\mu{}_\nu(\vec{p})e^\nu(0,\sigma) \qquad e^\mu(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad e^\mu(0,1) = \frac{-1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ i \\ 0 \end{bmatrix}, \quad e^\mu(0,-1) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -i \\ 0 \end{bmatrix}$$

$$L^i_k(p) = \delta_{ik} + (\gamma - 1)\hat{p}_i\hat{p}_k \qquad L^i{}_0(p) = L^0{}_i(p) = \frac{p^i}{M} \qquad L^0{}_0(p) = \gamma \equiv \frac{\sqrt{\vec{p}^2 + M^2}}{M} \qquad \hat{p}_i \equiv \frac{p^i}{|\vec{p}|}$$

$$p_\mu L^\mu{}_k(\vec{p}) = p^0\frac{p^k}{M} - p^i\left(\delta_{ik} + (\gamma - 1)\frac{p^i p^k}{\vec{p}\cdot\vec{p}}\right) = p^0\frac{p^k}{M} - p^k + (1-\gamma)p^k = 0$$

$$p_\mu e^\mu(\vec{p},\sigma) = p_\mu L^\mu{}_k(\vec{p})e^k(0,\sigma) + p_\mu L^\mu{}_0(\vec{p})e^0(0,\sigma) = 0 \quad \Rightarrow \quad \partial_\mu v^\mu(x) = 0$$

Note: 虽然 $e^0(0,\sigma) = 0$，但$e^0(\vec{p},\sigma) \neq 0$, 因此$v^0(x) \neq 0$

$v^\mu(x)$的四个分量不都是独立分量，通常取三个空间分量为独立变量

自由矢量场的场方程、哈密顿量和作用量、正则对易关系

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

$$p^\mu p_\mu = M^2 \quad (\partial^2 + M^2)v^\mu(x) = 0$$

有质量的矢量场: $p_\mu e^\mu(\vec{p},\sigma) = 0 \quad \Rightarrow \quad \partial_\mu v^\mu(x) = 0$

$v^\mu(x)$四个分量不都是独立的，通常取三个空间分量为独立变量.三个独立自由度

无质量的矢量场: $e^\mu(\vec{p},\sigma) = L(p)^\mu{}_\nu e^\nu(\vec{k},\sigma) \quad p^\mu = L(p)^\mu{}_\nu k^\nu \quad L(p) = R(\hat{p})B(\frac{|\vec{p}|}{\kappa})$

$B^\mu{}_\nu(\frac{|\vec{p}|}{\kappa})e^\nu(\vec{k},\sigma) = e^\mu(\vec{k},\sigma)$ B是沿z轴的boost,不影响只有x,y分量的$e^\mu(\vec{k},\sigma)$ $\quad e^\mu(\vec{k},\pm 1) = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$

$$e^\mu(\vec{p},\sigma) = [R(\hat{p})B(\frac{|\vec{p}|}{\kappa})]^\mu{}_\nu e^\nu(\vec{k},\sigma) = R^\mu{}_\nu(\hat{p})e^\nu(\vec{k},\sigma) \qquad R^0{}_0 = 1 \quad R^0{}_i = R^i{}_0 = 0$$

$$e^0(\vec{p},\sigma) = R^0{}_\nu(\hat{p})e^\nu(\vec{k},\sigma) = e^0(\vec{k},\sigma) = 0 \quad \Rightarrow \quad v^0(x) = 0$$

$$p_\mu e^\mu(\vec{p},\sigma) = (L(p)k)_\mu(L(p)e(\vec{k},\sigma))^\mu = k_\mu e^\mu(\vec{k},\sigma) = 0 \quad \Rightarrow \quad \vec{p}\cdot\vec{e}(\vec{p},\sigma) = 0 \quad \Rightarrow \quad \nabla\cdot\vec{v}(x) = 0$$

现在约束条件是两个, 比有质量的矢量场多了一个约束条件.两个独立自由度

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

自共轭自由矢量场作业27的哈密顿量和拉格朗日量:

$$H_0' = \sum_\sigma \int d\vec{p}\ \ a^\dagger(\vec{p},\sigma)a(\vec{p},\sigma)\sqrt{\vec{p}^2 + M^2} = H_0$$ 作业32

$$= \sum_\sigma \int d\vec{p}\ \frac{1}{2}\sqrt{\vec{p}^2 + M^2} : \{a^\dagger(\vec{p},\sigma)a(\vec{p},\sigma) + a(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)\} :$$

$$\dot{v}^\mu(x) = i\sum_\sigma\int\frac{d\vec{p}}{(2\pi)^{3/2}}\sqrt{\frac{p^0}{2}}[-e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$\nabla v^\mu(x) = i\sum_\sigma\int\frac{d\vec{p}}{(2\pi)^{3/2}}\frac{\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} - e^{\mu *}(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$H_0 = -\int d\vec{x}\,\frac{1}{2} : [\dot{v}^2(x) + (\nabla v_\mu(x))\cdot(\nabla v^\mu(x)) + M^2 v^2(x)] :$$

$$e^\mu(\vec{p},\sigma)e^*_\mu(\vec{p},\sigma') = -\delta_{\sigma\sigma'}$$ 作业28

$$= \int d\vec{x}\,\frac{-1}{2} : \{-\dot{\vec{v}}^2(x) + [\nabla\cdot\vec{v}(x)]^2 + [\nabla v_\mu(x)]\cdot[\nabla v^\mu(x)] + M^2 v^2(x)\} :$$

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu*}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

自共轭自由矢量场的哈密顿量和拉格朗日量: $(\partial^2 + M^2)v^\mu(x) = 0$

$$\dot{v}^\mu(x) = i\sum_\sigma\int\frac{d\vec{p}}{(2\pi)^{3/2}}\sqrt{\frac{p^0}{2}}[-e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu*}(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$\nabla v^\mu(x) = i\sum_\sigma\int\frac{d\vec{p}}{(2\pi)^{3/2}}\frac{\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} - e^{\mu*}(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$H_0 = \int d\vec{x}\,(-1/2) : \{-\dot{\vec{v}}^2(x) + [\nabla\cdot\vec{v}(x)]^2 + [\nabla v_\mu(x)]\cdot[\nabla v^\mu(x)] + M^2v^2(x)\} :$$

$$-\dot{\vec{\pi}}(x) \equiv \frac{\delta H_0}{\delta\vec{v}(x)}\Big|_{\pi\text{固定}} = -\nabla[\nabla\cdot\vec{v}(x)] - (\nabla^2 - M^2)\vec{v}(x) = \nabla[\dot{v}_0(x)] - \ddot{\vec{v}}(x) \Rightarrow \vec{\pi}(x) = \dot{\vec{v}}(x) - \nabla v_0(x)$$

$$S_0 = \int d^4x\ :\vec{\pi}(x)\cdot\dot{\vec{v}}(x): - \int dt\, H_0 = \int d^4x\,\frac{1}{2} : \{-[\partial_\mu v_\nu(x)][\partial^\mu v^\nu(x)] + M^2v^2(x)\} :$$

$$= \int d^4x\ : \{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}[\partial_\mu v^\mu(x)]^2 + \frac{1}{2}M^2v^2(x)\} : \qquad F_{\mu\nu}(x) \equiv \partial_\mu v_\nu(x) - \partial_\nu v_\mu(x)$$

略去边界积分，将S_0取极值就得到场方程 $(\partial^2 + M^2)v^\mu(x) = 0$

自由矢量场的场方程、哈密顿量和作用量、正则对易关系

$$\vec{\pi}(x) = \dot{\vec{v}}(x) - \nabla v_0(x)$$

$$H_0 = \int d\vec{x}\, (-1/2) : \{-\dot{\vec{v}}^2(x) + [\nabla\cdot\vec{v}(x)]^2 + [\nabla v_\mu(x)]\cdot[\nabla v^\mu(x)] + M^2 v^2(x)\} :$$

$$= -\frac{1}{2}\int d\vec{x}\ : \{-\dot{\vec{v}}^2(x) + [\nabla v_0(x)]\cdot[\nabla v_0(x)] + [\nabla\cdot\vec{v}(x)]^2 + [\nabla v_i(x)]\cdot[\nabla v^i(x)] + M^2 v^2(x)\} :$$

$$S_0 = \int d^4x\ : \vec{\pi}(x)\cdot\dot{\vec{v}}(x) : - \int dt\, H_0$$

$$= \frac{1}{2}\int d^4x\ : \{2\dot{\vec{v}}^2(x) - 2\dot{\vec{v}}(x)\cdot\nabla v_0(x) - \dot{\vec{v}}^2(x) + [\nabla\cdot\vec{v}(x)]^2 + [\nabla v_\mu(x)]\cdot[\nabla v^\mu(x)] + M^2 v^2(x)\} :$$

$$= \frac{1}{2}\int d^4x\ : \{\dot{\vec{v}}^2(x) - 2\nabla\cdot\vec{v}(x)\dot{v}_0(x) + [\nabla\cdot\vec{v}(x)]^2 + [\nabla v_\mu(x)]\cdot[\nabla v^\mu(x)] + M^2 v^2(x)\} :$$

$$= \frac{1}{2}\int d^4x\ : \{\dot{\vec{v}}^2(x) - \dot{v}_0^2(x) + [\nabla v_\mu(x)]\cdot[\nabla v^\mu(x)] + M^2 v^2(x)\} :$$

$$= \frac{1}{2}\int d^4x\ : \{-[\partial_0 v_\mu(x)][\partial_0 v^\mu(x)] + [\nabla v_\mu(x)]\cdot[\nabla v^\mu(x)] + M^2 v^2(x)\} :$$

$$= \frac{1}{2}\int d^4x\ : \{-[\partial_\mu v_\nu(x)][\partial^\mu v^\nu(x)] + M^2 v^2(x)\} :$$

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu*}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

自共轭自由矢量场的哈密顿量和拉格朗日量:

$$\dot{v}^\mu(x) = i\sum_\sigma\int\frac{d\vec{p}}{(2\pi)^{3/2}}\sqrt{\frac{p^0}{2}}[-e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu*}(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$\nabla v^\mu(x) = i\sum_\sigma\int\frac{d\vec{p}}{(2\pi)^{3/2}}\frac{\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} - e^{\mu*}(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$S_0 = \int d^4x \ :\{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}[\partial_\mu v^\mu(x)]^2 + \frac{1}{2}M^2v^2(x)\}: \qquad F_{\mu\nu}(x)\equiv\partial_\mu v_\nu(x)-\partial_\nu v_\mu(x)$$

$$S_0 = \int d^4x\,\mathcal{L}_0 \qquad \mathcal{L}_0 =: \{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}[\partial_\mu v^\mu(x)]^2 + \frac{1}{2}M^2v^2(x)\}:$$

$$\mathcal{L}_0 = 1/2 : \{[\dot{\vec{v}}^2(x)] - \dot{\vec{v}}(x)\cdot\nabla v_0(x) + \nabla v^0(x)\cdot\nabla v^0(x) - [\partial_i v_j(x)][\partial^i v^j(x)] + M^2v^2(x)\}:$$

$$\pi^i(x) = \frac{\partial\mathcal{L}_0}{\partial\dot{v}^i(x)} = \dot{v}^i(x) - \partial^i v_0(x)$$

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

自共轭自由矢量场的哈密顿量和拉格朗日量:

$$\dot{v}^\mu(x) = i\sum_\sigma\int\frac{d\vec{p}}{(2\pi)^{3/2}}\sqrt{\frac{p^0}{2}}[-e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$\nabla v^\mu(x) = i\sum_\sigma\int\frac{d\vec{p}}{(2\pi)^{3/2}}\frac{\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} - e^{\mu *}(\vec{p},\sigma)a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$S_0 = \int d^4x\,\mathcal{L}_0 \qquad \mathcal{L}_0 =: \{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}[\partial_\mu v^\mu(x)]^2 + \frac{1}{2}M^2v^2(x)\}:$$

$$F_{\mu\nu}(x)\equiv\partial_\mu v_\nu(x)-\partial_\nu v_\mu(x) \qquad \pi^i(x) = \frac{\partial\mathcal{L}_0}{\partial\dot{v}^i(x)} = \dot{v}^i(x) - \partial^i v_0(x)$$

$$[v^i(\vec{x},t),\pi^j(\vec{y},t)]_- = i\delta^{ij}\delta(\vec{x}-\vec{y}) \qquad [v^i(\vec{x},t),v^j(\vec{y},t)]_- = [\pi^i(\vec{x},t),\pi^j(\vec{y},t)]_- = 0$$ 作业29

