Duality in Condensed Matter Physics and Quantum Field Theory

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Motivation

- Dualities in CM and QFT
- Particle-Vortex duality
- Applications to the Fractional Quantum Hall Effect
- Conjectured dualities, bosonization and fermionization
- Loop models: flux attachment, duality and periodicity
- Periodicity vs Fractional Spin
- Implications for Fractional Quantum Hall fluids

Dualities

- EM duality: E $\Longleftrightarrow$ B, electric charges $\Longleftrightarrow$ magnetic monopoles$\Rightarrow$Dirac quantization
- 2D Ising Model: Kramers-Wannier duality, high T $\Longleftrightarrow$ low T, order $\Longleftrightarrow$ disorder
- Duality of the 3D $\mathbb{Z}_2$ gauge theory $\Longleftrightarrow$ 3D Ising model, order $\Longleftrightarrow$ confinement
- Particle-Vortex duality: electric charge $\Longleftrightarrow$ vortex (magnetic charge)
- Mappings between phases of matter, most often between different theories
- Conjectured web of dualities between CFTs in 2+1 dimensions

Electromagnetic Duality

Electric-magnetic asymmetry of Maxwell's equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j}, \qquad \boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

- EM duality: **E** ⟺ **B**
- electric charge *e* ⟺ magnetic monopole *m*
- Dirac quantization $em = 2\pi$

Duality of Forms

- Geometric duality
- *p* forms in *D* dimensions are dual to *D-p* forms
- In *D=2* the dual of a vector is a vector, $J_\mu^*=\varepsilon_{\mu\nu} J^\nu$, and the dual of a 2nd rank tensor is a scalar, $F_{\mu\nu}=\varepsilon_{\mu\nu}\,\theta$
- In *D*=*3* the dual of a vector is a 2nd rank tensor, $J_\mu^*=1/2\ \varepsilon_{\mu\nu\lambda} F^{\nu\lambda}$ (and the dual of a 2nd rank tensor is a vector), etc. Duality exchanges the vector potential A_μ with a *compactified scalar* θ
- In *D*=*4* duality exchanges $F_{\mu\nu} \Leftrightarrow F_{\mu\nu}^*=1/2\ \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}$, $\boldsymbol{E} \Leftrightarrow \boldsymbol{B}$
- Lattice duality: in *D*=*2* the dual of a link is a link, and the dual of a plaquette is a site
- In *D*=*3* the dual of a link is an (oriented) plaquette (and viceversa), and the dual of a 3 volume is a site (and viceversa)

Duality the Maxwell field in D=2+1

- Canonical quantization of the Maxwell field in the gauge *A_0=0*
- In 2+1 dimensions there is only *one* transverse degree of freedom
- It is *equivalent* (dual!) to a *compactified* scalar
- The compactified scalar is a Goldstone field
- Charge quantization implies compactification (periodicity) of the dual scalar field

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \Rightarrow E_i = \epsilon_{ij}\partial_j \theta$$

$$[E_i(\boldsymbol{x}), A_j(\boldsymbol{y})] = i\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{y})$$

$$[E_i(\boldsymbol{x}), B(\boldsymbol{y})] = i\epsilon_{ij}\partial_j\delta(\boldsymbol{x}-\boldsymbol{y})$$

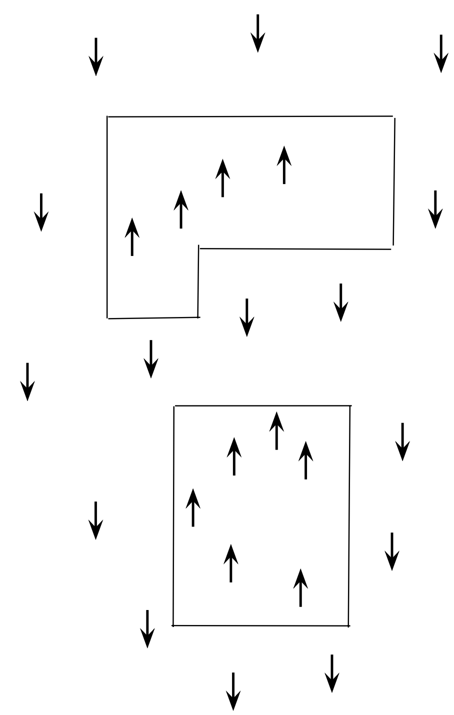
$$[\theta(\boldsymbol{x}), B(\boldsymbol{y})] = i\delta(\boldsymbol{x}-\boldsymbol{y}) \Rightarrow \theta \equiv \Pi$$

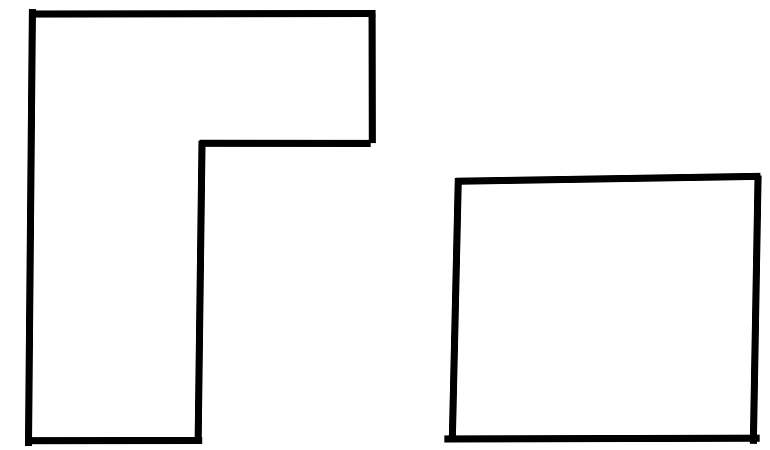
$$\begin{aligned}\mathcal{H} =& \frac{1}{2}\left(\boldsymbol{E}^2 + B^2\right)\\ =& \frac{1}{2}\left((\nabla\theta)^2 + \Pi^2\right)\end{aligned}$$

$$\nabla \cdot \boldsymbol{E} = 2\pi n \delta(\boldsymbol{x}) \Rightarrow \Delta\theta_\gamma = \oint_\gamma dx_i \partial_i \theta = 2\pi n$$

Duality in Classical Statistical Mechanics

- 2D Ising Model: Kramers-Wannier (self) duality
- Partition function as a sum over closed domain walls in the low T expansion
- Partition function as a sum over loops of the high T expansion
- high T ⟺ low T
- 2D: order ⟺ disorder
- 3D: Duality of the $\mathbb{Z}_2$ gauge theory ⟺ Ising model
- high T loops and 3D surfaces of domains
- 3D: order ⟺ confinement; disorder ⟺ deconfinement

