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Irreducible Representations of the Five-Dimensional Rotation Group. I*

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Explicit matrix elements are found for the generators of the group $R(5)$ in an arbitrary irreducible representation using the "natural basis" in which the representation of $R(5)$ is fully reduced with respect to the subgroup $R(4) = SU(2) \otimes SU(2)$. The technique used is based on the well-known Racah algebra. The dimension formula is derived and the invariants are found. A family of identities is established which relates various polynomials of degree four in the generators and which holds in any representation of the group.

INTRODUCTION

Recently, interest has been revived in describing the collective states of certain even–even nuclei by means of a five-dimensional isotropic harmonic oscillator arising out of the quadrupole vibrations of the nuclear surface about a spherical equilibrium shape. This model predicts[1] that the second excited state should be a degenerate triplet of angular momenta ($L^\pi = 0^+, 2^+, 4^+$) occurring at twice the excitation of the first excited state which has angular momentum and parity $L^\pi = 2^+$. As this prediction is not observed to hold in actual nuclei, the five-dimensional oscillator model is only an approximate description of the excited states of these nuclei. This description has nonetheless proved to be a convenient starting point in describing the coupling of the collective modes to the giant dipole oscillations resulting in the splitting of the giant dipole resonance.[2,3]

For the five-dimensional oscillator, only the totally symmetric irreducible representations of $SU(5)$ occur, and these may be considered to be fully reduced with respect to the subgroup $R(5)$. For application to the above problem, it is then convenient to reduce the $R(5)$ irreducible representations with respect to the physical $R(3)$.

Of course, all the main properties of the classical groups are already well known and may be found by mining in such classic works as the books by Murnaghan, Weyl, and Littlewood.[4] For practical applications, however, it is necessary to realize the irreducible representation of the group in an explicit way. This introduces the problem of labeling the states within an irreducible representation in a manner whose physical meaning is transparent. For application to the physical problem in mind, we have already indicated that the $R(5)$ representations should be explicitly reduced with respect to the physical $R(3)$ subgroup; however, it is very hard to obtain suitable explicit representation matrices directly using such a fully reduced basis. Instead, we adopt the "natural" labeling in which an irreducible representation of $R(5)$ is considered to be fully reduced with respect to its subgroup $R(4) = SU(2) \otimes SU(2)$, and a state is labeled by the particular weight of the particular irreducible representation of $R(4)$ to which it belongs. The problem of relating the natural labeling to that in which $R(5)$ is reduced with respect to the physical $R(3)$ subgroup will be the subject of our second paper.

The main original results in the present work are the development of the explicit representation matrices in the natural basis,[5] and the discussion of the well-known dimension formula and Casimir-type operators by means of our algebraic approach. So far as we know, the fourth-order identities discussed in Sec. 5 are completely new. They are analogous to those found by Pursey[6] for $SU(3)$. Much of this work was developed in embryonic form some years ago by two of us (N. K. and D. L. P.), but was not published at that time. The present treatment closely follows Pursey's treatment of $SU(3)$ in its whole-hearted exploitation of Racah algebra.

Because $R(5)$ is compact, we know that all the

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[1] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **26**, No. 14 (1952).

[2] J. Le Tourneux, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **34**, No. 11 (1965).

[3] M. G. Huber, H. J. Weber, and W. Greiner (to be published).

[4] F. D. Murnaghan, *The Theory of Group Representations* (Johns Hopkins Press, Baltimore, 1938); H. Weyl, *The Classical Groups* (Princeton Univ. Press, Princeton, N.J., 1946); D. E. Littlewood, *The Theory of Group Characters* (Oxford Univ. Press, London, 1940).

[5] Explicit representation matrices have been found for the generators of the rotation groups in arbitrarily many dimensions by I. M. Gel'fand and M. L. Zetlin, Dokl. Akad. Nauk. SSSR **71**, 1017 (1950). We believe our treatment is a simpler approach to the problem in the particular case of $R(5)$.

[6] D. L. Pursey, Proc. Roy. Soc. (London) **A275**, 284 (1963).

irreducible representations may be taken to be unitary and are finite-dimensional. It will not be necessary to make use of this latter property, however, since the finite dimensionality of the unitary representations will emerge from the algebraic formalism. It is perhaps of interest to note that this approach will also provide, via the unitary trick, the finite nonunitary irreducible representations of the de Sitter group.

