Sine Gordon model Renormalization

$$L[\varphi] = \frac{1}{2v}(\partial_t\varphi)^2 - \frac{v}{2}(\partial_x\varphi)^2 + \hat{g}\cos\beta\varphi$$

integrations by part

$$S[\varphi] = \iint dt\,dx \Big[\underbrace{\frac{v}{2}\varphi\,\partial_x^2\varphi - \frac{1}{2v}\varphi(\partial_t\varphi)^2}_{S_0[\varphi]} + \underbrace{\hat{g}\cos\beta\varphi}_{S_I[\varphi]} \Big]$$

split into fast and slow modes:

$$\varphi(x) = \varphi^s(x) + \delta\varphi(x)$$

$$(\varphi^s(x) + \delta\varphi(x))(\partial_x^2\varphi^s(x) + \partial_x^2\delta\varphi(x))$$

$$\int dx\, \delta\varphi\,\nabla_x^2\varphi^s = \int dx\,\varphi^s\,\nabla_x^2\delta\varphi \xrightarrow{\text{Fourier transform}}$$

$$= \int dx \int_{\text{bulk}} \frac{d^2q}{(2\pi)^2} \int_{\text{shell}} \frac{d^2q'}{(2\pi)^2}\, \varphi(q)\left(-q'^2 - \frac{\omega^2}{v^2}\right)\varphi(q')$$

$$e^{i(q+q')x}$$

$$= \int dx \int_{\text{bulk}} \frac{d^2q}{(2\pi)^2}\, \varphi(q)\left(-q^2 - \frac{\omega^2}{v^2}\right)\theta\left(|q| - \frac{\Lambda}{s}\right)$$

Definition;

$$\frac{1}{2}\nabla_x^2 G_0(x,x') = \delta(x-x')$$

$$\frac{1}{2}\nabla_x^2 \int \frac{d^2q}{(2\pi)^2} G_0(q) e^{iq(x-x')} = \int \frac{d^2q}{(2\pi)^2} e^{iq(x-x')}$$

$$= \int \frac{d^3q}{(2\pi)^2} G_0(q) \cdot \frac{1}{2}\left(-q^2 - \frac{\omega^2}{\gamma^2}\right) e^{iq(x-x')}$$

$$G_0(q) = -\frac{2\gamma^2}{q^2\gamma^2+\omega^2}$$

Dyson eq:

$$G^{-1}(x,x') = G_0^{-1}(x,x') - \Sigma(x,x')$$

reverse

$$\frac{1}{2}\nabla_x^2 G(x-x') - \int dx'' \Sigma(x,x'') G(x'',x')$$

$$= \frac{1}{(2\pi)^2}\delta(x-x')$$

$\Rightarrow$

$$\int \frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2} G(q,q')\frac{1}{2}\left(-q^2 - \frac{\omega^2}{\gamma^2}\right) e^{iqx+iq'x'}$$

$$-\int dx'' \int \frac{d^2q}{(2\pi)^2}\frac{d^2k}{(2\pi)^2}\frac{d^2k'}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\,\Sigma(q,k)\,G(k',q')$$

$$e^{iqx+\underbrace{i(k+k')x''}_{\Downarrow\ \text{integrate out}}+iq'x'} = \int \frac{d^2q}{(2\pi)^4}\, e^{iq(x-x')}$$

$$\int \frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\Big[G(q,q')G_0^{-1}(q) - \int \frac{d^2q''}{(2\pi)^2}\Sigma(q,q'')$$

$$G(-q'',q')\Big]e^{iqx+iq'x'} = \int \frac{d^2q}{(2\pi)^2}\frac{d^2q'}{(2\pi)^2}\,\delta(q+q')$$

$$e^{iqx+iq'x'}$$

$$\boxed{\Rightarrow\ G(q,q') = G_0(q)\,\delta(q+q') + G_0(q)\int\frac{d^2q''}{(2\pi)^2}\Sigma(q,q'')\,G(-q'',q')}$$

