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sets of irreducible tensors. Recoupling leads to the identities

$$T^{[j_1\lambda_1,j_2\lambda_2;JL]} = \sum_{\substack{j_1',\lambda_1',\\ j_2',\lambda_2'}} [(2j_1+1)(2\lambda_1+1)(2j_2+1)(2\lambda_2+1) \times (2j_1'+1)(2\lambda_1'+1)(2j_2'+1)(2\lambda_2'+1)]^{\frac{1}{2}} \times \begin{Bmatrix} p_1p_2j_1 \\ p_3p_4j_2 \\ j_1'j_2'J \end{Bmatrix} \begin{Bmatrix} q_1q_2\lambda_1 \\ q_3q_4\lambda_2 \\ \lambda_1\lambda_2'L \end{Bmatrix} T^{[j_1'\lambda_1',j_2'\lambda_2';JL]}, \quad (64)$$

where { } is the usual $9j$ symbol. We are specifically interested in the tensor $T^{[11,11;00]}$ formed from four bispinor components. From Eq. (64) we find

$$T^{[11,11;00]} = \sum_{j\lambda} [(2j+1)(2\lambda+1)]^{\frac{1}{2}}(-1)^{j+\lambda} \times W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2};1j)W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2};1\lambda)T^{[j\lambda,j\lambda;00]}. \quad (65)$$

When we use explicit forms for the Racah coefficients, Eq. (65) becomes

$$T^{[11,11;00]} = 9T^{[00,00;00]} - 3\sqrt{3}(T^{[10,10;00]} + T^{[01,01;00]}). \quad (66)$$

Now, from Eq. (63) with all $T_{\alpha\beta} \equiv T_{\alpha\beta}^{[\frac{1}{2}\frac{1}{2}]}$, we have

$$T^{[11,11;00]} = T^4, \quad (67a)$$
$$T^{[00,00;00]} = (T^2)^2, \quad (67b)$$
$$T^{[10,10;00]} = -p^2, \quad (67c)$$

and

$$T^{[01,01;00]} = -q^2. \quad (67d)$$

Equations (67) together with Eq. (66) provide the identity we seek, namely,

$$T^4 = 9(T^2)^2 + 3\sqrt{3}(p^2+q^2). \quad (68)$$

Thus, the invariant of Eq. (60) can be written in terms of fourth-order quantities as

$$M^4 = 12B^4 + (p^2-q^2)^2 + 2(p^2+q^2)T^2 + \tfrac{2}{9}\sqrt{3}[T^4 - 9(T^2)^2]. \quad (69)$$

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Irreducible Representations of the Five-Dimensional Rotation Group. II*

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A systematic study is made of the relationship between the generators of $R(5)$ expressed in the "natural basis," as discussed in I [J. Math. Phys. **9**, 1224 (1968)], and the same generators in the "physical basis" in which representations of $R(5)$ are fully reduced with respect to the physical three-dimensional rotation group. In this paper, attention is confined to the traceless symmetric tensors of $R(5)$ which are the representations appropriate to the discussion of quadrupole vibrations of the nuclear surface. For these representations, one quantum number in addition to the angular momentum and its projection is required to specify a state within a representation. The required extra label is found through the definition of "intrinsic states" in the natural basis, and a complete set of states in the physical basis is projected out of these intrinsic states by integrations over the physical rotation group manifold. Members of this set of physical states are not orthonormal; however, the overlap integrals are presented in two simple algebraic forms convenient for computer programming. The construction of the explicit representation matrices for the generators of $R(5)$ is completed by giving the reduced matrix elements of the octopole generator between physical states in terms of the overlap integrals.

INTRODUCTION

In I[1] the irreducible representations of $R(5)$ were built up directly from the generator algebra in a manner closely analogous to that usually done for $SU(2)$. For this purpose, the irreducible representations of $R(5)$ were reduced with respect to the subgroup $R(4) = SU(2) \otimes SU(2)$. We shall call the state labeling so developed the *natural labeling*. Unfortunately, neither of these two $SU(2)$ subgroups corresponds to the physical angular momentum. For physical application it is essential that the irreducible representations of $R(5)$ be decomposed into irreducible representations of the physical $R(3)$.

The particular physical application we have in mind is the five-dimensional harmonic oscillator which has been used[2] to describe quadrupole vibrations of the

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[1] N. Kemmer, D. L. Pursey, and S. A. Williams, J. Math. Phys. **9**, 1224 (1968) (preceding paper).

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nuclear surface about a spherical equilibrium shape. Such a description is, of course, only very approximate, but it has proved to be a convenient starting point for describing the coupling of these nuclear surface oscillations to the oscillations of the giant dipole resonance.[3,4] The state functions for the five-dimensional isotropic harmonic oscillator form the bases for the totally symmetric irreducible representations of $SU(5)$ and these may be considered to be fully reduced with respect to the subgroup $R(5)$. In this paper, we shall therefore confine ourselves to the problem of the decomposition of the symmetric irreducible representations of $R(5)$ with respect to $R(3)$.

The irreducible representations of $R(3)$ contained in a given irreducible representation of $R(5)$ may of course be obtained by the well-known methods of the reduction of product representations for both $R(5)$ and $R(3)$.[5,6] This technique, while useful for some purposes, does not allow one to perform detailed calculations using the basis functions involved, since the method does not yield explicit representations matrices for all the generators. A further difficulty arises in that, within a given irreducible representation of $R(5)$, a particular irreducible representation of $R(3)$ may occur more than once. When we have specified the generators, it will become clear that no proper subgroup of $R(5)$ [apart from the physical $R(3)$ itself] contains the particular $R(3)$ (the physical angular-momentum group) in which we are interested. Therefore, we must seek additional labels, which cannot be obtained by the aforementioned technique, to completely specify the basis functions. For the symmetric irreducible representations of $R(5)$ we shall find that only one additional label is necessary.

This additional label is obtained in a manner closely analogous to that of Elliott[7] for $SU(3)$. Specifically, the $R(5)$–$R(3)$ basis functions will be projected from a small subset of the natural basis functions by Hill–Wheeler[8]-type integrals. This will lead, as it did in Elliott's case, to $R(5)$–$R(3)$ basis functions which are not orthogonal. The fact that the solution to the problem is simpler in terms of nonorthogonal functions is not too surprising in view of Elliott's work and Racah's[9] comments about the $SU(3)$ problem. This lack of orthogonality presents no serious difficulties in general, and for convenience for machine coding, the set of linearly independent functions given could of course be orthogonalized.

In Sec. 1, we shall give the connection between the natural generators and those which explicitly exhibit the $R(3)$ subgroup. This is done in preparation for Sec. 2, in which we shall give a formula for the irreducible representations of $R(3)$ which occur in a given symmetric irreducible representation of $R(5)$. This formula serves to introduce the additional quantum number in an empirical way. In Sec. 3, we shall relate this additional quantum number to the natural basis functions and show that only a small subset of the natural basis function are required for projecting out the nonorthonormal $R(5)$–$R(3)$ basis functions. In Sec. 4, we will explicitly determine the normalization and overlap integrals for the $R(5)$–$R(3)$ basis functions. These quantities are used in Sec. 5, in which we shall give expressions for the matrix elements of the group generators expressed in the $R(5)$–$R(3)$ basis.