1. CHOICE OF GENERATORS

It is well known that the generators M_{jk} of $R(5)$ satisfy the commutation relations

$$[M_{jk}, M_{lm}] = i(\delta_{jl}M_{km} + \delta_{km}M_{jl} - \delta_{jm}M_{kl} - \delta_{kl}M_{jm}), \quad (1)$$

where the indices run from 1 to 5. M_{jk} is the generalization to five dimensions of the angular-momentum tensor $\epsilon_{jkl}J_l$ in three dimensions. In particular, if A_l is a vector, then

$$[M_{jk}, A_l] = i(\delta_{jl}A_k - \delta_{kl}A_j). \quad (2)$$

It will be convenient to replace the ten linearly independent generators of Eq. (1) by linear combinations which explicitly display the $SU(2) \otimes SU(2) = R(4)$ subgroup of $R(5)$. This may be done by defining

$$\not{p}_\sigma = \tfrac{1}{4}\epsilon_{\sigma jk}M_{jk} + \tfrac{1}{2}M_{\sigma 4}, \quad (3a)$$

$$\not{q}_\sigma = \tfrac{1}{4}\epsilon_{\sigma jk}M_{jk} - \tfrac{1}{2}M_{\sigma 4}, \quad (3b)$$

where $\sigma, j, k = 1, 2, 3$ only, and we use the summation convention for repeated indices. Then we have the commutators

$$[\not{p}_\alpha, \not{p}_\beta] = i\epsilon_{\alpha\beta\gamma}\not{p}_\gamma, \quad (4a)$$

$$[\not{q}_\alpha, \not{q}_\beta] = i\epsilon_{\alpha\beta\gamma}\not{q}_\gamma, \quad (4b)$$

and

$$[\not{p}_\alpha, \not{q}_\beta] = 0. \quad (4c)$$

The remaining four generators are then conveniently grouped to display their transformation properties under the $SU(2) \otimes SU(2)$ subgroup generated by $\mathbf{p}$ and $\mathbf{q}$. They form a bispinor $T^{[pq]}_{\alpha\beta}$ with components

$$T^{[\frac12\frac12]}_{\frac12\frac12} = -2^{-\frac12}(M_{15} + iM_{25}), \quad (5a)$$

$$T^{[\frac12\frac12]}_{-\frac12-\frac12} = 2^{-\frac12}(M_{15} - iM_{25}), \quad (5b)$$

$$T^{[\frac12\frac12]}_{\frac12-\frac12} = 2^{-\frac12}(M_{35} - iM_{45}), \quad (5c)$$

$$T^{[\frac12\frac12]}_{-\frac12\frac12} = 2^{-\frac12}(M_{35} + iM_{45}). \quad (5d)$$

It is also convenient to replace the Cartesian generators of the two $SU(2)$ subgroups by tensors irreducible with respect to the product group. Thus we use

$$\begin{aligned} p_1 &\equiv T^{[10]}_{10} = -2^{-\frac12}(\not{p}_1 + i\not{p}_2), \\ p_0 &\equiv T^{[10]}_{00} = \not{p}_3, \\ p_{-1} &\equiv T^{[10]}_{-10} = 2^{-\frac12}(\not{p}_1 - i\not{p}_2), \end{aligned} \quad (6)$$

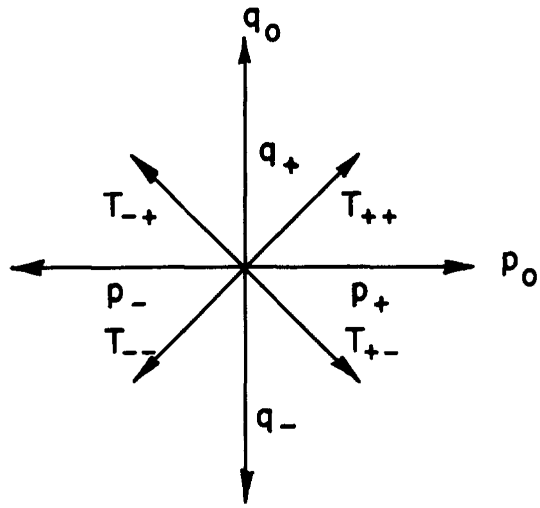

FIG. 1. The root diagram corresponding to the choice of generators of $R(5)$ given in the text. For simplicity the superscripts on the bispinor have been omitted and the $\pm\frac12$ components denoted by $\pm$ only. Similarly, the ± 1 components of $\mathbf{p}$ and $\mathbf{q}$ are denoted $\pm$.

and similarly,

$$\begin{aligned} q_1 &\equiv T^{[01]}_{01} = -2^{-\frac12}(\not{q}_1 + i\not{q}_2), \\ q_0 &\equiv T^{[01]}_{00} = \not{q}_3, \\ q_{-1} &\equiv T^{[01]}_{0-1} = 2^{-\frac12}(\not{q}_1 - i\not{q}_2). \end{aligned} \quad (7)$$

This choice of generators is conveniently displayed on the root diagram of Fig. 1. The commutation properties of the p's, q's, and the bispinor are then given by[7]

$$\begin{aligned} [p_\mu, p_\nu] &= 2^{-\frac12}C(111; \nu\mu)p_{\mu+\nu}, \\ [q_\mu, q_\nu] &= 2^{-\frac12}C(111; \nu\mu)q_{\mu+\nu}, \\ [p_\mu, q_\nu] &= 0, \\ [p_\mu, T^{[\frac12\frac12]}_{\alpha\beta}] &= \frac{3^{\frac12}}{2} C(\tfrac12 1 \tfrac12; \alpha\mu)T^{[\frac12\frac12]}_{\alpha+\mu,\beta}, \end{aligned} \quad (8)$$

and

$$[q_\nu, T^{[\frac12\frac12]}_{\alpha\beta}] = \frac{3^{\frac12}}{2} C(\tfrac12 1 \tfrac12; \beta\nu)T^{[\frac12\frac12]}_{\alpha,\beta+\nu}.$$