利用对易关系, 可以证明量子情形下的正则场方程 作业30

$$\dot{\vec{v}}(\vec{x},t) = i[H_0,\vec{v}(\vec{x},t)] = \frac{\delta H_0}{\delta\vec{\pi}(\vec{x},t)} \qquad \dot{\vec{\pi}}(\vec{x},t) = i[H_0,\vec{\pi}(\vec{x},t)] = -\frac{\delta H_0}{\delta\vec{v}(\vec{x},t)}$$

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu*}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

自共轭自由矢量场的哈密顿量和拉格朗日量:

$$S_0 = \int d^4x\,\mathcal{L}_0 \qquad \mathcal{L}_0 =: \{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}[\partial_\mu v^\mu(x)]^2 + \frac{1}{2}M^2v^2(x)\}:$$

$$F_{\mu\nu}(x) \equiv \partial_\mu v_\nu(x) - \partial_\nu v_\mu(x) \qquad \pi^i(x) = \frac{\partial\mathcal{L}_0}{\partial\dot{v}^i(x)} = \dot{v}^i(x) - \partial^i v_0(x)$$

关于约束条件 $\partial_\mu v^\mu(x) = 0$ 从原拉格朗日量无法得到 及 π^0

$$\mathcal{L}_0' =: \{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2}M^2v^2(x)\}: \qquad \mathcal{L}_0' \text{ 不含 } \dot{v}_0 \Rightarrow \pi^0 = \frac{\partial\mathcal{L}_0}{\partial\dot{v}^0(x)} = 0$$

v^0不是动力学变量!

$$\frac{\partial\mathcal{L}_0'}{\partial v_\mu(x)} - \partial_\nu\frac{\partial\mathcal{L}_0'}{\partial\partial_\nu v_\mu(x)} = 0 \Rightarrow M^2v^\mu + \partial_\nu(\partial^\nu v^\mu - \partial^\mu v^\nu) = 0 \Rightarrow M^2\partial_\mu v^\mu = 0$$

$$\xRightarrow{\text{非零质量}} \partial_\mu v^\mu = 0 \qquad (\partial^2 + M^2)v^\mu = 0$$

v^μ作为基本场

$$\xRightarrow{\text{零质量}} \partial_\mu F^{\mu\nu} = 0 \qquad F^{\mu\nu} = \partial^\mu v^\nu - \partial^\nu v^\mu \qquad v^\mu \to v^{\mu\prime} = v^\mu + \partial^\mu\Omega$$

v^μ不显式出现，$F_{\mu\nu}$作为基本场

$$v^\mu(x) = (2\pi)^{-3/2}\sum_\sigma\int\frac{d\vec{p}}{\sqrt{2p^0}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x} + e^{\mu*}(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}]$$

自共轭自由矢量场的哈密顿量和拉格朗日量:

$$S_0 = \int d^4x\,\mathcal{L}_0 \qquad \mathcal{L}_0 =: \{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}[\partial_\mu v^\mu(x)]^2 + \frac{1}{2}M^2v^2(x)\} :$$

$$F_{\mu\nu}(x) \equiv \partial_\mu v_\nu(x) - \partial_\nu v_\mu(x) \qquad \pi^i(x) = \frac{\partial\mathcal{L}_0}{\partial\dot{v}^i(x)} = \dot{v}^i(x) - \partial^i v_0(x)$$

推广的处理： 关于约束条件 $\partial_\mu v^\mu(x) = 0$ 从原拉格朗日量无法得到

$$\mathcal{L}_0\text{''} =: \{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{\lambda}{2}[\partial_\mu v^\mu(x)]^2 + \frac{1}{2}M^2v^2(x)\} : \qquad \frac{\partial\mathcal{L}_0\text{''}}{\partial\lambda} = 0 \Rightarrow \partial_\mu v^\mu(x) = 0$$

$$\frac{\partial\mathcal{L}_0\text{''}}{\partial v_\mu(x)} - \partial_\nu\frac{\partial\mathcal{L}_0\text{''}}{\partial\partial_\nu v_\mu(x)} = 0 \Rightarrow M^2v^\mu + \partial_\nu[\partial^\nu v^\mu - (1-\lambda)\partial^\mu v^\nu] = 0 \Rightarrow (\partial^2 + M^2)v^\mu = 0$$

原始导出的拉氏量$\mathcal{L}_0$对应 $\lambda = 1$ 但导不出$v^0 = 0$； 修改的拉氏量$\mathcal{L}_0'$对应 $\lambda = 0$

矢量场的地位:

♣ 是各种非平庸量子场中有经典对应的最简单的场!

♢ 零质量矢量场要求规范对称性!

♡ 有质矢量场对质量零点连续要求规范对称性的破坏与质量相关!

♠ 现实世界已发现除引力外所有物质间相互作用都由矢量场描述!

¶ 规范对称性是决定物质基本相互作用的对称性!

✠ 时空对称性决定物质的相互作用?

如何理解标量、旋量场无质量奇异，而矢量场有质量奇异性？

♣ 因为有无质量的标量、旋量场对应的态的数目无差别!

♠ 而有无质量的矢量场对应的态的数目是有差别的!

✠ 更高自旋的量子场有无质量的态的数目都有差别！

¶ 更高自旋的量子场应该都具有质量的奇异性！

♡ 因此更高自旋的零质量量子场需要规范对称性 消除这种奇异性！

♢ 高自旋的零质量量子场的在壳纲领是量子场论新发展的核心！

非奇次洛伦兹群的一般不可约表示

$$i[\mathcal{J}^{\mu\nu},\mathcal{J}^{\rho\sigma}]=g^{\nu\rho}\mathcal{J}^{\mu\sigma}-g^{\mu\rho}\mathcal{J}^{\nu\sigma}-g^{\sigma\mu}\mathcal{J}^{\rho\nu}+g^{\sigma\nu}\mathcal{J}^{\rho\mu}$$

$$\vec{\mathcal{J}}\equiv\{\mathcal{J}^1,\mathcal{J}^2,\mathcal{J}^3\}=\{-\mathcal{J}_1,-\mathcal{J}_2,-\mathcal{J}_3\}=\{-\mathcal{J}^{23},-\mathcal{J}^{31},-\mathcal{J}^{12}\}$$

$$\vec{\mathcal{K}}\equiv\{\mathcal{K}^1,\mathcal{K}^2,\mathcal{K}^3\}=\{-\mathcal{K}_1,-\mathcal{K}_2,-\mathcal{K}_3\}=\{-\mathcal{J}^{10},-\mathcal{J}^{20},-\mathcal{J}^{30}\}$$

$$[\mathcal{J}^i,\mathcal{J}^j]=i\epsilon_{ijk}\mathcal{J}^k \qquad [\mathcal{J}^i,\mathcal{K}^j]=i\epsilon_{ijk}\mathcal{K}^k \qquad [\mathcal{K}^i,\mathcal{K}^j]=-i\epsilon_{ijk}\mathcal{J}^k$$

$$\mathcal{A}^i\equiv\frac{1}{2}(\mathcal{J}^i+i\mathcal{K}^i) \qquad \mathcal{B}^i\equiv\frac{1}{2}(\mathcal{J}^i-i\mathcal{K}^i)$$

若要求$\mathcal{A},\mathcal{B}$是厄米的，则 $\mathcal{J}$必须厄米，$\mathcal{K}$必须反厄米

可证，上节矢量场给出的厄米$\vec{\mathcal{J}}_{\rm vec}$和$\vec{\mathcal{K}}_{\rm vec}$与本节定义的厄米$\vec{\mathcal{J}}$反厄米$\vec{\mathcal{K}}$之间满足：$\vec{\mathcal{J}}_{\rm vec}=Q\vec{\mathcal{J}}Q^{-1}$, $\vec{\mathcal{K}}_{\rm vec}=Q\vec{\mathcal{K}}Q^{-1}$

$$[\mathcal{A}^i,\mathcal{A}^j]=i\epsilon_{ijk}\mathcal{A}^k \qquad [\mathcal{B}^i,\mathcal{B}^j]=i\epsilon_{ijk}\mathcal{B}^k \qquad [\mathcal{A}^i,\mathcal{B}^j]=0$$

两个相互独立的自旋分别为A和B整数或半整数 的不耦合的粒子对的直和

$$(\vec{\mathcal{A}})_{a'b',ab}=\delta_{b'b}\vec{J}^{(A)}_{a'a} \qquad (\vec{\mathcal{B}})_{a'b',ab}=\delta_{a'a}\vec{J}^{(B)}_{b'b}$$

(A,B)表示的维数$(2A+1)(2B+1)$

$$(J^{(A),3})_{a'a}=a\delta_{a'a} \qquad (J^{(A),1}\pm iJ^{(A),2})_{a'a}=\delta_{a',a\pm1}\sqrt{(A\mp a)(A\pm a+1)} \qquad a=-A,-A+1,\cdots,+A$$

$$(J^{(B),3})_{b'b}=b\delta_{b'b} \qquad (J^{(B),1}\pm iJ^{(B),2})_{b'b}=\delta_{b',b\pm1}\sqrt{(B\mp b)(B\pm b+1)} \qquad b=-B,-B+1,\cdots,+B$$

宇称变换：$(A,B)\overset{\text{宇称变换}}{\Longleftarrow\!\!=\!\!=\!\!=\!\!\Longrightarrow}(B,A)$

$$\beta\mathcal{J}\beta^{-1}=\mathcal{J} \qquad \beta\mathcal{K}\beta^{-1}=-\mathcal{K} \qquad \Rightarrow \qquad \beta\mathcal{A}\beta^{-1}=\mathcal{B} \qquad \beta\mathcal{B}\beta^{-1}=\mathcal{A}$$

非奇次洛伦兹群的一般不可约表示

$\mathcal{A}^i \equiv \frac{1}{2}(\mathcal{J}^i + i\mathcal{K}^i)$ $\mathcal{B}^i \equiv \frac{1}{2}(\mathcal{J}^i - i\mathcal{K}^i)$ 若要求$\mathcal{A}, \mathcal{B}$是厄米的，则$\mathcal{J}$必须厄米，$\mathcal{K}$必须反厄米

$[\mathcal{A}^i, \mathcal{A}^j] = i\epsilon_{ijk}\mathcal{A}^k$ $[\mathcal{B}^i, \mathcal{B}^j] = i\epsilon_{ijk}\mathcal{B}^k$ $[\mathcal{A}^i, \mathcal{B}^j] = 0$

$(A, B) \overset{\text{宇称变换}}{\Longleftarrow\!=\!=\!=\!=\!\Longrightarrow} (B, A) \quad \Leftrightarrow \quad \beta\mathcal{A}\beta^{-1} = \mathcal{B} \quad \beta\mathcal{B}\beta^{-1} = \mathcal{A}$

$\vec{\mathcal{J}} = \vec{\mathcal{A}} + \vec{\mathcal{B}} \Rightarrow$ (A, B)表示的空间转动表现为具有自旋 j 的场 $j = A+B, A+B-1, \cdots, |A-B|$

$j = 0$ $(0, 0)$场

$j = \frac{1}{2}$ $(\frac{1}{2}, 0)$或$(0, \frac{1}{2})$场讨论见后 **Dirac**旋量$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$具有确定的宇称

$j = 1$ $(\frac{1}{2}, \frac{1}{2})$ 讨论见后，$(1, 0)$或$(0, 1)$场

- $(\frac{1}{2}, \frac{1}{2})$由$j=1$和$j=0$的场组成,分别对应$(\vec{v}, v^0)$与标量场有关系吗？有确定宇称
- $(1, 0)$和$(0, 1)$分别对应自对偶和反自对偶的反对称二阶张量
- $(1, 0) \oplus (0, 1)$具有确定的宇称