1. GENERATORS

In I we utilized the natural generators of $R(5)$, namely, p_μ, q_ν, and $T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}$. The p_μ and q_ν are the generators of the two commuting $SU(2)$ subgroups and the $T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}$ are the remaining generators which are expressed as a bispinor under the product group $SU(2)\otimes SU(2)$. These generators satisfy the commutation rules:

$$\begin{aligned}
[p_\mu, p_\nu] &= -\sqrt{2}\, C(111;\mu\nu)p_{\mu+\nu},\\
[q_\mu, q_\nu] &= -\sqrt{2}\, C(111;\mu\nu)q_{\mu+\nu},\\
[p_\mu, q_\nu] &= 0,\\
[p_\mu, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}] &= \tfrac{1}{2}\sqrt{3}\, C(\tfrac{1}{2}1\tfrac{1}{2};\alpha\mu)T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha+\mu,\beta},
\end{aligned} \tag{1}$$

and

$$[q_\nu, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}] = \tfrac{1}{2}\sqrt{3}\, C(\tfrac{1}{2}1\tfrac{1}{2};\beta\nu)T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha,\beta+\nu}.$$

In Eqs. (1), the $SU(2)$ Clebsch–Gordan coefficients are in the notation of Rose.[10]

The $R(5)$ generators may also be grouped as the generators of $R(3)$ together with a third-rank irreducible tensor under $R(3)$. These we write as J_μ and $Q^{[3]}_\nu$. The latter is abbreviated as Q_ν. When expressed in spherical tensor form, these generators satisfy the commutation relationships:

$$\begin{aligned}
[J_\mu, J_\nu] &= -\sqrt{2}\, C(111;\mu\nu)J_{\mu+\nu},\\
[J_\mu, Q_\nu] &= -2\sqrt{3}\, C(133;\mu\nu)Q_{\mu+\nu},
\end{aligned} \tag{2}$$

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and

$$[Q_\mu, Q_\nu] = -2\sqrt{7}\, C(331;\mu\nu)J_{\mu+\nu} + \sqrt{6}\, C(333;\mu\nu)Q_{\mu+\nu}.$$

From the commutation algebra among the generators we may identify the natural basis generators in terms of the $R(5)$–$R(3)$ basis generators. We find[11]

$$p_1 = 10^{-\frac{1}{2}}Q_3, \quad (3a)$$

$$p_0 = 10^{-1}(3J_0 - Q_0), \quad (3b)$$

and

$$p_{-1} = 10^{-\frac{1}{2}}Q_{-3}; \quad (3c)$$

also

$$q_1 = 5^{-1}(J_1 + \tfrac{1}{2}6^{\frac{1}{2}}Q_1), \quad (4a)$$

$$q_0 = 10^{-1}(J_0 + 3Q_0), \quad (4b)$$

$$q_{-1} = 5^{-1}(J_{-1} + \tfrac{1}{2}6^{\frac{1}{2}}Q_{-1}); \quad (4c)$$

and, finally,

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2}\frac{1}{2}} = 5^{-\frac{1}{2}}Q_2, \quad (5a)$$

$$T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2}\frac{1}{2}} = 5^{-1}(3^{\frac{1}{2}}J_{-1} - 2^{\frac{1}{2}}Q_{-1}), \quad (5b)$$

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2},-\frac{1}{2}} = -5^{-1}(3^{\frac{1}{2}}J_1 - 2^{\frac{1}{2}}Q_1), \quad (5c)$$

and

$$T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2},-\frac{1}{2}} = -5^{-\frac{1}{2}}Q_{-2}. \quad (5d)$$

In Eqs. (5), the subscripts on the bispinor are, as usual, p, q ordered. From the second of Eqs. (3) and (4) we have the primary equation which relates the natural basis to the $R(5)$–$R(3)$ basis; namely,

$$J_0 = 3p_0 + q_0. \quad (6)$$

2. BASIS FUNCTIONS

In I we showed that the irreducible representations of $R(5)$ were characterized by two nonnegative numbers (l, k) which are either both integers or both half integers. For given (l, k), $s \equiv p + q$ ranges from k to l by steps of 1 and $t \equiv p - q$ from $-k$ to k by steps of 1. The symmetric representations of $R(5)$ are $(l, 0)$, in which case, since $k = 0$, $p = q$. Then, since $s = 0, 1, 2, \cdots, l$, we have that for the symmetric representations $(l, 0)$

$$p = q = 0, \tfrac{1}{2}, 1, \ldots\ldots, \tfrac{1}{2}l. \quad (7)$$

In general, to completely specify a basis function for an irreducible representation of $R(5)$ requires six labels. In the natural basis these are (l, k), p, q, λ, μ. The p and q label the two commuting $SU(2)$ subgroups, and λ and μ are the eigenvalues of p_0 and q_0, respectively. The (l, k) are related to the eigenvalues of the two invariant operators constructed from the generators of $R(5)$. These operators and their eigenvalues are

$$A^2 \equiv -[T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]}]^{[00]} + p^2 + q^2 = \tfrac{1}{2}[l(l+3) + k(k+1)]$$

and

$$M^4 \equiv 12[[pq]^{[11]}[T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]}]^{[11]}]^{[00]} + (p^2 - q^2)^2 + 2(p^2+q^2)\{-[T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]}]^{[00]} + 1\} = (l+1)(l+2)k(k+1). \quad (8)$$

For the symmetric representations, only the second-order $R(5)$ invariant is, in general, nonzero and its value from the first of Eq. (8) is $\frac{1}{2}l(l+3)$. We also require p, λ, and μ—four labels in all. Thus for the symmetric representations we may denote the natural basis functions as $\chi(lp\lambda\mu)$.

In the $R(5)$–$R(3)$ basis we shall also require four labels for the symmetric-representation basis functions. Three of these are l, J, and M, which are the simultaneous eigenvalues of

$$A^2 = \tfrac{1}{2}l(l+3),$$
$$J^2 = J(J+1), \quad (9)$$
$$J_0 = M.$$

From Eqs. (2), it is clear that no proper subgroup of $R(5)$ contains $R(3)$ as a proper subgroup. Further, although Q^2 commutes with A^2, J^2, and J_0, $J^2 + Q^2$ is essentially A^2, and hence Q^2 does not provide a new label. We shall label the $R(5)$–$R(3)$ basis functions as $\psi(l\nu JM)$ where the additional label ν is as yet unspecified.

We may determine the possible values of J within a representation by a method similar to the well-known derivation of the Clebsh–Gordan series for $R(3)$. The procedure is illustrated in Fig. 1 for the representations $(1, 0)$ and $(6, 0)$. A grid of points (λ, μ), where λ and μ are the eigenvalues of p_0 and q_0, respectively, is set up, and at each point of the grid we write the degeneracy of the corresponding pair (λ, μ) of eigenvalues. We next draw the lines $3\lambda + \mu = \text{const} = M$ and label each line by $[M, d(M)]$, where $d(M)$ is the total degeneracy of the eigenvalue M of J_0. The possible values of J, together with their degeneracies, are then found by a simple counting procedure: the angular momentum J occurs $d(J) - d(J+1)$ times.

From Fig. 1 it is clear that the extra quantum number ν is indeed necessary to distinguish between the two occurrences of $J = 6$ in the irreducible

[11] To obtain Eqs. (3)–(5) it is easiest first to convince oneself of the truth of Eq. (6). Apart from factors, this is sufficient to establish Eqs. (3a), (3c), (5a), and (5d). The form (3b) and the numerical factors in Eqs. (3) follow from the $SU(2)$ commutation rules obeyed by p_μ. Equations (3b) and (6) then yield (4b). Equation (5b) is derived from Eqs. (5a), (3c) and similarly Eq. (5c) comes from Eqs. (5d) and (3a). The commutation properties of the bispinor fix the constants in Eqs. (5) and also yield Eqs. (4).

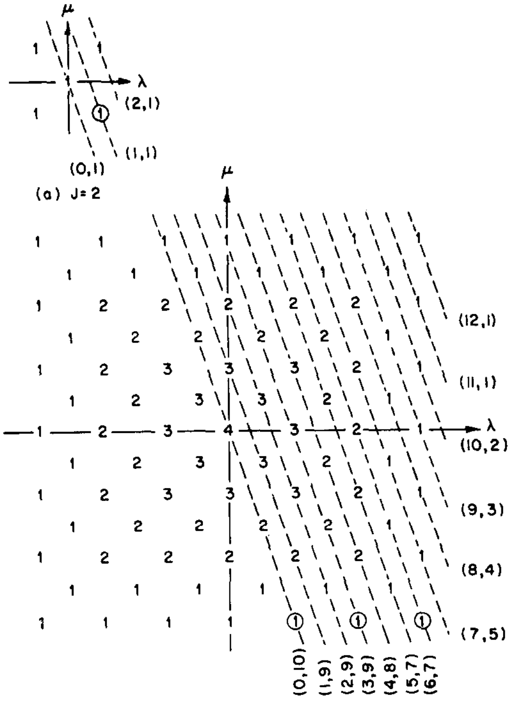

FIG. 1. Degeneracy diagrams for the (a) (1, 0) and (b) (6, 0) irreducible representations of $R(5)$. The figure is discussed in the main text.

representation (6, 0). Here we shall introduce a suitable extra label and state the general rule for the range of possible J values associated with the label ν in the irreducible representation $(l, 0)$. The proof of the rule will be postponed until the next section.

