Notes for Group Representations

Chris Blair

Personal notes on course 424. Cover various bits and pieces in a haphazard sort of way.

Contents

1 Compact groups

1.1 Definitions

A **topological group** is a topological space with a group structure defined on it so that the group operations of multiplication and inversion are continuous. A space X is said to be *Hausdorff* if given $x, y \in X$ there exist open sets U, V in X such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. A space X is said to be **compact** if it is Hausdorff and every open covering $X = \bigcup_{i \in I} U_i$ has a finite subcovering, $X = U_{i_1} \cup \cdots \cup U_{i_r}$. A subspace X of n-dimensional Euclidean space is compact iff X is closed and bounded.

Before defining representations of a compact topological group (or compact group for brevity) we must set up integration on compact groups.

1.2 Haar's theorem

If X is a compact space, $C(X, k)$ denotes the vector space of functions $f : X \to k$, where k is a field (which we always take to be $\mathbb{R}$ or $\mathbb{C}$). A **measure** μ on X is a constant linear functional $\mu : C(X, k) \to k$, i.e. a mapping from the space of functions on X to the field k, which is linear, $\mu(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \mu(f_1) + \lambda_2 \mu(f_2)$, and continuous, given $\epsilon > 0$ there exists $\delta > 0$ such that $|f| < \delta \Rightarrow |\mu(f)| < \epsilon$. We need only treat real measures, as a complex measure will split into real and imaginary parts, $\mu_c = \mu_1 + i\mu_2$, with μ_1, μ_2 real.

Theorem (Haar measure) *Suppose G is a compact group. Then there exists a unique real measure μ on G such that μ is invariant on G,*

$$\int_G (gf) d\mu = \int_G f d\mu \qquad \forall g \in G, f \in C(G, \mathbb{R})$$

and μ is normalised to that G has volume 1,

$$\int_G d\mu = 1$$

Also, μ is strictly positive,

$$f(x) \geq 0 \, \forall x \Rightarrow \int_G f d\mu \geq 0$$

with equality if and only if $f = 0$, and

$$\left| \int_G f d\mu \right| \leq |f|$$

Proof. This is only a sketch of the proof. The idea is to use a sequence of *averages* of $f(x) \in C(G, \mathbb{R})$. An average $F(x)$ of $f(x)$ is a weighted average of transforms of f, i.e.

$$F(x) = \lambda_1 f(g_1 x) + \cdots + \lambda_r f(g_r x) \qquad g_i \in G, 0 \leq \lambda_1 \leq \cdots \leq \lambda_r \leq 1, \sum_i \lambda_i = 1$$

We have that the average of an average is also an average, and if there exists an invariant measure on G then $\mu(F) = \mu(f)$, or $\int f d\mu = \int F d\mu$. We also have that averaging smoothes: $\min f \leq \min F \leq \max F \leq \max F$. We define $\operatorname{var} f = \max f - \min f$, then $\operatorname{var} F \leq \operatorname{var} f$.

The idea then is that if a positive invariant measure exists we can find a sequence of averages with $\operatorname{var} F$ tending to zero - in other words the averages tend towards a (unique) real number. As

$$\int f d\mu = \int F d\mu$$

we can define this number to be the value of $\mu(f)$.

To smooth f, let $m = \min f$, $M = \max f$ and define $U = \{x \in G : f(x) < \frac{1}{2}(m+M)\}$, i.e. the set of points in G where f is less than average. Now, the transforms of U (i.e. gU, $g \in G$) cover X ($x \in G, x_0 \in U \Rightarrow x \in (xx_0^{-1})U$). As U is open and X is compact, a finite number of these transforms cover X, $X \subset g_1U \cup \cdots \cup g_rU_r$. Consider the average

$$F = \frac{1}{r}\left(g_1 f + \cdots + g_r f\right) \Rightarrow F(x) = \frac{1}{r}\left(f(g_1^{-1}x) + \cdots + f(g_r^{-1}x)\right)$$

Now as $x \in X, x \in g_iU$ for some i, and so $g_i^{-1}x \in U$, so $f(g_i^{-1}x) < \frac{1}{2}(m+M)$. Thus,

$$F(x) < \frac{1}{r}\left((r-1)M + \frac{1}{2}(m+M)\right) = \left(1 - \frac{1}{2r}M\right) + \frac{1}{2r}m$$

while $F(x) > \frac{1}{r}(rm) = m$, so

$$\operatorname{var} F < \left(1 - \frac{1}{2r}\right)(M-m) < \operatorname{var} f$$

Hence we can reduce $\operatorname{var} F$, and so construct a sequence of averages, $F_0 = f, F_1, F_2 \ldots$ with $\operatorname{var} f > \operatorname{var} F_1 > \mathrm{F}\, F_2 > \ldots$ We need to also show that $\operatorname{var} F_i \to 0$; however to do this we need to use the fact that any function $f \in C(G, \mathbb{R})$ is *uniformly continuous.*

This is the basic idea behind the proof - the rest is rather technical and (hopefully) won't be asked on an exam. □

Example: The unique (up to normalisation) measure on U(1) is $\frac{1}{2\pi} d\theta$.

2 U(1) and SU(2)

It is easy to see using boundedness that the only 1-dimensional representations of a compact group G over $\mathbb{C}$ are homomorphisms from G to U(1) (for if α is a 1-dimensional representation of G then $|\alpha(g)| > 1 \Rightarrow |\alpha(g^n)| = |\alpha(g)|^n \to \infty$ contradicting the fact that αG, the image of a compact group, must be compact and so bounded, while if $|\alpha(g)| < 1$ then $|\alpha(g^{-1})| = |\alpha(g)|^{-1} > 1$, leading to the same conclusion).

In particular the representations of U(1) are the homomorphisms from U(1) to itself.