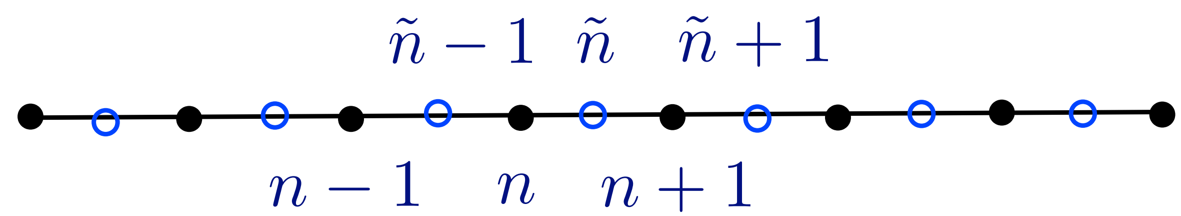




$$Z_{\text{domains}}[\exp(-2/T)] = Z_{\text{loops}}[\tanh(1/T)]$$

- Closed loops of the high T expansion: Euclidean worldlines of massive neutral scalar particles of the symmetric phase
- In D=2 the closed domain walls represent the Euclidean evolution of *kinks* (solitons)
- in D=3 the closed domain walls represent the evolution of closed *strings*

Duality and the d=1 Quantum Ising Model



- Define a Pauli (Clifford) algebra in terms of (the kink operator) τ_3 and τ_1 defined on the dual lattice
- Maps λ to $1/\lambda$ (strong coupling and weak coupling)
- Order and disorder
- Disordered phase is a kink condensate

$$H = -\sum_n \sigma_1(n) - \lambda \sum_n \sigma_3(n)\sigma_3(n)$$

$$\tau_3(\tilde{n}) = \prod_{j\leq n} \sigma_1(j)$$

$$\{\sigma_1(n), \sigma_3(k)\} = 0 \Rightarrow \{\tau_1(n), \tau_3(k)\} = 0$$

$$\sigma_3(n)^2 = \sigma_1(n)^2 = 1 \Rightarrow \tau_3(n)^2 = \tau_1(n)^2 = 1$$

$$\tau_1(\tilde{n}) = \sigma_3(n)\sigma_3(n+1)$$

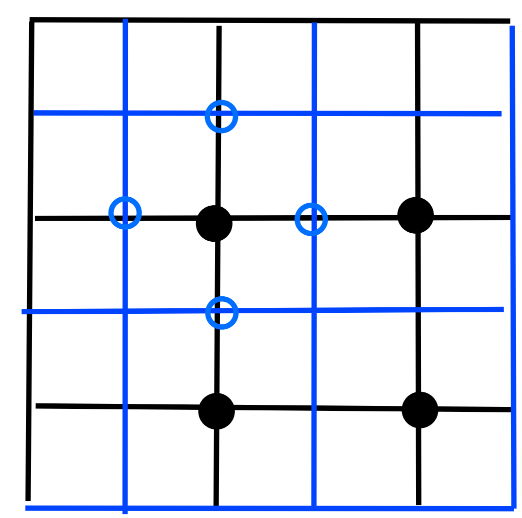
$$\tau_3(\tilde{n}-1)\tau_3(\tilde{n}) = \sigma_1(n)$$

$$H = -\sum_{\tilde{n}} \tau_3(\tilde{n})\tau_3(\tilde{n}+1) - \lambda \sum_{\tilde{n}} \tau_1(\tilde{n})$$

What happens in 2+1 dimensions?

- The dual of the 2d quantum Ising model is the 2d $\mathbb{Z}_2$ gauge theory
- The gauge fields reside on the links of the *dual* lattice
- Duality maps order to confinement and disorder to deconfinement
- Ising order parameter maps onto a $\mathbb{Z}_2$ magnetic charge ("monopole")

$$H = -\sum_{\boldsymbol{r}} \sigma_1(\boldsymbol{r}) - \lambda \sum_{\boldsymbol{r}, j=1,2} \sigma_3(\boldsymbol{r})\sigma_3(\boldsymbol{r}+\boldsymbol{e}_j)$$



$$\tau_1(\tilde{\boldsymbol{r}}, 1) = \sigma_3(\boldsymbol{r})\sigma_3(\boldsymbol{r}+\boldsymbol{e}_2)$$

$$\sigma_1(\boldsymbol{r}) = \tau_3(\tilde{\boldsymbol{r}}, 1)\tau_3(\tilde{\boldsymbol{r}}+\boldsymbol{e}_1, 2)\tau_3(\tilde{\boldsymbol{r}}, 2)\tau_3(\tilde{\boldsymbol{r}}+\boldsymbol{e}_2, 1)$$

$$H = -\lambda \sum_{\tilde{\boldsymbol{r}}, j} \tau_1(\boldsymbol{r}, j) - \sum_{\tilde{\boldsymbol{r}}} \tau_3(\tilde{\boldsymbol{r}}, 1)\tau_3(\tilde{\boldsymbol{r}}+\boldsymbol{e}_1, 2)\tau_3(\tilde{\boldsymbol{r}}, 2)\tau_3(\tilde{\boldsymbol{r}}_1+\boldsymbol{e}_2)$$

$$\text{Gauss Law}: \quad \tau_1(\tilde{\boldsymbol{r}}, 1)\tau_1(\tilde{\boldsymbol{r}}-\boldsymbol{e}_1, 1)\tau_1(\tilde{\boldsymbol{r}}, 2)\tau_1(\tilde{\boldsymbol{r}}-\boldsymbol{e}_2, 2) = 1$$

Bosonization in 1+1 dimensions as duality

- 1d free fermions at low energies are equivalent to a free massless Dirac field, $\psi(x)=(\psi_R(x), \psi_L(x))$
- Conserved current j_μ
- Current algebra
- Equivalent to the algebra of a canonical massless compactified boson
- Addition of one fermion Q=1, implies that the boson must obey twisted boundary conditions
- Axial current $j_\mu^5=\varepsilon_{\mu\nu} j^\nu$ is not conserved (axial anomaly)

$$\partial^\mu j_\mu^5 = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \Leftrightarrow \partial^2\phi = \frac{e}{\sqrt{\pi}}F^*$$

$$\mathcal{H} = -i\psi_R^\dagger\partial_x\psi_R + i\psi_L^\dagger\partial_x\psi_L$$

$$j_0 = :\psi_R^\dagger\psi_R: + :\psi_L^\dagger\psi_L:$$

$$j_1 = :\psi_R^\dagger\psi_R: - :\psi_L^\dagger\psi_L:$$

$$[j_0(x), j_1(y)] = -\frac{i}{\pi}\delta'(x-y)$$

$$j_0 \equiv \frac{1}{\sqrt{\pi}}\partial_x\phi, \quad j_1 \equiv -\frac{1}{\sqrt{\pi}}\Pi$$

$$j_\mu = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi, \quad \partial_\mu j^\mu = 0$$

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_x\phi)^2$$

$$Q = \int j_0(x)dx = \frac{1}{\sqrt{\pi}}\Delta\phi$$

Duality in the Classical 3D XY Model

At high T the partition function is a sum over closed particle loops

$$Z_{XY} = \prod_{\boldsymbol{r}} \int_0^{2\pi} \frac{d\theta(\boldsymbol{r})}{2\pi} \exp(\beta \sum_{\boldsymbol{r},\mu} \cos(\Delta_\mu \theta(\boldsymbol{r}))$$

$$\propto \sum_{\ell_\mu(\boldsymbol{r}) \in \mathbb{Z}} \prod_{\boldsymbol{r}} \delta(\Delta_\mu \ell_\mu(\boldsymbol{r})) \exp(-\sum_{\boldsymbol{r},\mu} \frac{\ell_\mu(\boldsymbol{r})^2}{2\beta})$$