We define *intrinsic states* $\chi(l, \nu)$ by

$$\chi(l, \nu) \equiv \chi(l, \tfrac{1}{2}l, \tfrac{1}{2}l - \nu, -\tfrac{1}{2}l), \tag{10}$$

where

$$\nu = 0, 1, 2, \cdots, [l/3] \tag{11}$$

and $[l/3]$ is the integral part of $l/3$. The points corresponding to intrinsic states are circled in Fig. 1. We next introduce the numbers

$$K = l - 3\nu, \tag{12}$$

which are the values of M for the intrinsic states. K takes on the values $l, l-3, l-6, \cdots, 0$ or 1 or 2. Then, corresponding to any ν or, equivalently, any K, the possible values of J are

$$J = 2K, 2K-2, 2K-3, \cdots, K; \tag{13}$$

that is, J can take on all values from K through $2K$ except for $2K-1$. Furthermore, use of the label ν is sufficient to resolve the degeneracy in J.

We are now faced not only with the problem of proving this general result, but also with that of constructing from the intrinsic states $\chi(l, \nu)$ a complete (but not necessarily orthonormal) set of states $\psi(l\nu JM)$. We shall discuss these problems in the next section, and content ourselves here with noting that, as proved in Appendix A, our general rule reproduces the dimension formula

$$d(l, 0) = \tfrac{1}{6}(l+1)(l+2)(2l+3) \tag{14}$$

found in I.

3. MAIN THEOREM

In this section, we shall prove the general result expressed in Eqs. (11) and (13). To do so, we shall define the states

$$\psi(l\nu JM) = \int D^{J*}_{M,K}(\Omega)\chi_\Omega(l, \nu)\, d\Omega, \tag{15}$$

where

$$\nu = 0, 1, 2, \cdots, [l/3],$$
$$K = l - 3\nu,$$

and

$$J = 2K, 2K-2, 2K-3, \cdots, K;$$

$\chi(l, \nu) \equiv \chi(l, \frac{1}{2}l, \frac{1}{2}l - \nu, -\frac{1}{2}l)$ and $D^{J}_{M,K}(\Omega)$ is an ordinary rotation matrix; finally, the integral is the invariant integral over the group manifold of $R(3)$. Thus Eq. (15) defines $\psi(l\nu JM)$ as the state of angular momentum J, z component of angular momentum M, projected out of the intrinsic state $\chi(l, \nu)$ by the Hill–Wheeler[8] technique. We now state our main theorem:

Theorem: The functions $\psi(l\nu JM)$, defined by Eq. (15), span the representation space of the irreducible representation $(l, 0)$ of $R(5)$.

Before we start on the proof, we note that the results in the previous section on the possible values of J follow as a trivial corollary.

We first prove the following:

Lemma: Any angular momentum J which is not represented by at least one of the functions $\psi(l\nu JM)$ defined by Eq. (15) is entirely absent from the irreducible representation $(l, 0)$ of $R(5)$.

This lemma is of course much weaker than the result asserted in the last section and implied by the main theorem and is consequently easier to prove. To establish the lemma, we must determine which values of J are absent according to Eqs. (12) and (13), and then verify that these J values are indeed missing from the irreducible representation $(l, 0)$ of $R(5)$. Two cases arise:

(a) $l = 3n$: The missing values of J are $J = 1, 2, 5$, $2l-1$, and $J > 2l$.

(b) $l \neq 3n$: The missing values of J are $J = 0, 1, 3, 2l - 1$, and $J > 2l$.

From the degeneracy diagrams introduced in the previous section, it is clear that $J > 2l$ and $J = 2l - 1$ are indeed absent from the irreducible representation $(l, 0)$. It remains to show that $J = 1, 2, 5$ are missing if $l = 3n$ and $J = 0, 1, 3$ are missing if $l \neq 3n$. We notice that, to conclude that a particular value of J is missing, it is sufficient to show that $d(J) = d(J + 1)$ in the notation of the previous section. A detailed formal proof of the lemma would be tedious and not particularly illuminating; instead we shall sketch the bare bones of a proof, making full use of insights gained from the degeneracy diagrams.

By comparing the degeneracies of the pairs of eigenvalues (p_0, q_0) and $(p_0 + 1, q_0)$ it is clear that, for sufficiently small M,

$$d(M) - d(M + 3) = [M/2] + 1, \qquad (16)$$

where, as before, $[M/2]$ is the integral part of $M/2$. In particular,

$$\begin{aligned} d(0) - d(3) &= 1, \\ d(1) - d(4) &= 1, \\ d(2) - d(5) &= 2, \\ d(3) - d(6) &= 2. \end{aligned}$$

From this it follows that one, but only one, of the J values 0, 1, 2 occurs and that this J value is nondegenerate. The same conclusion applies to the possible values 1, 2, and 3 of J. Of the set of values 2, 3, and 4, either only one occurs with degeneracy 2 or else two distinct values occur; a similar conclusion applies to the values 3, 4, and 5.

Now, either $J = 0$ occurs in the representation $(l, 0)$ or it does not. If $J = 0$ does occur, then $J = 1$ or 2 must be absent. Hence, $J = 3$ must occur nondegenerately and therefore 4 must also occur. Consequently, $J = 5$ must be absent. Thus if $J = 0$ occurs in $(l, 0)$, then $J = 1, 2, 5$ must be missing. On the other hand, if $J = 0$ does not occur in $(l, 0)$, then either $J = 1$ is present or $J = 1$ is missing. If $J = 1$ is missing, one concludes that $J = 3$ is also missing and $J = 2, 4, 5$ each occur without degeneracy.

These results are sufficient to prove the lemma as soon as we have seen that $J = 0$ occurs for $l = 3n$ and not otherwise, and that $J = 1$ never occurs. It is easy to convince oneself of these special results by consideration of degeneracy diagrams.

We are now in a position to prove the main theorem. The method of proof is an adaptation of that used by Elliott[7] in considering the $SU(3)$–$R(3)$ reduction problem and proceeds by *reductio ad absurdum.*

We suppose that the set of functions $\psi(l\nu JM)$ does *not* span the representation space for the irreducible representation $(l, 0)$. It follows that there must exist a function $\varphi(J', M')$ in the representation space of $(l, 0)$ which is orthogonal to all the $\psi(l\nu JM)$. Of course, this would be automatically true if J' were different from any of the J values represented among the functions $\psi(l\nu JM)$, but this possibility is excluded by the lemma. Hence we conclude that there exists a function $\varphi(JM)$ such that the Hilbert space scalar product

$$(\varphi(JM), \psi(l\nu J'M')) = 0$$

for all ν, J', M', and the result is nontrivial only when $J' = J$, $M' = M$.

Hence our hypothesis shows that

$$\begin{aligned} &\int d\Omega\, D^{J*}_{MK}(\Omega)(\varphi(JM), \Omega\chi(l\nu)) \\ &\quad = \int d\Omega\, D^{J*}_{MK}(\Omega)(\Omega^{-1}\varphi(JM), \chi(l, \nu)) \\ &\quad = \sum_{M'} \int d\Omega\, D^{J*}_{MK}(\Omega)\, D^{J}_{MM'}(\Omega)(\varphi(JM'), \chi(l\nu)) \\ &\quad = (2J + 1)^{-1}(\varphi(J, K), \chi(l, \nu)) = 0 \qquad (17) \end{aligned}$$

for a suitable choice of the volume element $d\Omega$ in the $R(3)$ parameter space. Hence, since $J_0\chi(l, \nu) = K\chi(l, \nu)$, we must have

$$[\varphi(J, M), \chi(l, \nu)] = 0 \qquad (18)$$

for all intrinsic states $\chi(l, \nu)$ and all values of M, not necessarily equal to K.