We shall not explicitly require the commutators of the elements of the bispinor among themselves. Rather we take linear combinations of these commutators with vector coupling coefficients to construct, in spherical-tensor form, the vector scalar $[T^{[\frac12\frac12]}, T^{[\frac12\frac12]}]^{[10]}_{\mu 0}$. Clearly one has

$$[T^{[\frac12\frac12]}, T^{[\frac12\frac12]}]^{[10]}_{\mu 0} = \lambda p_\mu, \quad (9)$$

and in order to find λ, we need merely consider one component, say $\mu = 1$; this yields $\lambda = -2$. Similarly,

[7] The vector coupling coefficients here are in the notation of M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

one has

$$[T^{[\frac{1}{2}\frac{1}{2}]}, T^{[\frac{1}{2}\frac{1}{2}]}]^{[01]}_{0\nu} = -2q_\nu. \tag{10}$$

We then use

$$[T^{[\frac{1}{2}\frac{1}{2}]}, T^{[\frac{1}{2}\frac{1}{2}]}]^{[10]}_{\mu 0} = 2(T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]})^{[10]}_{\mu 0}$$

to find

$$(T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]})^{[10]}_{\mu 0} = -T^{[10]}_{\mu 0} \tag{11a}$$

and

$$(T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]})^{[01]}_{0\nu} = -T^{[01]}_{0\nu}. \tag{11b}$$

2. BASIS STATES AND REDUCED MATRIX ELEMENTS

Each irreducible representation of $R(5)$ is considered to be fully reduced with respect to the product subgroup $SU(2) \otimes SU(2)$. Therefore the state vectors will bear the labels $|p\lambda q\mu\rangle$, where $p(p+1)$, λ, $q(q+1)$, and μ are the eigenvalues of p^2, p_0, q^2, and q_0, respectively. The basic problem then is to determine the ranges of p and q within a given irreducible representation of $R(5)$. We do this by finding the reduced matrix elements of the bispinor for our choice of basis states. In the notation of Fano and Racah,[8] the matrix element of $T^{[j_1 j_2]}_{\alpha_1\alpha_2}$ is given by the Wigner–Eckart theorem

$$\begin{aligned}\langle p'\lambda' q'\mu'|\, T^{[j_1 j_2]}_{\alpha_1\alpha_2}\, |p\lambda q\mu\rangle &= [(2p'+1)(2q'+1)]^{-\frac{1}{2}} C(pj_1p';\lambda\alpha_1\lambda') \\ &\quad \times C(qj_2q';\mu\alpha_2\mu')\,\langle p'q'\|\, T^{[j_1j_2]}\,\|pq\rangle. \end{aligned}\tag{12}$$

We have then from Eqs. (11) and (12), together with Fano and Racah's equation (15.15) and the reduced matrix elements of $T^{[10]}$ and $T^{[01]}$,

$$\begin{aligned}\sum_{p''q''}\sqrt{3}\, W(pp'\tfrac{1}{2}\tfrac{1}{2};1p'')W(qq'\tfrac{1}{2}\tfrac{1}{2};0q'') &\\ \times \langle p'q' \,\|\, p''q''\rangle\langle p''q''\,\|\, pq\rangle &\\ = -\delta_{p'p}\delta_{q'q}[p(p+1)(2p+1)(2q+1)]^{\frac{1}{2}} \end{aligned}\tag{13a}$$

and

$$\begin{aligned}\sum_{p''q''}\sqrt{3}\, W(pp'\tfrac{1}{2}\tfrac{1}{2};0p'')W(qq'\tfrac{1}{2}\tfrac{1}{2};1q'') &\\ \times \langle p'q' \,\|\, p''q''\rangle\langle p''q''\,\|\, pq\rangle &\\ = -\delta_{p'p}\delta_{q'q}[q(q+1)(2q+1)(2p+1)]^{\frac{1}{2}}, \end{aligned}\tag{13b}$$

where, in Eqs. (13), we have used Racah's notation[9] for the recoupling coefficients and have abbreviated

$$\langle p'q'\|\, T^{[\frac{1}{2}\frac{1}{2}]}\,\|pq\rangle \quad \text{by} \quad \langle p'q'\,\|\,pq\rangle.$$