At low T it can be written a sum over vortex loops

$$Z_{XY} \propto \sum_{\ell_\mu(\boldsymbol{r}) \in \mathbb{Z}} \prod_{\boldsymbol{r}} \delta(\Delta_\mu \ell_\mu(\boldsymbol{r})) \exp(-\sum_{\boldsymbol{r},\mu} \frac{\ell_\mu(\boldsymbol{r})^2}{2\beta})$$

$$= \sum_{s_\mu(\tilde{\boldsymbol{r}}) \in \mathbb{Z}} \prod_{\tilde{\boldsymbol{r}}} \delta(\Delta_\mu s_\mu) \exp(-\frac{1}{2\beta} \sum_{\tilde{\boldsymbol{r}},\mu} (\epsilon_{\mu\nu\lambda} \Delta_\nu s_\lambda)^2)$$

$$= \sum_{m_\mu(\boldsymbol{r})} \prod_{\tilde{\boldsymbol{r}},\mu} d\phi_\mu(\tilde{\boldsymbol{r}}) \exp(-\frac{1}{2\beta} \sum_{\tilde{\boldsymbol{r}},\mu} (\epsilon_{\mu\nu\lambda} \Delta_\nu \phi_\lambda)^2 + i2\pi \sum_{\tilde{\boldsymbol{r}},\mu} m_\mu(\boldsymbol{r}) \phi_\mu(\boldsymbol{r}))$$

$$= \sum_{m_\mu(\boldsymbol{r})} \exp(-2\pi^2 \beta \sum m_\mu(\tilde{\boldsymbol{r}}) G_{\mu\nu}(\tilde{\boldsymbol{r}} - \tilde{\boldsymbol{r}}') m_\nu(\tilde{\boldsymbol{r}}'))$$

$$\text{where } G_{\mu\nu}(\tilde{\boldsymbol{r}} - \tilde{\boldsymbol{r}}') = \langle \phi_\mu(\tilde{\boldsymbol{r}}) \phi_\nu(\tilde{\boldsymbol{r}}') \rangle$$

Particle-Vortex Duality

- Theories with a global U(1) symmetry, e.g. the 3D XY model
- High T expansion loop gas: worldlines of charged particles with short-range interactions
- Low T expansion: closed vortex loops with Biot-Savart long-range interactions
- Particle-Vortex duality: electric charge $\Leftrightarrow$ vortex (magnetic charge)
- The situation reverses for a *XY* model is coupled to a fluctuating Maxwell field: Particle loops have long range Coulomb interactions, and vortex loops have short range interactions (Higgs mechanism)
- The two models are dual to each other! $Z(\beta, e) \simeq Z\left(\frac{e^2}{4\pi}, \frac{1}{2\beta}\right)$
- In field theory language

$$|(\partial_\mu + iA_\mu)\phi|^2 + m^2|\phi|^2 + \lambda|\phi|^4 \leftrightarrow |(\partial_\mu + ia_\mu)\tilde{\phi}|^2 - m^2|\tilde{\phi}|^2 + \lambda|\tilde{\phi}|^4 + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu A_\lambda$$

$$j_\mu \leftrightarrow \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$$

The Fractional Quantum Hall Effect

- Two-dimensional system of N_e electrons in Landau levels created by a large external uniform magnetic field with N_Φ fluxes
- Filling fraction: $\nu = N_e/N_\Phi$
- Quantized Hall conductivity σ_{xy}=p/(2np±1) e^2/h (Jain fractions) (p, n $\in \mathbb{Z}$)
- Laughlin states: ν=1/m (m $\in \mathbb{Z}$) (m odd for fermions, even for bosons)
- Laughlin wavefunction: $\Psi(z_1, \ldots, z_N) = \prod_{i<j=1}^{N} (z_i - z_j)^m \ \exp(-\frac{1}{4\ell_0^2} \sum_{i=1}^{N} |z_i|^2)$
- Statistical transmutation of charge-flux composites (Wilczek)
- Composite bosons: m fluxes attached to bosons (Zhang, Hansson and Kivelson; Read)
- Composite fermions: (m-1) fluxes attached to fermions (Jain)
- Field theory: Chern-Simons gauge field encodes flux attachment

$$\mathcal{L} = \frac{m}{4\pi} \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda - j_\mu a^\mu \Rightarrow j_0 = \frac{m}{2\pi} \epsilon_{ij} \partial_i a_j \text{ and } [a_i(x), a_j(y)] = i \frac{2\pi}{m} \epsilon_{ij} \delta(x - y)$$

Composite Boson Picture of the FQHE

- Landau-Ginzburg theory (Zhang, Hansson and Kivelson): Non-Relativistic abelian-Higgs model with a Chern-Simons term: composite bosons coupled to *m* fluxes

$$\mathcal{S}_{\mathrm{B}} = \int d^3z \, \left\{ \phi^*(z)[iD_0 + \mu]\phi(z) + \frac{\hbar^2}{2M}|\boldsymbol{D}\phi(z)|^2 + \frac{1}{4\pi m}\epsilon_{\mu\nu\lambda}a^\mu \partial^\nu a^\lambda \right\}$$
$$- \frac{1}{2}\int d^3z \int d^3z' \, (|\phi(z)|^2 - \rho_0)V(|\boldsymbol{z} - \boldsymbol{z}'|)(|\phi(z')|^2 - \rho_0)$$

$$D_\mu = \partial_\mu + i(A_\mu + a_\mu) \qquad \phi(x) = \sqrt{\rho(x)}e^{i\omega(x)}$$

$$\frac{1}{2\pi m}\epsilon_{ij}\partial_i a_j + |\phi(x)|^2 = 0, \int d^3x |\phi(x)|^2 = \rho_0 L^2 T$$

- FQH plateau: composite bosons condense

$$\langle a_i \rangle + A_i = 0 \qquad |\phi|^2 = \rho_0 = \frac{\nu}{2\pi \ell_0^2} \qquad \nu = \frac{1}{m}, \ell_0^2 = \frac{1}{B}$$

Fluctuations and the FQHE

- Effective action for quantum fluctuations: $a_\mu = \langle a_\mu \rangle + \delta a_\mu$; probe field δA_i

$$\mathcal{L}_{\text{eff}} = \frac{\kappa}{2}\left(\partial_0 \omega - \delta a_0 - e\delta A_0\right)^2 - \frac{\rho_s}{2}\left(\nabla\omega - \delta \boldsymbol{a} - e\delta \boldsymbol{A}\right)^2 + \frac{1}{2\pi m}\epsilon_{\mu\nu\lambda}\delta a^\mu \partial^\nu \delta a^\lambda$$

- Integrating out the fluctuations δa_μ

$$\mathcal{L}\text{eff}[\delta A_\mu] = \frac{e^2}{4\pi m}\epsilon_{\mu\nu\lambda}\delta A^\mu \partial^\nu \delta A^\lambda + \ldots \Rightarrow \sigma_{xy} = \frac{e^2}{2\pi m} = \frac{1}{m}\left(\frac{e^2}{h}\right)$$