(A, A)包含$j = 2A, 2A-1, \cdots, 0$的场，只对有质量非平庸。无质量讨论见后

判断我们讨论过的旋量场：

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad \mathcal{J}^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad P_L \equiv \frac{1-\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \qquad P_R \equiv \frac{1+\gamma_5}{2} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma^j\gamma^k = \begin{pmatrix} -\sigma^j\sigma^k & 0 \\ 0 & -\sigma^j\sigma^k \end{pmatrix} \Rightarrow \mathcal{J}^i = -\frac{1}{2}\epsilon_{ijk}\mathcal{J}^{jk} = \frac{i}{8}\epsilon_{ijk}[\gamma^j, \gamma^k] = \frac{1}{4}\epsilon_{ijk}\epsilon_{ljk}\begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix} = \begin{pmatrix} \frac{\sigma^i}{2} & 0 \\ 0 & \frac{\sigma^i}{2} \end{pmatrix}$$

$$[\gamma^0, \gamma^i] = 2\gamma^0\gamma^i = \begin{pmatrix} 2\sigma^i & 0 \\ 0 & -2\sigma^i \end{pmatrix} \qquad \mathcal{K}^i = \mathcal{J}^{0i} = -\frac{i}{4}[\gamma^0, \gamma^i] = \begin{pmatrix} -i\frac{\sigma^i}{2} & 0 \\ 0 & i\frac{\sigma^i}{2} \end{pmatrix}$$

$$\mathcal{A}^i \equiv \frac{1}{2}(\mathcal{J}^i + i\mathcal{K}^i) = \begin{pmatrix} \frac{\sigma^i}{2} & 0 \\ 0 & 0 \end{pmatrix} = P_R\mathcal{J}^i \qquad \mathcal{B}^i \equiv \frac{1}{2}(\mathcal{J}^i - i\mathcal{K}^i) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma^i}{2} \end{pmatrix} = P_L\mathcal{J}^i$$

$$\mathcal{J}^2 = \frac{3}{4}I = \mathcal{A}^2 + \mathcal{B}^2 \qquad \mathcal{A}^2 = P_R\mathcal{J}^2 = P_R\frac{1}{2}(\frac{1}{2}+1) \qquad \mathcal{B}^2 = P_L\mathcal{J}^2 = P_L\frac{1}{2}(\frac{1}{2}+1) \qquad \mathcal{A}^i\mathcal{B}^j = 0$$

我们讨论过的旋量场属于$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$场

判断我们讨论过的矢量场：

$$\begin{matrix}(\mathcal{A}^2)^\mu{}_\nu \\ (\mathcal{B}^2)^\mu{}_\nu\end{matrix} = \frac{1}{4}(\vec{\mathcal{J}}^2 - \vec{\mathcal{K}}^2 \pm 2i\vec{\mathcal{J}}\cdot\vec{\mathcal{K}})^\mu{}_\nu \qquad (\mathcal{J}^{\sigma\rho})^\mu{}_\nu = i(g^\sigma{}_\nu g^{\rho\mu} - g^\rho{}_\nu g^{\sigma\mu})$$

$$= \frac{1}{4}[(\mathcal{J}^{23})^2+(\mathcal{J}^{31})^2+(\mathcal{J}^{12})^2-(\mathcal{J}^{10})^2-(\mathcal{J}^{20})^2-(\mathcal{J}^{30})^2\pm 2i(\mathcal{J}^{23}\mathcal{J}^{10}+\mathcal{J}^{31}\mathcal{J}^{20}+\mathcal{J}^{12}\mathcal{J}^{30})]^\mu{}_\nu$$

$$= \frac{1}{16}\epsilon_{ijk}\epsilon_{ilm}(\mathcal{J}^{jk})^\mu{}_\lambda(\mathcal{J}^{lm})^\lambda{}_\nu - \frac{1}{4}(\mathcal{J}^{i0})^\mu{}_\lambda(\mathcal{J}^{i0})^\lambda{}_\nu \pm \frac{i}{4}\epsilon_{ijk}(\mathcal{J}^{i0})^\mu{}_\lambda(\mathcal{J}^{jk})^\lambda{}_\nu$$

$$= \frac{1}{8}(\mathcal{J}^{jk})^\mu{}_\lambda(\mathcal{J}^{jk})^\lambda{}_\nu - \frac{1}{4}(\mathcal{J}^{i0})^\mu{}_\lambda(\mathcal{J}^{i0})^\lambda{}_\nu \pm \frac{i}{4}\epsilon_{ijk}(\mathcal{J}^{i0})^\mu{}_\lambda(\mathcal{J}^{jk})^\lambda{}_\nu$$

$$= -\frac{1}{8}(g^j{}_\lambda g^{k\mu} - g^k{}_\lambda g^{j\mu})(g^j{}_\nu g^{k\lambda} - g^k{}_\nu g^{j\lambda}) + \frac{1}{4}(g^i{}_\lambda g^{0\mu} - g^0{}_\lambda g^{i\mu})(g^i{}_\nu g^{0\lambda} - g^0{}_\nu g^{i\lambda})$$
$$\mp i\epsilon_{ijk}(g^i{}_\lambda g^{0\mu} - g^0{}_\lambda g^{i\mu})(g^j{}_\nu g^{k\lambda} - g^k{}_\nu g^{j\lambda})$$

$$= \frac{1}{8}[-(g^{jk}g^{k\mu}-g^{kk}g^{j\mu})g^j{}_\nu+(g^{jj}g^{k\mu}-g^{kj}g^{j\mu})g^k{}_\nu+2(g^{i0}g^{0\mu}-g^{00}g^{i\mu})g^i{}_\nu-2(g^{ii}g^{0\mu}-g^{0i}g^{i\mu})g^0{}_\nu]$$
$$\mp 2i\epsilon_{ijk}[(g^{ik}g^{0\mu}-g^{0k}g^{i\mu})g^j{}_\nu-(g^{ij}g^{0\mu}-g^{0j}g^{i\mu})g^k{}_\nu]$$

$$= \frac{1}{8}[g^k{}_\nu g^{k\mu}-3g^{j\mu}g^j{}_\nu-3g^{k\mu}g^k{}_\nu+g^j{}_\nu g^{j\mu}-2g^{i\mu}g^i{}_\nu+6g^{0\mu}g^0{}_\nu] = \frac{3}{4}g^{\rho\mu}g^\rho{}_\nu = \frac{1}{2}(\frac{1}{2}+1)g^\mu{}_\nu$$

我们讨论过的矢量场属于$(\frac{1}{2},\frac{1}{2})$场，它包含$j=1,0$两部分： $\mathcal{A}^2=\mathcal{B}^2=\frac{1}{2}(\frac{1}{2}+1)I$

有质量的任意自旋量子场

$$\psi_{ab}(x) = (2\pi)^{-3/2}\sum_{\sigma}\int d^3p[\kappa a(\vec{p},\sigma)e^{-ip\cdot x}u_{ab}(\vec{p},\sigma) + \lambda a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}v_{ab}(\vec{p},\sigma)]$$

$$\sum_a \vec{J}^{(A)}_{\bar{a}a}u_{ab}(0,\sigma) + \sum_b \vec{J}^{(B)}_{\bar{b}b}u_{\bar{a}b}(0,\sigma) \xLeftarrow{\vec{\mathcal{J}}=\vec{\mathcal{A}}+\vec{\mathcal{B}}} \sum_{a,b}\vec{\mathcal{J}}_{\bar{a}\bar{b},ab}u_{ab}(0,\sigma) = \sum_{\bar{\sigma}}u_{\bar{a}\bar{b}}(0,\bar{\sigma})\vec{J}^{(j)}_{\bar{\sigma}\sigma}$$

$$\sum_a \vec{J}^{(A)}_{\bar{a}a}v_{ab}(0,\sigma) + \sum_b \vec{J}^{(B)}_{\bar{b}b}v_{\bar{a}b}(0,\sigma) \xLeftarrow{\vec{\mathcal{J}}=\vec{\mathcal{A}}+\vec{\mathcal{B}}} \sum_{a,b}\vec{\mathcal{J}}_{\bar{a}\bar{b},ab}v_{ab}(0,\sigma) = -\sum_{\bar{\sigma}}v_{\bar{a}\bar{b}}(0,\bar{\sigma})\vec{J}^{(j)*}_{\bar{\sigma}\sigma}$$

CG系数分解: $\Psi^j{}_\sigma \equiv \sum_{ab} C_{AB}(j\sigma;ab)\Psi_{ab}$ 标量场的微商构成的矢量场？在无穷小空间转动下

$$\delta\Psi^j{}_\sigma = i\sum_{\bar{\sigma}}\vec{\theta}\cdot\vec{J}^{(j)}_{\bar{\sigma}\sigma}\Psi^j{}_{\bar{\sigma}} \qquad \delta\Psi_{ab} = i\sum_{\bar{a}}\vec{\theta}\cdot\vec{J}^{(A)}_{\bar{a}a}\Psi_{\bar{a}b} + i\sum_{\bar{b}}\vec{\theta}\cdot\vec{J}^{(B)}_{\bar{b}b}\Psi_{a\bar{b}} \Rightarrow u_{ab}(0,\sigma) = (2m)^{-\frac{1}{2}}C_{AB}(j\sigma;ab)$$

$$\sum_{\sigma\bar{a}\bar{b}}\vec{J}^{(j)}_{\bar{\sigma}\sigma}C_{AB}(j\bar{\sigma};\bar{a}\bar{b})\Psi_{\bar{a}\bar{b}} = \sum_{\bar{\sigma}}\vec{J}^{(j)}_{\bar{\sigma}\sigma}\Psi^j{}_{\bar{\sigma}} = \sum_{\bar{a}\bar{b}}C_{AB}(j\sigma;\bar{a}\bar{b})[\sum_a \vec{J}^{(A)}_{\bar{a}a}\Psi_{a\bar{b}} + \sum_b \vec{J}^{(B)}_{\bar{b}b}\Psi_{\bar{a}b}] = \sum_{\bar{a}\bar{b}}[\sum_a \vec{J}^{(A)}_{\bar{a}a}C_{AB}(j\sigma;a\bar{b}) + \sum_b \vec{J}^{(B)}_{\bar{b}b}C_{AB}(j\sigma;\bar{a}b)]\Psi_{\bar{a}\bar{b}}$$

从矩阵元得出: $-\vec{J}^{(j)*}_{\sigma\sigma'} = (-1)^{-\sigma+\sigma'}\vec{J}^{(j)}_{-\sigma,-\sigma'} \Rightarrow v_{ab}(0,\sigma) = (-1)^{j+\sigma}u_{ab}(0,-\sigma)$ ⇑去掉$\Psi_{\bar{a}\bar{b}}$正是第二行公式

推进变换: $\cosh\theta = \sqrt{\vec{p}^2+m^2}/m \quad \sinh\theta = |\vec{p}|/m$

$$L^i{}_k(\theta) = \delta_{ik} + (\cosh\theta - 1)\hat{p}_i\hat{p}_k \qquad L^i{}_0(\theta) = L^0{}_i(\theta) = \hat{p}_i\sinh\theta \qquad L^0{}_0(\theta) = \cosh\theta$$

无穷小θ: $[L(\theta)]^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu \Rightarrow \omega^i{}_0 = \omega^0{}_i = \hat{p}_i\theta \qquad \omega^i{}_j = \omega^0{}_0 = 0$

有质量的任意自旋量子场

$$\psi_{ab}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p[\kappa a(\vec{p},\sigma)e^{-ip\cdot x}u_{ab}(\vec{p},\sigma) + \lambda a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}v_{ab}(\vec{p},\sigma)]$$

$$\sum_a \vec{J}^{(A)}_{\bar{a}a} u_{a\bar{b}}(0,\sigma) + \sum_b \vec{J}^{(B)}_{\bar{b}b} u_{\bar{a}b}(0,\sigma) \overset{\vec{\mathcal{J}}=\vec{\mathcal{A}}+\vec{\mathcal{B}}}{=\!=\!=\!=\!=\!=\!=} \sum_{a,b} \vec{\mathcal{J}}_{\bar{a}\bar{b},ab} u_{ab}(0,\sigma) = \sum_{\bar{\sigma}} u_{\bar{a}\bar{b}}(0,\bar{\sigma})\vec{J}^{(j)}_{\bar{\sigma}\sigma}$$

$$\sum_a \vec{J}^{(A)}_{\bar{a}a} v_{a\bar{b}}(0,\sigma) + \sum_b \vec{J}^{(B)}_{\bar{b}b} v_{\bar{a}b}(0,\sigma) \overset{\vec{\mathcal{J}}=\vec{\mathcal{A}}+\vec{\mathcal{B}}}{=\!=\!=\!=\!=\!=\!=} \sum_{a,b} \vec{\mathcal{J}}_{\bar{a}\bar{b},ab} v_{ab}(0,\sigma) = -\sum_{\bar{\sigma}} v_{\bar{a}\bar{b}}(0,\bar{\sigma})\vec{J}^{(j)*}_{\bar{\sigma}\sigma}$$

$$u_{ab}(0,\sigma) = (2m)^{-\frac{1}{2}} C_{AB}(j\sigma;ab)$$

推进变换： $\cosh\theta = \sqrt{\vec{p}^2+m^2}/m \quad \sinh\theta = |\vec{p}|/m$

$$L^i{}_k(\theta) = \delta_{ik} + (\cosh\theta - 1)\hat{p}_i\hat{p}_k \qquad L^i{}_0(\theta) = L^0{}_i(\theta) = \hat{p}_i \sinh\theta \qquad L^0{}_0(\theta) = \cosh\theta$$