In Eq. (13a), the left-hand side vanishes identically unless $p'' = p \pm \frac{1}{2}$, $p'' = p' \pm \frac{1}{2}$, $p' = p$, $p \pm 1$, and $q'' = q \pm \frac{1}{2}$. First of all, take $p' = p \pm 1$, which requires that $p'' = p \pm \frac{1}{2} = p' \mp \frac{1}{2}$. Then Eq. (13a) yields

$$\begin{aligned}\langle p\pm 1, q \,\|\, p \pm \tfrac{1}{2}, q + \tfrac{1}{2}\rangle\langle p\pm\tfrac{1}{2}, q+\tfrac{1}{2}\,\|\,pq\rangle &\\ = \langle p\pm 1, q \,\|\, p \pm \tfrac{1}{2}, q - \tfrac{1}{2}\rangle\langle p\pm\tfrac{1}{2}, q-\tfrac{1}{2}\,\|\,pq\rangle. \end{aligned}\tag{14}$$

Let us now define

$$s = p + q, \quad t = p - q, \tag{15}$$

so that Eq. (14) becomes

$$\begin{aligned}\langle s\pm 1, t\pm 1 \,\|\, s\pm 1, t\rangle\langle s\pm 1, t\,\|\, s, t\rangle &\\ = \langle s\pm 1, t\pm 1\,\|\, s, t\pm 1\rangle\langle s, t\pm 1\,\|\, s, t\rangle. \end{aligned}\tag{16}$$

Similarly, in Eq. (13b) we take $q' = q \pm 1$ and thus $q'' = q \pm \frac{1}{2} = q' \pm \frac{1}{2}$ to yield

$$\begin{aligned}\langle s\pm 1, t\mp 1 \,\|\, s\pm 1, t\rangle\langle s\pm 1, t\,\|\, s, t\rangle &\\ = \langle s\pm 1, t\mp 1\,\|\, s, t\mp 1\rangle\langle s, t\mp 1\,\|\, st\rangle. \end{aligned}\tag{17}$$

In Eq. (17) with the upper sign, we replace t by $t+1$ and multiply the resulting equation into Eq. (16) also with the upper sign. Then one has, after some cancellations,

$$\begin{aligned}\langle s+1, t+1 \,\|\, s+1, t\rangle\langle s+1, t\,\|\, s+1, t+1\rangle &\\ = \langle s, t+1\,\|\, s, t\rangle\langle s, t\,\|\, s, t+1\rangle. \end{aligned}\tag{18}$$

Since we seek unitary representations, we have so defined our generators that

$$T^{[\frac{1}{2}\frac{1}{2}]\dagger}_{\alpha\beta} = (-1)^{\alpha+\beta}T^{[\frac{1}{2}\frac{1}{2}]}_{-\alpha-\beta}, \tag{19}$$

and thus it follows from Eq. (12) that

$$\langle p'q'\,\|\,pq\rangle^* = (-1)^{p+q-p'-q'}\langle pq\,\|\,p'q'\rangle, \tag{20a}$$

or, in terms of s and t,

$$\langle s't'\,\|\,st\rangle^* = (-1)^{s-s'}\langle st\,\|\,s't'\rangle. \tag{20b}$$

Upon using Eqs. (20b) and (18), we find

$$|\langle s+1, t+1\,\|\, s+1, t\rangle|^2 = |\langle s, t+1\,\|\, s, t\rangle|^2 \equiv g(t), \tag{21}$$

which is clearly independent of s.

By a similar procedure, using opposite signs in Eqs. (16) and (17), we obtain

$$\begin{aligned}|\langle s+1, t+1\,\|\, s, t+1\rangle|^2 &\\ = |\langle s+1, t\,\|\, s, t\rangle|^2 \equiv f(s). \end{aligned}\tag{22}$$

Finally, we use Eqs. (13) with $p' = p$, $q' = q$ to obtain the induction equations for $f(s)$ and $g(t)$. From Eq. (13a), we have

$$\begin{aligned}(s+t)[f(s)+g(t)] - (s+t+2) &\\ \times [f(s-1)+g(t-1)] &\\ = -(s+t)(s+t+1)(s+t+2) &\\ \times (s-t+1), \end{aligned}\tag{23}$$

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[8] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York, 1959).

[9] See, for example, M. E. Rose, *Elementary Theory of Angular Momenta* (John Wiley & Sons, Inc., New York, 1957).

while Eq. (13b) yields

$$(s-t)[f(s)+g(t-1)]-(s-t+2)\times[f(s-1)+g(t)] = -(s-t)(s-t+1)(s-t+2)\times(s+t+1). \quad (24)$$

3. SOLUTION OF THE INDUCTION EQUATIONS

We may note in passing that from Eqs. (23) and (24) it is obvious that $g(t) = g(-t-1)$, but it will be unnecessary to use this symmetry property to solve these induction equations. Similarly, while it is very straightforward to use the compactness of $R(5)$ and therefore the knowledge that all unitary representations are finite, it is necessary only to use the fact that all irreducible representations may be taken as unitary. The finite dimensionality of the representations arises in a very natural way from the solution to the induction equations which themselves have come from the group algebra.