Theorem *The simple representations of* U(1) *are given by the homomorphisms* $E_n : e^{i\theta} \to e^{in\theta}$*, and these are the only simple representations of* U(1)*.*

Dense proof. Let $U = \{e^{i\theta} : -\frac{\pi}{2} < \theta < \frac{\pi}{2}\} \subset \mathrm{U}(1)$. Note that every element in U has a unique square root (in U). Let α be a simple representation of U(1). As a homomorphism, α takes 1 to 1, hence for small enough $\delta > 0$, $-\delta < \theta < \delta \Rightarrow \alpha(e^{i\theta}) \in U$, by continuity. Let N be large enough that $\frac{1}{N} < \delta$, and let $\omega = e^{\frac{2\pi i}{N}}$, then $\alpha(\omega) \in U$. We have that $\omega^N = 1 \Rightarrow \alpha(\omega)^N = 1 \Rightarrow \alpha(\omega) = e^{\frac{2\pi i n}{N}}$, where $-\frac{N}{2} < n < \frac{N}{2}$ and $n \in \mathbb{Z}$.

Let $\omega_1 = e^{\frac{\pi i}{N}}$, then $\omega_1^2 = \omega$, so $\alpha(\omega_1^2) = \alpha(\omega) \Rightarrow \alpha(\omega_1) = e^{\frac{\pi i n}{N}} = \omega_1^n$ (using that elements of U have unique square roots in U). This holds for elements of the form $\omega_j = e^{\frac{2\pi i}{2^j N}}$, and for their powers. This establishes the desired result for all elements of the form $\theta = \frac{2\pi k}{2^j}$, but these are dense in U(1), and the complete result follows by continuity. □

Fourier proof. Let α be a simple representation of U(1) distinct from the E_n. Then $I(\alpha, E_n) = 0$ for all $n \in \mathbb{Z}$, or

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\, \alpha(e^{i\theta})e^{in\theta} = 0 \qquad \forall\, n$$

which says that all the Fourier coefficients of $\alpha(e^{i\theta})$ vanish and hence that $\alpha(e^{i\theta})$ is identically zero. However we know that $\alpha(1) = 1$ giving a contradiction. □

Now conisder the group SU(2) of two by two unitary matrices with positive determinant, $\mathrm{SU}(2) = \{U \in \mathrm{Mat}\,(2, \mathbb{C}) : U^\dagger U = \mathbb{I}\}$. Before we can find the representations of SU(2) we must find its conjugacy classes. We know that if $U \sim V$ in SU(2) and so in $\mathrm{GL}(2, \mathbb{C})$ then U and V have the same eigenvalues. In fact the converse is true, if U and $V \in \mathrm{SU}(2)$ have the same eigenvalues then $U \sim V$ in SU(2).

First note that the eigenvalues of a unitary matrix have absolute value one, for $Uv = \lambda v \Rightarrow v^\dagger U^\dagger = \overline{\lambda} v^\dagger$ so $v^\dagger U^\dagger U v = v^\dagger v = \overline{\lambda}\lambda v v^\dagger$ hence we must have $|\lambda|^2 = 1$. As an element of SU(2) has determinant equal to 1, and the product of the eigenvalues must equal the determinant, we have that $U \in \mathrm{SU}(2)$ has eigenvalues $e^{\pm i\theta}$. It follows from this that the diagonal matrices

$$U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

in SU(2) form a subgroup isomorphic to U(1).

Now let v be an eigenvector of $U \in \mathrm{SU}(2)$ with eigenvalue $e^{i\theta}$ normalised such that $v^*v = 1$, and w an eigenvector of U with eigenvalue $e^{-i\theta}$ and $w^*w = 1$. Then w is orthogonal to v, i.e $v^*w = 0$ and the matrix

$$V = (vw) = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

is unitary and in fact

$$V^{-1}UV = U(\theta)$$

Now as V is unitary, $|\det V| = 1 \Rightarrow \det V = e^{i\phi}$, so let $V' = e^{-i\phi/2}V$ so that $V' \in \mathrm{SU}(2)$ and

$$V'^{-1}UV' = U(\theta)$$

So we have shown that every matrix in SU(2) is conjugate in SU(2) to a diagonal matrix of the form

$$U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Now as $U(\theta) \sim U(-\theta)$ by permuting coordinates so we find that the conjugacy classes in SU(2) are

$$C(\theta) = \{\, U \in \mathrm{SU}(2) \sim U(\theta) \,:\, 0 \leq \theta \leq \pi \}$$

To find the representations of SU(2) consider that a general element in SU(2) can be written

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

and consider the action of U on the vector $(z\ w)^t$ where $z, w \in \mathbb{C}$:

$$U \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az + bw \\ -\bar{b}z + \bar{a}w \end{pmatrix}$$

This gives an action of SU(2) on the space of homogeneous polynomials in z and w. Let V_j denote the space of such polynomials of degree $2j$:

$$V_j = \langle z^{2j}, z^{2j-1}w, \ldots, w^{2j} \rangle$$

and denote by D_j the action of SU(2) in this space. We claim that the D_j, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$ of degree $2j+1$, are the only simple representations of SU(2).

To see this let us restrict to the diagonal elements $U(\theta)$ in SU(2). These act on z, w by $z \mapsto e^{i\theta}z$, $w \mapsto e^{-i\theta}w$, so in general $z^{2j-k}w^k \mapsto e^{2i(j-k)\theta}z^{2j-k}w^k$. It follows that the character of the representation is given by

$$\chi(U(\theta)) = e^{2ij\theta} + e^{2i(j-1)\theta} + \cdots + e^{-2ij\theta}$$

and that the restriction of D_j to the diagonal subgroup U(1) is given by

$$D_j|_{\mathrm{U}(1)} = E(2j) + E(2j-2) + \ldots E(-2j)$$

where $E(n)$ denotes the representation of U(1) where $e^{i\theta} \mapsto e^{in\theta}$. As all the simple representations of U(1) are of this form it follows that the splitting of the space V_j as a direct sum of U(1) spaces is unique:

$$V_j = \langle z^{2j} \rangle \oplus \langle z^{2j-1}w \rangle \oplus \cdots \oplus \langle w^{2j} \rangle$$

Now suppose that the representation D_j is a direct sum of simple representations of SU(2), so that

$$V_j = W_1 \oplus W_2$$

say. But then W_1 and W_2 can be expressed uniquely as direct sums of simple U(1) spaces, and in particular $z^{2j} \in W_1$ and not W_2 for instance. But take

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in \mathrm{SU}(2)$$

Under the action of T, $z^{2j} \mapsto \frac{1}{\sqrt{2}}(z+w)^{2j}$ and so each $z^{2j-k}w^k$ term in fact appears in W_1. Under the hypothesis that W_1 is simple, we must have that in fact $V_j = W_1$ and so D_j is a simple representation of SU(2).