无穷小θ: $[L(\theta)]^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu \Rightarrow \omega^i{}_0 = \omega^0{}_i = \hat{p}_i\theta \qquad \omega^i{}_j = \omega^0{}_0 = 0$

$$D(L(p)) = e^{-i\hat{p}\cdot\vec{\mathcal{K}}\theta} \overset{i\mathcal{K}=\mathcal{A}-\mathcal{B}}{=\!=\!=\!=\!=\!=\!=} e^{-\hat{p}\cdot\vec{J}^{(A)}\theta} e^{\hat{p}\cdot\vec{J}^{(B)}\theta} \Rightarrow D(L(p))_{a'b',ab} = \left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)_{a'a}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)_{b'b}$$

$$u_{ab}(\vec{p},\sigma) = \frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)_{bb'} C_{AB}(j\sigma;a'b') \qquad v_{ab}(\vec{p},\sigma) = (-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma)$$

有质量的任意自旋量子场

$$\psi_{ab}(x) = (2\pi)^{-3/2}\sum_{\sigma}\int d^3p[\kappa a(\vec{p},\sigma)e^{-ip\cdot x}u_{ab}(\vec{p},\sigma) + \lambda a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}v_{ab}(\vec{p},\sigma)]$$

$$u_{ab}(0,\sigma) = (2m)^{-1/2}C_{AB}(j\sigma;ab)$$

$$u_{ab}(\vec{p},\sigma) = \frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{\mathcal{J}}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{\mathcal{J}}^{(B)}\theta}\right)_{bb'}C_{AB}(j\sigma;a'b') \qquad v_{ab}(\vec{p},\sigma) = (-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma)$$

$$[\psi_{ab}(x),\tilde{\psi}^{\dagger}_{ab}(y)]_{\mp} = (2\pi)^{-3}\int\frac{d^3p}{2p^0}\pi_{ab,\tilde{a}\tilde{b}}(\vec{p})[\kappa\tilde{\kappa}^*e^{-ip\cdot(x-y)} \mp \lambda\tilde{\lambda}^*e^{ip\cdot(x-y)}]$$

$$\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) \equiv 2p^0\sum_{\sigma}u_{ab}(\vec{p},\sigma)\tilde{u}^*_{\tilde{a}\tilde{b}}(\vec{p},\sigma) = 2p^0\sum_{\sigma}v_{ab}(\vec{p},\sigma)\tilde{v}^*_{\tilde{a}\tilde{b}}(\vec{p},\sigma) \qquad \tilde{j}=j,\ \ \tilde{\sigma}=\sigma$$

$$= \sum_{a'b'}\sum_{\tilde{a}'\tilde{b}'}\sum_{\sigma}C_{AB}(j\sigma;a'b')C_{\tilde{A}\tilde{B}}(j\sigma;\tilde{a}'\tilde{b}')\left(e^{-\hat{p}\cdot\vec{\mathcal{J}}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{\mathcal{J}}^{(B)}\theta}\right)_{bb'}\left(e^{-\hat{p}\cdot\vec{\mathcal{J}}^{(A)}\theta}\right)^*_{\tilde{a}\tilde{a}'}\left(e^{\hat{p}\cdot\vec{\mathcal{J}}^{(B)}\theta}\right)^*_{\tilde{b}\tilde{b}'}$$

$$= \underbrace{P_{ab,\tilde{a}\tilde{b}}(\vec{p})}_{p^0\text{偶幂次}} + 2\sqrt{\vec{p}^2+m^2}\underbrace{Q_{ab,\tilde{a}\tilde{b}}(\vec{p})}_{p^0\text{偶幂次}}$$

例子见后 $P(-\vec{p}) = (-1)^{2A+2\tilde{B}}P(\vec{p}) \quad Q(-\vec{p}) = -(-1)^{2A+2\tilde{B}}Q(\vec{p})$

$$[\psi_{ab}(x),\tilde{\psi}^{\dagger}_{ab}(y)]_{\mp} \overset{x^0=y^0}{=\!=\!=\!=} [\kappa\tilde{\kappa}^* \mp (-1)^{2A+2\tilde{B}}\lambda\tilde{\lambda}^*]P_{ab,\tilde{a}\tilde{b}}(i\nabla)\Delta_+(\vec{x}-\vec{y},0)$$

$$+ [\kappa\tilde{\kappa}^* \pm (-1)^{2A+2\tilde{B}}\lambda\tilde{\lambda}^*]Q_{ab,\tilde{a}\tilde{b}}(i\nabla)\delta^3(\vec{x}-\vec{y})$$

类空间隔只限于$\vec{x}\neq\vec{y}$, $\vec{x}=\vec{y}$是类时间隔

有质量的任意自旋量子场

$$\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) \equiv 2p^0\sum_\sigma u_{ab}(\vec{p},\sigma)\tilde{u}^*_{\tilde{a}\tilde{b}}(\vec{p},\sigma) = 2p^0\sum_\sigma v_{ab}(\vec{p},\sigma)\tilde{v}^*_{\tilde{a}\tilde{b}}(\vec{p},\sigma)$$

$$= \sum_{a'b'}\sum_{\tilde{a}'\tilde{b}'}\sum_\sigma C_{AB}(j\sigma;a'b')C_{\tilde{A}\tilde{B}}(j\sigma;\tilde{a}'\tilde{b}')\left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)_{bb'}\left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)^*_{\tilde{a}\tilde{a}'}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)^*_{\tilde{b}\tilde{b}'}$$

$$= \underbrace{P_{ab,\tilde{a}\tilde{b}}(\vec{p})}_{p^0\text{偶幂次}} + 2\sqrt{\vec{p}^2+m^2}\,\underbrace{Q_{ab,\tilde{a}\tilde{b}}(\vec{p})}_{p^0\text{偶幂次}}$$

例子见下 $P(-\vec{p}) = (-1)^{2A+2\tilde{B}}P(\vec{p}) \quad Q(-\vec{p}) = -(-1)^{2A+2\tilde{B}}Q(\vec{p})$

考虑$\vec{p}$沿z轴的情况： $\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) = \sum_\sigma C_{AB}(j\sigma;ab)C_{\tilde{A}\tilde{B}}(j\sigma;\tilde{a}\tilde{b})e^{(-a+b-\tilde{a}+\tilde{b})\theta}$

$$C_{AB}(j\sigma;ab)\Big|_{\sigma\neq a+b} = C_{\tilde{A}\tilde{B}}(j\sigma;\tilde{a}\tilde{b})\Big|_{\sigma\neq\tilde{a}+\tilde{b}} = 0 \quad\Rightarrow\quad -a+b-\tilde{a}+\tilde{b} = -2a+\sigma+2\tilde{b}-\sigma = 2\tilde{b}-2a$$

$$\cosh\theta = \frac{p^0}{m} \qquad \sinh\theta = \frac{p^3}{m} \quad\Rightarrow\quad e^{\pm\theta} = \cosh\theta \pm \sinh\theta = \frac{p^0\pm p^3}{m}$$

$$\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) = \sum_\sigma C_{AB}(j\sigma;ab)C_{\tilde{A}\tilde{B}}(j\sigma;\tilde{a}\tilde{b})\times\begin{cases}\left[\frac{p^0+p^3}{m}\right]^{2\tilde{b}-2a} & \tilde{b}\geq a\\ \left[\frac{p^0-p^3}{m}\right]^{2a-2\tilde{b}} & a\geq\tilde{b}\end{cases} \xRightarrow{\vec{p}\to-\vec{p};\ p_0\to-p_0} (-1)^{2A+2\tilde{B}}\pi_{ab,\tilde{a}\tilde{b}}(\vec{p})$$

$$(-1)^{2\tilde{b}-2a} = (-1)^{2A+2\tilde{B}}(-1)^{2\tilde{b}-2\tilde{B}-2a-2A} = (-1)^{2A+2\tilde{B}} \qquad (-1)^{2a-2\tilde{b}} = (-1)^{2A+2\tilde{B}}(-1)^{-2\tilde{b}-2\tilde{B}+2a-2A} = (-1)^{2A+2\tilde{B}}$$

有质量的任意自旋量子场

$$\psi_{ab}(x)=(2\pi)^{-3/2}\sum_{\sigma}\int d^3p[\kappa a(\vec{p},\sigma)e^{-ip\cdot x}u_{ab}(\vec{p},\sigma)+\lambda a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}v_{ab}(\vec{p},\sigma)]$$

$$u_{ab}(0,\sigma)=(2m)^{-1/2}C_{AB}(j\sigma;ab)$$

$$u_{ab}(\vec{p},\sigma)=\frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)_{bb'}C_{AB}(j\sigma;a'b')\qquad v_{ab}(\vec{p},\sigma)=(-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma)$$

$$[\psi_{ab}(x),\tilde{\psi}^{\dagger}_{ab}(y)]_{\mp}=(2\pi)^{-3}\int\frac{d^3p}{2p^0}\pi_{ab,\tilde{a}\tilde{b}}(\vec{p})[\kappa\tilde{\kappa}^*e^{-ip\cdot(x-y)}\mp\lambda\tilde{\lambda}^*e^{ip\cdot(x-y)}]$$

$$\pi_{ab,\tilde{a}\tilde{b}}(\vec{p})=P_{ab,\tilde{a}\tilde{b}}(\vec{p})+2\sqrt{\vec{p}^2+m^2}Q_{ab,\tilde{a}\tilde{b}}(\vec{p})\qquad P(-\vec{p})=(-1)^{2A+2\tilde{B}}P(\vec{p})\qquad Q(-\vec{p})=-(-1)^{2A+2\tilde{B}}Q(\vec{p})$$

$$[\psi_{ab}(x),\tilde{\psi}^{\dagger}_{ab}(y)]_{\mp}\overset{x^0=y^0}{=\!=\!=\!=}[\kappa\tilde{\kappa}^*\mp(-1)^{2A+2\tilde{B}}\lambda\tilde{\lambda}^*]P_{ab,\tilde{a}\tilde{b}}(i\nabla)\Delta_+(\vec{x}-\vec{y},0)$$

$$+[\kappa\tilde{\kappa}^*\pm(-1)^{2A+2\tilde{B}}\lambda\tilde{\lambda}^*]Q_{ab,\tilde{a}\tilde{b}}(i\nabla)\delta^3(\vec{x}-\vec{y})\qquad\kappa\tilde{\kappa}^*=\pm(-1)^{2A+2\tilde{B}}\lambda\tilde{\lambda}^*$$

自旋统计关系：$j=$整数$\Leftrightarrow$玻色子；$j=$半整数$\Leftrightarrow$费米子

$$A=\tilde{A}\quad B=\tilde{B}\quad\Rightarrow\quad|\kappa|^2=\pm(-1)^{2A+2B}|\lambda|^2\xrightarrow{2A+2B-2j=\text{偶数}}\pm(-1)^{2j}=1\quad|\kappa|^2=|\lambda|^2$$

$$|\tilde{\kappa}|^2=|\tilde{\lambda}|^2\Rightarrow\frac{\kappa}{\tilde{\kappa}}=\pm(-1)^{2A+2\tilde{B}}\frac{\lambda}{\tilde{\lambda}}=(-1)^{-2B+2\tilde{B}}\frac{\lambda}{\tilde{\lambda}}\Rightarrow\lambda=(-1)^{2B}c\kappa\xrightarrow{|\kappa|^2=|\lambda|^2}c=1\Rightarrow\lambda=(-1)^{2B}\kappa$$

无质量的任意自旋量子场

$$\psi_{ab}(x) = (2\pi)^{-3/2}\sum_{\sigma}\int d^3p[\kappa a(\vec{p},\sigma)e^{-ip\cdot x}u_{ab}(\vec{p},\sigma) + \lambda a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}v_{ab}(\vec{p},\sigma)]$$

$$u_{\bar{a}\bar{b}}(\vec{k},\sigma)e^{i\theta\sigma} = \sum_{ab} D_{\bar{a}\bar{b},ab}(R(\theta))u_{ab}(\vec{k},\sigma) \qquad u_{\bar{a}\bar{b}}(\vec{k},\sigma) = \sum_{ab} D_{\bar{a}\bar{b},ab}(S(\alpha,\beta))u_{ab}(\vec{k},\sigma)$$

$$v_{\bar{a}\bar{b}}(\vec{k},\sigma)e^{-i\theta\sigma} = \sum_{ab} D_{\bar{a}\bar{b},ab}(R(\theta))v_{ab}(\vec{k},\sigma) \qquad v_{\bar{a}\bar{b}}(\vec{k},\sigma) = \sum_{ab} D_{\bar{a}\bar{b},ab}(S(\alpha,\beta))v_{ab}(\vec{k},\sigma)$$

$$S^{\mu}{}_{\nu}(\alpha,\beta) = \begin{bmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{bmatrix} \qquad R^{\mu}{}_{\nu}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