We now proceed to show that Eq. (18) implies

$$[\varphi(J, M), \mathcal{O}\chi(l, \nu)] = 0, \qquad (19)$$

where $\mathcal{O}$ is an arbitrary element of $R(5)$ acting on the intrinsic state $\chi(l, \nu)$. We note that $\mathcal{O}$ can be expressed as a power series in the generators of the group, and that the generators in any particular term may be ordered in any desired manner provided we included a compensating term of lower degree derived from the commutation rules. We divide the generators appropriate to the natural basis into two sets

$$\text{A:}\quad p_{\pm1},\quad p_0,\quad q_{-1},\quad q_0,\quad T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},-\frac{1}{2}}$$

and

$$\text{B:}\quad q_1,\quad T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\frac{1}{2}}.$$

When a generator of the set A acts on an intrinsic state, the result is either an intrinsic state (if the generator is one of $p_{\pm1}$, p_0, q_0) or zero (for q_{-1}, $T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},-\frac{1}{2}}$, and if $\nu = 0, p_{+1}$). Generators of set B do not reproduce intrinsic states but are equivalent in their effect to the operators $J_{\pm1}$ acting on intrinsic states.

To see this, we note from Eqs. (3)–(5) that

$$J_1 = 2q_1 - \sqrt{3}\, T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2},-\frac{1}{2}}$$

and

$$J_{-1} = 2q_{-1} + \sqrt{3}\, T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2},\frac{1}{2}}. \tag{20}$$

From these, together with the fact that $T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2},-\frac{1}{2}}$ and q_{-1} annihilate intrinsic states, it follows that

$$q_1\chi(l, \nu) = \tfrac{1}{2}J_1\chi(l, \nu) \tag{21}$$

and

$$T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2},\frac{1}{2}}\chi(l, \nu) = 3^{-\frac{1}{2}}J_{-1}\chi(l, \nu). \tag{22}$$

The operator $T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2}\frac{1}{2}}$ is more tricky to deal with, and success depends on the fact that for intrinsic states $p = q = \frac{1}{2}l$, i.e., p attains its maximum value. From the explicit matrix elements of $T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\frac{1}{2}}$ which were found in I, specialized to intrinsic states, we find

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2}\frac{1}{2}}\chi(l, \nu) = T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2},\frac{1}{2}}\chi(l, \nu - 1) = 3^{-\frac{1}{2}}J_{-1}\chi(l, \nu - 1), \tag{23}$$

where the first step utilizes $p = \frac{1}{2}l = p_{\max}$ and the second step follows from Eq. (22).

We now see that

$$(\varphi(J, M), \mathcal{O}\chi(l, \nu)) = \sum_r (\varphi(J, M), \pi_{B_r}\pi_{A_r}\chi(l, \nu)), \tag{24}$$

where π_{A_r}, π_{B_r} are products of generators of sets A and B, respectively. Each factor π_{A_r} merely reproduces an intrinsic state [in general, different from $\chi(l, \nu)$], while the factor π_{B_r} is then equivalent to a product of $R(3)$ generators operating on an intrinsic state. However, the $R(3)$ generators may be taken to act on $\varphi(J, M)$, in which case only the M value can be changed. Thus we obtain, finally,

$$\begin{aligned}(\varphi(J, M), \mathcal{O}\chi(l, \nu)) &= \sum_r (\varphi(J, M), \pi_{B_r}\pi_{A_r}\chi(l, \nu)) \\ &= \sum_{rM'\nu'} C(r, M', \nu')(\varphi(J, M'), \chi(l, \nu')) = 0\end{aligned} \tag{25}$$

by Eq. (19).

We are now in a position to complete the *reductio ad absurdum* proof of the theorem. By our hypothesis that the theorem is false, we have found that there exist functions $\varphi(J, M)$ belonging to the representation space for the irreducible representation $(l, 0)$ of $R(5)$ which are orthogonal to all the intrinsic states $\chi(l, \nu)$. Then, by Eq. (25), we see that the $\varphi(J, M)$ are orthogonal to all states of the form $\mathcal{O}\chi(l, \nu)$, where $\mathcal{O}$ is an arbitrary element of the group $R(5)$. However, from the irreducibility of the representation $(l, 0)$, which implies by definition that the representation space possesses no proper subspace invariant under the group, it follows that from the set of states $\mathcal{O}\chi(l, \nu)$ we can find a subset which spans the complete representation space. Hence, $\varphi(J, M)$ is orthogonal to all states in the representation space of $(l, 0)$, which contradicts the hypothesis that $\varphi(J, M)$ belongs to this space. This contradiction is sufficient to prove the theorem.

4. NORMALIZATION AND OVERLAP INTEGRALS

The functions $\psi(l\nu JM)$ defined by Eq. (15) have been shown to be the basis functions for the irreducible representation $(l, 0)$ of $R(5)$ fully reduced specifically with respect to the physical $R(3)$. However, if two of these basis functions differ only in the value of ν, they are not orthogonal, nor are any normalized. We shall define the Hilbert-space integral

$$A^l_J(\nu', \nu) \equiv (\psi(l\nu'JM), \psi(l\nu JM)). \tag{26}$$

When $\nu' = \nu$, $A^l_J(\nu, \nu)$ is the square of the normalization constant and we adopt the convention of taking the positive square root. For $\nu' \neq \nu$, Eq. (26) is the overlap integral for states of common J but different ν belonging to the irreducible representation $(l, 0)$.

Before we proceed to compute $A^l_J(\nu', \nu)$, let us indicate specifically which J's are involved in the overlap integrals. We shall consider $\nu' \geq \nu$ or, equivalently, $K \geq K'$. From Eq. (12) it follows that $\psi(l\nu'JM)$ and $\psi(l\nu JM)$ have common values of J only for

$$K' = K - 3n. \tag{27}$$

The maximum common J value is $2K'$ and the minimum is K. Therefore,

$$2K' \geq K,$$

which implies that the $A^l_J(\nu', \nu)$ are zero unless

$$\begin{aligned}K' &= K - 3n,\\ n &= 0, 1, 2, \cdots, [K/6].\end{aligned} \tag{28}$$

The common values of J are the set

$$J = 2K', 2K' - 2, 2K' - 3, \cdots, K; \tag{29}$$

that is, J runs from K to $2K'$ in steps of 1 excluding $2K' - 1$.