Now suppose that α is a simple representation of SU(2) distinct from the D_j. When restricted to the U(1) subgroup in SU(2) we must have

$$\alpha|_{\mathrm{U}(1)} = e_r E(r) + e_{r-1} E(r-1) + \cdots + e_{-r} E(-r)$$

i.e. $\alpha|_{\mathrm{U}(1)}$ is a sum of simple representations of U(1) (where again $E(n)$ is the representation of U(1) under which $e^{i\theta} \mapsto e^{in\theta}$). Now, as $U(\theta) \sim U(-\theta)$ we have $e_r = e_{-r}$. But it follows from this that $\chi_\alpha(U(\theta))$ is expressible as a linear combination of the $\chi_j(U(\theta))$ with integer coefficients:

$$\alpha|_{\mathrm{U}(1)} = a_0\chi_0(U(\theta)) + a_{1/2}\chi_{1/2}(U(\theta)) + \cdots + a_s\chi_s(U(\theta))$$

and taking the intertwining number, $I(\alpha, \alpha) = a_0^2 + a_{1/2}^2 + \cdots + a_s^2 = 1$ which implies that $a_i = 1$ for some i and all other coefficients zero, so in fact $\alpha = D_i$ for some i.

We conclude that the representations D_j are the only simple representatons of SU(2).

Finally, we have the following formula which expresses products of simple representations of SU(2) as a sum of simple representations:

$$D_j D_k = D_{j+k} + D_{j+k-1} + \cdots + D_{|j-k|}$$

Note that in quantum mechanics this corresponds to the formula for the addition of angular momenta.

It suffices to prove this result for the characters, namely

$$\chi_j \chi_k = \chi_{j+k} + \ldots \chi_{j-k}$$

where

$$\chi_j(U) = e^{2ij\theta} + \cdots + e^{-2ij\theta}$$

for $U \in \mathrm{SU}(2)$ with eigenvalues $e^{\pm i\theta}$. We use induction on k. Note that for $k = 0$ the result is trivial, while for $k = \frac{1}{2}$ we have

$$\begin{aligned}
\chi_j(U)\chi_{1/2}(U) &= (e^{2ij\theta} + \cdots + e^{-2ij\theta})(e^{2i\frac{1}{2}\theta} + e^{-2i\frac{1}{2}\theta}) \\
&= e^{2i\left(j+\frac{1}{2}\right)\theta} + \cdots + e^{-2i\left(j+\frac{1}{2}\right)\theta} + e^{2i\left(j-\frac{1}{2}\right)\theta} + \cdots + e^{-2i\left(j-\frac{1}{2}\right)\theta} \\
&= \chi_{j+1/2}(U) + \chi_{j-1/2}(U)
\end{aligned}$$

Now suppose the result holds for $k - 1$, i.e.

$$\chi_j \chi_{k-1} = \chi_{j+k-1} + \ldots \chi_{j-k+1}$$

and consider

$$\chi_k(U) = e^{2ik\theta} + e^{-2ik\theta} + \chi_{k-1}(U) \Rightarrow \chi_{k-1} = \chi_k(U) - e^{2ik\theta} - e^{-2ik\theta}$$

hence

$$\chi_j(U)\chi_k(U) = \chi_j(U)(e^{2ik\theta} + e^{-2ik\theta}) + \chi_{j+k-1} + \ldots \chi_{j-k+1}$$

but

$$\begin{aligned}
\chi_j(U)(e^{2ik\theta} + e^{-2ik\theta}) &= (e^{2ij\theta} + \cdots + e^{-2ij\theta})(e^{2ik\theta} + e^{-2ik\theta}) \\
&= e^{2i(j+k)\theta} + \cdots + e^{-2i(j+k)\theta} + e^{2i(j-k)\theta} + \cdots + e^{-2i(j-k)\theta} \\
&= \chi_{j+k}(U) + \chi_{j-k}(U)
\end{aligned}$$

giving us the result.

3 SU(2) and SO(3)

The essential step in determining the simple representations of SO(3) is to establish the existence of a double covering of SO(3) by SU(2). Recall that a map $f : G \to H$ of topological groups is said to be a n-fold covering if f is surjective and has a discrete kernel of order n.

3.1 Covering map: quaternions

First, let us note that $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$, the group of unit quaternions. To see this let us consider quaternions as pairs of complex numbers: $q = z + wj$. Now, note that for $z = a + ib$, $jz = ja + ijb = (a - ib)j = \bar{z}j$. Now consider the right multiplication of $q = a + bj \in \mathrm{Sp1}$ on $z + wj$. We have

$$(z + wj)(a + bj) = az - w\bar{b} + waj + z\bar{b}j$$

i.e.

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

and we have $|a|^2 + |b|^2 = 1$ as $q \in \mathrm{Sp}(1)$, hence this gives an isomorphism between Sp(1) and SU(2).

Now, note that $q^{-1} = q^* = \bar{a} - bj$ (as $(bj)^* = \bar{j}\bar{b} = -j\bar{b} = -bj$), and consider the space V of pure imaginary quaternions: $V = \{xi + yj + zk : x, y, z \in \mathbb{R}\} \cong \mathbb{R}^3$. An element v in V satisfies $v^* = -v$. We define an action of Sp(1) on this space by

$$v \mapsto qvq^* \qquad (qvq^*)^* = qv^*q^* = -qvq^* \Rightarrow qvq^* \in V$$

so V is stable under this action and so carries a representation of Sp(1), i.e. a homomorphism $\Theta : \mathrm{Sp}(1) \to \mathrm{GL}(3, \mathbb{R})$, $\Theta(q)v = qvq^*$. Now, $vv^* = x^2 + y^2 + z^2$ and

$$|\Theta(q)v|^2 = (qvq^*)(qvq^*)^* = qvq^*qv^*q^* = qvv^*q^* = vv^*qq^* = vv^* = |v|^2$$

so the quadratic form $x^2 + y^2 + z^2$ is preserved under the action of Sp(1), and hence $\mathrm{Im}\,\Theta \subseteq \mathrm{O}(3)$. In fact Sp(1) is isomorphic to S^3 and so is connected, hence $\mathrm{Im}\,\Theta \subseteq \mathrm{SO}(3)$.

