Bethe Equation

①

Classical nonlinear sigma model:

$$S_0 = \frac{1}{2g} \int dx^2 \, \partial^\mu g^{-1} \partial_\mu g$$

variation: $g \to g + \delta g$

$$\partial^\mu \delta g^{-1} \partial_\mu g + \partial^\mu g^{-1} \partial_\mu \delta g$$

- $\partial^\mu g^{-1} \partial_\mu \delta g = -g^{-1} \partial^\mu g \, g^{-1} \partial_\mu \delta g = -\partial_\mu (g^{-1} \partial^\mu g \, g^{-1} \delta g) + \underline{\partial_\mu (g^{-1} \partial^\mu g \, g^{-1}) \delta g}$

$= \partial_\mu (g^{-1} \partial^\mu g) g^{-1} \delta g + g^{-1} \partial^\mu g \, \partial_\mu g^{-1} \delta g$

- $\partial^\mu \delta g^{-1} \partial_\mu g = -\partial^\mu (g^{-1} \delta g \, g^{-1}) \partial_\mu g = -\partial^\mu (g^{-1} \delta g \, g^{-1} \partial_\mu g) + g^{-1} \delta g \, g^{-1} \partial^\mu \partial_\mu g$

- $g^{-1} \partial^\mu g \, \partial_\mu g^{-1} \delta g = -g^{-1} \partial^\mu g \, g^{-1} \partial_\mu g \, g^{-1} \delta g$

$= -g^{-1} \delta g \, g^{-1} \partial^\mu g \, g^{-1} \partial_\mu g = g^{-1} \delta g \, \partial^\mu g^{-1} \partial_\mu g$

The motion of implies conservation of currents: ⑧

$$\delta g = \frac{1}{2g}\int dx^2 \left\{ \mathrm{tr}\left(g^{-1}\delta g\, \partial_\mu (g^{-1}\partial^\mu g)\right) + \mathrm{tr}\left(g^{-1}\delta g\, \partial^\mu (g^{-1}\partial_\mu g)\right)\right\}$$

- Lemma: $\partial^\mu (g^{-1}\partial_\mu g) \neq \partial_\mu (g^{-1}\partial^\mu g)$

$$\partial^\mu g^{-1}\,\partial_\mu g + g^{-1}\partial^\mu\partial_\mu g$$

$$= g^{-1}\partial^\mu g\,\partial_\mu g^{-1} g + g^{-1}\partial^\mu\partial_\mu g$$

$\cancel{\mathrm{tr}(\partial^\mu (g^{-1}}$

$$\mathrm{tr}\left(g^{-1}\delta g\, g^{-1}\partial^\mu g\,\partial_\mu g^{-1} g\right) = \mathrm{tr}\left(\delta g\, g^{-1}\partial^\mu g\,\partial_\mu g^{-1}\right)$$

$$= \mathrm{tr}\left(\delta g\, g^{-1} g\,\partial^\mu g^{-1} g\, g^{-1}\partial_\mu g\, g^{-1}\right)$$

conclusion: $\partial^\mu (g^{-1}\partial_\mu g) \neq \partial_\mu (g^{-1}\partial^\mu g)$

The motion eq implies conservation of currents:

$$J_\mu = g^{-1}\partial_\mu g \quad \to \quad \partial^\mu J_\mu = 0$$

if we introduce holomorphic and anti-holomorphic current:

$$\partial_z J_{\bar z} + \partial_{\bar z} J_z = 0$$

$$(\partial_1 - i\partial_2)(J_1 + iJ_2) + (\partial_1 + i\partial_2)(J_1 - iJ_2)$$

$$= 0 \quad \Rightarrow \quad \partial_1 J_1 + \partial_2 J_2 = 0$$

chiral current: $\epsilon^{\mu\nu} J_\nu = \tilde{J}_\mu$

- $[D_\mu, D_\nu] = \partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0$

$\Rightarrow$ J_μ is the pure gauge current

WZW model

$$\Gamma = \frac{-i}{24\pi}\int_B dy^3 \, \epsilon_{\alpha\beta\gamma} \, tr(g^{-1}\partial_\alpha g \, g^{-1}\partial_\beta g \, g^{-1}\partial_\gamma g)$$

④

- then we consider action

$$S = S_0 + k\Gamma$$

variation of WZW term:

$$\frac{-i}{24\pi}\int_B \delta\left[\epsilon^{\alpha\beta\gamma}\,\mathrm{tr}\left(g^{-1}\partial_\alpha g\, g^{-1}\partial_\beta g\, g^{-1}\partial_\gamma g\right)\right]$$

$$= -\frac{i}{24\pi}\int_B 3\,\epsilon^{\alpha\beta\gamma}\left[\delta(g^{-1}\partial_\alpha g)\left[-\partial_\beta g^{-1}\partial_\gamma g - g^{-1}\partial_\beta\partial_\gamma g\right]\right]$$

$$= \frac{i}{8\pi}\int_B \epsilon^{\alpha\beta\gamma}\left[-g^{-1}\delta g\, g^{-1}\partial_\alpha g + g^{-1}\partial_\alpha \delta g\right]$$