- perturbative theory in powers of g

$$G^{(0)}(q,q') = G_0(q)\,\delta(q+q') \quad \text{(zero-th order)}$$

up to first order

$$G^{(1)}(q,q') = G_0(q)\,\delta(q+q') - G_0(q)\int\frac{d^2q''}{(2\pi)^2}\,b^s(q,q'')$$
$$G(-q'')\,\delta(q - q'')$$

$$G^{(1)}(q,q') = G_0(q)\,\delta(q+q') - \frac{1}{(2\pi)^2}\,G_0(q)\,b^s(q,q')\,G_0(-q')$$

second order

$$G^{(2)}(q,q') = G_0(q)\,\delta(q+q') - G_0(q)\int\frac{dq''^2}{(2\pi)^2}\,b^s(q,q'')$$

$$\left[\,G_0(-q'')\,\delta(-q''+q') - \frac{1}{(2\pi)^2}\,G_0(-q'')\,b^s(-q'',q')\,G_0(-q')\right]$$

$$\Rightarrow\ G^{(2)}(q,q') = G_0(q)\,\delta(q+q') - \frac{1}{(2\pi)^2}\,G_0(q)\,b^s(q,q')$$

$$G_0(-q') + \frac{1}{(2\pi)^4}\,G_0(q)\int\frac{dq''^2}{(2\pi)^2}\,b^s(q,q'')\,G_0(-q')$$

$$b^s(-q'',q')\,G_0(-q')$$

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with

$$\delta S[\varphi^s,\delta\varphi] = S_0[\delta\varphi] + \int dx\,a^s(x)\,\delta\varphi(x) +$$

$$\int dx'dx\;\delta\varphi(x)\,b^s(x,x')\,\delta\varphi(x')$$

$$\to \int dx\,dx'\;\delta\varphi(x)\left[\,\delta(x-x')\,\frac{1}{2}\nabla_x^2 + b^s(x,x')\,\right]\delta\varphi(x')$$

$$+\int dx\;a^s(x)\,\delta\varphi(x)$$

$$- \int dx'' \int \frac{d^2 q}{(2\pi)^2} \frac{d^2 k}{(2\pi)^2} \frac{d^2 k'}{(2\pi)^2} \frac{d^2 p'}{(2\pi)^2} \Sigma(q, k) G(k', q')$$

$$\underbrace{e^{iqx + i(k+k')x'' + iq'x'}}_{\Downarrow \text{ integrate out}} = \int \frac{d^2 q}{(2\pi)^4} e^{iq(x-x')}$$

$$\int \frac{d^2 q}{(2\pi)^2} \frac{d^2 q'}{(2\pi)^2} \left[G(q, q') G_0^{-1}(q) - \int \frac{d^2 q''}{(2\pi)^2} \Sigma(q, q'') \right.$$

$$\left. G(-q'', q') \right] e^{iqx + iq'x'} = \int \frac{d^2 q}{(2\pi)^2} \frac{d^2 q'}{(2\pi)^2} \delta(q+q')$$

$$e^{iqx + iq'x'}$$

$$\boxed{\Rightarrow \quad G(q, q') = G_0(q) \delta(q+q') + G_0(q) \int \frac{d^2 q''}{(2\pi)^2} \Sigma(q, q'') G(-q'', q')}$$

- perturbative theory in powers of g

$$G^{(0)}(q, q') = G_0(q) \delta(q+q') \quad \text{(zero-th order)}$$

up to first order

$$G^{(1)}(q, q') = G_0(q) \delta(q+q') - G_0(q) \int \frac{d^2 q''}{(2\pi)^2} b^S(q, q'')$$
$$G(-q'') \delta(q + q'')$$

$$G^{(1)}(q,q') = G_0(q)\,\delta(q+q') - \frac{1}{(2\pi)^2} G_0(q)\, b^s(q,q')\, G_0(-q')$$

second order

$$G^{(2)}(q,q') = G_0(q)\,\delta(q+q') - G_0(q)\int \frac{d^2q''}{(2\pi)^2} b^s(q,q'')$$

$$\left[G_0(-q'')\,\delta(-q''+q') - \frac{1}{(2\pi)^2} G_0(q'')\, b^s(-q'',q')\, G_0(-q') \right]$$

$$\Rightarrow G^{(2)}(q,q') = G_0(q)\,\delta(q+q') - \frac{1}{(2\pi)^2} G_0(q)\, b^s(q,q')$$

$$G_0(-q') + \frac{1}{(2\pi)^4} G_0(q) \int \frac{dq''^2}{(2\pi)^2} b^s(q,q'')\, G_0(-q')$$

$$b^s(-q'',q')\, G_0(-q')$$

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with

$$\delta S[\Phi^s, \delta\Phi] = S_0[\delta\Phi] + \int dx\, a^s(x)\, \delta\Phi(x) +$$