Equation (23) is rewritten as

$$\frac{f(s)+g(t)}{(s+t+2)(s-t+1)} - \frac{f(s-1)+g(t-1)}{(s+t)(s-t+1)} = -(s+t+1) = -\tfrac{1}{4}[(s+t+2)^2-(s+t)^2].$$

Therefore, noting that on both sides the second term is obtained from the first by replacing s by $s-1$ and t by $t-1$, we must have

$$f(s)+g(t) = -\tfrac{1}{4}(s+t+2)(s-t+1)\times[(s+t+2)^2+\alpha(s-t)], \quad (25)$$

where α is a function of $s-t$ as yet to be determined. Equation (24) is then rewritten as

$$\frac{f(s)+g(t-1)}{(s-t+2)(s+t+1)} - \frac{f(s-1)+g(t)}{(s-t)(s+t+1)} = -(s-t+1) = -\tfrac{1}{4}[(s-t+2)^2-(s-t)^2],$$

from which we find in a similar way that

$$(s)+g(t-1) = -\tfrac{1}{4}(s-t+2)(s+t+1)\times[(s-t+2)^2+\beta(s+t)]. \quad (26)$$

In Eq. (26) we replace t by $t+1$ and compare the result with Eq. (25). This gives

$$\beta(s+t) = (s+t+1)^2+2\gamma,$$
$$\alpha(s-t) = (s-t+1)^2+2\gamma. \quad (27)$$

On substituting these results into Eq. (25) or Eq. (26), we find

$$f(s)+g(t) = -\tfrac{1}{4}(s+t+2)(s-t+1)\times[(s+t+2)^2+(s-t+1)^2+2\gamma] = -\tfrac{1}{2}[(s+\tfrac{3}{2})^2-(t+\tfrac{1}{2})^2]\times[(s+\tfrac{3}{2})^2+(t+\tfrac{1}{2})^2+\gamma].$$

Therefore,

$$f(s) = -\tfrac{1}{2}\{(s+\tfrac{3}{2})^2[(s+\tfrac{3}{2})^2+\gamma]+\delta\} \quad (28a)$$

and

$$g(t) = +\tfrac{1}{2}\{(t+\tfrac{1}{2})^2[(t+\tfrac{1}{2})^2+\gamma]+\delta\}. \quad (28b)$$

Let us now introduce $x = (s+\frac{3}{2})^2$ and $y = (t+\frac{1}{2})^2$. Since $x > 0$, $y \geq 0$, and $|t| \leq |s|$, we have

$$0 \leq y < x. \quad (29)$$

Furthermore, since $f(s)$ and $g(t)$ are intrinsically positive, we have

$$-\tfrac{1}{2}x^2-\tfrac{1}{2}\gamma x-\tfrac{1}{2}\delta \geq 0 \quad (30a)$$

and

$$\tfrac{1}{2}y^2+\tfrac{1}{2}\gamma y+\tfrac{1}{2}\delta \geq 0. \quad (30b)$$

From Eq. (30a) it follows that x must lie between the roots of $x^2+\gamma x+\delta = 0$ and, from Eq. (30b), y must lie outside the roots. Thus we have

$$0 \leq y \leq -\tfrac{1}{2}\gamma-(\tfrac{1}{4}\gamma^2-\delta)^{\frac{1}{2}} \leq x \leq -\tfrac{1}{2}\gamma+(\tfrac{1}{4}\gamma^2-\delta)^{\frac{1}{2}}. \quad (31)$$

It therefore follows that $\gamma \leq 0$ and $0 \leq \delta \leq (\gamma/2)^2$.

It is clear that the bispinor has the ladder property for s and t, in that it steps either s or t up or down by 1. Hence y must reach its upper bound, for only then will $g(t)$ vanish and terminate the ladder. Similarly, x must attain its upper bound. Thus

$$f(s_{\max}) = 0 \quad (32a)$$

and

$$g(t_{\max}) = 0. \quad (32b)$$

Let us denote the upper bounds of s and t by l and k, respectively. Then we have

$$(l+\tfrac{3}{2})^2 = -\tfrac{1}{2}\gamma+(\tfrac{1}{4}\gamma^2-\delta)^{\frac{1}{2}} \quad (33a)$$

and

$$(k+\tfrac{1}{2})^2 = -\tfrac{1}{2}\gamma-(\tfrac{1}{4}\gamma^2-\delta)^{\frac{1}{2}}, \quad (33b)$$

from which it follows that

$$\gamma = -(l+\tfrac{3}{2})^2-(k+\tfrac{1}{2})^2 \quad (34)$$

and

$$\delta = (l+\tfrac{3}{2})^2(k+\tfrac{1}{2})^2. \quad (35)$$

Using Eqs. (34) and (35) with Eqs. (28), we find

$$f(s) = \tfrac{1}{2}(l-s)(l+s+3)(s-k+1)(s+k+2) \quad (36)$$

and

$$g(t) = \tfrac{1}{2}(l-t+1)(l+t+2)(k-t)(k+t+1). \quad (37)$$

To recapitulate we have

$$
\begin{aligned}
f(s) &= |\langle s+1, t\| T^{[\frac{1}{2}\frac{1}{2}]} \|s, t\rangle|^2 \\
&= |\langle p+\tfrac{1}{2}, q+\tfrac{1}{2}\| T^{[\frac{1}{2}\frac{1}{2}]} \|pq\rangle|^2, \\
g(t) &= |\langle s, t+1\| T^{[\frac{1}{2}\frac{1}{2}]} \|s, t\rangle|^2 \qquad (38)\\
&= |\langle p+\tfrac{1}{2}, q-\tfrac{1}{2}\| T^{[\frac{1}{2}\frac{1}{2}]} \|pq\rangle|^2, \\
s &= p+q, \quad t = p-q,
\end{aligned}
$$

and the solution to the induction equations is completed once we adopt the phase convention that the reduced matrix elements are the positive square roots of $f(s)$ and $g(t)$.

