Final Exam: Phy239 Group Theory (II)

In this final exam, we will do a mini-project on the algebra spanned by 4-component, spin-3/2, fermions. You will see that it naturally gives rise to a variety of Lie algebra structure of Sp(4)(SO(5)), SU(4)(SO(6)), SO(7) and SO(8). Actually it also has the G_2 structure, which will not be tested here. If you want to study physics of spin-3/2 fermions, you need to understand these groups. On the other hand, 4-component fermions are realistic: They can be realized in terms of various ways, including holes in semi-conductors; electrons with 2-band, 2-leg ladders, bi-layers; nucleon with spin and isospin, cold atoms, Dirac fermion with 4 components etc.

The following problems are designed based on "C. Wu, J. P. Hu, and S. C. Zhang, *Exact SO(5) symmetry in the spin 3/2 fermionic system*, Phys. Rev. Lett. 91 , 186402(2003)" (the link https://wucj.physics.ucsd.edu/publication/PRL86402.pdf) and "C. Wu, *Hidden symmetry and quantum phases in spin-3/2 cold atomic systems*, Mod. Phys. Lett. B 20, 1707 (2006)" (The link https://wucj.physics.ucsd.edu/publication/spin32_rvw.pdf) You cannot directly quote the results therein, but need to derive them. These papers are condensed and have typos! The problems below are actually designed easier for you to follow.

In class, we have constructed the Sp(4), or, isomorphically, the SO(5) algebra in terms of the 4-component spin-3/2 fermions. The following notation is used:

$$\Gamma_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \ \Gamma_{2,3,4} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}, \ \Gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{1}$$

and

$$\Gamma_{ab} = -\frac{i}{2}[\Gamma_a, \Gamma_b], \ \ 1 \le a < b \le 5. \tag{2}$$

We will consider a lattice model. On each site i, there is a 4-component spinor $\psi(i) = (c_{\frac{3}{2}}(i), c_{\frac{1}{2}}(i), c_{-\frac{1}{2}}(i), c_{-\frac{3}{2}}(i))^T$. The 16 fermion bi-linears on each site in the particle-hole channel are decomposed into SO(5)'s channels of scalar, 5-vector, and 10-generator (the rank-2 antisymmetric tensor):

$$n(i) = \psi^\dagger(i)\psi(i), \ \ n_a(i) = \frac{1}{2}\psi^\dagger(i)\Gamma_a\psi(i), \ \ L_{ab}(i) = \frac{1}{2}\psi^\dagger(i)\Gamma_{ab}\psi(i). \tag{3}$$

As we learned in class, they are particle number, spin quadrople, spin plus spin octupole operators, respectively. Below for simplicity, let me suppress the site index i for problems from (a) to (f).

(a) Charge conjugation matrix R is defined as the product of the two purely imaginary
5 Γ-matrices as $R = \Gamma_1\Gamma_3$. What is the relation between $R\Gamma^a R$ and $\Gamma^{a,*}$ and also that between $R\Gamma^{ab}R$ and $\Gamma^{ab,*}$? Here * means complex conjugation. **Please note that** $R^T = R^\dagger = R^{-1} = -R$**.**

The time-reversal transformation is defined as $T = RC$ **where** C **is complex conjugation. Show that** $T^2 = -1$**. The time-reversal operation for fermions is defined as**

$\hat{T}\psi\hat{T}^{-1} = R\psi$, and that of a fermion bilinear is defined as $\hat{T}(\psi^\dagger M\psi)T^{-1} = \psi^\dagger R^\dagger M^* R\psi$, where M is the matrix kernel. Calculate the time-reversal transformation: $\hat{T}n\hat{T}^{-1}$, $\hat{T}n_a\hat{T}^{-1}$, and $\hat{T}L_{ab}\hat{T}^{-1}$.