- Vortices: $$\lim_{|\boldsymbol{x}|\to\infty} \phi(\boldsymbol{x}) = \sqrt{\rho_0}\, e^{i\varphi(\boldsymbol{x})},\ \delta a_0 = 0,\ \varphi(\boldsymbol{x}) = \tan^{-1}(y/x)$$

$$\lim_{|\boldsymbol{x}|\to\infty} \delta a_i = \pm\, \partial_i \varphi = \pm \epsilon_{ij}\frac{x_j}{|\boldsymbol{x}|^2}$$

$$\Rightarrow \oint_\gamma \delta \boldsymbol{a}\cdot d\boldsymbol{x} = \pm\, 2\pi$$

- Vortices have finite energy (not logarithmic!) and fractional charge:

$$Q = \frac{e}{2\pi m}\int_\Sigma d^2x\ \epsilon_{ij}\partial_i \delta a_j = \frac{e}{2\pi m}\oint_{\partial\Sigma} d\boldsymbol{x}\cdot\delta\boldsymbol{a} = \pm\frac{e}{m}$$

Vortex Partition Function

- Using the identity $\mathcal{L} = \frac{m}{4\pi}\epsilon_{\mu\nu\lambda}b^\mu\partial^\nu b^\lambda + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}b^\mu\partial^\nu a^\lambda \equiv -\frac{1}{4\pi m}\epsilon_{\mu\nu\lambda}a^\mu\partial^\nu a^\lambda$

  - Effective topological field theory in terms of the hydrodynamic field b_μ (Wen)

$$\mathcal{L}_{\text{eff}} = -\frac{m}{4\pi}\epsilon_{\mu\nu\lambda}b^\mu\partial^\nu b^\lambda + \frac{e}{2\pi}\epsilon_{\mu\nu\lambda}\delta A^\mu\partial^\nu b^\lambda + j^v_\mu b^\mu$$

- $j^v{}_\mu$ is a current that represents slowly-varying vortex *worldlines*

- We recover the correct Hall conductivity and the fractional charge of the vortex

- Integrating out the field b_μ we obtain the partition function of the worldlines of the vortices whose action is i times the *Hopf invariant (or linking number)*

- Fractional statistics! $S_{\text{eff}}[j^v_\mu] = \frac{\pi}{m}\int d^3x \int d^3y \; j^v_\mu(x)\epsilon^{\mu\nu\lambda}\langle x|\frac{1}{\partial^2}|x'\rangle \; \partial^y_\lambda \; j^v_\nu(y)$

- The vortex excitations of a FQH state are represented by a the worldlines of vortices with this effective action (a loop model!)

- In FQH insulator one obtains a similar expression for the fermions (with m=1)

Duality in the FQHE

- Both vortices and fermions are described by a model of loops that close in imaginary time
- Both sides of the plateau transition are described by worldlines representing massive particles
- The transition between FQH states can then be thought as the condensation of some anyons with the two phases being related by duality
- Other states can be thought of being obtained by “addition of Landau levels”
- Duality and Landau level addition do not commute as operations
- $SL(2,\mathbb{Z})$ symmetry: Universal phase diagram for the FQH states based on particle-vortex duality (Kivelson, Lee and Zhang) with “super-universal” transitions (superconductor-insulator transition)
- Suggests that there is self-duality at the plateau transitions (I $\longleftrightarrow$ V) (Shimshoni, Sondhi and Shahar)

Composite Fermion Perspective

- We can also use flux attachment to map *fermions* to *composite fermions* by attaching an even number of fluxes
- Non-relativistic composite fermions at finite density coupled to a Chern-Simons gauge field with prefactor $1/2\pi(m-1)$ (López and Fradkin)
- For the Jain electron filling fractions $\nu_{\pm}=p/(2np\pm1)$, the composite fermions fill *p* Landau levels of a reduced effective magnetic field with a gap $\sim 1/(2np\pm1)$
- These are the fractions seen in experiment!
- Upon the computation of quantum fluctuations at the quadratic level one obtains a FQHE with $\sigma_{xy}=\nu\, e^2/h$.
- Composite fermions become anyons with fractional statistics $\pi/(2np\pm1)$, and charge $e/(2np\pm1)$
- The hydrodynamic (topological) field theory has the same general form (Wen)
- This theory predicts that the FQH fluid becomes *compressible* for $\nu=1/2n$ (as $p \mapsto \infty$)!

Topology and Geometry

- Both theories lead to a unified description of the FQH states as topological fluids
- Hall conductivity and the quantum numbers of the vortices
- On a closed surface of genus *g* (g=0 for a sphere, 1 for a torus, 2 for a pretzel, etc) the fluid has a topological degeneracy of m^g
- For of *non-abelian FQH* states this leads to the concept of a topological *qubit*
- In addition, the fluid can also sense the *geometry* (i.e. the *curvature*) of the surface through the coupling to the *spin connection* ω_μ through the topological spin *s=m/2* of the vortices
- New “universal” numbers: the Hall viscosity $\eta_H = s\rho_0/2 = m\rho_0/4$, the shift (Wen-Zee term), and a gravitational Chern-Simons term (edge thermal conductivity) (c=1)

$$
\begin{aligned}
\mathcal{L} = \rho_0 \delta A_0 + \frac{m}{2}\rho_0 \omega_0 - \frac{m}{4\pi}\epsilon_{\mu\nu\lambda} b^\mu \partial^\nu b^\lambda - \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\delta A^\mu \partial^\nu b^\lambda \\
- \frac{m}{2}\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\omega^\mu \partial^\nu b^\lambda - \frac{1}{48\pi}\epsilon_{\mu\nu\lambda}\omega^\mu \partial^\nu \omega^\lambda
\end{aligned}
$$

Compressible States and the (Non) Fermi Liquid

- Non-relativistic composite fermions at fixed density with a Fermi surface (Halperin, Lee, and Read) as $p \mapsto \infty$ (the FQH gap collapses)
- At fixed electron density the compressible state is reach at a field B_c
- In mean field theory it is a Fermi liquid
- Successful to explain several experiments
- Predicts quantum oscillations as a function of $B-B_c$ (seen in experiment)
- Compressible states seen at ν=1/2, 1/4, 3/4
- Pairing of composite fermions in the p+ip channel leads to the Moore-Read non-abelian FQH state (formally at ν=1/2 but works for ν=5/2)

Problems at ν=1/2

- In the high magnetic field limit, at ν=1/2 we expect to see particle-hole symmetry
- The HLR Fermi liquid is manifestly not particle-hole symmetric (Kivelson and DH Lee)
- It also has a large amount of Landau level mixing (largest in the compressible states!)
- The theory also has dynamical Chern-Simons gauge fields ⇒ non-Fermi liquid!
- The Jain fractions predict that all compressible states are limits of two converging sequences with $\nu_{\pm}$=p/(2np±1): "mirror symmetry"?
- DT Son proposed to to describe the compressible states in terms of *relativistic spinor (Dirac)* field ψ, which is particle-hole symmetric