Now, the kernel of Θ consists of elements that are mapped to the identity in SO(3), i.e. elements that satisfy $\Theta(q)v = qvq^* = v$, so $qv = vq$. But as any quaternion may be written as $t1 + v$ for $v \in V, t \in \mathbb{R}$ it follows that $q \in \ker\Theta$ commutes with all quaternions, and so $\ker\Theta = \{\pm 1\}$ as these are the only elements of Sp(1) with this property, and so is discrete.

Finally, note that SO(3) is generated by half-turns about an axis. Let v a unit vector in $\mathbb{R}^3$ and so an element of V, then $\Theta(v)$ is a rotation which satisfies $\Theta(v)v = vvv^* = v$, and hence must be a rotation about an axis containing v. But as v is a unit vector, $v^2 = -vv^* = -1$ and $\Theta(v)\Theta(v) = \Theta(v^2) = \Theta(-1) = 1$ so $\Theta(v)$ is a half-turn about an axis through v. In this way we see $\mathrm{Im}\,\Theta$ contains all half-turns through all axes, and so must be the whole of SO(3). We conclude that Θ is a surjective homomorphism with a finite kernel of order 2, and so establishes a double covering of SO(3) by $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

3.2 Representations of SO(3)

Now, suppose $\Theta : C \to G$ is a surjective and continuous homomorphism of topological groups. Then given a representation α of G in V we can define a representation β of C in V by $\beta = \alpha\Theta$. Now such a representation β arises from a representation α if and only if it is trivial on $\ker\Theta$. In our case a representation α of SO(3) gives a representation β of SU(2) if and only if β acts trivially on $\ker\Theta = \{\pm\mathbb{I}\}$. But we know that the simple representations of SU(2) are given by the action of SU(2) on polynomials $P(z, w) = c_0 z^{2j} + c_1 z^{2j-1} w + \cdots + c_{2j} w^{2j}$, and clearly $P(-z, -w) = P(z, w)$ only if j is an integer, i.e. if $2j$ is even. We want the action of $\pm I$ on $P(z, w)$ to be trivial, and this is guaranteed if $2j$ is even, hence we see that the representations of SO(3) are given by the representations D_j for integral j.

4 Linear groups

4.1 Definitions, basic properties and Lie algebras

We list the definitions, basic properties and Lie algebras of the more common linear groups (i.e. matrix groups).

Recall that a subset of Euclidean space E^n is compact iff it is closed and bounded. We can identify the space of $n \times n$ real matrices, $\mathrm{Mat}\,(n, \mathbb{R})$ with E^{n^2}, and similarly the space of $n \times n$ complex matrices, $\mathrm{Mat}\,(n, \mathbb{C})$ with E^{2n^2}. There is a natural embedding of the complex matrices $\mathrm{Mat}\,(n, \mathbb{C})$ into $\mathrm{Mat}\,(2n, \mathbb{R})$ giving by letting each entry $z_{jk} = x_{jk} + iy_{jk}$ of the complex matrix Z go to the two-by-two matrix

$$\begin{pmatrix} x_{jk} & -y_{jk} \\ y_{jk} & x_{jk} \end{pmatrix}$$

The matrix iI is mapped to

$$J = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

As $X \in \mathrm{Mat}\,(n, \mathbb{C}) \Rightarrow (iI)X = X(iI)$ we can then say that $\mathrm{Mat}\,(n, \mathbb{C}) = \{X \in \mathrm{Mat}\,(2n, \mathbb{R}) : XJ = JX\}$.

A similar construction allows us to view $\mathrm{Mat}\,(n, \mathbb{H})$ ($n \times n$ quaternionic matrices) as a subgroup of $\mathrm{Mat}\,(2n, \mathbb{C})$, with the quaternionic entry $q_{rs} = z_{rs} + jw_{rs}$ going to the two-by-two matrix

$$\begin{pmatrix} z_{rs} & -w_{rs} \\ \overline{w_{rs}} & \overline{z_{rs}} \end{pmatrix}$$

As $jqj^{-1} = \overline{q}$, the matrix map $Q \to JQJ^{-1}$ for $J = \mathrm{diag}\,(j, \ldots, j) \in \mathrm{Mat}\,(n, \mathbb{H})$ sends $Q \to \overline{Q}$. The complex version of J is

$$\mathbb{C}J = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

and we can define $\mathrm{Mat}\,(n, \mathbb{H}) = \{X \in \mathrm{Mat}\,(2n, \mathbb{C}) : (\mathbb{C}J)X(\mathbb{C}J)^{-1} = \overline{X}\}$.

A topological space X is called **disconnected** if it can be partitioned into 2 non-empty open sets, $X = U \cup V$, $U \cap V = \emptyset$. The space X is called **connected** if it is not disconnected. We have the following result:

Theorem *Let a compact group G act transitively on the compact space X. Let $x_0 \in X$ and consider the corresponding stabiliser subgroup, $S(x_0) = \{g \in G : gx_0 = x_0\}$. If X is connected and $S(x_0)$ is connected, then G is connected.*

A **linear group** G is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. The **Lie algebra** $\mathcal{L}G$ of a linear group G is the vector space defined as $\mathcal{L}G = \{X \in \mathrm{Mat}\,(n, \mathbb{R}) : e^X \in G\}$, where e^X is the **exponential** of the matrix X, $e^X = I + X + \frac{1}{2!}X^2 + \ldots$ We have:

Theorem *Let $T(t) \in G$ for small enough t, and suppose $T(t) = I + tX + o(t)$ for some $X \in Mat\,(n, \mathbb{R})$, then $X \in \mathcal{L}G$.*

The **dimension** of a linear group G is defined to be the (real) dimension of the corresponding Lie algebra.