5 (b) Prove that $R_{\alpha\beta}\psi^\dagger_\beta$ transforms in the same way as ψ_α under a general Sp(4) transformation $\hat{U} = \exp(i\sum_{1\le a<b\le 5} L_{ab}\theta_{ab})$, where θ_{ab} are real parameters. (Hint: You may read the appendix of my updated Lect 14 from page 14 to 16, if you feel that you need some help on this part.)

5 (c) Consider the pairing operators $\eta^\dagger = \frac{1}{2}\psi^\dagger R\psi^\dagger$, $\chi^\dagger_a = -\frac{i}{2}\psi^\dagger\Gamma_a R\psi^\dagger$. Prove that $\eta^\dagger$ is an [illegible] $\chi^\dagger_a$'s transform as an SO(5) vector, i.e.,

$$[L_{ab}, \eta^+] = 0, \quad [L_{ab}, \chi^\dagger_c] = i(\delta_{ac}\chi^\dagger_b - \delta_{bc}\chi^\dagger_a). \tag{4}$$

10 (d) Define $\mathrm{Re}\chi_a(i) = \frac{1}{2}(\chi^\dagger_a(i) + \chi_a(i))$, $\mathrm{Im}\chi_a(i) = \frac{1}{2i}(\chi^\dagger_a(i) - \chi_a(i))$ with $l \le a \le 5$. Define $M_{0a} = \mathrm{Re}\chi_\mathrm{a}$, $M_{a6} = \mathrm{Im}\chi_\mathrm{a}$, $M_{ab} = L_{ab}$ for $1 \le a < b \le 5$ and $M_{06} = N = \frac{1}{2}(n-2)$. Prove that the generators M_{ab} with $0 \le a < b \le 6$ span the SO(7) algebra as

$$M_{ab} = \begin{pmatrix} 0 & \mathrm{Re}\chi_1 \sim \mathrm{Re}\chi_5 & N \\ & & \mathrm{Im}\chi_1 \\ & L_{ab} & \sim \\ & & \mathrm{Im}\chi_5 \\ & & 0 \end{pmatrix}. \tag{5}$$

In other words, they satisfy

$$[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{ad}M_{bc} - \delta_{bc}M_{ad}), \tag{6}$$

for $0 \le a < b \le 6$.

5 (e) Define $\mathrm{Re}\eta = \frac{1}{2}(\eta^\dagger + \eta)$ and $\mathrm{Im}\eta = \frac{1}{2i}(\eta^\dagger - \eta)$. Show that $\mathrm{Re}\eta$, $\mathrm{Im}\eta$ and n_a with $a = 1 \sim 5$ can be organized into a 7-vector V such that

$$[M_{ab}, V_c] = i(\delta_{ac}V_b - \delta_{bc}V_a). \tag{7}$$

Please determine the concrete expressions of V_a with $a = 0 \sim 6$. Show that all these 28 operators can be arranged as $N_{ab} = M_{ab}$ with $0 \le a < b \le 6$, and $N_{a7} = V_a$, such that N_{ab} span the SO(8) algebra.

$$[N_{ab}, N_{cd}] = i(\delta_{ac}N_{bd} + \delta_{bd}N_{ac} - \delta_{ad}N_{bc} - \delta_{bc}N_{ad}), \tag{8}$$

for $0 \le a < b \le 7$. (You may need to adjust the sign of some operators to make the algebra close.)

(f) Now we construct the SO(7) representations based on the representation theory of Lie algebra. Consider the M_{ab} with $0 \le a < b \le 6$ defined in Eq. **??**. We take the Carton

10

sub-algebra as the set of (N,Q,S_z) defined as

$$N = M_{06} = \frac{1}{2}(n-2),\ Q = -M_{15} = \frac{1}{2}(n_{\frac{3}{2}} + n_{\frac{1}{2}} - n_{-\frac{1}{2}} - n_{-\frac{3}{2}}), \tag{9}$$

$$S_z = M_{23} = \frac{1}{2}(n_{\frac{3}{2}} - n_{\frac{1}{2}} + n_{-\frac{1}{2}} - n_{-\frac{3}{2}}). \tag{10}$$

Draw the Dynkin diagram of SO(7), and explain its meaning: the number of simple roots, the lengths, and angles. Write down its Cartan matrix. Denote its simple root vectors s as $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,1,-1)$, and $\alpha_3 = (0,0,1)$. Based on the Cartan matrix, please construct all the other 9 positive root vectors. (For simplicity, you do not need to work out the operator forms of roots.)