In passing, we also note that these considerations will yield an explicit formula for the number of times $N(J)$ that J occurs in the irreducible representation $(l, 0)$. Again, with $\nu \leq \nu'$, which implies $K \geq K'$, J occurs for both values of ν provided

$$K \leq J \leq 2K'$$

or

$$l - 3\nu \leq J \leq 2l - 6\nu'.$$

This gives

$$\nu \geq \tfrac{1}{3}(l - J)$$

and

$$\nu' \leq \tfrac{1}{6}(2l - J),$$

or, since ν and ν' are integers,

$$[\tfrac{1}{3}(l - J + 2)] \leq \nu \leq \nu' \leq [\tfrac{1}{6}(2l - J)].$$

Hence, the number of ν values associated with J cannot exceed $[\frac{1}{6}(2l - J)] - [\frac{1}{3}(l - J + 2)] + 1$. This will be $N(J)$ provided we interpret the [ ] as being zero whenever its argument is negative, and unless $2l - J - 1$ is $6n$, where n is an integer, in which case $N(J)$ will be smaller by 1. Hence we conclude that

$$N(J) = \begin{cases} [\frac{1}{6}(2l - J)] - [\frac{1}{3}(l - J + 2)] + 1, \\ \qquad\qquad 2l - J - 1 \neq 6n, \\ [\frac{1}{6}(2l - j)] - [\frac{1}{3}(l - J + 2)], \\ \qquad\qquad 2l - J - 1 = 6n, \end{cases} \tag{30}$$

where n is an integer. Equation (30) is the solution to the induction equations given by Weber *et al.*[12]

From Eq. (26) together with Eq. (15) we have

$$\begin{aligned} A^l_J(\nu', \nu) &= \int d\Omega\, D^J_{MK'}(\Omega)(\chi(l\nu'), \Omega^{-1}\psi(l\nu JM)) \\ &= \sum_{M'} \int d\Omega\, D^J_{MK'}(\Omega)(\chi(l\nu'), \psi(l\nu JM'))\, D^{J*}_{MM'}(\Omega) \\ &= (2J + 1)^{-1}(\chi(l\nu'), \psi(l\nu JK')). \end{aligned} \tag{31}$$

Again, we use Eq. (15) and find

$$A^l_J(\nu', \nu) = (2J + 1)^{-1} \int d\Omega\, D^{J*}_{K'K}(\Omega)(\chi(l\nu'), \Omega\chi(l\nu)). \tag{32}$$

The problem then is finding $[\chi(l\nu'), \Omega\chi(l\nu)]$. To obtain the required matrix elements, we may construct states with the same transformation properties as $\chi(l\nu)$ in any manner we like. Now $\chi(l\nu)$ is a state in the $R(4) = SU(2) \otimes SU(2)$ space corresponding to

$$(p, q, p_0, q_0) = \left(\frac{l}{2}, \frac{l}{2}, \frac{l}{2} - \nu, -\frac{l}{2}\right).$$

As is well known, the state constructed from functions $\chi_\pm$, corresponding to $(p, q, p_0, q_0) = (\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2})$, by

$$\bar{\chi}(l, \nu) = [(l - \nu)!\, \nu!]^{-\frac{1}{2}} \chi_+^{l-\nu} \chi_-^{\nu},$$

has the same transformation properties as $\chi(l\nu)$ and is normalized provided

$$(\chi^a_\alpha, \chi^b_\beta) = \delta_{\alpha\beta}\delta_{ab}\, a!.$$

But $\chi_\pm$ may be taken as suitable states of the vector representation of $R(5)$, which we know contains only the $J = 2$ representation of $R(3)$. In particular, since $M = 3p_0 + q_0$, we can identify χ_+ with $J = 2$, $M = 1$ and χ_- with $J = 2$, $M = -2$. Similarly, the other states of the vector representation are identified. Clearly $\Omega\chi(l\nu)$ is the same function of $\Omega\chi_+$ and $\Omega\chi_-$ as $\chi(l\nu)$ is of χ_+ and χ_-. But $\Omega\chi_+$ and $\Omega\chi_-$ may be written as linear combinations of χ_+, χ_-, and the other states of the vector representation with appropriate matrix elements of the representation D^2 of $R(3)$ as coefficients. Only the χ_+ and χ_- coefficients need concern us because of the Hilbert-space integral in Eq. (32). Thus, if $\nu' \geq \nu$, we obtain

$$\begin{aligned} &(\chi(l\nu'), \Omega\chi(l\nu)) \\ &= [(l - \nu)!\, \nu!\, (l - \nu')!\, \nu'!]^{\frac{1}{2}} \\ &\qquad \times \sum_\beta \frac{(D^2_{11})^{l-\nu-\nu'+\beta}(D^2_{-2,1})^{\nu'-\beta}(D^2_{1,-2})^{\nu-\beta}(D^2_{-2,-2})^\beta}{(l - \nu - \nu' + \beta)!\, (\nu' - \beta)!\, (\nu - \beta)!\, \beta!}, \\ &\qquad\qquad \nu' \geq \nu. \end{aligned} \tag{33}$$

There are several ways to evaluate the integral in Eq. (32) using Eq. (33). Here we merely quote two equivalent forms for $A^l_J(\nu', \nu)$ and relegate the derivations to Appendix B. The first form, suitable for machine computation when one has already available a fast program for Clebsch–Gordan coefficients is

$$\begin{aligned} A^l_J(\nu', \nu) &= (2J + 1)^{-2}[(l - \nu)!\, (l - \nu')!\, \nu!\, \nu'!]^{\frac{1}{2}} \\ &\times \sum_{\beta J' J''} \frac{1}{(l - \nu - \nu' + \beta)!\, (\nu' - \beta)!} \\ &\times \frac{C(l - \nu - \nu' + \beta, J')K(\nu' - \beta)K(\nu - \beta)}{(\nu - \beta)!\, \beta!} \\ &\times C(2\nu - 2\beta, 2\beta, 2\nu'; \nu - \beta, -2\beta) \\ &\times C(2\nu' - 2\beta, 2\nu, J''; -2\nu' + 2\beta, \nu - 3\beta) \\ &\times C(2\nu' - 2\beta, 2\nu, J''; \nu' - \beta, -2\nu) \\ &\times C(J'J''J; l - \nu - \nu' + \beta, \nu - 2\nu' - \beta) \\ &\times C(J'J''J; l - \nu - \nu' + \beta, \nu' - 2\nu - \beta), \\ &\qquad \nu' \geq \nu, \end{aligned} \tag{34}$$

where

$$K(x) = 2^x \left[\frac{(3x)!\, x!}{(4x)!}\right]^{\frac{1}{2}}, \tag{35}$$

and

$$\begin{aligned} C(l, J) &= \frac{4^{J-l} l!\, (J + l)!}{(2l - J)!(2J)!} \\ &\quad \times {}_2F_1(J - 2l, J - l + 1; 2J + 2; 4). \end{aligned} \tag{36}$$

Alternatively, the $C(l, J)$ may be given by the recursion relationship

$$C(l, J) = \sum_{J'} C(l - 1, J')[C(J'2J; l - 1, 1)]^2$$

with

$$C(0, J) = \delta_{J,0}. \tag{37}$$

The J values belonging to $C(l, J)$ are those values of J in the $(l, 0)$ representation with $K = l$ [Eq. (13)].

[12] H. J. Weber, M. G. Haker, and W. Greiner, Z. Physik **190**, 25 1 966).

Our second form for $A^l_J(\nu', \nu)$ is

$$A^l_J(\nu', \nu) = \frac{2^{\nu'-\nu}}{(2J+1)}\left[\frac{(l-\nu)!\,(l-\nu')!\,\nu!\,\nu'!\,(J-K)!\,(J-K')!}{(J+K)!\,(J+K')!}\right]^{\frac{1}{2}}$$
$$\times \sum_{\alpha\beta\gamma} \frac{4^{\alpha}(-1)^{\alpha+\gamma}(J+K+\gamma)!}{(l-\nu'-\alpha)!\,(\alpha-\beta)!\,(\nu'-\nu+\beta)!\,\beta!\,(\nu-\beta)!}$$
$$\times \frac{(2l-2\nu'-2\beta)!\,(3\nu'-3\nu+2\beta+\alpha+\gamma)!}{(J-K-\gamma)!\,(K-K'+\gamma)!\,(2l+\nu'-3\nu+\alpha+\gamma+1)!}, \quad \nu' \geq \nu. \tag{38}$$

At first sight, this would appear to be no simpler than Eq. (34). However, when it is remembered that the $C(l, J)$ and each Clebsch–Gordan coefficient in Eq. (34) is itself a sum, Eq. (38) then appears to be a considerable simplification. If ν' is smaller than ν, Eqs. (34) and (38) still apply, but with $\nu' \leftrightarrow \nu$ and an additional factor $(-1)^{\nu-\nu'}$. This is equivalent to the symmetry rule

$$A^l_J(\nu', \nu) = (-1)^{\nu-\nu'} A^l_J(\nu, \nu'). \tag{39}$$

5. REDUCED MATRIX ELEMENTS OF $R(5)$ GENERATORS IN THE PHYSICAL BASIS

So far, we have managed to define a complete set of states $\psi(l\nu JM)$, where J is the physical angular momentum. It remains to obtain explicit matrix elements between such states for all generators of $R(5)$. This is trivial for those generators, which are also the generators of $R(3)$. We have

$$\langle\psi(l\nu' JM')|\, J_\lambda\, |\psi(l\nu JM)\rangle = A^l_J(\nu', \nu)[J(J+1)]^{\frac{1}{2}} C(J1J; M\lambda M'). \tag{40}$$