$$\int dx' dx\; \delta\Phi(x)\, b^s(x,x')\, \delta\Phi(x')$$

$$\rightarrow \int dx\, dx'\; \delta\Phi(x) \left[\delta(x-x')\frac{1}{2}\nabla_x^2 + b^s(x,x') \right] \delta\Phi(x')$$

$$+ \int dx\; a^s(x)\, \delta\Phi(x)$$

$$\to \quad \delta S[\Psi^S, \delta\Psi] = \int dx\, dx'\, \delta\Psi(x)\, G^{-1}(x,x')\, \delta\Psi(x')$$

$$+ \int dx\, a^S(x)\, \Psi(x)$$

$$\delta\Psi(x) = \hat{\Psi}(x) + r(x)$$

$$\delta S[\Psi^S, \delta\Psi] = \int dx dx' \{ \hat{\Psi}(x)\, G^{-1}(x,x')\, \hat{\Psi}(x)$$

$$+ \hat{\Psi}(x)[\, 2G^{-1}(x,x')\, r(x') + a^S(x')\, \delta(x-x')\,]$$

$$+ r(x)[\, G^{-1}(x,x')\, r(x') + a^S(x')\, \delta(x-x')\,]$$

Now if

$$\int dx'\, G^{-1}(x,x')\, r(x') = -\frac{1}{2} a^S(x)$$

$$\delta S[\Psi^S, \delta\Psi] = \int dx dx' [\, \hat{\Psi}(x)\, G^{-1}(x,x')\, \hat{\Psi}(x')$$

$$- \frac{1}{4} a^S(x)\, G(x,x')\, a^S(x')\,]$$

$$\Rightarrow \int dx \int dx'\, \underbrace{G(x'',x)\, G^{-1}(x,x')}_{\delta(x'-x'')}\, r(x')$$

$$= r(x'') = -\frac{1}{2}\int dx'\, G(x,x')\, a^S(x')$$

$$\Psi(x) = \Psi^{s}(x) + r(x) + \tilde{\Psi}(x) = \tilde{\Psi}(x) + \Psi^{s}(x)$$

$$H_0[\Pi, \Psi] = \int dx \frac{v}{2}\left[\Pi^2 + (\partial_x \Psi)^2\right] \longrightarrow H_0[\Pi^{s}, \Psi^{s}]$$

$$+ H_0[\hat{\Pi}, \Psi]$$

unperturbed Green function

$$G_0(x, x') = {}^{\Psi}\langle 0 | \Psi(x)\, \Psi(x') | 0 \rangle^{\Psi}$$

$$= \langle 0^{\Psi} | (\tilde{\Psi}(x) + \Psi^{s}(x))(\tilde{\Psi}(x') + \Psi^{s}(x')) | 0 \rangle^{\Psi}$$

$$G_0(x, x') = 2\, {}^{\Psi}\langle 0 | \tilde{\Psi}(x)\, \tilde{\Psi}(x') | 0 \rangle^{\Psi}$$

$$= 2\, {}^{\hat{\Psi}}\langle 0 | \hat{\Psi}(x)\, \hat{\Psi}(x') | 0 \rangle^{\hat{\Psi}}$$

the effective slow mode residual contribution to the action

$$\delta S_{eff}[\Psi^{s}] = {}^{\Psi}\langle 0 | \delta S[\Psi^{s}, \Psi] | 0 \rangle^{\Psi}$$

$$\delta S_{eff}[\Psi^{s}] = \int dx\, dx' \left[\frac{1}{2} G_0(x, x')\, G^{-1}(x', x') - \frac{1}{4} a^{s}(x)\, G(x, x')\, a^{s}(x')\right]$$

$$\delta S_{eff}[\Psi^0] = \int dx\,dx' \Big[\frac{1}{2} G_0(x,x') \big[G_0^{-1}(x,x') + b^s(x,x') \big]$$

$$- \frac{1}{4} a^s(x) G(x,x') a^s(x') \Big]$$

$$= {}^{\Phi}\langle 0 | S_0[\Phi] | 0 \rangle^{\Phi} + \int dx\,dx' \Big[\frac{1}{2} G_0(x,x') b^s(x,x')$$

$$- \frac{1}{4} a^s(x) G(x,x') a^s(x') \Big] \longleftarrow \delta S_{eff}[\Psi^0].$$

