Review of $SO(5)$ Group Theory

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A. $SO(5)$ Group

An element R in the orthogonal group $O(5)$ is a real linear orthogonal transformation in five dimension space. Supposing X, Y are two vectors in five dimension space, the scalar product

$$XY = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5$$

is invariant under the transformation $X' = RX, Y' = RY$. Therefore, the matrix R must satisfy the following equation, $R^* = R$, $\tilde{R}R = 1$, $det(R) = \pm 1$. $SO(5)$ is a subgroup of $O(5)$. The elements in $SO(5)$ satisfy $det(R) = 1$. $det(R) = 1$.

Any elements in $SO(5)$ can be diagonalized by a unitary matrix U. The eigenvalue λ should satisfy $|\lambda| = 1$; thus, λ^* is also an eigenvalue. Generally, the element R in $SO(5)$ can be written as

$$R = U\Lambda U^{-1} = e^{-iH}$$

Λ is a diagonal matrix. Since $R^* = R$, the matrix H satisfies $H^* = -H, tr(H) = 0$. H is a traceless and pure matrix, that kind of matrix requires ten independent parameters . We choose ten basis matrixes $L_{ab}, a < b = 1, 2, 3, 4, 5$, and $(L_{ab})_{cd} = -i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$ and H can be written as

$$H = \sum_{a<b} \omega_{ab} L_{ab}, a, b = 1, 2, 3, 4, 5,$$

where ω_{ab} are the real parameters.

The above ten matrixes zl_{ab} are called the generators of $SO(5)$. The commutation relation between them defines the Lie algebra of $SO(5)$,

$$[T_{ab}, T_{cd}] = -i(\delta_{bc}T_{ad} + \delta_{ad}T_{bc} - \delta_{ac}T_{bd} - \delta_{bd}T_{ac}). \tag{1}$$

The diagonal matrix Λ can be written as

$$\Lambda = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 & 0 & 0 \\ 0 & e^{-i\varphi_1} & 0 & 0 & 0 \\ 0 & 0 & e^{i\varphi_2} & 0 & 0 \\ 0 & 0 & 0 & e^{-i\varphi_2} & 0 \\ 0 & 0 & O & 0 & 1 \end{pmatrix}.$$

It is convenient to convert above matrix to real matrix, which can be written as

$$V^{-1}\Lambda V = A = \begin{pmatrix} cos\varphi_1 & -sin\varphi_1 & 0 & 0 & 0 \\ sin\varphi_1 & cos\varphi_1 & 0 & 0 & 0 \\ 0 & 0 & cos\varphi_2 & -sin\varphi_2 & 0 \\ 0 & 0 & sin\varphi_2 & cos\varphi_2 & 0 \\ 0 & 0 & O & 0 & 1 \end{pmatrix}$$

, where

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & O & 0 & 1 \end{pmatrix}.$$

It is easily to check that the matrix UV is a real and unitary matrix. UV is an element of $SO(5)$. So the equation $(UV)^{-1}R(UV) = A$ define the class of $SO(5)$. The class in $SO(5)$ can be described by two parameters φ_1, φ_2. The class concepts is very useful in discussing the representation of group.

B. The Symplectic Group $SP(4)$

The symplectic group $SP(4)$ is the group of real linear transformation P in four dimensional space, which leave $skew-symmetric$ bilinear form defined as

$$XY = x_1 y_1 - x_1' y_1' + x_2 y_2 - x_2' y_2' \tag{2}$$

invariant. Here we choose the components of a vector X in four dimensional space as (x_1, x_2, x_1', x_2'). Simply, the $skew - symmetric$ bilinear form can be written as

$$XY = \varepsilon_{ij} x_i y_j,$$

where the ε matrix is

$$\varepsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The invariance of above equation defines an symplectic transformation P which satisfies

$$\tilde{P} \varepsilon P = \varepsilon \tag{3}$$

The symplectic transformation P is unimodular, which means $det(P) = 1$. This property can be easily derived from above equation. To derive the generators of $SP(4)$, we consider the infinitesimal symplectic transformation $P = I - iA$, where I is the identical transformation and A is the infinitesimal part. By using Eq.**??**, we immediately obtain that the infinitesimal matrix A has the following form

$$A = \begin{pmatrix} A_1 & A_3 \\ A_2 & -\tilde{A}_1 \end{pmatrix},$$

and $A_2 = \tilde{A}_2$, $A_3 = \tilde{A}_3$, $A^* = A$. Therefore, each of matrixes A_2 and A_3 has three independent parameters, and the matrix A_1 has four. There are totally ten parameters for A. The matrix A can be expressed by ten basis matrixes $E_{\alpha\beta}, \alpha, \beta = \pm 1, \pm 2$. Here we choose these ten independent matrixes as $(E_{\alpha\beta})_{\gamma\sigma} = i(sign(\alpha)\delta_{\alpha\gamma}\delta_{\beta\sigma} - sign(\beta)\delta_{-\beta\gamma}\delta_{-\alpha\sigma})$, and $E_{\alpha\beta} = E_{-\beta-\alpha}$. The commutation relation between the generators is