4. CHARACTERIZATION OF THE IRREDUCIBLE REPRESENTATIONS AND THE DIMENSION FORMULA

We have solved the induction equations in terms of the maximum values l and k of s and t in the representation. We must still determine the permissible values of l and k. We have made use of the essential nonnegative character of $f(s)$ and $g(t)$, which implies that the induction equation shall not lead to values of x greater than $(l+\frac{3}{2})^2$ nor less than $(k+\frac{1}{2})^2$ nor to values of y greater than $(k+\frac{1}{2})^2$. From the definitions of l and k, it is clear that $l \geq k \geq 0$. From the necessary symmetry of the irreducible representations under interchange of p and q, it is clear that $-k \leq t \leq k$ so that k must be an integer or half-integer; and since $l-k$ must be an integer, l is correspondingly an integer or half integer.

Thus we conclude that *an irreducible representation of $R(5)$ is characterized by two nonnegative numbers (l, k) such that both are either integers or half-integers and $l \geq k$. For given (l, k), s ranges from k to l by steps of* 1, *and t from $-k$ to k by steps of* 1.

We may express these results in terms of p and q as follows: *In an irreducible representation (l, k) of $R(5)$, p and q range from* 0 *by steps of* $\frac{1}{2}$ *to* $\frac{1}{2}(k+l)$. *For a fixed value of q, p ranges from $|k-q|$ by steps of* 1 *to the minimum of $[l-q, k+q]$. For a fixed value of p, q ranges from $|k-p|$ by steps of* 1 *to the minimum of* $[l-p, k+p]$.

The dimension of an irreducible representation (l, k) may be easily computed by summing $(2p+1)(2q+1)$ over the possible simultaneous values of p and q. Alternatively, we may sum $(s+t+1)(s-t+1)$ over the permissible values of s and t to obtain

$$d(l, k) = \tfrac{1}{6}(2k+1)(2l+3)(l+k+2)(l-k+1). \qquad (39)$$

The irreducible representations of $R(5)$ may also be characterized using the well-known isomorphism with $Sp(4)$[10] for which the irreducible representations are put into one to one correspondence with the two-rowed Young tableaux labeled by σ_1 and σ_2, which are the numbers of boxes in the first and second rows, respectively. This corresponds to classifying tensors under $Sp(4)$ according to the symmetry properties of their indices. The vector representation of $Sp(4)$ is the spinor representation of $R(5)$ and the connection between the characterizations is

$$\sigma_1 = l+k, \quad \sigma_2 = l-k. \qquad (40)$$

The substitution of Eqs. (40) into Eq. (39) gives Weyl's result.[11]

In yet another characterization, based on weight diagrams, Speiser[12] labels the irreducible representations by (L_1, L_2), which are related to σ_1, σ_2, l, and k by

$$L_1 = \sigma_2 = l-k, \quad L_2 = \sigma_1 - \sigma_2 = 2k. \qquad (41)$$

Finally, we make connection with the characterization given by Hecht[13] and by Parikh,[14] in which the labels are (p_m, q_m) which are the values of λ and μ for the state of maximum weight. The relationships are

$$p_m = \tfrac{1}{2}(l+k), \quad q_m = \tfrac{1}{2}(l-k) \qquad (42)$$

or

$$p_m + q_m = s_{\max} = l, \quad p_m - q_m = t_{\max} = k. \qquad (43)$$

5. INVARIANTS

The most direct approach to finding the invariants is undoubtedly to directly construct operators which commute with all the group generators. The matrix elements of these operators must then be expressible as a function of only l and k. Since there are but two numbers required to characterize a representation, there exist only two independent invariants. For the group $R(5)$ there will be a second-order and a fourth-order invariant since the third-order invariant $M_{ij}M_{jk}M_{ki}$ obviously vanishes. Thus, we could construct $M_{ij}M_{ji}$ and $M_{ij}M_{jk}M_{kl}M_{li}$ directly. However, this is not the most convenient way to proceed.