对无穷小θ: $D(R(\theta)) = 1 + i\theta\mathcal{J}^3$

$$\sigma u_{ab}(\vec{k},\sigma) = (a+b)u_{ab}(\vec{k},\sigma) \quad\Rightarrow\quad u_{ab}(\vec{k},\sigma) \overset{\sigma\neq a+b}{=\!=\!=\!=\!=} 0$$

$$-\sigma v_{ab}(\vec{k},\sigma) = (a+b)v_{ab}(\vec{k},\sigma) \quad\Rightarrow\quad v_{ab}(\vec{k},\sigma) \overset{\sigma\neq -a-b}{=\!=\!=\!=\!=} 0$$

从方程看可取 $v_{ab}(\vec{k},\sigma) = u_{ab}(\vec{k},-\sigma)$

无质量的任意自旋量子场

$$\psi_{ab}(x) = (2\pi)^{-3/2}\sum_{\sigma}\int d^3p[\kappa a(\vec{p},\sigma)e^{-ip\cdot x}u_{ab}(\vec{p},\sigma) + \lambda a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}v_{ab}(\vec{p},\sigma)]$$

$$u_{\bar{a}\bar{b}}(\vec{k},\sigma) = \sum_{ab} D_{\bar{a}\bar{b},ab}(S(\alpha,\beta))u_{ab}(\vec{k},\sigma) \qquad v_{\bar{a}\bar{b}}(\vec{k},\sigma) = \sum_{ab} D_{\bar{a}\bar{b},ab}(S(\alpha,\beta))v_{ab}(\vec{k},\sigma)$$

$$S^{\mu}{}_{\nu}(\alpha,\beta) = \begin{bmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{bmatrix} \qquad u_{ab}(\vec{k},\sigma) \overset{\sigma\neq a+b}{=\!=\!=\!=\!=} 0 \qquad v_{ab}(\vec{k},\sigma) \overset{\sigma\neq -a-b}{=\!=\!=\!=\!=} 0$$

对无穷小α, β: $D(S(\alpha,\beta)) = 1 - i\alpha(\mathcal{J}^{01}-\mathcal{J}^{13}) - i\beta(\mathcal{J}^{02}-\mathcal{J}^{23})$

$$\mathcal{J}^{01}-\mathcal{J}^{13}=\mathcal{K}^1-\mathcal{J}^2=-i\mathcal{A}^1+i\mathcal{B}^1-\mathcal{A}^2-\mathcal{B}^2 \qquad \mathcal{J}^{02}-\mathcal{J}^{23}=\mathcal{K}^2+\mathcal{J}^1=-i\mathcal{A}^2+i\mathcal{B}^2+\mathcal{A}^1+\mathcal{B}^1$$

$$0 = (J^{(A),2}+iJ^{(A),1})_{aa'}u_{a'b}(\vec{k},\sigma) + (J^{(B),2}-iJ^{(B),1})_{bb'}u_{ab'}(\vec{k},\sigma)$$

$$0 = (J^{(A),1}-iJ^{(A),2})_{aa'}u_{a'b}(\vec{k},\sigma) + (J^{(B),1}+iJ^{(B),2})_{bb'}u_{ab'}(\vec{k},\sigma)$$

$$(J^{(A),1}-iJ^{(A),2})_{aa'}u_{a'b}(\vec{k},\sigma)=0 \qquad (J^{(B),1}+iJ^{(B),2})_{bb'}u_{ab'}(\vec{k},\sigma)=0$$

$u_{ab}(\vec{k},\sigma) \overset{a\neq -A,\ b\neq B}{=\!=\!=\!=\!=\!=} 0$ $\sigma=B-A$ 类似地 $v_{ab}(\vec{k},\sigma) \overset{a\neq -A,\ b\neq B}{=\!=\!=\!=\!=\!=} 0$ $\sigma=A-B$ 反粒子具有相反的螺旋度！

任意自旋的零质量量子场： (A,B)

$u_{ab}(\vec{k},\sigma) \overset{a\neq -A,\ b\neq B}{=\!=\!=\!=\!=\!=} 0 \quad \sigma = B-A = \pm j$ 类似地 $v_{ab}(\vec{k},\sigma) \overset{a\neq -A,\ b\neq B}{=\!=\!=\!=\!=\!=} 0 \quad \sigma = A-B = \mp j$

$(A,B) \overset{\text{宇称变换}}{\Longleftarrow\!=\!=\!=\!\Longrightarrow} (B,A) \quad \Leftrightarrow \quad \beta\mathcal{A}\beta^{-1} = \mathcal{B} \quad \beta\mathcal{B}\beta^{-1} = \mathcal{A}$

$j=0, \sigma = 0$ $(0,0)$场； $(\frac{1}{2},\frac{1}{2})$ 对应$\partial_\mu \phi$,见后； (A,A)场 红色是零质量特殊的

$j=\frac{1}{2}$ $(\frac{1}{2},0)$或$(0,\frac{1}{2})$场 $\sigma = \pm\frac{1}{2}$ Dirac旋量$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$具有确定的宇称

$j=1$ $(1,0)$或$(0,1)$场 $\sigma=\pm 1$ $(1,0)\oplus(0,1)$有确定宇称;对应$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\mu v_\mu$ v_μ不显式出现

$j=A$ $(A,0)$或$(0,A)$场 $\sigma = \pm A$ $(A,0)\oplus(0,A)$具有确定的宇称

可以证明：$(A,A+j)$是对$(0,j)$进行$2A$次微商；$(B+j,B)$是对$(j,0)$进行$2B$次微商

只要讨论 $(j,0)$ 只有$u_{-j,0}(\vec{k},-j)$和$v_{-j,0}(\vec{k},j)$非零 和 $(0,j)$ 只有$u_{0,j}(\vec{k},j)$和$v_{0,j}(\vec{k},-j)$非零 场！

无质量的任意自旋量子场

$$\psi_{-j,0}(x) = (2\pi)^{-3/2} \int d^3p[\tilde{\kappa} a(\vec{p},j)e^{-ip\cdot x}u_{-j,0}(\vec{p},j) + \tilde{\lambda} a^{c\dagger}(\vec{p},-j)e^{ip\cdot x}v_{-j,0}(\vec{p},-j)]$$

$$\psi_{0,j}(x) = (2\pi)^{-3/2} \int d^3p[\kappa' a(\vec{p},j)e^{-ip\cdot x}u_{0,j}(\vec{p},j) + \lambda' a^{c\dagger}(\vec{p},-j)e^{ip\cdot x}v_{0,j}(\vec{p},-j)]$$

$$u_{\bar{a}\bar{b}}(\vec{p},\sigma) = \sqrt{\frac{\kappa}{p^0}} \sum_{a,b} D_{\bar{a}\bar{b}\,ab}(L(p))u_{ab}(\vec{k},\sigma) \qquad v_{\bar{a}\bar{b}}(\vec{p},\sigma) = \sqrt{\frac{\kappa}{p^0}} \sum_{a,b} D_{\bar{a}\bar{b}\,ab}(L(p))v_{ab}(\vec{k},\sigma)$$

$$L(p) = R(\hat{p})B(\frac{|\vec{p}|}{\kappa}) \qquad D(L(p)) = D(R(\hat{p}))D(B(\frac{|\vec{p}|}{\kappa})) \qquad B(\frac{|\vec{p}|}{\kappa}) = e^{\omega}$$

$$B(u) = \begin{pmatrix} \frac{u^2+1}{2u} & 0 & 0 & \frac{u^2-1}{2u} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{u^2-1}{2u} & 0 & 0 & \frac{u^2+1}{2u} \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & 0 & \text{arc}\cosh(\frac{u^2+1}{2u}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \text{arc}\cosh(\frac{u^2+1}{2u}) & 0 & 0 & 0 \end{pmatrix}$$

$$\omega_{03} = \text{arc}\cosh(\frac{u^2+1}{2u})\Big|_{u=\frac{|\vec{p}|}{\kappa}} = \ln u\Big|_{u=\frac{|\vec{p}|}{\kappa}} = \ln\frac{|\vec{p}|}{\kappa} \quad D(B(\frac{|\vec{p}|}{\kappa})) = e^{-i\omega_{03}\mathcal{K}^3} \overset{i\mathcal{K}^3=\mathcal{A}^3-\mathcal{B}^3}{=\!=\!=\!=\!=\!=} \left(\frac{|\vec{p}|}{\kappa}\right)^{-\mathcal{A}^3+\mathcal{B}^3} = \left(\frac{p^0}{\kappa}\right)^j$$

$$u_{\bar{a}\bar{b}}(\vec{p},\sigma) = \left(\frac{p^0}{\kappa}\right)^{j-1/2} \sum_{a,b} D_{\bar{a}\bar{b}\,ab}(R(\hat{p}))u_{ab}(\vec{k},\sigma) \qquad v_{\bar{a}\bar{b}}(\vec{p},\sigma) = \left(\frac{p^0}{\kappa}\right)^{j-1/2} \sum_{a,b} D_{\bar{a}\bar{b}\,ab}(R(\hat{p}))v_{ab}(\vec{k},\sigma)$$

无质量的任意自旋量子场

$$\psi_{-j,0}(x) = (2\pi)^{-\frac{3}{2}}\int d^3p\left(\frac{p^0}{\kappa}\right)^{j-1/2}[\tilde{\kappa}a(\vec{p},j)e^{-ip\cdot x}(Du)_{-j,0}(\vec{p},j)+\tilde{\lambda}a^{c\dagger}(\vec{p},-j)e^{ip\cdot x}(Dv)_{-j,0}(\vec{p},-j)]$$

$$\psi_{0,j}(x) = (2\pi)^{-\frac{3}{2}}\int d^3p\left(\frac{p^0}{\kappa}\right)^{j-1/2}[\kappa' a(\vec{p},j)e^{-ip\cdot x}(Du)_{0,j}(\vec{p},j)+\lambda' a^{c\dagger}(\vec{p},-j)e^{ip\cdot x}(Dv)_{0,j}(\vec{p},-j)]$$

$$(Du)_{\bar{a}\bar{b}}(\vec{p},\sigma)\equiv\sum_{a,b}D_{\bar{a}\bar{b}\,ab}(R(\hat{p}))u_{ab}(\vec{k},\sigma)\qquad (Dv)_{\bar{a}\bar{b}}(\vec{p},\sigma)\equiv\sum_{a,b}D_{\bar{a}\bar{b}\,ab}(R(\hat{p}))v_{ab}(\vec{k},\sigma)$$

$$[\psi_{-j,0}(x),\psi^\dagger_{-j,0}(y)]_\mp=\int\frac{d^3p}{(2\pi)^3}\frac{\pi(p)}{2p^0}\left[|\tilde{\kappa}|^2e^{-ip\cdot(x-y)}\mp|\tilde{\lambda}|^2e^{ip\cdot(x-y)}\right]$$

$$[\psi_{0,j}(x),\psi^\dagger_{0,j}(y)]_\mp=(2\pi)^{-3}\int\frac{d^3p}{2p^0}\pi'(p)\left[|\kappa'|^2e^{-ip\cdot(x-y)}\mp|\lambda'|^2e^{ip\cdot(x-y)}\right]$$

$$\pi(p)\xlongequal{\text{利用}v_{ab}(\vec{k},\sigma)=u_{ab}(\vec{k},-\sigma)}(p^0/\kappa)^{2j}(Du)_{-j,0}(\vec{p},j)(Du)^*_{-j,0}(\vec{p},j)\xRightarrow{p^0\to-p^0;\ \vec{p}\to-\vec{p}}(-1)^{2j}\pi(p)$$

$$\pi'(p)\xlongequal{\text{利用}v_{ab}(\vec{k},\sigma)=u_{ab}(\vec{k},-\sigma)}(p^0/\kappa)^{2j}(Du)_{0,j}(\vec{p},j)(Du)^*_{0,j}(\vec{p},j)\xRightarrow{p^0\to-p^0;\ \vec{p}\to-\vec{p}}(-1)^{2j}\pi'(p)$$ 后面证

$$[\psi_{-j,0}(x),\psi^\dagger_{-j,0}(y)]_\mp=\pi(-i\partial_x)[(-)^{2j}|\tilde{\kappa}|^2\Delta_+(x-y)\mp|\tilde{\lambda}|^2\Delta_+(y-x)]\xRightarrow{\text{类似有质量}}$$ 自旋统计关系

$$[\psi_{0,j}(x),\psi^\dagger_{0,j}(y)]_\mp=\pi'(-i\partial_x)[(-)^{2j}|\kappa'|^2\Delta_+(x-y)\mp|\lambda'|^2\Delta_+(y-x)]$$