4.1.1 $\mathrm{GL}(n, \mathbb{R})$

The **general linear group** is defined as

$$\mathrm{GL}(n, \mathbb{R}) = \{X \in \mathrm{Mat}\,(n, \mathbb{R}) : \det X \neq 0\}$$

Regarding it as a subspace of E^{n^2} we see that $\mathrm{GL}(n, \mathbb{R})$ is not compact, as $\det X$ is unbounded. It is not connected, as it can be partitioned into the sets of matrices with positive determinant and matrices with negative determinant. The Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ consists of all $n \times n$ matrices (as e^X is always invertible). It follows that $\dim \mathrm{GL}(n, \mathbb{R}) = n^2$.

4.1.2 $\mathrm{GL}(n, \mathbb{C})$

The **complex general linear group** is defined as

$$\mathrm{GL}(n, \mathbb{C}) = \{X \in \mathrm{Mat}\,(n, \mathbb{C}) : \det X \neq 0\}$$

It is not compact, but is connected, and its Lie algebra consists of all $n \times n$ complex matrices, so that $\dim \mathrm{GL}(n, \mathbb{C}) = 2n^2$.

4.1.3 $\mathrm{SL}(n, \mathbb{R})$

The **special linear group** is defined as

$$\mathrm{SL}(n, \mathbb{R}) = \{X \in \mathrm{GL}(n, \mathbb{R}) : \det X = 1\}$$

It is not compact, as for instance the matrix

$$\begin{pmatrix} t & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \frac{1}{t} \end{pmatrix}$$

(with 1s down the diagonal) is in $\mathrm{SL}(n,\mathbb{R})$ but is not bounded.

Let $T \in \mathrm{SL}(n,\mathbb{R})$ be given by $T = I + tX$, then to first order in t, $\det T = 1 + t \operatorname{tr} X$, implying that the Lie algebra of $\mathrm{SL}(n,\mathbb{R})$ is given by traceless $n \times n$ matrices. Checking, let $\operatorname{tr} X = 0$, then $\det e^X = e^{\operatorname{tr} X} = 1$, so $e^X \in \mathrm{SL}(n,\mathbb{R})$. The dimension of $\mathrm{SL}(n,\mathbb{R})$ is then $n^2 - 1$ (as there is one constraint on the entries of X).

To see that $\mathrm{SL}(n,\mathbb{R})$ is connected, let $T \in \mathrm{SL}(n,\mathbb{R})$, then T^tT is positive definite and so has a positive definite square root, Q, i.e. $T^tT = Q^2$. Now, let $O = TQ^{-1}$, then $O^tO = I$ (as $Q = Q^t$), so $O \in \mathrm{SO}(n)$ (using that $\det T = 1$ and $\det Q > 0$). Hence $T = OQ$ for $O \in \mathrm{SO}(n)$ and $Q \in P$, the set of positive definite matrices. So $\mathrm{SL}(n,\mathbb{R}) = \mathrm{SO}(n)P$. Now $\mathrm{SO}(n)$ is connected (see below) and so is P, and hence so is $\mathrm{SL}(n,\mathbb{R})$.

4.1.4 $\mathrm{SL}(n,\mathbb{C})$

The **complex special linear group** is defined as

$$\mathrm{SL}(n,\mathbb{C}) = \{X \in \mathrm{GL}(n,\mathbb{C}) : \det X = 1\}$$

It is not compact, and its Lie algebra consists of traceless $n \times n$ complex matrices, so $\dim \mathrm{SL}(n,\mathbb{C}) = 2(n^2-1)$ (one complex constraint = 2 real constraints). It is connected.

4.1.5 $\mathrm{O}(n)$

The **orthogonal group** is defined as

$$\mathrm{O}(n) = \{X \in \mathrm{GL}(n,\mathbb{R}) : X^tX = I\}$$

This group is compact: it is closed as its entries are the points satisfying the simultaneous equations defining the matrix identity $X^tX = I$, and bounded as the entries satisfy $|x_{ij}| \leq 1$ (we have $(X^tX)_{ii} = 1 \Rightarrow \sum_j x_{ij}^2 = 1$ for each i). It is not connected as it can be partitioned into sets of matrices with determinant 1 and matrices with determinant -1. Writing $T = I + tX$, we see that $T^tT = I + t(X^t + X)$ to first order in t, hence $X = X^t$ and the Lie algebra $\mathfrak{o}(n)$ consists of all skew-symmetric $n \times n$ matrices (also see that $(e^X)(e^{X^t}) = (e^{X^t})(e^X) = I \Rightarrow X + X^t = 0$). As the diagonal entries of such a matrix are zero, and those below the diagonal are minus those above the diagonal, the dimension of $\mathrm{O}(n)$ is $\frac{1}{2}n(n-1)$.

4.1.6 $\mathrm{SO}(n)$

The **special orthogonal group** is defined as

$$\mathrm{SO}(n) = \{X \in \mathrm{GL}(n,\mathbb{R}) : X^tX = I, \det X = 1\} = \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R})$$

As a subgroup of $\mathrm{O}(n)$, $\mathrm{SO}(n)$ is compact. It is also connected as can be seen by considering the action of $\mathrm{SO}(n)$ on the connected space S^{n-1}. It is clear that $\mathrm{SO}(n)$ sends S^{n-1} into itself; consider the action of $\mathrm{SO}(n)$ on the point $(1, 0, \ldots, 0)$. The stabiliser subgroup of this point clearly consists of matrices in $\mathrm{SO}(n)$ with the 1^{st} row and column containing all zeros except for the $(1,1)$ entry, and so is isomorphic to $\mathrm{SO}(n-1)$. The theorem stated above then tells us that $\mathrm{SO}(n-1)$ connected implies $\mathrm{SO}(n)$ is connected. As $\mathrm{SO}(1) = \{1\}$ is connected, this implies that $\mathrm{SO}(n)$ is connected for all n.

The Lie algebra of $\mathrm{SO}(n)$ is the same as that of $\mathrm{O}(n)$.