We shall find it most convenient to define the operator

$$T^2 = -(T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]})^{[00]}. \qquad (44)$$

Then, since this obviously commutes with p^2 and q^2, we need consider only its diagonal reduced matrix

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elements, which are given by

$$\langle (lk)pq \| T^2 \| (lk)pq \rangle = -\sum_{p'q'} W(pp\tfrac{1}{2}\tfrac{1}{2}; 0p')W(qq\tfrac{1}{2}\tfrac{1}{2}; 0q') \times \langle pq \| p'q' \rangle \langle p'q' \| pq \rangle. \quad (45)$$

From this, together with Eqs. (20a), (21), and (22), we obtain

$$\langle (lk)pq \| T^2 \| (lk)pq \rangle = \tfrac{1}{2}[(2p+1)(2q+1)]^{-\frac{1}{2}} \times \{f(s) + f(s-1) + g(t) + g(t-1)\}. \quad (46)$$

Then, using Eqs. (36) and (37), we find after some simplification

$$\langle (lk)pq \| T^2 \| (lk)pq \rangle = [(2p+1)(2q+1)]^{\frac{1}{2}} \times \{a(l,k) - p(p+1) - q(q+1)\}, \quad (47)$$

in which $a(l, k)$ is defined by

$$2a(l,k) = l(l+3) + k(k+1). \quad (48)$$

Therefore, we obtain an invariant A^2 defined by

$$A^2 = T^2 + p^2 + q^2, \quad (49)$$

whose value in the representation (l, k) is

$$A^2 = a(l,k). \quad (50)$$

This is essentially the second-order Casimir operator.

To obtain the fourth-order invariant we construct two bilinear operators

$$\tau^{[11]} \equiv (T^{[10]}T^{[01]})^{[11]}, \quad (51a)$$

or

$$\tau^{[11]}_{\alpha\beta} = p_\alpha q_\beta, \quad (51b)$$

and

$$T^{[11]} = (T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]})^{[11]}. \quad (52)$$

Then we consider the reduced matrix elements of

$$B^4 \equiv (\tau^{[11]}T^{[11]})^{[00]}. \quad (53)$$

Since

$$\langle (lk)p'q' \| \tau^{[11]} \| (lk)pq \rangle = \delta_{pp'}\delta_{qq'}[p(p+1)(2p+1)q(q+1)(2q+1)]^{\frac{1}{2}}, \quad (54)$$

we need only consider the diagonal elements of $T^{[11]}$, which are

$$\langle (lk)pq \| T^{[11]} \| (lk)pq \rangle = -\tfrac{1}{2}[p(p+1)(2p+1)q(q+1)(2q+1)]^{-\frac{1}{2}} \times \sum_{p'q'} [p(p+1) - p'(p'+1) + \tfrac{3}{4}] \times [q(q+1) - q'(q'+1) + \tfrac{3}{4}] |\langle p'q' \| pq \rangle|^2. \quad (55)$$

Then we find

$$\langle (kl)pq \| B^4 \| (kl)pq \rangle = \tfrac{1}{6}[(2p+1)(2q+1)]^{\frac{1}{2}} \times \{p(q+1)g(t) + q(p+1)g(t-1) - pqf(s) - (p+1)(q+1)f(s-1)\}. \quad (56)$$

To reduce this it is convenient to note that we may write

$$g(t) = \tfrac{1}{2}\{b(l,k) - 2t(t+1)a(l,k) + (t-1)t(t+1)(t+2)\} \quad (57a)$$

and

$$f(s) = -\tfrac{1}{2}\{b(l,k) - 2(s+1)(s+2)a(l,k) + s(s+1)(s+2)(s+3)\}, \quad (57b)$$

in which

$$b(l,k) = (l+1)(l+2)k(k+1). \quad (58)$$

Using Eqs. (57), after some simplification one has

$$\langle (kl)pq \| B^4 \| (kl)pq \rangle = \tfrac{1}{12}[(2p+1)(2q+1)]^{\frac{1}{2}}\{b(l,k) - 2a(l,k)[p(p+1) + q(q+1)] + p^2(p+1)^2 + q^2(q+1)^2 - 2p(p+1) - 2q(q+1) + 6p(p+1)q(q+1)\}. \quad (59)$$

Therefore, the operator

$$M^4 = 12B^4 + (p^2 - q^2)^2 + 2(p^2 + q^2)(T^2 + 1) \quad (60)$$

is invariant and in the representation (l, k) has the value

$$M^4 = b(l,k) = (l+1)(l+2)k(k+1). \quad (61)$$

At first sight, the invariant of Eqs. (60) and (61) seems curious in that the usual way of forming the second invariant operator would have it symmetric in all the Cartesian generators and of fourth order. M^4 contains second-order terms and does not involve

$$T^4 \equiv (T^{[11]}T^{[11]})^{[00]}. \quad (62)$$

One would therefore expect that there exists an identity relating T^4 to p^2, q^2, and possibly other quantities. This indeed is the case as may be readily demonstrated by considering

$$T_{(\alpha_1\beta_1)(\alpha_2\beta_2)(\alpha_3\beta_3)(\alpha_4\beta_4)} \equiv T_{\alpha_1\beta_1}[T_{\alpha_2\beta_2}, T_{\alpha_3\beta_3}]_+ T_{\alpha_4\beta_4}, \quad (63)$$