无质量的任意自旋量子场

$$\delta_{\bar{a},-j}\delta_{\bar{a}',-j} = \frac{1\times 2\times\cdots\times(2j-1)\times 2j}{(2j)!}\delta_{\bar{a},-j}\delta_{\bar{a}',-j} = \frac{1}{(2j)!}\left[\Pi^{j}_{\lambda=-j+1}(\lambda - J^{(A),3})\right]_{\bar{a}\bar{a}'}\underbrace{\delta_{\bar{a},-j}\delta_{\bar{a}',-j}}_{\text{可不要}}$$

$$\delta_{\bar{b},j}\delta_{\bar{b}',j} = \frac{2j\times(2j-1)\times\cdots\times 2\times 1}{(2j)!}\delta_{\bar{b},j}\delta_{\bar{b}',j} = \frac{1}{(2j)!}\left[\Pi^{j-1}_{\lambda=-j}(J^{(B),3} - \lambda)\right]_{\bar{b}\bar{b}'}\delta_{\bar{b},j}\delta_{\bar{b}',j}$$

$$\pi(p) = \left(\frac{p^0}{\kappa}\right)^{2j}\left[u^{\dagger}(\vec{k},j)D^{\dagger}(R(\hat{p}))\right]_{-j,0}\left[D(R(\hat{p}))u(\vec{k},j)\right]_{-j,0}$$

$$= \left(\frac{p^0}{\kappa}\right)^{2j}\left[u^{\dagger}(\vec{k},j)D^{\dagger}(R(\hat{p}))\right]_{\bar{a},0}\delta_{\bar{a},-j}\delta_{\bar{a}',-j}\left[D(R(\hat{p}))u(\vec{k},j)\right]_{\bar{a}',0}$$

$$= \frac{1}{(2j)!\kappa^{2j}}\left[u^{\dagger}(\vec{k},j)D^{\dagger}(R(\hat{p}))\right]_{\bar{a},0}\left[\Pi^{j}_{\lambda=-j+1}(p^0\lambda - J^{(A),3}p^0)\right]_{\bar{a},\bar{a}'}\left[D(R(\hat{p}))u(\vec{k},j)\right]_{\bar{a}',0}$$

$$\overset{D\text{幺正}}{=\!=\!=\!=} \frac{1}{(2j)!\kappa^{2j}}u^{\dagger}_{\bar{a},0}(\vec{k},j)\left[\Pi^{j}_{\lambda=-j+1}(p^0\lambda - \vec{J}^{(A)}\cdot\vec{p})\right]_{\bar{a},\bar{a}'}u_{\bar{a}',0}(\vec{k},j) \qquad D^{\dagger}(R(\hat{p}))J^{(A),3}D(R(\hat{p})) = \vec{J}^{(A)}\cdot\hat{p}$$

$$\pi'(p) = \left(\frac{p^0}{\kappa}\right)^{2j}\left[u^{\dagger}(\vec{k},j)D^{\dagger}(R(\hat{p}))\right]_{0,j}\left[D(R(\hat{p}))u(\vec{k},j)\right]_{0,j}$$

$$\overset{D\text{幺正}}{=\!=\!=\!=} \frac{1}{(2j)!\kappa^{2j}}u^{\dagger}_{0,\bar{b}}(\vec{k},j)\left[\Pi^{j-1}_{\lambda=-j}(\vec{J}^{(B)}\cdot\vec{p} - p^0\lambda)\right]_{\bar{b},\bar{b}'}u_{0,\bar{b}'}(\vec{k},j)$$

有质量： $\psi_{ab}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p[a(\vec{p},\sigma)e^{-ip\cdot x}u_{ab}(\vec{p},\sigma) + (-1)^{2B}a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot x}v_{ab}(\vec{p},\sigma)]$

$$u_{ab}(\vec{p},\sigma) = \frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)_{bb'}C_{AB}(j\sigma;a'b') \qquad v_{ab}(\vec{p},\sigma) = (-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma)$$

$$Pa(\vec{p},\sigma)P^{-1} = \eta^*a(-\vec{p},\sigma) \qquad Pa^{c\dagger}(\vec{p},\sigma)P^{-1} = \eta^c a^{c\dagger}(-\vec{p},\sigma)$$

$$P\psi^{AB}_{ab}(x)P^{-1} = (2\pi)^{-3/2}\sum_\sigma\int d^3p[\eta^*a(-\vec{p},\sigma)e^{-ip\cdot x}u^{AB}_{ab}(\vec{p},\sigma) + \eta^c(-1)^{2B}a^{c\dagger}(-\vec{p},\sigma)e^{ip\cdot x}v^{AB}_{ab}(\vec{p},\sigma)]$$

$$C_{AB}(j\sigma;ab) = (-1)^{A+B-j}C_{BA}(j\sigma;ba)$$

$$\Rightarrow \quad u^{AB}_{ab}(-\vec{p},\sigma) = (-1)^{A+B-j}u^{BA}_{ba}(\vec{p},\sigma) \qquad v^{AB}_{ab}(-\vec{p},\sigma) = (-1)^{A+B-j}v^{BA}_{ba}(\vec{p},\sigma)$$

$$P\psi^{AB}_{ab}(x)P^{-1} = (2\pi)^{-3/2}\sum_\sigma\int d^3p(-1)^{A+B-j}[\eta^*a(\vec{p},\sigma)e^{-ip\cdot\mathcal{P}x}u^{BA}_{ba}(\vec{p},\sigma)$$
$$+ \eta^c(-1)^{2B}a^{c\dagger}(\vec{p},\sigma)e^{ip\cdot\mathcal{P}x}v^{BA}_{ba}(\vec{p},\sigma)]$$

$$\eta^c(-1)^{2B}/\eta^* = (-1)^{2A} \quad\Rightarrow\quad \eta^c = \eta^*(-1)^{2j} \qquad P\psi^{AB}_{ab}(x)P^{-1} = \eta^*(-1)^{A+B-j}\psi^{BA}_{ba}(-\vec{x},x^0)$$

分立对称性变换性质

无质量：

$$\psi_{-j,0}(x)=(2\pi)^{-\frac{3}{2}}\int d^3p(p^0/\kappa)^{j-1/2}[a(\vec{p},j)e^{-ip\cdot x}(Du)_{-j,0}(\vec{p},j)+a^{c\dagger}(\vec{p},-j)e^{ip\cdot x}(Dv)_{-j,0}(\vec{p},-j)]$$

$$\psi_{0,j}(x)=(2\pi)^{-\frac{3}{2}}\int d^3p(p^0/\kappa)^{j-1/2}[a(\vec{p},j)e^{-ip\cdot x}(Du)_{0,j}(\vec{p},j)+a^{c\dagger}(\vec{p},-j)e^{ip\cdot x}(Dv)_{0,j}(\vec{p},-j)]$$

$$(Du)_{\bar{a}\bar{b}}(\vec{p},\sigma)\equiv\sum_{a,b}D_{\bar{a}\bar{b}\,ab}(R(\hat{p}))u_{ab}(\vec{k},\sigma)\qquad (Dv)_{\bar{a}\bar{b}}(\vec{p},\sigma)\equiv\sum_{a,b}D_{\bar{a}\bar{b}\,ab}(R(\hat{p}))v_{ab}(\vec{k},\sigma)$$

$$Pa^\dagger(\vec{p},\sigma,n)P^{-1}=\eta_\sigma e^{\mp i\pi\sigma}a^\dagger(\mathcal{P}\vec{p},-\sigma,n)\qquad Pa(\vec{p},\sigma,n)P^{-1}=\eta_\sigma^* e^{\pm i\pi\sigma}a(\mathcal{P}\vec{p},-\sigma,n)$$

负号: $0\leq\phi<\pi$， 正号: $\pi\leq\phi<2\pi$ （ϕ是$\vec{p}$在xy平面上的投影与x轴的夹角）

$$P\psi_{-j,0}(x)P^{-1}=(2\pi)^{-\frac{3}{2}}\int d^3p\left(\frac{p^0}{\kappa}\right)^{j-\frac{1}{2}}[\eta_j^* e^{\pm i\pi j}a(-\vec{p},-j)e^{-ip\cdot x}(Du)_{-j,0}(\vec{p},j)+\eta_{-j}^c e^{\pm i\pi j}a^{c\dagger}(-\vec{p},j)e^{ip\cdot x}(Dv)_{-j,0}(\vec{p},-j)]$$

$$\overset{\vec{p}\to-\vec{p}}{=\!=\!=\!=}\eta_j^*(-1)^{A+B-j}\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\left(\frac{p^0}{\kappa}\right)^{j-\frac{1}{2}}[a(\vec{p},-j)e^{-ip\cdot\mathcal{P}x}(Du)_{0,j}(\vec{p},-j)+\eta_j\eta_{-j}^c a^{c\dagger}(\vec{p},j)e^{ip\cdot\mathcal{P}x}(Dv)_{0,j}(\vec{p},-j)]\overset{\eta_j^*=\eta_{-j}^c}{=\!=\!=\!=\!=}\eta_j^*(-1)^{A+B-j}\psi_{0,j}(\mathcal{P}x)$$

$$U(R(\hat{p})R_2)=U(R(-\hat{p}))e^{\pm i\pi J^3}\qquad D_{a'b'\,ab}(R_2)=(e^{-i\pi\mathcal{A}^2})_{a'a}(e^{-i\pi\mathcal{B}^2})_{b'b}=(-1)^{A+a}\delta_{a',-a}(-1)^{B+b}\delta_{b',-b}$$

$$\Rightarrow D_{\bar{a}\bar{b}\,ab}(R(-\hat{p}))e^{\pm i\pi(a+b)}=D_{\bar{a}\bar{b}\,ab}(R(\hat{p})R_2)=(-1)^{A+B+a+b}D_{\bar{a}\bar{b}\,-a-b}(R(\hat{p}))$$

类似地： $P\psi_{0,j}(x)P^{-1}=\eta_j^*\eta^*(-1)^{A+B-j}\psi_{-j,0}(\mathcal{P}x)\xrightarrow{\text{与有质量结果一致}}P\psi_{ab}^{AB}(x)P^{-1}=\eta^*(-1)^{A+B-j}\psi_{ba}^{BA}(-\vec{x},x^0)$

分立对称性变换性质

$$\psi_{ab}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u_{ab}(\vec{p},\sigma)[a(\vec{p},\sigma)e^{-ip\cdot x} + (-1)^{2B}(-1)^{j-\sigma} a^{c\dagger}(\vec{p},-\sigma)e^{ip\cdot x}]$$

$$u_{ab}(\vec{p},\sigma) = \frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{\mathcal{J}}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{\mathcal{J}}^{(B)}\theta}\right)_{bb'} C_{AB}(j\sigma;a'b') \qquad v_{ab}(\vec{p},\sigma) = (-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma)$$

$$Ca(\vec{p},\sigma)C^{-1} = \xi^* a^c(\vec{p},\sigma) \qquad Ca^{c\dagger}(\vec{p},\sigma)C^{-1} = \xi^c a^\dagger(\vec{p},\sigma)$$

$$C\psi_{ab}^{AB}(x)C^{-1} = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u_{ab}^{AB}(\vec{p},\sigma)[\xi^* a^c(\vec{p},\sigma)e^{-ip\cdot x} + \xi^c(-1)^{2B}a^\dagger(\vec{p},-\sigma)(-1)^{j-\sigma}e^{ip\cdot x}]$$

$$\psi_{ba}^{BA\dagger}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u_{ba}^{BA*}(\vec{p},\sigma)[(-1)^{2A}(-1)^{j-\sigma}a^c(\vec{p},-\sigma)e^{-ip\cdot x} + a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$-\vec{J}^{(j)*}_{\sigma\sigma'} = (-1)^{-\sigma+\sigma'}\vec{J}^{(j)}_{-\sigma,-\sigma'} \quad\Rightarrow\quad \vec{J}^{(j)*} = -\mathcal{B}\vec{J}^{(j)}\mathcal{B}^{-1} \quad \mathcal{B}_{\bar\sigma\sigma} = (-1)^{j+\sigma}\delta_{\bar\sigma,-\sigma} \quad \mathcal{B}^{-1}_{\bar\sigma\sigma} = (-1)^{-j+\sigma}\delta_{\bar\sigma,-\sigma}$$

$$u_{ba}^{BA}(\vec{p},\sigma)^* = \frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{\mathcal{J}}^{(A)}\theta}\right)_{-a,-a'}\left(e^{\hat{p}\cdot\vec{\mathcal{J}}^{(B)}\theta}\right)_{-b,-b'}(-1)^{a'-a}(-1)^{b'-b}C_{BA}(j\sigma;b'a')$$

$$C_{BA}(j,-\sigma;-b',-a') = C_{AB}(j\sigma;a'b')\delta_{a'+b',\sigma} \qquad\Rightarrow\quad u_{-b,-a}^{BA}(\vec{p},-\sigma)^* = (-1)^{a+b-\sigma}u_{ab}^{AB}(\vec{p},\sigma)$$

$$\psi_{-b,-a}^{BA\dagger}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p(-1)^{a+b-\sigma}u_{ab}^{AB}(\vec{p},\sigma)[(-1)^{2A}(-1)^{j+\sigma}a^c(\vec{p},\sigma)e^{-ip\cdot x} + a^\dagger(\vec{p},-\sigma)e^{ip\cdot x}]$$

$$(-1)^{-2A-a-b-j}\psi_{-b,-a}^{BA\dagger}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u_{ab}^{AB}(\vec{p},\sigma)[a^c(\vec{p},\sigma)e^{-ip\cdot x} + (-1)^{j-\sigma+2B}a^\dagger(\vec{p},-\sigma)e^{ip\cdot x}]$$