4.1.7 U(n)

The **unitary group** is defined as

$$\mathrm{U}(n) = \{X \in \mathrm{GL}(n, \mathbb{C}) : X^\dagger X = I\}$$

It is compact for the same reasons as $\mathrm{O}(n)$, viewing $\mathrm{U}(n)$ as a closed, bounded subspace of E^{2n^2}. We can show it is connected by considering the action of $\mathrm{U}(n)$ on the sphere $S^{2n-1} = \{x \in \mathbb{C} : |x| = 1\}$ and arguing as for $\mathrm{SO}(n)$. The Lie algebra is also similarly derived, and consists of $n \times n$ skew hermitian complex matrices. Such matrices have pure imaginary terms on the diagonal, so the complex dimension is $\frac{1}{2}(n^2 - n) + n$, while the real dimension is $\frac{1}{2}(2n^2 - 2n) + n = n^2$.

4.1.8 SU(n)

The **special unitary group** is defined as

$$\mathrm{SU}(n) = \{X \in \mathrm{GL}(n, \mathbb{C}) : X^\dagger X = I, \det X = 1\} = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$$

As a subgroup of $\mathrm{U}(n)$, $\mathrm{SU}(n)$ is compact. We can show it is connected by considering the action of $\mathrm{SU}(n)$ on the sphere $S^{2n-1} = \{x \in \mathbb{C} : |x| = 1\}$ and arguing as before. The Lie algebra consists of traceless skew hermitian complex matrices (as it is the intersection of the Lie algebras of $\mathrm{U}(n)$ and $\mathrm{SL}(n, \mathbb{C})$), and so has real dimension $n^2 - 1$ (same as $\mathrm{U}(n)$ but with one additional constraint).

4.1.9 Sp(n)

The **symplectic group** is defined as

$$\mathrm{Sp}(n) = \{X \in \mathrm{GL}\,(n, \mathbb{H}) : X^* X = I\}$$

It is compact, and can be shown to be connected by considering the action of $\mathrm{Sp}(n)$ on the sphere S^{4n-1} of unit quaternions, using that $Sp(1) \cong S^3$. The Lie algebra consists of $n \times n$ quaternionic matrices such that $X^* + X = 0$, and has quaternionic dimension $\frac{1}{2}(n^2 - n) + n$ and hence real dimension $\frac{1}{2}(4n^2 - 4n) + 3n = n(2n+1)$.

4.1.10 E(n)

The **Euclidean group** is the group of rotations and translations in Euclidean space. We could write it as:

$$\mathrm{E}(n) = \left\{X \in \mathrm{Mat}\,(n+1, \mathbb{R}) : X = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix}, R \in \mathrm{O}(n), v \in \mathbb{R}^n \right\}$$

It is clearly not compact. Note that if we expand the determinant of X along the bottom row, we see that $\det X = \det R = \pm 1$, hence it is not connected. The Lie algeba consists of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \dots & 1 \\ & & 0 \\ & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

with $A \in \mathfrak{o}(n)$ in the first matrix and 1s appearing successively in the first n rows of the last column of the matrices of the second form. Hence the dimension of $E(n)$ is $\frac{1}{2}(n^2 - n) + n = \frac{1}{2}n(n+1)$.

4.2 General properties of Lie algebras

4.2.1 The exponential map

We begin with some properties of the exponential map $\exp : \mathrm{Mat}\,(n,k) \to \mathrm{GL}(n,k)$:

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

If X, Y commute then

$$e^X e^Y = e^Y e^X = e^{X+Y}$$

The exponential of a matrix is always invertible, with inverse given by e^{-X} (as X and $-X$ commute). We have that $e^{TXT^{-1}} = Te^X T^{-1}$, and that if X has eigenvalues λ_i then e^X has eigenvalues e^{λ_i} (apply e^X to an eigenvector v of X with eigenvector λ and use $X^k v = \lambda^k v$). The determinant of e^X is the exponential of the trace, $\det e^X = e^{\operatorname{tr} X}$.

The exponential map has an inverse called the logarithmic map, or more precisely, there exists an open neighbourhood of 0 in $\mathrm{Mat}\,(n,\mathbb{R})$ which is mapped homeomorphically by the exponential map onto an open neighbourhood of $\mathbb{I}$ in $\mathrm{GL}(n,\mathbb{R})$. Note that it is only for $X \in \mathrm{GL}(n,\mathbb{R})$ small enough (using the matrix norm) that the logarithm and exponential are mutual inverses. The open neighbourhood of 0 for which this is so is called the **logarithmic zone**.

We have the following rather nice theorem:

Theorem *For each $X \in Mat\,(n,\mathbb{R})$ the map from $\mathbb{R}$ to $\mathrm{GL}(n,\mathbb{R})$ given by $t \mapsto e^{tX}$ is a continuous homomorphism, and in fact every continuous homomorphism from $\mathbb{R}$ to $\mathrm{GL}(n,\mathbb{R})$ is of this form.*

Proof. Firstly, since sX and tX commute, $s, t \in \mathbb{R}$, we have that $e^{sX}e^{tX} = e^{(s+t)X}$, hence the map is a homomorphism. Now let $T : \mathbb{R} \to \mathrm{GL}(n,\mathbb{R})$ be a continuous homomorphism. It t is sufficiently small, say $t \in J = [-c, c]$, $T(t)$ must lie in the logarithmic zone, and so we can set $X(t) = \log T(t)$ for $t \in J$. Now, as T is a homomorphism, $T(s)T(t) = T(s+t) = T(t)T(s)$, hence $T(s)$ and $T(t)$ commute and so too do $X(t), X(s)$ (as they can be expressed as power series in $T(s), T(t)$). So if $s, t, s+t \in J$,

$$e^{X(s)+X(t)} = e^{X(s)}e^{X(t)} = T(s)T(t) = T(s+t) = e^{X(s+t)}$$

Now if s, t small enough then $X(s), X(t), X(s+t)$ and $X(s)+X(t)$ will all lie in the logarithmic zone and we find