The remaining generators are components $Q^{[3]}_\lambda$ of a third rank tensor with respect to $R(3)$, and their matrix elements may be written

$$\langle\psi(l\nu' J'M')|\, Q^{[3]}_\lambda\, |\psi(l\nu JM)\rangle = C(J3J'; M\lambda M')\,\langle l\nu' J'\|\, Q\,\| l\nu J\rangle, \tag{41}$$

where the sole remaining problem is the evaluation of the reduced matrix element $\langle l\nu' J'\|\, Q\,\| l\nu J\rangle$, which we obtain by considering the particular component $Q^{[3]}_0$ or, more briefly, Q_0. Then

$$Q_0\psi(l\nu JM) = \int D^{J*}_{MK}(\Omega) Q_0 \chi_\Omega(l, \nu)\, d\Omega. \tag{42}$$

We make use of the explicit properties of the $Q^{[3]}_\mu$ under $R(3)$ and rewrite Eq. (42) as

$$Q_0\psi(l\nu JM) = \sum_\rho \int D^{J*}_{MK}(\Omega) D^{3*}_{0\rho}(\Omega) Q_\rho(\Omega)\chi_\Omega(l, \nu)\, d\Omega$$
$$= \sum_{\rho J'} C(J3J'; M0)C(J3J'; K\rho)$$
$$\times \int D^{J'*}_{M,K+\rho}(\Omega) Q_\rho(\Omega)\chi_\Omega(l, \nu)\, d\Omega. \tag{43}$$

This has reduced the problem to finding the effect of $Q_\rho(\Omega)$ upon $\chi_\Omega(l, \nu)$ or, what amounts to the same thing, the effect of Q_ρ upon $\chi(l, \nu)$. The Q_ρ may be expressed in terms of the natural basis generators, using Eqs. (3)–(5), as

$$\begin{aligned} Q_{\pm 3} &= 10^{\frac{1}{2}} p_{\pm 1}, \\ Q_{\pm 2} &= \pm 5^{\frac{1}{2}} T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\pm\frac{1}{2}}, \\ Q_{\pm 1} &= 6^{\frac{1}{2}} q_{\pm 1} \pm 2^{\frac{1}{2}} T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\mp\frac{1}{2}}, \end{aligned} \tag{44}$$

and

$$Q_0 = 3q_0 - p_0.$$

Then, operating upon the intrinsic states and using the results of I, we find

$$\begin{aligned} Q_3\chi(l, \nu) &= -[5\nu(l-\nu+1)]^{\frac{1}{2}}\chi(l, \nu-1), \\ Q_{-3}\chi(l, \nu) &= [5(l-\nu)(\nu+1)]^{\frac{1}{2}}\chi(l, \nu+1), \\ Q_{-2}\chi(l, \nu) &= 0, \\ Q_0\chi(l, \nu) &= -(2l-\nu)\chi(l, \nu). \end{aligned} \tag{45}$$

For the remaining components, we have

$$\begin{aligned} Q_2\chi(l, \nu) &= [\tfrac{5}{3}]^{\frac{1}{2}} J_{-1}\chi(l, \nu-1), \\ Q_1\chi(l, \nu) &= \tfrac{1}{2}6^{\frac{1}{2}} J_1\chi(l, \nu), \\ Q_{-1}\chi(l, \nu) &= -[\tfrac{2}{3}]^{\frac{1}{2}} J_{-1}\chi(l, \nu). \end{aligned} \tag{46}$$

Now, from Eq. (15),

$$\chi(l, \nu) = \sum_J (2J+1)\psi(l\nu JK)$$

follows as a trivial corollary. We shall use this to evaluate

$$\mathfrak{J}_\pm \equiv \int D^{J'*}_{M,K'}(\Omega) J_{\pm 1}(\Omega)\chi_\Omega(l, \nu)\, d\Omega. \tag{47}$$

Then from Eq. (15) we have

$$\begin{aligned} &J_{\pm 1}(\Omega)\chi_\Omega(l, \nu) \\ &= \sum_J (2J+1) J_{\pm 1}(\Omega)\psi_\Omega(l\nu JK) \\ &= \sum_J (2J+1)[J(J+1)]^{\frac{1}{2}} C(J1J; K, \pm 1)\psi_\Omega(l\nu JK \pm 1) \\ &= \sum_{J\rho} (2J+1)[J(J+1)]^{\frac{1}{2}} C(J1J; K, \pm 1) D^J_{\rho, K\pm 1}(\Omega) \\ &\qquad \times \psi(l\nu J\rho). \end{aligned} \tag{48}$$

Thus,

$$\mathfrak{J}_{\pm} = [J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K' \mp 1, \pm 1, K)\psi(l\nu J'M). \tag{49}$$

Finally then, we find

$$\begin{aligned} Q_0\psi(l\nu JM) &= \sum_{J'} C(J3J'; M, 0)\{-C(J3J'; K, 3)[5\nu(l-\nu+1)]^{\frac{1}{2}} \\ &\quad \times \psi(l, \nu-1, J', M) + C(J3J'; K, 2) \\ &\quad \times [\tfrac{5}{3}J'(J+1)]^{\frac{1}{2}} C(J'1J'; K+3, -1) \\ &\quad \times \psi(l, \nu-1, J', M) + C(J3J'; K, 1) \\ &\quad \times [\tfrac{3}{2}J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K, 1)\, \psi(l, \nu, J', M) \\ &\quad - (2l-\nu)C(J3J'; K, 0)\psi(l, \nu, J', M) \\ &\quad - [\tfrac{3}{2}J'(J'+1)]^{\frac{1}{2}} C(J3J'; K, -1)C(J'1J'; K, -1) \\ &\quad \times \psi(l, \nu, J', M) + [5(l-\nu)(\nu+1)]^{\frac{1}{2}} \\ &\quad \times C(J3J'; K, -3)\psi(l, \nu+1, J', M)\}. \end{aligned} \tag{50}$$

In interpreting Eq. (50), we note that if $\nu = [l/3]$, then $\chi(l, \nu+1)$ is not an intrinsic state as we have previously defined it. In this case we continue to define $\psi(l, \nu+1, J, M)$ by the projection equation (15) from $\chi(l, \nu+1)$ and note that the evaluation of the overlap integrals in Appendix B remains valid. From Eqs. (50) and (41) we now see that

$$\begin{aligned} &\langle l\nu'J' \| \, Q \, \| l\nu J\rangle \\ &= \{-[5\nu(l-\nu+1)]^{\frac{1}{2}} C(J3J'; K, 3) \\ &\quad + [\tfrac{5}{3}J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K+3, -1)C(J3J'; K, 2)\} \\ &\quad \times A^l_{J'}(\nu', \nu-1) \\ &\quad + [5(l-\nu)(\nu+1)]^{\frac{1}{2}} C(J3J'; K, -3)A^l_{J'}(\nu', \nu+1) \\ &\quad + \{[\tfrac{3}{2}J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K, 1)C(J3J'; K, 1) \\ &\quad - [\tfrac{3}{2}J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K, -1)C(J3J'; K, -1) \\ &\quad - (2l-\nu)C(J3J'; K, 0)\}A^l_{J'}(\nu', \nu). \end{aligned} \tag{51}$$