$$[E_{\alpha\beta}, E_{\gamma\sigma}] = i(sign(\beta)\delta_{\beta\gamma}E_{\alpha\sigma} - sign(\alpha)\delta_{\alpha\sigma}E_{\gamma\beta} + sign(\beta)\delta_{\beta-\sigma}E_{\alpha-\gamma} - sign(\alpha)\delta_{\alpha-\gamma}E_{-\beta\sigma}). \tag{4}$$

The above commutator defines the Lie algebra for $SP(4)$.

C. The equivalence of $SO(5)$ and $SP(4)$

One can check that the Lie algebra of $SO(5)$ is the same as that of $SP(4)$ by comparing the generators as shown in (table **??**). Later we will see the Dynkin diagram for $SP(4)$,C_2 and $SO(5)$,B_2 are similarly related. Since these two groups have the same Lie algebra structure, their finite irreducible representations (irreps) are also identical. In this paper, we focus on the $SO(5)$. However, sometimes it is more convenient to use the notation of $SP(4)$. For example, we can obtain all of irreps of $SP(4)$ by making use of the $SO(4)$ tensors, however, if we discuss $SO(5)$, we need to discuss three different kind representation: tensor spinor and spinor tensor representations.

$E(SP(4))$	Root$(SO(5))$	Operator	$L(SO(5))$	$T(SO(4))$
$\frac{1}{2}E_{11}$	$\frac{1}{2}(H_1+H_2)$	$\frac{1}{2}(S_z-Q)$	$\frac{1}{2}(L_{12}+L_{34})$	T^{10}_{00}
E_{1-1}	E_3	$\pi_y-i\pi_x$	$\frac{1}{2}(L_{14}+L_{23}+i(L_{31}+L_{24}))$	$-\sqrt{2}T^{10}_{10}$
E_{-11}	E_{-3}	$\pi_y^+ + i\pi_x^+$	$\frac{1}{2}(L_{14}+L_{23}-i(L_{31}+L_{24}))$	$\sqrt{2}T^{10}_{-10}$
$\frac{1}{2}E_{22}$	$\frac{1}{2}(H_1-H_2)$	$-\frac{1}{2}(S_z+Q)$	$\frac{1}{2}(L_{12}-L_{34})$	T^{01}_{00}
E_{2-2}	E_1	$-\pi_y-i\pi_x$	$\frac{1}{2}(-L_{14}+L_{23}+i(L_{31}-L_{24}))$	$-\sqrt{2}T^{01}_{01}$
E_{-22}	E_{-1}	$-\pi_y^+ + i\pi_x^+$	$\frac{1}{2}(-L_{14}+L_{23}-i(L_{31}-L_{24}))$	$\sqrt{2}T^{01}_{0-1}$
$-\frac{1}{\sqrt{2}}E_{-12}$	E_{-4}	$-i\sqrt{2}\pi_z^+$	$\frac{1}{\sqrt{2}}(L_{25}+iL_{15})$	$T^{\frac{1}{2}\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}}$
$-\frac{1}{\sqrt{2}}E_{-21}$	E_4	$i\sqrt{2}\pi_z^+$	$\frac{1}{\sqrt{2}}(L_{25}-iL_{15})$	$T^{\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}$
$-\frac{1}{\sqrt{2}}E_{2-1}$	E_{-2}	$-\frac{1}{\sqrt{2}}S_-$	$\frac{1}{\sqrt{2}}(L_{45}+iL_{35})$	$T^{\frac{1}{2}\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}}$
$-\frac{1}{\sqrt{2}}E_{1-2}$	E_2	$\frac{1}{\sqrt{2}}S_+$	$\frac{1}{\sqrt{2}}(-L_{45}+iL_{35})$	$T^{\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}}$

TABLE I. Mapping of infinitesimal operators between $Sp(4)$ and $SO(5)$. The third column responds to the irreducible tensor operators of $SO(4)$ (see Section **??**)

I. REPRESENTATION OF $SO(5)$

In this Chapter, three different methods with respect to the representation of $SO(5)$ are given. Readers who have different favors can readily choose one of them and follow it. All concepts we use in this chapter are briefly discussed. In section (**??**), our discussion is based on tensor and spinor. It is easily to be understood by those who are not familiar with the general concepts in Lie Algebra. Readers are supposed to know some basic concepts of Young Operator and Young Pattern. The second section is devoted to the general highest weight representation of $SO(5)$. We briefly express the concepts and omit the mathematical proof. The third section is rather special only for $SO(5)$. The method can not be generalized to all of other simple Lie Group. We decompose $SO(5)$ to its subgroup $SO(4)$, which is equal to $SU(2) \times SU(2)$. The states in the representation space are naturally labeled by representation in $SU(2)$. For practical application in physical model, this method may be more convenient.