$$X(s) + X(t) = X(s+t)$$

Now we may suppose that $s, t \in [-1, 1]$ by for instance replacing t by ct and X by $c^{-1}X$. We then wish to show that $X(t) = tX(1)$. Say s a positive integer, then

$$sX\left(\frac{1}{s}\right) = X\left(\frac{1}{s}\right) + \cdots + X\left(\frac{1}{s}\right) = X\left(\frac{1}{s} + \cdots + \frac{1}{s}\right) = X(1)$$

so

$$X\left(\frac{1}{s}\right) = \frac{1}{s}X(1)$$

and if $0 \leq r \leq s$,

$$X\left(\frac{r}{s}\right) = \frac{r}{s}X(1)$$

so the relation holds for $t = r/s \in [0, 1]$, i.e. for rational numbers in that interval, but the rational numbers lie dense among the reals and hence by continuity $X(t) = tX(1)$ for all $t \in [0, 1]$. We can extend this to the negative interval using

$$X(0) + X(0) = X(0) \Rightarrow X(0) = 0$$

and so as $X(t) + X(-t) = X(0)$ we have $X(-t) = -X(t)$. Then we have

$$T(t) = e^{X(t)} = e^{tX}$$

for $t \in J$, and by taking powers we extend the result to $t \in \mathbb{R}$. □

4.2.2 Definition of a Lie algebra

A **Lie algebra** $\mathcal{L}$ over a field k (with $k = \mathbb{R}$ or $\mathbb{C}$) is a finite-dimensional vector space $\mathcal{L}$ over k equipped with a skew-symmetric bilinear map $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ denoted by $[X, Y]$ satisfying the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

The Lie algebra of a linear group G is defined to be

$$\mathcal{L}G = \{X \in \mathrm{Mat}\,(n, \mathbb{R}) : e^X \in G\}$$

We can show that this is indeed a Lie algebra for instance using

$$e^{tX}e^{tY} = (I + tX + \dots)(I + tY + \dots) = I + t(X+Y) + O(t^2)$$

hence $X+Y \in \mathcal{L}G$ for $X, Y \in \mathcal{L}G$, and

$$e^{tX}e^{tY}e^{-tX}e^{-tY} = \left(I + tX + \frac{t^2}{2}X^2 + \dots\right)\left(I + tY + \frac{t^2}{2}Y^2 + \dots\right) \times \left(I - tX + \frac{t^2}{2}X^2 + \dots\right)\left(I - tY + \frac{t^2}{2}Y^2 + \dots\right)$$

so

$$e^{tX}e^{tY}e^{-tX}e^{-tY} = I + t^2[X,Y] + O(t^3)$$

and hence $[X,Y] \in \mathcal{L}G$ (where we are using the result that if to first order $T(t) = I + tX$, then $X \in \mathcal{L}G$). Alternatively, one can consider that

$$\left(e^{X/n}e^{Y/n}\right)^n \to e^{X+Y}$$

as $n \to \infty$ (by taking the nth root and taking logarithms of both sides), and from which $X, Y \in \mathcal{L}G$ implies $X + Y \in \mathcal{L}G$, and also

$$\left(e^{X/n}e^{Y/n}e^{-X/n}e^{-Y/n}\right)^{n^2} \to e^{[X,Y]}$$

giving that $[X,Y] \in \mathcal{L}G$.

4.2.3 Miscellaneous useful properties

Every linear group has a logarithmic zone (i.e. has a local neighbourhood of the identity where it is homeomorphic to its Lie algebra).

The **connected component** of a linear group G is a normal open subgroup G_0, generated by the exponentials e^X for $X \in \mathcal{L}G$:

$$G_0 = \{e^{X_1}e^{X_2}\dots e^{X_r} : X_1, X_2, \dots X_r \in \mathcal{L}G\}$$

In particular if G is connected then G is generated by exponentials.

4.2.4 Abelian linear groups

A Lie algebra is said to be abelian if $[X,Y] = 0$ for all X, Y in the Lie algebra. We have that G abelian implies $\mathcal{L}G$ abelian, and if G is connected then $\mathcal{L}G$ is abelian if and only if G is abelian. A connected abelian linear group G is isomorphic to $T^r \times \mathbb{R}^s$ where $T = \mathbb{R}/\mathbb{Z}$.

4.2.5 Homomorphisms and representations

A **homomorphism of Lie algebras** is a linear map $f : \mathcal{L} \to \mathcal{M}$ preserving the Lie product, i.e.

$$f(aX) = af(X) \qquad f(X+Y) = f(X) + f(Y) \qquad f([X,Y]) = [f(X), f(Y)]$$

If $F : G \to H$ is a continuous homomorphism of linear groups then there exists a unique homomorphism $f = \mathcal{L}F : \mathcal{L}G \to \mathcal{H}$ of the corresponding Lie algebras, with

$$e^{fX} = F(e^X)$$

If we have $E : G \to H$ and $F : H \to K$ then $\mathcal{L}(FE) = (\mathcal{L}F)(\mathcal{L}E)$ and the identity map on G induces the identity map on $\mathcal{L}G$. It follows that isomorphic linear groups have isomorphic Lie algebras (which allows us to find the Lie algebras of topological groups which are not themselves linear groups but are isomorphic to some linear group).

If G is connected and $F_1, F_2 : G \to H$ are continuous homomorphisms of linear groups then $\mathcal{L}F_1 = \mathcal{L}F_2 \Rightarrow F_1 = F_2$.

If $F : G \to H$ a continuous homomorphism of linear groups then the kernel of F is a linear group, and its Lie algebra is the kernel of $\mathcal{L}F$.

If $F : G \to H$ a continuous homomorphism of linear groups then $\mathcal{L}F$ is injective if and only if $\ker F$ is discrete. In particular, F injective implies $\mathcal{L}F$ injective.

If $F : G \to H$ a continuous homomorphism of linear groups and H connected then $\mathcal{L}F$ is surjective if and only if F is surjective.