APPENDIX A

In this appendix we want to show that the rules given by Eqs. (11)–(13) produce the dimension formula

$$d(l) = \tfrac{1}{6}(l+1)(l+2)(2l+3). \tag{A1}$$

Now, according to Eq. (13) for given K, the total number of states is

$$d_K = \sum_{J=K}^{2K}{}'(2J+1), \tag{A2}$$

where we have specifically indicated by the prime that the value $2K-1$ is excluded from the sum. Therefore,

$$\begin{aligned} d_K &= (4K+1) + \sum_{J=K}^{2K-2}(2J+1) \\ &= 4K + 1 + (3K-1)(K-1) \\ &= 3K^2 + 2. \end{aligned} \tag{A3}$$

This is clearly valid so long as $K \neq 0$. For $K = 0$ we have $d_0 = 1$. Thus for all K we have

$$d_K = 3K^2 + 2 - \delta_{K,0}. \tag{A4}$$

Now let us write

$$l = 3n + m, \quad n = 0, 1, 2 \cdots, \quad m = 0, 1, 2. \tag{A5}$$

Then with $K = l - 3\nu$ we have $\nu = 0, 1, 2, \cdots, n$. Thus, in terms of ν

$$d_\nu = 3(l-3\nu)^2 + 2 - \delta_{m,0}\delta_{\nu,n}. \tag{A6}$$

The dimension of $(l, 0)$ is therefore given by

$$\begin{aligned} d(l) = \sum_{\nu=0}^{n} d_\nu &= (3l^2+2)(n+1) - \delta_{m,0} \\ &\quad - 9ln(n+1) + \tfrac{9}{2}n(n+1)(2n+1). \end{aligned} \tag{A7}$$

We use $n = (l-m)/3$ and after some rearrangement find

$$\begin{aligned} d(l) &= \tfrac{1}{6}\{(l+1)(l+2)(2l+3) \\ &\quad - (m-1)(m-2)(2m-3) - 6\delta_{m,0}\}. \end{aligned} \tag{A8}$$

But since $m = 0$, 1, or 2 only, we find the desired result.

APPENDIX B

We want to evaluate the quantity given in Eq. (32) for $\nu' \geq \nu$:

$$\begin{aligned} &A^l_J(\nu', \nu) \\ &= (2J+1)^{-1}\int d\Omega\, D^{J*}_{K'K}(\Omega)(X(l, \nu'), \Omega X(l, \nu)), \end{aligned} \tag{B1}$$

where

$$(X(l,\nu'), \Omega X(l,\nu)) = [(l-\nu)!\,\nu!\,(l-\nu')!\,\nu'!]^{\frac{1}{2}} \sum_{\beta} \frac{[D^2_{11}(\Omega)]^{l-\nu-\nu'+\beta}[D^2_{-2,1}(\Omega)]^{\nu'-\beta}[D^2_{1,-2}(\Omega)]^{\nu-\beta}[D^2_{-2,-2}(\Omega)]^{\beta}}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!}. \tag{B2}$$

Now,

$$[D^2_{11}(\Omega)]^{\alpha} \equiv \sum_{J'} C(\alpha, J')D^{J'}_{\alpha\alpha}(\Omega), \tag{B3}$$

where the J' values in the sum are those from α to 2α in steps of 1 with the exception of $2\alpha - 1$, i.e., those in $(\alpha, 0)$ with $K = \alpha$. Also

$$[D^2_{-2,1}(\Omega)]^{\beta} \equiv K(\beta)D^{2\beta}_{-2\beta,\beta}(\Omega), \tag{B4}$$

and then clearly,

$$[D^2_{1,-2}(\Omega)]^{\gamma} = K(\gamma)D^{2\gamma}_{\gamma,-2\gamma}(\Omega);$$

finally,

$$[D^2_{-2,-2}(\Omega)]^\delta = D^{2\delta}_{-2\delta,-2\delta}(\Omega). \quad \text{(B5)}$$

Then we have

$$\begin{aligned}(X(l,\nu'),\Omega X(l\nu)) &= [(l-\nu)!\,\nu!\,(l-\nu')!\,\nu'!]^{\frac{1}{2}} \\ &\times \sum_{\beta J'} \frac{C(l-\nu-\nu'+\beta,J')K(\nu'-\beta)K(\nu-\beta)}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!} \\ &\times D^{J'}_{l-\nu-\nu'+\beta,l-\nu-\nu'+\beta}(\Omega)\,D^{2(\nu'-\beta)}_{-2(\nu'-\beta),\nu'-\beta}(\Omega) \\ &\times D^{2(\nu-\beta)}_{\nu-\beta,-2(\nu-\beta)}(\Omega)\,D^{2\beta}_{-2\beta,-2\beta}(\Omega). \end{aligned} \quad \text{(B6)}$$

In Eq. (B6), we couple the last two rotation matrices to 2ν, couple the result with $D^{2(\nu'-\beta)}(\Omega)$, and, finally, couple the result to $D^{J'}(\Omega)$. We find

$$\begin{aligned}(X(l\nu'),\Omega X(l\nu)) &= [(l-\nu)!\,\nu!\,(l-\nu')!\,\nu'!]^{\frac{1}{2}} \\ &\times \sum_{\beta J' J'' J'''} \frac{C(l-\nu-\nu'+\beta,J')K(\nu'-\beta)K(\nu-\beta)}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!} \\ &\times C(2(\nu-\beta)2\beta 2\nu;\nu-\beta,-2\beta) \\ &\times C(2(\nu'-\beta)2\nu J'';-2\nu'+2\beta,\nu-3\beta) \\ &\times C(2(\nu'-\beta)2\nu J'';\nu'-\beta,-2\nu) \\ &\times C(J'J''J''';l-\nu-\nu'+\beta,\nu-2\nu'-\beta) \\ &\times C(J'J''J''';l-\nu-\nu'+\beta,\nu'-2\nu-\beta) \\ &\times D^{J'''}_{K',K}(\Omega). \end{aligned} \quad \text{(B7)}$$

When Eq. (B7) is inserted into Eq. (B1) and the resulting integration performed, we find the result given in Eq. (34).

Now, the quantity $K(\beta)$ is defined by Eq. (B4). Thus we see that $K(0) = 1$. From

$$(D^2_{-2,1}(\Omega))^\beta = (D^2_{-2,1}(\Omega))^{\beta-1} D^2_{-2,1}(\Omega)$$

it follows that

$$\begin{aligned}K(\beta) &= K(\beta-1)C(2(\beta-1),2,2\beta;\beta-1,1) \\ &= K(\beta-1)\left[\frac{(3\beta-2)(3\beta-1)(3\beta)}{(4\beta-3)(4\beta-2)(4\beta-1)}\right]^{\frac{1}{2}}. \end{aligned} \quad \text{(B8)}$$

This induction equation can clearly be iterated to yield

$$K(\beta) = 2^x\left[\frac{(3\beta)!\,[4(\beta-x)]!\,\beta!}{[3(\beta-x)]!\,(4\beta)!\,(\beta-x)!}\right]^{\frac{1}{2}} K(\beta-x) \quad \text{(B9)}$$

and, upon setting $x = \beta$ in Eq. (B9), we find the result given in Eq. (35).