Let's compute first term:

$$F_1[\Psi^0] = \frac{1}{2} \int dx\,dx' \int_{\text{shell}} \frac{d^2q}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} G_0(q)$$

$$b^s(q', q'') \, e^{iq(x-x') + iq'x + iq''x'}$$

$$\Big\downarrow \text{integrate out } x, x'$$

$$= \frac{1}{2} \int_{\text{shell}} \frac{d^2q}{(2\pi)^2} G_0(q) \, b_s(-q, q)$$

$$b^s(x,x') = -\frac{\beta^2}{2} g \cos\beta\Psi \, \delta(x-x')$$

$$= -\frac{\beta^2}{2} l_I[\Psi^0] \, \delta(x-x')$$

$$b^s(q,q') = \int dx\, dx'\, \cancel{b^s(x,x')}\, e^{iqx+iq'x'}\, \delta(x-x')$$

$$-\frac{\beta^2}{2}\, g \cos\beta\Phi^s$$

$$= b_s(q+q')$$

- $$F_1[\Phi^s] = \frac{1}{2} b^s(q=0) \int_{\text{shape}} \frac{d^2q}{(2\pi)^2}\, G_0(q)$$

$$= -b^s(q=0) \int_{\text{shape}} \frac{d^2q}{(2\pi)^2}\, \frac{1}{q^2+\omega^2/V^2}$$

$$= -\frac{1}{2\pi} b^s(q=0) \int_{\Lambda/s}^{\Lambda} \frac{|q|}{q^2}\, dq$$

$$= -\frac{1}{2\pi} b^s(q=0) \log s$$

$$\rightarrow \frac{\beta^2}{4\pi} \log s \int dx\, f_I[\Phi^s] = -\frac{g\,dl}{4\pi} \beta^2 \int dx \cos\beta\Phi^s$$

$$F_2[\Phi^s] = -\int d^2x\, d^2x'\, \frac{1}{4} G_0(x,x')\, a^s(x)\, a^s(x')$$

$$=$$

$$S_R[\Psi] = \int dx \left[\frac{1}{2\beta^2}\left(1 + \frac{3\beta^4 g^2 dl}{4\pi\Lambda^3}\right) \Psi \partial_x^2 \Psi + g s^{-2} \left(1 - \frac{\beta^2 dl}{4\pi}\right) \cos\Psi \right]$$

$$S[\Psi] = \int dx \left[\frac{1}{2\beta^2} \Psi \nabla_x^2 \Psi + g \cos\Psi \right]$$

$$\Rightarrow \begin{cases} \beta_R^{-2} = \beta^{-2}\left(1 + \dfrac{3\beta^4 g^2 dl}{4\pi\Lambda^3}\right) \\ g_R = g s^{-2}\left(1 - \dfrac{\beta^2 dl}{4\pi}\right) \end{cases}$$

$$\Rightarrow \quad \frac{d\beta^{-2}}{dl} = \frac{3\beta^2 g^2}{4\pi\Lambda^3} = \frac{-2}{\beta^{-3}} \frac{d\beta}{dl}$$

$$\Rightarrow \quad \boxed{\frac{d\beta^2}{dl} = -\frac{3\beta^6 g^2}{4\pi\Lambda^3}}$$

$$g_R = g(1 + 2dl)\left(1 - \frac{\beta^2 dl}{4\pi}\right) = g + g\left(2 - \frac{\beta^2}{4\pi}\right) dl$$

$$\Rightarrow \quad \boxed{\frac{dg}{dl} = 2g\left(1 - \frac{\beta^2}{8\pi}\right)}$$

Let: $K = \dfrac{\beta^2}{8\pi}$, $\quad u = 4\sqrt{\dfrac{3\pi}{\Lambda^3}}\, g$

$$\boxed{\begin{aligned} \frac{du}{dl} &= 2u(1-K) \\ \frac{dK}{dl} &= -u^2K^2 \end{aligned}}$$

$K = 1+v$

$$\begin{cases} \dfrac{du}{dl} = -2uv \\ \dfrac{dv}{dl} = -u^2(1+2v) \end{cases}$$

$$\Rightarrow \quad 2u\frac{du}{dl} = \frac{du^2}{dl} = -4u^2v$$

$$2v\frac{dv}{dl} = -2u^2v \underbrace{-4u^2v^2}_{\text{second order}} = \frac{dv^2}{dl}$$

$$\Rightarrow \frac{d}{dl}(u^2-2v^2) = \frac{d}{dl}\left(u^2-2(K-1)^2\right) = 0$$

Crossorver Regime