分立对称性变换性质

$$\psi_{ab}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u_{ab}(\vec{p},\sigma)[a(\vec{p},\sigma)e^{-ip\cdot x} + (-1)^{2B}(-1)^{j-\sigma}a^{c\dagger}(\vec{p},-\sigma)e^{ip\cdot x}]$$

$$u_{ab}(\vec{p},\sigma) = \frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)_{bb'} C_{AB}(j\sigma;a'b') \qquad v_{ab}(\vec{p},\sigma) = (-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma)$$

$$Ca(\vec{p},\sigma)C^{-1} = \xi^* a^c(\vec{p},\sigma) \qquad Ca^{c\dagger}(\vec{p},\sigma)C^{-1} = \xi^c a^\dagger(\vec{p},\sigma)$$

$$C\psi^{AB}_{ab}(x)C^{-1} = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u^{AB}_{ab}(\vec{p},\sigma)[\xi^* a^c(\vec{p},\sigma)e^{-ip\cdot x} + \xi^c(-1)^{2B}a^\dagger(\vec{p},-\sigma)(-1)^{j-\sigma}e^{ip\cdot x}]$$

$$\psi^{BA\dagger}_{ba}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u^{BA*}_{ba}(\vec{p},\sigma)[(-1)^{2A}(-1)^{j-\sigma}a^c(\vec{p},-\sigma)e^{-ip\cdot x} + a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$\psi^{BA\dagger}_{-b,-a}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, (-1)^{a+b-\sigma}u^{AB}_{ab}(\vec{p},\sigma)[(-1)^{2A}(-1)^{j+\sigma}a^c(\vec{p},\sigma)e^{-ip\cdot x} + a^\dagger(\vec{p},\sigma)e^{ip\cdot x}]$$

$$(-1)^{-2A-a-b-j}\psi^{BA\dagger}_{-b,-a}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u^{AB}_{ab}(\vec{p},\sigma)[a^c(\vec{p},\sigma)e^{-ip\cdot x} + (-1)^{j-\sigma+2B}a^\dagger(\vec{p},-\sigma)e^{ip\cdot x}]$$

$$\xi^* = \xi^c \qquad C\psi^{AB}_{ab}(x)C^{-1} = \xi^*(-1)^{-2A-a-b-j}\psi^{BA\dagger}_{-b,-a}(x)$$

$$\xrightarrow{\text{粒子=反粒子}} \psi^{AB}_{ab}(x) = (-1)^{-2A-a-b-j}\psi^{BA\dagger}_{-b,-a}(x)$$

分立对称性变换性质

$$\psi_{ab}(x) = (2\pi)^{-3/2}\sum_\sigma \int d^3p\, u_{ab}(\vec{p},\sigma)[a(\vec{p},\sigma)e^{-ip\cdot x} + (-1)^{2B}(-1)^{j-\sigma}a^{c\dagger}(\vec{p},-\sigma)e^{ip\cdot x}]$$

$$u_{ab}(\vec{p},\sigma) = \frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)_{bb'}C_{AB}(j\sigma;a'b') \qquad v_{ab}(\vec{p},\sigma) = (-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma)$$

$$Ta(\vec{p},\sigma)T^{-1} = \zeta^*(-1)^{j-\sigma}a(-\vec{p},-\sigma) \qquad Ta^{c\dagger}(\vec{p},\sigma)T^{-1} = \zeta^c(-1)^{j-\sigma}a^{c\dagger}(-\vec{p},-\sigma)$$

$$T\psi^{AB}_{ab}(x)T^{-1} = (2\pi)^{-\frac{3}{2}}\sum_\sigma\int d^3p\, u^{AB}_{ab}(\vec{p},\sigma)^*(-1)^{j-\sigma}[\zeta^* a(-\vec{p},-\sigma)e^{ip\cdot x} + \zeta^c(-1)^{2B+j-\sigma}a^{c\dagger}(-\vec{p},\sigma)e^{-ip\cdot x}]$$

$$C_{AB}(j\sigma;ab) = (-1)^{A+B-j}C_{AB}(j,-\sigma;-a,-b) \Rightarrow u^{AB*}_{ab}(-\vec{p},-\sigma) = (-1)^{a+b+\sigma+A+B-j}u^{AB}_{-a,-b}(\vec{p},\sigma)$$

$$\zeta^c = \zeta^* \qquad T\psi^{AB}_{ab}(x)T^{-1} = \zeta^*(-1)^{a+b+A+B-2j}\psi^{AB}_{-a,-b}(\vec{x},-x^0)$$

$$\underline{CPT\psi^{AB}_{ab}(x)[CPT]^{-1} = \xi^*\eta^*\zeta^*(-1)^{2A}\psi^{AB\dagger}_{ab}(-x)}$$

量子场小结 $\psi(x)=\sum_\sigma\int\frac{d^3p}{(2\pi)^{3/2}}[e^{-ip\cdot x}u(\vec{p},\sigma)a(\vec{p},\sigma)+e^{ip\cdot x}v(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)]$

标量场: $\sigma=0$ $u(\vec{p})=v(\vec{p})=1/\sqrt{2(\vec{p}^2+M^2)}$ **自旋统计关系**

$\mathcal{L}_{自共轭}=\frac{1}{2}:[(\partial_\mu\phi(x))^2-M^2\phi^2(x)]:$ $(\partial^2+M^2)\phi(x)=0$ 没发现质量与对称性的联系

旋量场: $\sigma=\pm1/2$ $u(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))u(0,\sigma)$ $v(\vec{p},\sigma)=\sqrt{M/p^0}D(L(p))v(0,\sigma)$

$\mathcal{L}_0=:\overline{\psi}(x)(i\gamma^\mu\partial_\mu-M)\psi(x):$ $(i\gamma^\mu\partial_\mu-M)\psi(x)=0$ 旋量场质量与手征对称性相联系

矢量场: $\sigma=0,\pm1$ or ±1 $u(\vec{p},\sigma)=v^*(\vec{p},\sigma)=(2p^0)^{-1/2}e(\vec{p},\sigma)$ $e^\mu(\vec{p},\sigma)\equiv L^\mu{}_\nu(\vec{p})e^\nu(0,\sigma)$

$\mathcal{L}_{自共轭}=:\{-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)+\frac{1}{2}M^2v^2(x)\}:$ $(\partial^2+M^2)v^\mu(x)=0$ $\partial_\mu v^\mu(x)=0$

矢量场质量与规范对称性相联系

一般情况: $a=-A,\cdots,A;\ b=-B,\cdots,B$ $u_{ab}(\vec{p},\sigma)=\frac{1}{\sqrt{2p^0}}\sum_{a'b'}\left(e^{-\hat{p}\cdot\vec{J}^{(A)}\theta}\right)_{aa'}\left(e^{\hat{p}\cdot\vec{J}^{(B)}\theta}\right)_{bb'}C_{AB}(j\sigma;a'b')=(-1)^{j+\sigma}v_{ab}(\vec{p},-\sigma)$

$[\psi_l(\vec{x},t),\psi_{l'}(\vec{y},t)]_\mp=[\pi_l(\vec{x},t),\pi_{l'}(\vec{y},t)]_\mp=0$ $[\psi_l(\vec{x},t),\pi_{l'}(\vec{y},t)]_\mp=i\delta(\vec{x}-\vec{y})$

量子场小结续

$$\psi(x) = \sum_{\sigma} \int \frac{d^3p}{(2\pi)^{3/2}} [e^{-ip\cdot x} u(\vec{p},\sigma) a(\vec{p},\sigma) + e^{ip\cdot x} v(\vec{p},\sigma) a^{c\dagger}(\vec{p},\sigma)]$$

自旋统计关系

我们虽然建立了自由场理论,但还没讨论相互作用,因此还不能讨论S矩阵!

现在我们应该能够理解如下问题:

为什么所有具有同样自旋的自由粒子具有同样的波函数?

或说满足同样的场方程. 不管它在什么微观物质层次上, 不管它有无内在结构.

因为相对性原理要求它们属于同一个洛伦兹群的表示

这个表示与体系的尺度无关, 完全将波函数确定下来, 因而导致确定的方程!

能动量算符

自由量子场: $\psi(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}[e^{-ip\cdot x}u(\vec{p},\sigma)a(\vec{p},\sigma) + e^{ip\cdot x}v(\vec{p},\sigma)a^{c\dagger}(\vec{p},\sigma)]$

$$\dot{\psi}(\vec{x},t) = i[H_0, \psi(\vec{x},t)] = \frac{\delta H_0}{\delta\pi(\vec{x},t)} \qquad \dot{\pi}(\vec{x},t) = i[H_0, \pi(\vec{x},t)] = -\frac{\delta H_0}{\delta\psi(\vec{x},t)}$$

$$[\psi_l(\vec{x},t), \psi_{l'}(\vec{y},t)]_\mp = [\pi_l(\vec{x},t), \pi_{l'}(\vec{y},t)]_\mp = 0 \qquad [\psi_l(\vec{x},t), \pi_{l'}(\vec{y},t)]_\mp = i\delta(\vec{x}-\vec{y})$$

自由场的能动量算符:

$$H_0 \equiv P_0^0 = \int d\vec{x} \ :[-\mathcal{L}_0 + \pi_l(x)\dot{\psi}_l(x)]: \qquad \vec{P}_0 \equiv \int d\vec{x} \ :\pi_l(x)\nabla\psi_l(x):$$ 可直接算!

$$\dot{\vec{P}}_0 = \int d\vec{x} \ :\{\dot{\pi}_l(x)\nabla\psi_l(x) + \pi_l(x)\nabla\dot{\psi}_l(x)\}: = i\int d\vec{x}\{[H_0, \pi_l(x)]\nabla\psi_l(x) + \pi_l(x)\nabla[H_0, \psi_l(x)]\}$$
$$= i[H_0, \vec{P}_0] = 0$$

$$i[\vec{P}_0, \psi(\vec{x},t)] = \pm i\int d\vec{y}\,[\pi_l(\vec{y},t), \psi(\vec{x},t)]_\mp \nabla_y\psi_l(\vec{y},t) = \nabla\psi(\vec{x},t)$$

$$i[\vec{P}_0, \pi(\vec{x},t)] = i\int d\vec{y}\,\pi_l(\vec{y},t)\nabla_y[\psi_l(\vec{y},t), \pi(\vec{x},t)]_\mp = \nabla\pi(\vec{x},t)$$

相互作用绘景和海森堡绘景

自由量子场: $\psi(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}} [e^{-ip\cdot x} u(\vec{p},\sigma) a(\vec{p},\sigma) + e^{ip\cdot x} v(\vec{p},\sigma) a^{c\dagger}(\vec{p},\sigma)]$

$$\dot{\psi}(\vec{x},t) = i[H_0, \psi(\vec{x},t)] = \frac{\delta H_0}{\delta\pi(\vec{x},t)} \qquad \dot{\pi}(\vec{x},t) = i[H_0, \pi(\vec{x},t)] = -\frac{\delta H_0}{\delta\psi(\vec{x},t)}$$

$$[\psi_l(\vec{x},t), \psi_{l'}(\vec{y},t)]_\mp = [\pi_l(\vec{x},t), \pi_{l'}(\vec{y},t)]_\mp = 0 \qquad [\psi_l(\vec{x},t), \pi_{l'}(\vec{y},t)]_\mp = i\delta(\vec{x}-\vec{y})$$