The quantities $C(\alpha, J)$ are defined by Eq. (B3). From the orthogonality of the irreducible representation of $R(3)$ it follows that

$$C(\alpha,J) = (2J+1)\int d\Omega\, D^{J*}_{\alpha\alpha}(\Omega)(D^2_{11}(\Omega))^\alpha. \quad \text{(B10)}$$

We use

$$D^J_{\alpha\beta}(\Omega) = e^{-i\alpha\theta_1} d^J_{\alpha\beta}(\theta_2) e^{-i\gamma\theta_3},$$

where the θ_i are Euler angles, and then we rewrite Eq. (B10) as

$$C(\alpha,J) = \frac{2J+1}{2}\int_0^\pi \sin\theta\, d\theta\, d^J_{\alpha\alpha}(\theta)(d^2_{11}(\theta))^\alpha. \quad \text{(B11)}$$

The $d^J_{\alpha\alpha}(\theta)$ are given by[10]

$$\begin{aligned}d^J_{\alpha\beta}(\theta) &= \frac{1}{(\alpha-\beta)!}\left[\frac{(J-\beta)!\,(J+\alpha)!}{(J-\alpha)!\,(J+\beta)!}\right]^{\frac{1}{2}} \\ &\times (\cos\theta/2)^{2J+\beta-\alpha}(-\sin\theta/2)^{\alpha-\beta} \\ &\times {}_2F_1(\alpha-J,-\beta-J;\alpha-\beta+1;-\tan^2\theta/2), \\ &\alpha \geq \beta. \end{aligned} \quad \text{(B12)}$$

We use Eq. (B12) in Eq. (B11) after changing the variable to $z = \frac{1}{2}(1-\cos\theta)$ and using the transformation of the ${}_2F_1$ function

$${}_2F_1(a,b;c;x) = (1-x)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right) \quad \text{(B13)}$$

to find

$$\begin{aligned}C(\alpha,J) &= (2J+1)\int_0^1 (1-z)^{2\alpha}(1-4z)^\alpha \\ &\times {}_2F_1(\alpha-J,\alpha+J+1;1;z)\,dz. \end{aligned} \quad \text{(B14)}$$

Now

$${}_2F_1(\alpha-J,\alpha+J+1;1;z) = \frac{1}{(J-\alpha)!}(1-z)^{-2\alpha}\frac{d^{J-\alpha}}{d^{J-\alpha}}[z^{J-\alpha}(1-z)^{J+\alpha}]$$

so we may do $J-\alpha$ partial integrations in Eq. (B14) and the integrated parts vanish. Thus

$$\begin{aligned}C(\alpha,J) &= \frac{(2J+1)}{(J-\alpha)!}4^{J-\alpha}\frac{\alpha!}{(2\alpha-J)!} \\ &\times \int_0^1 (1-4z)^{2\alpha-J} z^{J-\alpha}(1-z)^{J+\alpha}\,dz. \end{aligned} \quad \text{(B15)}$$

The integral in Eq. (B15) is a standard integral representation[13] for a hypergeometric ${}_2F_1$ function and so we have the result quoted in Eq. (36). We note that $C(\alpha, J)$ is specifically zero for $J < \alpha$, $J > 2\alpha$, or $J = 2\alpha - 1$. The recursion relationship for $C(\alpha, J)$ follows trivially from the definition Eq. (B3).

We shall now derive the second form for $A^l_J(\nu', \nu)$. Again we use Eq. (B2) in Eq. (B1), but now we

[13] M. Abramowitz and I. A. Stegum, *Handbook of Mathematical Functions* (National Bureau of Standards, AMS. 55, 1964).

specifically perform the azimuthal angle integrations to find

$$A^l_J(\nu',\nu)=\frac{1}{2(2J+1)}[(l-\nu)!\,(l-\nu')!\,\nu!\,\nu'!]^{\frac{1}{2}}\times\sum_\beta\frac{1}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!}\frac{(-1)^{\nu-\beta}}{(\nu-\beta)!\,\beta!}\times\int_0^\pi\sin\theta\,d\theta d^J_{K'K}(\theta)[d^2_{11}(\theta)]^{l-\nu-\nu'+\beta}\times[d^2_{-2,1}(\theta)]^{\nu'+\nu-2\beta}[d^2_{-2,-2}(\theta)]^\beta. \quad \text{(B16)}$$

Again, we use Eq. (B12), wherein we must take care that the left subscript is the larger; if it is not, we must use

$$d^J_{\alpha\beta}(\theta)=d^J_{\beta\alpha}(-\theta).$$

At the same time we shall also use Eq. (B13) and change the variable to $z=\frac{1}{2}(1-\cos\theta)$. Then we find

$$A^l_J(\nu',\nu)=\frac{2^{\nu'-\nu}}{(2J+1)(K-K')!}\times\left[\frac{(l-\nu)!(l-\nu')!\nu!\,\nu'!(J-K')!(J+K)!}{(J+K')!\,(J-K)!}\right]^{\frac{1}{2}}\times\sum_\beta\frac{(-4)^{\nu-\beta}}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!}\times\int_0^1dz(1-z)^{2(l-\nu-\nu'+\beta)}z^{3(\nu'-\beta)}(1-4z)^{l-\nu-\nu'+\beta}\times{}_2F_1(K-J,K+J+1;K-K'+1;z). \quad \text{(B17)}$$

We expand $(1-4z)^{l-\nu-\nu'+\beta}$ in the integral and find

$$A^l_J(\nu',\nu)=\frac{2^{\nu'-\nu}}{(2J+1)(K-K')!}\times\left[\frac{(l-\nu)!(l-\nu')!\,\nu!\,\nu'!\,(J-K')!\,(J+K)!}{(J+K')!\,(J-K)!}\right]^{\frac{1}{2}}\times\sum_{\alpha\beta}\frac{(-4)^{\nu-\beta+\alpha}}{(l-\nu-\nu'+\beta-\alpha)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!\,\alpha!}\times\int_0^1dz(1-z)^{2(l-\nu-\nu'+\beta)}z^{3\nu'-3\beta+\alpha}\times{}_2F_1(K-J,K+J+1;K-K'+1;z). \quad \text{(B18)}$$

But, from Eqs. B5.5.2(6) and B5.6(1) of the Bateman[14] papers, it follows that

$$\int_0^1y^{b-1}(1-y)^{c-b-1}{}_pF_q(a_1,\cdots,a_p;c_1,\cdots,c_q;xy)=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}{}_{p+1}F_{q+1}(a_1,\cdots,a_p,b;c_1,\cdots,c_q,c;x), \quad \text{(B19)}$$

provided $q<p+1$, $|\arg x|<\pi$, $\operatorname{Re}c>\operatorname{Re}b>0$. Thus we find

$$A^l_J(\nu',\nu)=\frac{2^{\nu'-\nu}}{(2J+1)(K-K')!}\times\left[\frac{(l-\nu)!\,(l-\nu')!\,\nu!\,\nu'!\,(J-K')!\,(J+K)!}{(J+K')!\,(J-K)!}\right]^{\frac{1}{2}}\times\sum_{\alpha\beta}\frac{(-4)^{\nu-\beta+\alpha}(3\nu'-3\beta+\alpha)!}{(l-\nu-\nu'+\beta-\alpha)!\,(\nu'-\beta)!\,(\nu-\beta)!}\times\frac{(2l-2\nu-2\nu'+2\beta)!}{\beta!\,\alpha!\,(2l+\nu'-2\nu-\beta+\alpha+1)!}\times{}_3F_2(K-J,K+J+1,3\nu'-3\beta+\alpha+1;K-K'+1,2l+\nu'-2\nu-\beta+\alpha+2;1). \quad \text{(B20)}$$

Unfortunately, the ${}_3F_2$ is neither well poised nor Saalschützian, so the final result is a triple sum:

$$A^l_J(\nu',\nu)=\frac{2^{\nu'-\nu}}{(2J+1)}\left[\frac{(l-\nu)!\,(l-\nu')!\,\nu!\,\nu'!(J-K)!\,(J-K')!}{(J+K)!\,(J+K')!}\right]\times\sum_{\alpha\beta\gamma}\frac{4^\alpha(-1)^{\alpha+\gamma}(J+K+\gamma)!\,(2l-2\nu'-2\beta)!}{(l-\nu'-\alpha)!\,(\alpha-\beta)!\,(\nu'-\nu+\beta)!\,\beta!\,(\nu-\beta)!}\times\frac{(3\nu'-3\nu+2\beta+\alpha+\gamma)!}{(J-K-\gamma)!\,(K-K'+\gamma)!\,(2l+\nu'-3\nu+\alpha+\gamma+1)!}. \quad \text{(B21)}$$

[14] A. Erdelyi *et al.*, *Bateman Manuscript Project, Higher Transcendental Functions* (McGraw-Hill Book Co., Inc., New York, 1953), Vol. I. We shall denote equations of this reference by section number, e.g., B5.6 and equation number therein, e.g., (1).