where the $T_{\alpha_i\beta_i}$ of Eq. (63) are indicated without superscripts and may be any one of the irreducible tensor generators of the group. From this tensor, which is clearly symmetric on its second and third pairs of indices, we may form irreducible tensors. For example, we may couple the first two pairs to $[j_1, \lambda_1]$ and the second two pairs to $[j_2, \lambda_2]$ and then couple the result to $[JL]$. We may also couple the first and third pairs to $[j_1'\lambda_1']$, the second and fourth pairs to $[j_2', \lambda_2']$ and then couple to $[JL]$. Because of the explicit symmetry between the second and third pairs of indices, the two couplings produce the same sets of irreducible tensors. Recoupling leads to the identities

$$T^{[j_1\lambda_1,j_2\lambda_2;JL]} = \sum_{\substack{j_1',\lambda_1',\\ j_2',\lambda_2'}} [(2j_1+1)(2\lambda_1+1)(2j_2+1)(2\lambda_2+1) \times (2j_1'+1)(2\lambda_1'+1)(2j_2'+1)(2\lambda_2'+1)]^{\frac{1}{2}} \times \begin{Bmatrix} p_1 p_2 j_1 \\ p_3 p_4 j_2 \\ j_1' j_2' J \end{Bmatrix} \begin{Bmatrix} q_1 q_2 \lambda_1 \\ q_3 q_4 \lambda_2 \\ \lambda_1 \lambda_2' L \end{Bmatrix} T^{[j_1'\lambda_1',j_2'\lambda_2';JL]}, \tag{64}$$

where { } is the usual $9j$ symbol. We are specifically interested in the tensor $T^{[11,11;00]}$ formed from four bispinor components. From Eq. (64) we find

$$T^{[11,11;00]} = \sum_{j\lambda} [(2j+1)(2\lambda+1)]^{\frac{1}{2}}(-1)^{j+\lambda} \times W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2};1j)W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2};1\lambda)T^{[j\lambda,j\lambda;00]}. \tag{65}$$

When we use explicit forms for the Racah coefficients, Eq. (65) becomes

$$T^{[11,11;00]} = 9T^{[00,00;00]} - 3\sqrt{3}(T^{[10,10;00]} + T^{[01,01;00]}). \tag{66}$$

Now, from Eq. (63) with all $T_{\alpha\beta} \equiv T_{\alpha\beta}^{[\frac{1}{2}\frac{1}{2}]}$, we have

$$T^{[11,11;00]} = T^4, \tag{67a}$$

$$T^{[00,00;00]} = (T^2)^2, \tag{67b}$$

$$T^{[10,10;00]} = -p^2, \tag{67c}$$

and

$$T^{[01,01;00]} = -q^2. \tag{67d}$$

Equations (67) together with Eq. (66) provide the identity we seek, namely,

$$T^4 = 9(T^2)^2 + 3\sqrt{3}(p^2+q^2). \tag{68}$$

Thus, the invariant of Eq. (60) can be written in terms of fourth-order quantities as

$$M^4 = 12B^4 + (p^2-q^2)^2 + 2(p^2+q^2)T^2 + \tfrac{2}{9}\sqrt{3}[T^4 - 9(T^2)^2]. \tag{69}$$

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Irreducible Representations of the Five-Dimensional Rotation Group. II*

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A systematic study is made of the relationship between the generators of $R(5)$ expressed in the "natural basis," as discussed in I [J. Math. Phys. **9**, 1224 (1968)], and the same generators in the "physical basis" in which representations of $R(5)$ are fully reduced with respect to the physical three-dimensional rotation group. In this paper, attention is confined to the traceless symmetric tensors of $R(5)$ which are the representations appropriate to the discussion of quadrupole vibrations of the nuclear surface. For these representations, one quantum number in addition to the angular momentum and its projection is required to specify a state within a representation. The required extra label is found through the definition of "intrinsic states" in the natural basis, and a complete set of states in the physical basis is projected out of these intrinsic states by integrations over the physical rotation group manifold. Members of this set of physical states are not orthonormal; however, the overlap integrals are presented in two simple algebraic forms convenient for computer programming. The construction of the explicit representation matrices for the generators of $R(5)$ is completed by giving the reduced matrix elements of the octopole generator between physical states in terms of the overlap integrals.

INTRODUCTION

In I[1] the irreducible representations of $R(5)$ were built up directly from the generator algebra in a manner closely analogous to that usually done for $SU(2)$. For this purpose, the irreducible representations of $R(5)$ were reduced with respect to the subgroup $R(4) = SU(2) \otimes SU(2)$. We shall call the state labeling so developed the *natural labeling*. Unfortunately, neither of these two $SU(2)$ subgroups corresponds to the physical angular momentum. For physical application it is essential that the irreducible representations of $R(5)$ be decomposed into irreducible representations of the physical $R(3)$.

The particular physical application we have in mind is the five-dimensional harmonic oscillator which has been used[2] to describe quadrupole vibrations of the

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