相互作用: $S = U(+\infty, -\infty)$ $\quad \Omega(\tau) = e^{iH\tau} e^{-iH_0\tau}$ $\quad \Psi^\pm_\alpha = \Omega(\mp\infty)\Phi_\alpha$

$$U(\tau,\tau_0) = \Omega^\dagger(\tau)\Omega(\tau_0) = e^{iH_0\tau} e^{-iH(\tau-\tau_0)} e^{-iH_0\tau_0} = \mathbf{T}\, e^{-i\int_{\tau_0}^{\tau} dt\ V(t)}$$

$$V(t) \equiv e^{iH_0 t} V e^{-iH_0 t} = \int d\vec{x}\, \mathcal{H}(\vec{r},t) \qquad V = H - H_0$$

海森堡绘景的场算符: $\psi_H(\vec{x},t) \equiv \Omega(t)\psi(\vec{x},t)\Omega^\dagger(t)$ $\quad \pi_H(\vec{x},t) \equiv \Omega(t)\pi(\vec{x},t)\Omega^\dagger(t)$

$$\begin{aligned}\dot{\psi}_H(\vec{x},t) &= \dot{\Omega}(t)\psi(\vec{x},t)\Omega^\dagger(t) + \Omega(t)\dot{\psi}(\vec{x},t)\Omega^\dagger(t) + \Omega(t)\psi(\vec{x},t)\dot{\Omega}^\dagger(t) \\ &= i[H\Omega(t) - \Omega(t)H_0]\psi(\vec{x},t)\Omega^\dagger(t) + i\Omega(t)[H_0, \psi(\vec{x},t)]\Omega^\dagger(t) + i\Omega(t)\psi(\vec{x},t)[H_0\Omega^\dagger(t) - \Omega^\dagger(t)H] \\ &= i[H, \psi_H(\vec{x},t)]\end{aligned}$$

$$\dot{\pi}_H(\vec{x},t) = i[H, \pi_H(\vec{x},t)]$$

$$[\psi_{H,l}(\vec{x},t), \psi_{H,l'}(\vec{y},t)]_\mp = [\pi_{H,l}(\vec{x},t), \pi_{H,l'}(\vec{y},t)]_\mp = 0 \qquad [\psi_{H,l}(\vec{x},t), \pi_{H,l'}(\vec{y},t)]_\mp = i\delta(\vec{x}-\vec{y})$$

海森堡绘景算符时空平移

含相互作用的能动量算符:

$$H \equiv P^0 = \int d\vec{x} \;:[-\mathcal{L} + \pi_l(x)\dot{\psi}_l(x)]: \qquad \vec{P} \equiv \int d\vec{x}\; :\pi_l(x)\nabla\psi_l(x):$$

$$\pi_l \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}_l} \Rightarrow \text{相互作用不含广义速度} \Rightarrow \vec{P} = \vec{P}_0$$

$$\Omega(\tau) = e^{iH\tau}e^{-iH_0\tau} \qquad \psi_H(\vec{x},t) \equiv \Omega(t)\psi(\vec{x},t)\Omega^\dagger(t) \quad \pi_H(\vec{x},t) \equiv \Omega(t)\pi(\vec{x},t)\Omega^\dagger(t)$$

$$\Omega(0) = 1 \qquad \psi_H(\vec{x},0) \equiv \psi(\vec{x},0) \qquad \pi_H(\vec{x},0) \equiv \pi(\vec{x},0)$$

$$U_0(a)\psi(x)U_0^{-1}(a) = \psi(x+a) \qquad U_0(a) = e^{ia_\mu P_0^\mu}$$

$$\vec{P} = \vec{P}_0 \qquad U(a) = e^{ia_\mu P^\mu}$$

$$\begin{aligned}
\psi_H(x+a) &= \Omega(t+a^0)\psi(x+a)\Omega^\dagger(t+a^0)\\
&= e^{iH(t+a^0)}e^{-iH_0(t+a^0)}e^{i(a^0H_0-\vec{a}\cdot\vec{P}_0)}\psi(x)e^{-i(a^0H_0-\vec{a}\cdot\vec{P}_0)}e^{iH_0(t+a^0)}e^{-iH(t+a^0)}\\
&= e^{iHt}e^{iHa^0}e^{-iH_0t}e^{-i\vec{a}\cdot\vec{P}_0}\psi(x)e^{i\vec{a}\cdot\vec{P}_0}e^{iH_0t}e^{-iHa^0}e^{-iHt}\\
&= e^{i(Ha^0-\vec{a}\cdot\vec{P})}e^{iHt}e^{-iH_0t}\psi(x)e^{iH_0t}e^{-iHt}e^{-i(Ha^0-\vec{a}\cdot\vec{P})} = U(a)\psi_H(x)U^{-1}(a)
\end{aligned}$$

相互作用绘景和薛定谔绘景

自由量子场: $\psi(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}} [e^{-ip\cdot x} u(\vec{p},\sigma) a(\vec{p},\sigma) + e^{ip\cdot x} v(\vec{p},\sigma) a^{c\dagger}(\vec{p},\sigma)]$

$$\dot{\psi}(\vec{x},t) = i[H_0, \psi(\vec{x},t)] = \frac{\delta H_0}{\delta \pi(\vec{x},t)} \qquad \dot{\pi}(\vec{x},t) = i[H_0, \pi(\vec{x},t)] = -\frac{\delta H_0}{\delta \psi(\vec{x},t)}$$

$$[\psi_l(\vec{x},t), \psi_{l'}(\vec{y},t)]_\mp = [\pi_l(\vec{x},t), \pi_{l'}(\vec{y},t)]_\mp = 0 \qquad [\psi_l(\vec{x},t), \pi_{l'}(\vec{y},t)]_\mp = i\delta(\vec{x}-\vec{y})$$

相互作用: $S = U(+\infty, -\infty)$ $\quad \Omega(\tau) = e^{iH\tau} e^{-iH_0\tau}$ $\quad \Psi^\pm_\alpha = \Omega(\mp\infty)\Phi_\alpha$

$$U(\tau,\tau_0) = \Omega^\dagger(\tau)\Omega(\tau_0) = e^{iH_0\tau} e^{-iH(\tau-\tau_0)} e^{-iH_0\tau_0} = \mathbf{T}\, e^{-i\int_{\tau_0}^{\tau} dt\, V(t)}$$

$$V(t) \equiv e^{iH_0 t} V e^{-iH_0 t} = \int d\vec{x}\, \mathcal{H}(\vec{r},t) \qquad V = H - H_0$$

薛定谔绘景的场算符: $\psi_S(\vec{x},t) \equiv e^{-iH_0 t}\psi(\vec{x},t)e^{iH_0 t}$ $\quad \pi_S(\vec{x},t) \equiv e^{-iH_0 t}\pi(\vec{x},t)e^{iH_0 t}$

$$\dot{\psi}_S(\vec{x},t) = \dot{e}^{-iH_0 t}\psi(\vec{x},t)e^{iH_0 t} + e^{-iH_0 t}\dot{\psi}(\vec{x},t)e^{iH_0 t} + e^{-iH_0 t}\psi(\vec{x},t)\dot{e}^{iH_0 t}$$
$$= -ie^{-iH_0 t}H_0\psi(\vec{x},t)e^{iH_0 t} + ie^{-iH_0 t}[H_0, \psi(\vec{x},t)]e^{iH_0 t} + ie^{-iH_0 t}\psi(\vec{x},t)H_0 e^{iH_0 t} = 0$$

$$\dot{\pi}_S(\vec{x},t) = 0$$

$$[\psi_{S,l}(\vec{x},t), \psi_{S,l'}(\vec{y},t)]_\mp = [\pi_{S,l}(\vec{x},t), \pi_{S,l'}(\vec{y},t)]_\mp = 0 \qquad [\psi_{S,l}(\vec{x},t), \pi_{S,l'}(\vec{y},t)]_\mp = i\delta(\vec{x}-\vec{y})$$

关于用场量表达的哈密顿量

$\dot{H}_0 = 0$ 明显验证

$$\psi_S(\vec{x},t) \equiv e^{-iH_0 t}\psi(\vec{x},t)e^{iH_0 t} \qquad \pi_S(\vec{x},t) \equiv e^{-iH_0 t}\pi(\vec{x},t)e^{iH_0 t}$$

$$\psi_S(\vec{x},0) = \psi(\vec{x},0) \qquad \pi_S(\vec{x},0) = \pi(\vec{x},0)$$

$$H \equiv H(\psi_S,\pi_S) = e^{-iH_0 t}H(\psi,\pi)e^{iH_0 t} \qquad H_0 = e^{-iH_0 t}H_0(\psi,\pi)e^{iH_0 t} \qquad V = e^{-iH_0 t}V(\psi,\pi)e^{iH_0 t}$$

相互作用： $S = U(+\infty,-\infty)$ $\quad \Omega(\tau) = e^{iH\tau}e^{-iH_0\tau}$ $\quad \Psi_\alpha^\pm = \Omega(\mp\infty)\Phi_\alpha$

$$U(\tau,\tau_0) = \Omega^\dagger(\tau)\Omega(\tau_0) = e^{iH_0\tau}e^{-iH(\tau-\tau_0)}e^{-iH_0\tau_0} = \mathbf{T}\ e^{-i\int_{\tau_0}^{\tau}dt\ V(t)}$$

$$V(t) \equiv e^{iH_0 t}Ve^{-iH_0 t} = \int d\vec{x}\ \mathcal{H}(\vec{r},t) \qquad V = H - H_0 \qquad [\mathcal{H}(x),\mathcal{H}(x')] \overset{\text{类空}}{=\!=\!=} 0$$

$$\int d\vec{x}\ \mathcal{H}(\vec{r},t) = e^{iH_0 t}Ve^{-iH_0 t} = V(\psi,\pi) \qquad S = U(\infty,-\infty) = \mathbf{T}e^{-i\int_{-\infty}^{\infty}dt\ V(\psi,\pi)} = \mathbf{T}e^{-i\int d^4x\ \mathcal{H}(x)}$$

给出$V(\psi,\pi)$,利用ψ,π的产生湮灭算符展开即可计算S矩阵元⇒正则计算体系!

Wick定理给出进一步的简化!

CPT定理

$$\int d\vec{x}\,\mathcal{H}(\vec{r},t) = e^{iH_0 t} V e^{-iH_0 t} = V(\psi,\pi) \qquad S = U(\infty,-\infty) = \mathbf{T}\, e^{-i\int d^4x\,\mathcal{H}(x)}$$

$\mathcal{H}(\vec{r},t)$是厄米的洛伦兹标量

$$CPT\psi_{ab}^{AB}(x)[CPT]^{-1} = \xi^*\eta^*\zeta^*(-1)^{2A}\psi_{ab}^{AB\dagger}(-x)$$

为使$\psi_{a_1b_1}^{A_1B_1}(x)\psi_{a_2b_2}^{A_2B_2}(x)\cdots$能耦合成标量$\mathcal{H}(\vec{r},t)$：

沿z轴的无穷小转动不变导致$a_1+a_2+\cdots+b_1+b_2+\cdots=0$；沿$z$轴的无穷小推进不变导致$a_1+a_2+\cdots-b_1-b_2-\cdots=0$

$$(-1)^{2a_1+2a_2+\cdots} = 1 \;\Rightarrow\; (-1)^{2(a_i-A_i)} = 1 \;\Rightarrow\; (-1)^{2A_1+2A_2+\cdots} = 1$$

为使$\psi_{a_1b_1}^{A_1B_1}(x)\psi_{a_2b_2}^{A_2B_2}(x)\cdots$能保证$\mathcal{H}(\vec{r},t)$厄米，$\psi_{a_1b_1}^{A_1B_1}(x)\psi_{a_2b_2}^{A_2B_2}(x)\cdots$必须与其共轭场同时出现

因而$\psi_{a_1b_1}^{A_1B_1}(x)\psi_{a_2b_2}^{A_2B_2}(x)\cdots$ 中**CPT**变换产生的复常数相角无贡献！

$$[CPT]\,\mathcal{H}(x)[CPT]^{-1} = \mathcal{H}(-x)$$

$$[CPT]\,S\,[CPT]^{-1} = \mathbf{T}\,e^{-i\int d^4x\,CPT\mathcal{H}(x)[CPT]^{-1}} = \mathbf{T}\,e^{-i\int d^4x\,\mathcal{H}(-x)} = \mathbf{T}\,e^{-i\int d^4x\,\mathcal{H}(-x)} = S$$