$$\partial_\beta\left[g^{-1}\partial_\gamma g\right]$$

$$= \frac{i}{8\pi}\int_B \epsilon^{\alpha\beta\gamma}\left[\partial_\alpha\left[g^{-1}\partial_\alpha \delta g\right]\right]\partial_\beta\left[g^{-1}\partial_\gamma g\right]$$

$$= \frac{i}{8\pi}\int_B \epsilon^{\alpha\beta\gamma}\,\partial_\alpha\left[g^{-1}\partial_\alpha \delta g\,\partial_\beta\left[g^{-1}\partial_\gamma g\right]\right]$$

$$= \frac{i}{8\pi}\int_{S^2} g^{-1}\partial_\alpha \delta g\,\partial_\beta\left[g^{-1}\partial_\gamma g\right]$$

• Full motion equation is:

$$\partial^{\mu}(g^{-1}\partial_\mu g) + \frac{g\,i k}{4\pi}\varepsilon^{\mu\nu}\partial^{\mu}(g^{-1}\partial_\nu g) = 0$$

$$\Rightarrow \left(1 + \frac{g k}{4\pi}\right)\partial_z(g^{-1}\partial_{\bar z} g) + \left(1 - \frac{g k}{4\pi}\right)\partial_{\bar z}(g^{-1}\partial_z g)$$

$= 0$

Let: $g = \frac{4\pi}{k}$, the current conservation law

$$\partial_z(g^{-1}\partial_{\bar z} g) = 0$$

$$\Rightarrow S_{WZ} = \frac{k}{16\pi}\int dx^2\left(\partial^\mu g^{-1}\partial_\mu g\right) + kT$$

• Current algebra

$$J(z) = -k\,\partial_z g\, g^{-1}$$

$$\bar J(\bar z) = k\, g^{-1}\partial_{\bar z} g$$

$$\delta S = -\frac{1}{2\pi}\int d^2x\left[\partial_{\bar z}\,\mathrm{Tr}(\omega(z)J(z)) + \partial_z \mathrm{Tr}(\bar\omega(\bar z)\bar J(\bar z))\right.$$

$$\int dx^2 \rightarrow \int \frac{-i}{2}\,dz\,d\bar z$$

⑥

where: $J = \sum_a J^a t^a \qquad \omega = \sum_a \omega^a t^a$

$$\delta_{\omega,\bar\omega} S = \frac{i}{4\pi}\oint dz\, Tr(\omega(z)\, J(z)) - \frac{i}{4\pi}\oint d\bar z\, Tr(\bar\omega(\bar z)\, \bar J(\bar z))$$

$$\downarrow$$

$$\delta_{\omega\bar\omega} S = -\frac{1}{2\pi i}\oint dz \sum_a \omega^a t^a + \frac{1}{2\pi i}\oint d\bar z \sum_a \bar\omega^a \bar J^a$$

ward identity

$$\delta_{\omega,\bar\omega}\langle X\rangle = -\frac{1}{2\pi i}\oint dz \sum_a \omega^a \langle J^a X\rangle + \frac{1}{2\pi i}\oint d\bar z \sum_a \bar\omega^a \langle \bar J^a X\rangle$$

we calculate $\delta_\omega J$

$$\delta_\omega J = -k\, \delta_\omega (\partial_z g\, g^{-1})$$

$$= -k\left[\partial_z(\delta_\omega g)\, g^{-1} - \partial_z g\, g^{-1}\delta_\omega g\, g^{-1}\right]$$

$$= -k\left[(\partial_z \omega g + \omega \partial_z g)\, g^{-1}\right] + k\, \partial_z g\, g^{-1}\, \omega]$$

where: $J = \sum_a J_a t^a$ $\quad \omega = \sum_a \omega_a t^a$

$$= [\omega, J] - k\partial_z \omega$$

it can be written as:

$$\delta_\omega J^a = i \sum_{b,c} f_{abc}\, \omega^b J^c - k\, \partial_z \omega^a$$

$\Rightarrow$ OPE expansion: ?

$$J^a(z)\, J^b(\omega) \sim \frac{k\,\delta_{ab}}{(z-\omega)^2} + \sum_c i f_{abc} \frac{J^c(\omega)}{z-w}$$

- Affine Kac-Moody algebra

$$J^a = \sum_{n\in\mathbb{Z}} J_n^a\, z^{-n-1}$$

$$[J_m^a, J_n^a] = \frac{1}{(2\pi i)^2}\left(\oint_{|z|>|\omega|} dz \oint d\omega - \oint_{|z|<|\omega|} dz \oint d\omega \right)$$

$z^m \omega^n J^a(z) J^b(\omega)$

$$= \frac{1}{(2\pi i)^2} \oint_0 d\omega \oint dz\; z^m z^n \left(\frac{k\,\delta^{ab}}{(z-\omega)^2} + \frac{i f_{abc} J^c(\omega)}{(z-\omega)} \right)$$

$$= \frac{1}{2\pi i} \oint_0 d\omega \left(k m\, \delta^{ab}\, \omega^{n+m-1} + i f_{abc} J^c(\omega)\, \omega^{m+n} \right)$$

$$= k m\, \delta^{ab}\, \delta_{m+n,0} + i f_{abc} J^c_{m+n}$$ (Kac-Moody Algebra)