Work out the fundamental weights $\mu_{1,2,3}$, and use μ_3 to denote the fundamental spinor representation. Consider a representation marked by its highest weight $\mu^* = \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3$, and denote it as $(\lambda_1,\lambda_2,\lambda_3)$. Work out the dimensions and the 2nd-order Casimir for the representations $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

(g) In class, we have constructed the spin-3/2 Hubbard model in the explicit SO(5) invariant form $H = H_0 + H_u$ as

5

$$H_0 = -t\sum_{\langle ij\rangle} c^\dagger_\sigma(i)c_\sigma(j) + h.c., \tag{11}$$

$$\begin{aligned} H_u &= U_0\sum_i P^\dagger_0(i)P_0(i) + U_2\sum_{i,m} P^\dagger_{2m}(i)P_{2m}(i) \\ &= \frac{3U_0+5U_2}{16}\sum_i (n(i)-2)^2 + \frac{U_0-U_2}{4}\sum_i n_a^2(i) - (\mu-\mu_0)\sum_i n(i), \end{aligned} \tag{12}$$

where $\sigma = \pm\frac{3}{2}, \pm\frac{1}{2}$, and $\mu_0 = \frac{U_0+5U_2}{4}$ is the chemical potential at the particle-hole symmetric point.

Just look at a single site and consider the case with $\mu = \mu_0$. Please use the energy level diagram of the 16 onsite states that I plotted in class, which is also shown in the note of Lect 15. Can you tune the ratio of U_0/U_2 to reach a degeneracy pattern that matches the SO(7) symmetry that you just worked out in part (f)? Can you prove that there is an exact SO(7) symmetry in this case?

(Hint: You may find the following Fietz identity useful

$$\sum_{0\le a<b\le 7} N^2_{ab}(i) = 7, \tag{13}$$

where M_{ab}'s are the SO(8) generators on each site i. You do not need to prove it here.)

(h) Now consider a square lattice, and in H_0 only the nearest neighboring hopping is kept. The chemical potential μ is set at μ_0. Now we generalize the SO(7) symmetry from a single site to the entire lattice.

Define the SO(7) generators of the entire system as follows: If M_{ab} conserves the particle

5

number, then

$$M_{ab} = \sum_i M_{ab}(i), \tag{14}$$

and if it is a pairing operator, then

$$M_{ab} = \sum_i (-)^i M_{ab}(i), \tag{15}$$

where $(-)^i = \pm 1$ for sites belonging to the two different sub-lattices, respectively. Prove that M_{ab} $(0 \le a < b \le 6)$ commutes with $H_0 + H_u$ at $\mu = \mu_0$.

(i) Define $\chi_a^\dagger = \sum_i (-)^i \chi_a^\dagger(i)$. Prove that when $\mu \neq \mu_0$, $\chi_a^\dagger$ remains an-eigen operator of $H = H_0 + H_u$ when the ratio of U_0/U_2 for the SO(7) symmetry is maintained. Then suppose we have a many-body ground state $|G\rangle$, prove that either states $\chi_a^\dagger|G\rangle$, or, $\chi_a|G\rangle$, are also eigenstates. What is the excitation energy? These excitations are denoted as χ-mode by me in analogous to the π-mesons in high energy. Suppose that the ground states are with the total momentum zero and a singlet of Sp(4). What are the momentum, and the Sp(4) quantum number (Q, S_z) of these χ-modes?

5