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - a\!\!\!/)\psi + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}A^{\mu}\partial^{\nu}a^{\lambda}$$

- Relativistic flux attachement
- FQH states: Dirac mass and a chemical potential
- PH symmetric paired state (Jackiw-Rossi ↦ Read-Green p+ip)
- One of the motivations of the web of dualities

Functional Bosonization

Early approach to bosonization of the fermion path integral deep in a massive phase (EF & F. Schaposnik; C. Burgess and F. Quevedo)

$$Z[A^{\rm ex}] = \int \mathcal{D}\left[\bar{\psi},\psi\right] \exp\left(iS_F[\bar{\psi},\psi,A^{\rm ex}]\right)$$

To compute current correlators

$$\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2)\cdots\rangle = \frac{1}{i}\frac{\delta}{\delta A^{\rm ex}_{\mu_1}(x_1)}\frac{1}{i}\frac{\delta}{\delta A^{\rm ex}_{\mu_2}(x_2)}\cdots \ln Z[A^{\rm ex}]$$

Use gauge invariance of the fermion path integral: shift A^{ex} to $A^{ex} + a$, where a is a gauge transformation: $f_{\mu\nu}[a] = 0$

$$Z[A^{\mathrm{ex}} + a] = Z[A^{\mathrm{ex}}]. \qquad\qquad Z[A^{\mathrm{ex}}] = \int \mathcal{D}[a]_{\mathrm{pure}} Z[A^{\mathrm{ex}} + a]$$

$$Z[A^{\mathrm{ex}}] = \int \mathcal{D}[a,b] Z[a] \times \exp\Big(-\frac{i}{2} \int d^D x \, b_{\mu\nu\ldots} \epsilon^{\mu\nu\cdots\alpha\beta} \left(f_{\alpha\beta}[a] - f_{\alpha\beta}[A^{\mathrm{ex}}]\right) \Big)$$

The form of the partition function Z[a] depends on the dimension (and regularization)

$$\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2) \cdots \rangle = \langle \epsilon^{\mu_1\nu_1\lambda_1\cdots} \partial_{\nu_1} b_{\lambda_1\ldots}(x_1) \epsilon^{\mu_2\nu_2\lambda_2\cdots} \partial_{\nu_2} b_{\lambda_2\ldots}(x_2) \cdots \rangle$$

$$j^{\mu}(x) \quad \equiv \quad \epsilon^{\mu\nu\lambda\rho\cdots} \partial_{\nu} b_{\lambda\rho\ldots}(x) \quad \Leftrightarrow \quad \partial_{\mu} j^{\mu} = 0$$

- This procedure is meaningful only if the effective action of the gauge field is local
- This works in 1+1 dimensions for massless relativistic fermions
- For D>1+1 it works only as en effective action for low energy degrees of freedom if the theory is massive
- For general dimension Z[a] can be computed only in the massive theory.
- The effective action is an expansion in 1/mass
- This approach does not work in a theory at (or even close to) a fixed point
- This leads to a hydrodynamic description of the massive phase
- For systems with a Fermi surface one obtains the Landau theory of the Fermi liquid

Example: Polyacetylene

- Fermions in d=1 with a spontaneously broken translation symmetry: broken chiral symmetry (Class AIII)
- It is a half-filled system of spin 1/2 fermions (the π electrons of the carbon atoms) coupled to an optical phonon vibration of the $(CH)_n$ chain
- As usual in d=1 we can decompose the electron field into its right and left moving components

$$\psi(x) = e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x)$$

A uniform displacement of the charge profile is equivalent to a chiral transformation

$$\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix} \to e^{i\sigma_3\theta}\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix} \qquad \rho(x) \to \rho(x+a) \Rightarrow \theta = k_F a$$

Charge Fractionalization

- At half-filling this system has a Peierls instability
- Spontaneous breaking of translation invariance: CDW on the bonds with wave vector $Q=2k_F=\pi$
- Gap in the spectrum of fermions
- Effective theory: Dirac (Weyl) fermions with a dynamically generated mass
- This system has soliton excitations which correspond to winding of θ

Topological invariant (Goldstone & Wilczek) $$\nu = \frac{\theta(+\infty) - \theta(-\infty)}{2\pi}$$

Charge conjugation (particle-hole) θ=nπ (mod 2π)

$$Z[A^{\text{ex}}] = \int \mathcal{D}[a,b] \exp\left(i \int d^D x \mathcal{L} \right)$$

$$\mathcal{L} = -b\epsilon^{\mu\nu}\partial_\mu(a_\nu - A^{\text{ex}}_\nu) + \frac{\theta}{2\pi}\epsilon^{\mu\nu}\partial_\mu a_\nu + \cdots$$

D=2+1 Chern Insulator (Class A or D)

- Free fermions with broken time reversal invariance: integer quantum Hall states and the quantum anomalous Hall state
- These states are characterized by a topological invariant, the Chern number Ch $\in \mathbb{Z}$
- The low energy effective theory is

$$\mathcal{L} = -b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A^{\rm ex}_\lambda) + \frac{\text{Ch}}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda .$$

- where we neglected terms in higher derivatives, e.g. a Maxwell term
- The first term is the *BF* Lagrangian
- The hydrodynamic field b_μ couples to flux tubes
- The statistical gauge field a_μ couples to quasiparticle worldlines
- Quantized Hall conductivity $\sigma_{xy} = \mathrm{Ch} \frac{e^2}{h}$

3D Topological Insulator (Class AIII and DIII)

- Example: massive relativistic fermions with a conserved U(1) charge.
- This system has a topological invariant: the winding number

$$\mathcal{L} = -b_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda(a_\rho - A^{\rm ex}_\rho) + \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu \partial_\lambda a_\rho - \frac{1}{4\pi^2 g^2}\partial_\mu a_\nu \partial^\mu a^\nu + \cdots$$

If time-reversal (particle-hole) is imposed, the topological class is $\mathbb{Z}_2$ with θ=$\nu\pi$ (mod 2π)

- The bulk gapped (massive) fermionic excitations are represented by their worldlines j_μ which are minimally coupled to the gauge field a_μ
- Flux tubes of a_μ are coupled minimally to the curl of $b_{\mu\nu}$
- The effective action for the external gauge field has an axion term
- (Qi, Hughes, Zhang, 2009)

Web of Dualities in 2+1 Dimensions

Seiberg, Senthil, Wang and Witten (2016)

- Recently conjectured dualities between *fixed points* (relativistic CFTs)
- A new look at particle-vortex duality of theory with a U(1) symmetry (Peskin; Thomas and Stone; Dasgupta and Halperin) (on the r.h.s. there is also a Maxwell term)

$$|D(A)\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \longleftrightarrow |D(a)\varphi|^2 + \tilde{m}^2|\varphi|^2 - \tilde{\lambda}|\varphi|^4 + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}a^\mu\partial^\nu A^\lambda$$