⑧

- Sugawara construction

We have to show that the energy momentum tensor satisfies the Virasoro algebra

$$T(z)\,T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

Classicaly:

$$\Rightarrow \; T_{\mu\nu} = \partial_\mu \frac{\partial L}{\partial(\partial^\nu \phi)} - g_{\mu\nu} L$$

$$T = \frac{1}{2k}\, \delta_{ab}\, J^a(z)\, J^b(z)$$

- As the first step, we compute the OPE of $T(z)$ and $J^a(w)$

$$J^a(z)\,T(w) = \frac{\gamma}{2\pi i} \oint_w \frac{1}{x-w}\, J^a(x)$$

$J^b(x)\; J^b(w)$

(9)

$$= \frac{\gamma}{2\pi i}\oint_w \frac{1}{x-w}\left(J^a(z)J^b(x)J^b(w) + J^b(x)J^b(z)J^a(w)\right)$$

$$= \frac{\gamma}{2\pi i}\oint_w \frac{1}{x-w}\left[\frac{k\delta^{ab}}{(z-x)^2} + \frac{i f_{abc}J^c(x)}{z-x}\right]J^b(w)$$

$$+ \frac{\gamma}{2\pi i}\oint_w \frac{J^b(x)}{x-w}\left[\frac{k\delta^{ab}}{(z-w)^2} + \frac{i f_{abd}J^d(w)}{z-w}\right]$$

$$= \gamma/2\pi i \oint_w \frac{1}{x-w}\left[\frac{k\delta^{ab}}{(z-x)^2} + \frac{k\delta^{ab}}{(z-w)^2}\right]J^b(w)$$

$$+ \frac{\gamma}{2\pi i}\oint_w \frac{1}{x-w}\, i f_{abc}\left[\frac{\delta^{cb}}{(x-w)^2} + \frac{i f_{cbd}J^d(w)}{x-w}\right]\frac{1}{z-x}$$

$$+ \frac{\gamma}{2\pi i}\oint_w \frac{1}{x-w}\, i f_{abc}\left[\frac{\delta^{cb}}{(x-w)^2} + \frac{i f_{cbd}J(w)}{(x-w)}\right]\frac{1}{z-w}$$

$$= \gamma\left[\frac{2k\delta^{ab}J^b(w)}{(w-x)^2} - \frac{f_{abc}f_{bcd}J^d(w)}{(w-x)^2}\right]$$

Dual Coxeter number $h^\vee$

$$[t^a, t^b] = i f_{abc} t^c$$

$$\Rightarrow [t^b,[t^a,t^b]] = i[t^b, t^c] f_{abc} = i f_{abc} f_{bcd} t^d$$

(18)

$$\mathrm{tr}(t^d, [t^b, [t^a, t^b]]) = \mathrm{tr}(t^d t^b t^a t^b) -$$
$$\mathrm{tr}(t^d t^b t^b t^a)$$

$$\Rightarrow \overline{T(z)\, J^a}(\omega) = 2\gamma(k + h^\vee)\left(\frac{J^a(\omega)}{(z-\omega)^2} + \frac{\partial J^a(\omega)}{z-\omega}\right)$$

$$\Rightarrow \quad \gamma = \frac{1}{2(k+h^\vee)}$$

- <u>demonstrate</u> $T(z)T(\omega)$ obey to virosura algebra:

$$\overline{T(z)\, T}(\omega) = \frac{1}{4\pi i(k+h^\wedge)} \oint \frac{dx}{x-\omega} \Big[\overline{T(z)\, J^a}(x)\, J^a(\omega)$$
$$+ \overline{T(z)\, J^a(x)\, J^a}(\omega) \Big]$$
$$= \frac{1}{4\pi i(k+h^\wedge)} \oint \frac{dx}{x-\omega} \left\{ \left[\frac{J^a(x)}{(z-x)^2} + \frac{\partial J^a(x)}{z-x} \right] J^a(\omega) \right.$$
$$\left. + J^a(x) \left[\frac{J^a(\omega)}{(z-\omega)^2} + \frac{\partial J^a(\omega)}{(z-\omega)} \right] \right\}$$

$$= \frac{1}{4\pi i(k+h^\vee)} \oint \frac{dx}{x-w} \left\{ \frac{k\dim g}{(z-x)^2(x-w)^2} - \frac{2k\dim g}{(z-x)(x-w)^3} \right.$$

$$\left. + \frac{k\dim g}{(x-w)^2(z-w)^2} + \frac{2k\dim g}{(z-x)(x-w)^3} \right\}$$

$$+ \frac{1}{4\pi i(k+h^\vee)} \oint \frac{dx}{x-w} \left(\frac{J^a J^a(w)}{(z-x)^2} + \frac{\partial J^a J^a(w)}{z-x} \right.$$

$$\left. + \frac{J^a J^a(w)}{(z-w)^2} + \frac{J^a \partial J^a(w)}{z-w} \right)$$

$$= \frac{k\dim g}{(2k+h^\vee)} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$