A **representation** of a Lie algebra $\mathcal{L}$ over k (where $k = \mathbb{R}$ or $\mathbb{C}$) in the vector space V over k is defined by a bilinear map $\mathcal{L} \times V \to V$ denoted by $(X, v) \mapsto Xv$ satisfying

$$[X, Y]v = X(Yv) - Y(Xv)$$

Note that we only consider real Lie algebras while we are most interested in complex representations. As a result we need to consider the **complexification** $\mathbb{C}\mathcal{L}$ of the real Lie algebra $\mathcal{L}$ derived by allowing multiplication by complex scalars.

Each real representation α of the linear group G in U givs rise to a representation $\mathcal{L}\alpha$ of the corresponding Lie algebra $\mathcal{L}G$ in U, uniquely characterised by the relation

$$\alpha(e^X) = e^{\mathcal{L}\alpha X} \qquad \forall X \in \mathcal{L}G$$

Similarly each complex representation α of G in V gives rise to a representation $\mathcal{L}\alpha$ of the complexified Lie algebra $\mathbb{C}\mathcal{L}G$ in V uniquely characterised by

$$\alpha(e^X) = e^{\mathcal{L}\alpha X} \qquad \forall X \in \mathcal{L}G$$

If G is connected then α is uniquely determined by $\mathcal{L}\alpha$, i.e. $\alpha = \beta \Rightarrow \mathcal{L}\alpha = \mathcal{L}\beta$.

It follows that if two Lie algebras have the same complexification then their complex representations are in one-to-one correspondence.

If G and H are linear groups and G is simply connected then every Lie algebra homomorphism $f : \mathcal{L}G \to \mathcal{L}H$ can be lifted to a unique group homomorphism $F : G \to H$ with $\mathcal{L}F = f$. It follows that if G is simply conencted then every representation α of $\mathcal{L}G$ can be lifted uniquely to a representation α' of G with $\alpha = \mathcal{L}\alpha'$.

4.3 Representations of $\mathrm{SL}(2,\mathbb{R})$

The group $\mathrm{SL}(2,\mathbb{R})$ consists of two-by-two real matrices with determinant equal to one. Its Lie algebra consists of traceless two-by-two matrices, and as a basis we can take

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

such that

$$\mathfrak{sl}(2,\mathbb{R}) = \langle H, E, F : [H,E] = 2E, [H,F] = -2F, [E,F] = H \rangle$$

Suppose we have a finite dimensional simple representation of $\mathfrak{sl}(2,\mathbb{R})$ in a vector space V, and let v be an eigenvector (or weightvector) of H with eigenvalue (or weight) λ: $Hv = \lambda v$, with corresponding weightspace $W(\lambda)$. Then we have

$$HEv = (2E + EH)v = (\lambda + 2)Ev \Rightarrow Ev \in W(\lambda + 2)$$

$$HFv = (-2F + FH)v = (\lambda - 2)Fv \Rightarrow Fv \in W(\lambda - 2)$$

So E is a sort of raising operator while F is a lowering operator. That is, if v is a weightvector with weight λ, then repeatedly applying E gives us weightvectors with weights $\lambda+2$, $\lambda+4, \ldots$ As the representation is finite-dimensional there are only a finite number of weights, so that eventually we will have $E^r v \neq 0, E^{r+1}v = 0$ for some r. Similarly in the other direction we will have $F^s v \neq 0$, $F^{s+1}v = 0$ for some s.

Let μ be the maximal weight, with corresponding weightvector e_μ. Define $e_{\mu-2} = Fe_\mu$, $e_{\mu-4} = F^2 e_\mu$ and so on, and suppose $F^{2j}e_\mu = 0$ for some half-integer j. In general we have

$$He_w = we_w \qquad Fe_w = e_{w-2}$$

Now note that as E raises the weight by two and F lowers it by two, if $v \in W(\lambda)$ then $EFv \in W(\lambda)$. We claim that

$$EFe_w = a(w)e_w \qquad FEe_w = b(w)e_w$$

i.e. acting by E and then F or vice versa gives us a scalar multiple of the vector e_w we started with. Note that if $EFe_w = a(w)$ then

$$FEe_w = (EF - H)e_w = (a(w) - w)e_w$$

so that $b(w) = a(w) - w$.

We prove the result using induction on w. At the maximal weight, $FEe_\mu = 0$ so the result holds with $b(w) = 0$ and $a(w) = -w$. Suppose it holds for $e_\mu, e_{\mu-2}, \ldots e_{w+2}$. Then

$$FEe_w = FEFe_{w+2} = Fa(w+2)e_{w+2} = a(w+2)e_w$$

so the result holds with

$$b(w) = a(w+2) \qquad a(w) = a(w+2) + w$$

This establishes the claim and gives a recursive formula for $a(w)$. Letting $w = \mu - 2k$, we have

$$a(\mu - 2k) = \mu + \mu - 2 + \cdots + \dot{\mu} - 2k = (k+1)\mu - 2\sum_{i=0}^{k} i$$

hence

$$a(\mu - 2k) = (k+1)\mu - 2\frac{1}{2}k(k-1) = (k+1)(\mu - k)$$

or as $k = \frac{1}{2}(\mu - w)$

$$a(w) = \frac{1}{4}(\mu - w + 2)(\mu + w)$$

and we have $b(w) = a(w+2)$ so

$$b(w) = \frac{1}{4}(\mu + w + 2)(\mu - w)$$

Finally note that as $F^{2j}e_\mu = Fe_{\mu-4j} = 0$, we have $a(\mu - 4j) = 0$ so

$$\frac{1}{4}(4+2)(2\mu - 4j) = 0$$

hence $\mu = 2j$. It follows that the weights are all integral and run from $\mu = 2j$ to $-2j$. We also see that the $2j+1$-dimensional subspace spanned by the weightvectors is stable under H, E, F and so must be the whole of V. We have thus established the existence of one simple representation of each dimension.

Note also that we know that $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2)$ have the same complexification and so the same complex representations. But SU(2) is simply connected, and so the representations of it and $\mathfrak{su}(2)$ are in one-to-one correspondence, and in particular as SU(2) is complex all its representations are semisimple, and hence so too are the representations of $\mathfrak{su}(2)$ and hence $\mathfrak{sl}(2, \mathbb{R})$.