  - $m^2<0$: l.h.s. in the broken symmetry phase with a Goldstone boson and quantized vortices with long range interactions; r.h.s. in the unbroken phase with a transverse photon and charged particle with long range interactions: compactified scalar ⟺ Maxwell field.
  - m2>0: l.h.s. in the unbroken phase with particle loops with short range interactions; r.h.s. in the Higgs phase with massive photons and vortices with short range interactions
  - $\phi \longleftrightarrow \varphi$ bound to a monopole of a_μ (end of a vortex), and $j_\mu \longleftrightarrow 1/2\pi\ \varepsilon_{\mu\nu\lambda}\ \partial_\nu\ a_\lambda$
  - Maps the Wilson-Fisher fixed point (l.h.s.) to the gauged Wilson-Fisher fixed point (r.h.s)

Bosonization Duality
(Conjectured by Seiberg, Senthil, Wang and Witten)

- Maps two different fixed point theories

$$i\bar{\psi}\not{D}(A)\psi - \frac{1}{8\pi}AdA \longleftrightarrow |D(a)\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA$$

- The 1/2 quantized Chern-Simons term can be regarded as due to have “fermion doublers” (lattice models) or as shorthand for the η invariant in a time-reversal invariant regularization
- Conjecture proven for $SU(N)_k$ gauge fields in the ’t Hooft limit: N, k $\mapsto\infty$, with N/k fixed (Minwalla et al; Aharony et al)
- Dirac fermion operator $\leftrightarrow$ magnetic monopole of the gauge field a_μ bound to the complex scalar field ϕ
- Dirac current: $j_\mu \leftrightarrow \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial^\nu a^\lambda$

Fermionic Particle-Vortex Duality

- Conjectured *fermionic* particle-vortex duality ("QED$_3$") (Son, Metlitski-Vishwanath):

$$i\bar{\psi}\not{D}(A)\psi + \frac{1}{8\pi}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda \longleftrightarrow$$

$$i\bar{\chi}\not{D}(a)\chi + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}a^\mu\partial^\nu b^\lambda - \frac{2}{4\pi}\epsilon_{\mu\nu\lambda}b^\mu\partial^\nu b^\lambda + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}a^\mu\partial^\nu A^\lambda - \frac{1}{8\pi}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda$$

- l.h.s: time reversal invariant free massless Dirac fermion
- r.h.s. charge conjugation invariant gauged Dirac fermion (QED3) (Maxwell terms com
- time reversal $\mathcal{T} \leftrightarrow$ charge conjugation $\mathcal{C}$ (PH); B $\leftrightarrow \mu$
- l.h.s. is $\mathcal{T}$ invariant and the r.h.s. is $\mathcal{C}$ invariant
- fermion masses will have opposite signs of the duality
- The 1/2 integer Chern-Simons terms can be viewed as coming from regularization ("doublers")
- Another interpretation is that this theory is at the *boundary* of a 3+1 dimensional top insulator which has $\theta=\pi$
- Duality maps a fermion to a fermion bound to a monopole

A way to “derive” the dualities with loop models

- “Derive” this web of dualities using quantum loop models near criticality, but still in the gapped phases.
- These models are related to modular invariant models we originally introduced with Kivelson

- Modular invariance cannot be kept close to the CFT.

- “Fractional spin” breaks modular invariance, and gives rise to Dirac fermions, leading to loop model based “proofs” of the CFT duality web.

Quantum Loop Models and Duality
(EF and Kivelson 1996)

- Non-intersecting linked loops [J_μ] in 3D Euclidean space-time (with no spin) with exact particle-hole symmetry
- flux attachment with fractional statistics θ, long ranged interactions with coupling g, and short-range repulsion (to avoid crossings)
- The imaginary part of the action is given in terms of the loops linking number

$$Z[g,\theta] = \sum_{\{J_\mu\}\in\mathbb{Z}} \delta(\Delta_\mu J^\mu) e^{-S[J_\mu]}$$

$$S[J_\mu] = \frac{g^2}{2}\sum_{x,y} J_\mu(x) G_{\mu\nu}(x-y) J_\nu(y) + i\theta \sum_{x,y} J_\mu(x) K_{\mu\nu}(x-y) J_\nu(y)$$

long ranged interactions

linking number $=\theta\Phi[J]$

$$G_{\mu\nu}(p) = \frac{1}{\sqrt{p^2}}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right), \; K_{\mu\nu}(p) = i\epsilon_{\mu\nu\lambda}\frac{p_\lambda}{p^2}$$

Field theory picture: 2+1 D complex scalar field coupled to 3+1 D Maxwell field with a θ term

$$\mathcal{L} = |D_\mu \phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4g^2} f_{\mu\nu}\frac{1}{\sqrt{-\partial^2}} f_{\mu\nu} + \frac{k}{4\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

Superconducting order parameter field at the boundary of a 3D topological insulator

Self-Duality and Modular Invariance

Modular parameter: $$\tau = \frac{\theta}{\pi} + i\frac{g^2}{2\pi}$$

- The partition functions of loop models regularized without self-linking (fractional spin) have the symmetries
- $\mathcal{S}$: duality: Z[τ]=Z[-1/τ], and $\mathcal{T}$: Periodicity: Z[τ]=Z[τ+1]
- $\mathcal{S}$ and $\mathcal{T}$ generate the modular group PSL(2,$\mathbb{Z}$)
- The partition function is self dual at the fixed points of the modular group
- Two types of PSL(2,$\mathbb{Z}$) fixed points: “bosonic” and “fermionic”
- FK showed that the finite modular fixed points are quantum critical points with $\sigma_{xx} \neq 0$ and $\sigma_{xy} = 0$
- The predicted conductivities are different in the FK loop models and the relativistic web of dualities

The Role of Fractional Spin

- The linking number of two separate loops l_1 and l_2 is

$$\Phi[J = \ell_1 + \ell_2] = 2 \times (\text{Linking number of } \ell_1 \text{ with } \ell_2) + W[\ell_1] + W[\ell_2]$$

"Writhe." Associated with self linking. Not necessarily a topological invariant (Hansson, Karlhede, et al)

- Witten: point-split the loops into ribbons so that the writhe is a frame-dependent topological invariant $W[l] = SL[l]$ = integer. Only consistent deep in the topological phase, not as the critical point is approached.

- Polyakov: no-point splitting and $W[l] = SL[l] - T[l]$ (writhe = self-linking - twist)

$$T[\ell] = \frac{1}{2\pi}\int_0^L ds \int_0^1 du\, \boldsymbol{e} \cdot \partial_s \boldsymbol{e} \times \partial_u \boldsymbol{e}$$

$T[l]$ is a Berry phase (fractional spin) and $\boldsymbol{e}$ is the tangent vector to the loop. The twist $T[l]$ is not quantized and depends on the metric.

Fractional Spin: Periodicity Lost, 3D Bosonization Regained

- $T[l]$ is not quantized. Means Duality $\mathcal{S}$ remains a symmetry, but periodicity $\mathcal{T}$ is lost
- Polyakov: fractional spin leads to the (IR) duality between a complex massive scalar with CS at k = 1 and a massive Dirac *spinor* (with a parity anomaly)
- Loop model representation

$$Z_{\mathrm{fermion}} = \det[i\partial\!\!\!/ - M] = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] - i\,\mathrm{sign}(M)\pi\Phi[J]}$$

$L[J]$: length of loop, $\Phi[J]$: linking number (including the spin factor)

For general statistical angle θ we have the loop model

$$Z = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] + i\theta\Phi[J]}$$

Fractional Spin: Periodicity Lost, 3D Bosonization Regained

- Can we use this to “derive” the web of dualities? Yes!

- First step: We introduce background fields to the boson side of Polyakov’s duality in the unbroken phase

$$\mathcal{L}_B = |D[a]\phi|^2 - m_0^2|\phi|^2 - |\phi|^4 + \tfrac{1}{4\pi} ada + \tfrac{1}{2\pi} adA$$

Exact rewriting as loop model coupled to gauge fields

$$Z[A] = \int \mathcal{D}J\mathcal{D}a\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J]+iS[J,a,A]}$$

$$S[J,a,A] = \int d^3x \left[J(a-A) + \frac{1}{4\pi} ada - \frac{1}{4\pi} AdA + \ldots\right]$$

Integrating-out *a* results in a term involving the linking number and the spin factor

$$-\pi\,\Phi[J] + \int d^3x \left[JA - \frac{1}{4\pi} AdA\right]$$

$$\mathcal{L}_F = \bar{\Psi}(i\not{D}[A] - M)\Psi - \frac{1}{8\pi} AdA$$

with M<0

Loop model representation

$$Z_{\mathrm{fermion}}[A; M<0]\, e^{-i\,\mathrm{CS}[A]/2} = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m| L[J] + i S_{\mathrm{fermion}}[J,A;M<0]} e^{-i\,\mathrm{CS}[A]/2}$$

$$S_{\mathrm{fermion}}[J, A; M<0] = \int d^3x \left(JA - \frac{1}{8\pi} AdA \right) - \pi \Phi[J]$$

The bosonization identity in the phase with broken time reversal, M>0, is obtained by a particle vortex duality in the bosonic theory

Loop Models: Tools for Deriving Dualities

1. Start with a proposed duality and write down boson loop models for each theory using Polyakov's duality.

2. Use path integral manipulations to equate the two loop model partition functions.

3. Match both sides of the critical point using bosonic particle-vortex duality. Relates superfluid of particles to insulator of vortices.

4. The dualities are IR identities

5. In the bosonic theories the short-distance repulsion between loops become the ϕ^4 coupling, which in the massless limit flow in the IR into the WF fixed point

Example: Fermion particle-vortex duality

Use loop models to derive the duality between free Dirac fermion and QED$_3$ with (quantized) Chern-Simons terms

$$i\bar{\Psi}\not{D}[A]\Psi - \tfrac{1}{8\pi}AdA \leftrightarrow i\bar{\psi}\not{D}[a]\psi + \tfrac{1}{8\pi}ada - \tfrac{1}{2\pi}adb + \tfrac{2}{4\pi}bdb - \tfrac{1}{2\pi}bdA$$

$$-M\bar{\Psi}\Psi, \quad M<0 \qquad\qquad -M'\bar{\psi}\psi, \quad M'>0$$

$$\int d^3x\, J_\mu A^\mu + \pi\Phi[J] \longleftarrow -\pi\Phi[J] + \int d^3x\left[J_\mu a^\mu - \frac{1}{2\pi}adb + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA\right]$$

Integrate out a, b

$$Z_{\mathrm{F}}[A; M<0] = Z_{\mathrm{QED}_3}[A; M'>0], Z_{\mathrm{F}}[A; M>0] = Z_{\mathrm{QED}_3}[A; M'<0]$$

- Case for opposite mass signs (QH phase) follows from the same logic
- Current mapping also natural upon integrating out b:

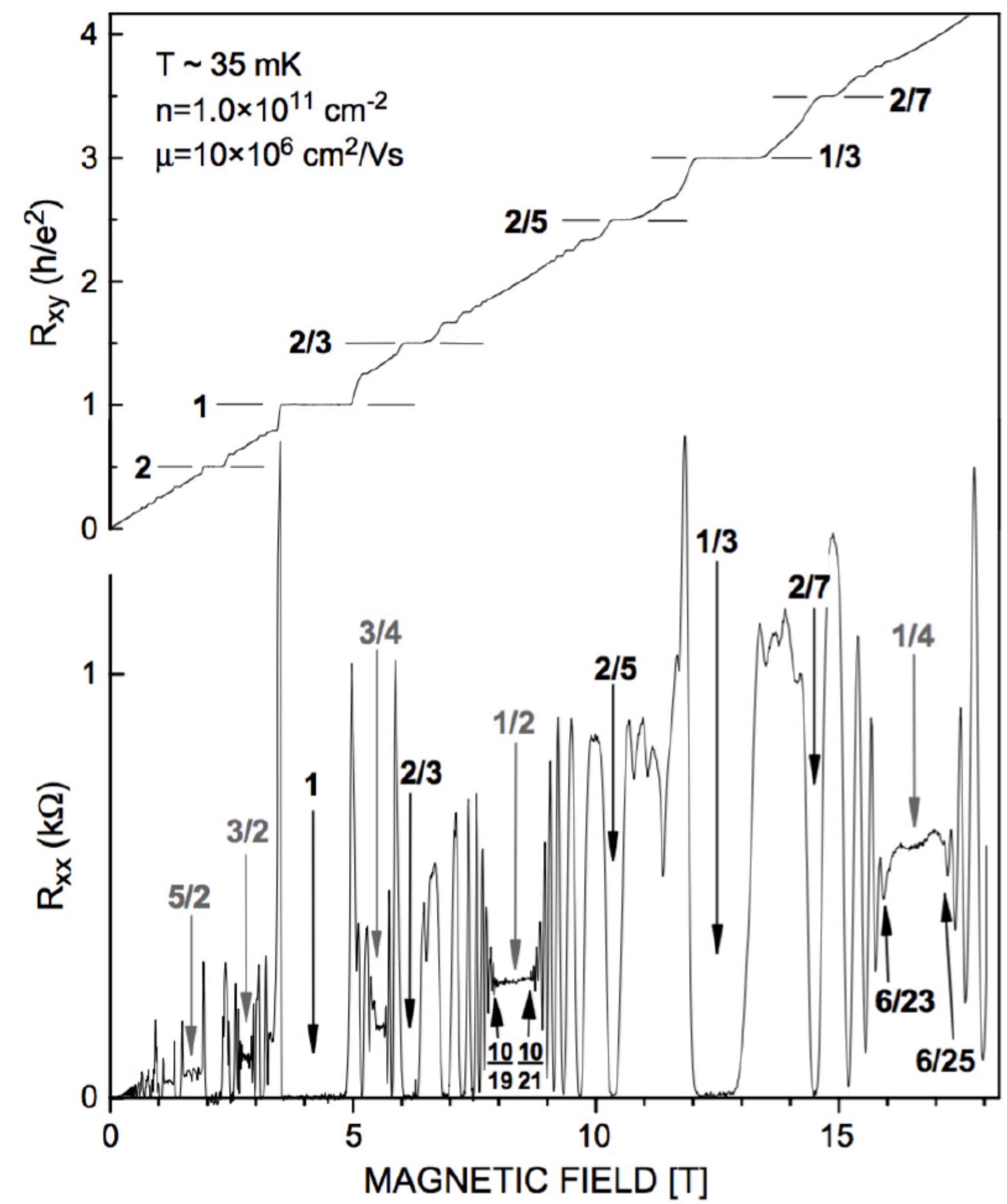
$$\bar{\Psi}\gamma^\mu\Psi \longleftrightarrow \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\partial^\nu a^\lambda$$

Compressible “FQH” states and Duality

- The Jain sequences of FQH states ν(p, n)=p/(2np±1) converge to ν=1/(2n) where the FQH gap vanishes ↦ Halperin, Lee, Read theory of a composite Fermi liquid
- This theory had great successes. It also has problems: in the simplest case, n=1, ν↦ 1/2 and PH symmetry is expected (for large B).
- HLR is not compatible with PH (DH Lee)
- The “Fermi liquid” is a “Non-Fermi liquid”
- Son proposed a relativistic version of HLR which satisfies PH
- At finite μ (Fermi surface!) this is still a “non-Fermi liquid”
- What about the ν=1/2n compressible states where PH should not hold?

ν=1/2n Compressible States

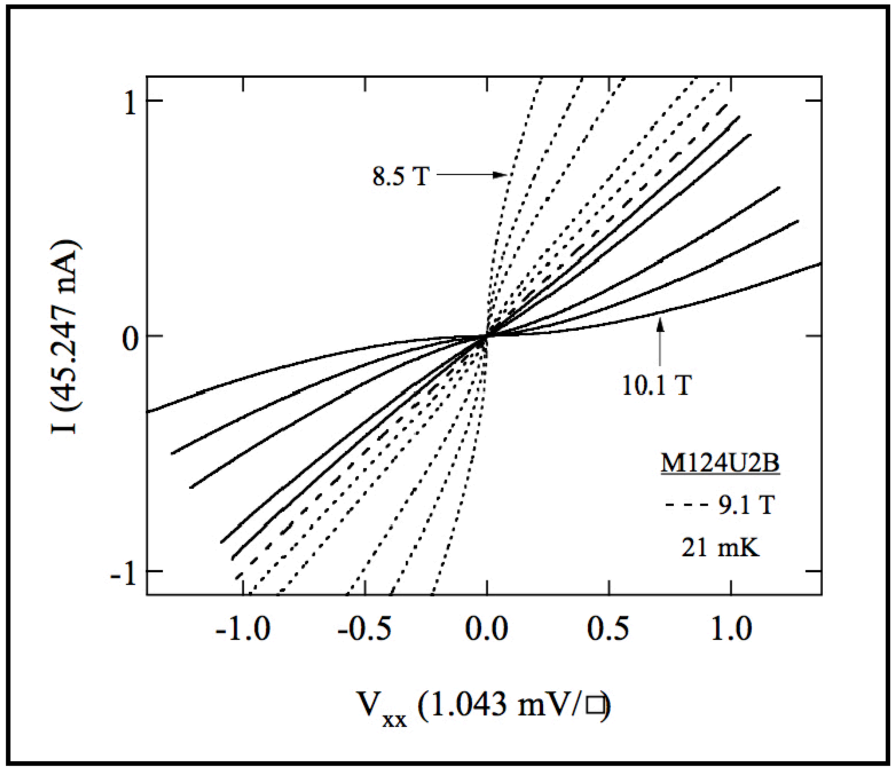
- Compressible states with ν=1/2n are predicted by the Jain sequences
- They are seen in experiment
- PH does not hold for general n
- Reflection symmetry of the I-V curves at plateau transitions
- Interpreted as evidence of particle-vortex duality (Shahar, Shimshoni, Sondhi)
- For ν=1/2 PH symmetry relates ρ_{xx} to σ_{xx} and $\nu \longleftrightarrow 1-\nu$



from [Stern, Ann. Phys. (2008)], data from W. Pan (Sandia)

Symmetries at 1/2n Compressible States

- The same reflection symmetry is seen at ν=1/4, locus of ν=1/3 $\mapsto$ 0 transition (where $\nu \nleftrightarrow$ 1-ν), with ρ_{xy}=-3e^2/h
- This is not PH symmetry!
- For ν=1/2n the symmetry is between the Jain states at ν=p/(2np+1) and ν'=(1+1)/(2n(1+p)-1), both converging to 1/2n
- For reflection symmetry to hold the HLR composite fermions must have σ_{xy}=-e^2/2h
- Flux attachment breaks PH and reflection explicitly
- Same problems in Son's theory which needs to be modified to treat ν and ν' equitably



"Charge-flux" symmetry at $\nu = 1/2n$.
[Shahar *et al.*, Science (1996)].

Reflection symmetry at ν=1/2n

$$\mathcal{L}_{1/2n} = i\bar{\psi}\not{D}_a\psi - \frac{1}{4\pi}\left(\frac{1}{2}-\frac{1}{2n}\right) ada + \frac{1}{2\pi}\frac{1}{2n} adA + \frac{1}{2n}\frac{1}{4\pi} AdA$$

- *A* is the external gauge field of strength *B*, and *a* is the Chern-Simons field (flux attachment); $b=\varepsilon_{ij}\partial_i a_j$

electron filling: $$\nu = \frac{2\pi}{B}\left\langle \frac{\delta \mathcal{L}_{\nu=1/2n}}{\delta A_0}\right\rangle = \frac{1}{2n}\left(1+\frac{b_*}{B}\right)$$

$$b_* = 0 \Rightarrow \nu = \frac{1}{2n}$$

Composite fermion ψ FS set by a_0: $$\rho_\psi = \frac{1}{2\pi}\left(\frac{1}{2}-\frac{1}{2n}\right) b_* - \frac{1}{2\pi}\frac{B}{2n}$$

$$\nu_\psi = 2\pi\frac{\rho_\psi}{b_*} = \frac{1}{2} + \frac{\nu}{1-2n\nu} \qquad \nu_\psi = p + \frac{1}{2} \Rightarrow \nu = \frac{p}{2np+1}$$

$$\nu_\psi \to -\nu_\psi \Leftrightarrow \nu = \frac{p}{2np+1} \to \frac{1+p}{2n(1+p)-1}$$

Reflection symmetry and boson self-duality

$$\mathcal{L}_{1/2n} \leftrightarrow |D_{g-A}\phi|^2 - |\phi|^4 + \frac{1}{4\pi}\frac{1}{2n-1} gdg$$
$$\leftrightarrow |D_h\varphi|^2 - |\varphi|^4 - \frac{2n-1}{4\pi} hdh + \frac{1}{2\pi} hdA$$

- First line: fermion-boson duality
- Second line: boson-vortex duality
- relates ν_ϕ to - $1/\nu_\varphi$
- ν=1/2n ⟺ ν_ϕ = - ν_φ=1
- Reflection related filling fractions ν_ϕ (ν) = - ν_φ(ν')
- Reflection symmetry is boson-vortex exchange
- Reflection symmetry at ν=1/2n ⟺ boson self-duality!

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