A. Young pattern, Young tableaux and $SO(5)$($SP(4)$) representation

In this section, we discuss how to use the Young pattern and Young tableaux to label the irreducible representations of $SO(5)$ and $SP(4)$. A brief summary about the irreducible tensor representation of $SU(N)$ is attached in appendix **??**. The representations of $SO(5)$ is a little more complicated. They can be divided into three kinds of representations: tensor, spinor, and spinor tensor representations. We will discuss those basic concepts. However, instead of detailing the representations of $SO(5)$, we will only focus on how to obtain the irreducible representation of $SP(4)$ by Young's technique in this section, although it is possible to study each type of representations of $SO(5)$, respectively. We will show how to derive the the general formula of an irreps dimension of $SP(4)$, a method to decompose a given irreps of $SU(4)$ into irrepses of $SP(4)$, and a generalized Littlewood-Richardson rule on $SP(4)$.

Similar to $SU(N)$, the n order tensor of $SO(5)$ is defined as

$$O_R T_{a_1 a_2 \ldots a_n} = R_{a_1 b_1} R_{a_2 b_2} \ldots R_{a_n b_n} T_{b_1 b_2 \ldots b_n}, \quad a_i, b_i = 1, 2, 3, 4, 5. \tag{5}$$

An essential difference here is that an constant tensor δ_{ab} is unvariant tensor of $SO(5)$, which mean

$$O_R \delta_{ab} = R_{aa'} R_{bb'} \delta_{a'b'}.$$

Therefore, given any tensor $T_{a_1 a_2 \ldots a_n}$, the tracing tensor on any two indexes,for example $\sum_a T_{aaa_3 \ldots a_n}$ by tracing the first two indexes, is an invariant tensor space of $SO(5)$. If we want to use Young pattern to describe the invariant tensor space of $SO(5)$, we must first separate the tracing tensor space. Then, we can use Young pattern to obtain the other invariant space for the tensor irreps of $SO(5)$.

The irreducible tensor representations obtained above are not complete . there exists the other irreducible representations on $SO(5)$. They are called spinor and spinor tensor representations. In Appendix (**??**), we summarize some basic concepts related to Clifford algebra, which is necessary in order to understand spinor and spinor tensor representation.

For $SO(5)$, the Clifford algebra is given by the famous Dirac matrixes(see Appendix (**??**), $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$,

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}.$$

The spinor representation of $SO(5)$ is defined as

$$D(R)^{-1} \gamma_a D(R) = \sum_b R_{ab} \gamma_b, R \in SO(5).$$

The spinor representation of $SO(5)$ is 4 dimensional.

The spinor tensor is defined as

$$O_R \Psi_{a_1 a_2 \ldots a_n, \alpha} == R_{a_1 b_1} R_{a_2 b_2} \ldots R_{a_n b_n} D(R)_{\alpha\beta} \Psi_{b_1 b_2 \ldots b_n, \beta}, a_i, b_i = 1, 2, 3, 4, 5; \alpha, \beta = 1, 2, 3, 4.$$

The irreducible representation gotten from the spinor tensor space is called the spinor tensor irreps. The tensor index in the invariant spinor tensor space must also satisfy the traceless condition.

So far, we give the basic concepts about the tensor, spinor and spinor tensor representation of $SO(5)$. Although it is possible to study each of these irrepses with Young pattern and tableux more deeply, the easiest way to apply Young technique is to discuss irrepses of $SP(4)$. Since the Lie algebras of $SP(4)$ are exactly the same as $SO(5)$, the irreps of $SP(4)$ are also the same as $SO(5)$. The whole finite irreps of $SP(4)$ can be obtained from its tensor invariant spaces; thus, the tensor and spinor representations of $SO(5)$ are automatically included in the irrepses of $SP(4)$. The following part of this section is devoted to the irreducible tensor representation of $SP(4)$.

A tensor of $SP(4)$ has the same definition as Eq.**??**. Considering the space of tensors of rank n, with components $T_{a_1 a_2 \ldots a_n}$, we can construct the specific trace in the $SP(4)$ tensor spaces by multiplying the matrix ε,

$$(trT)_{a_3 \ldots a_n} = \sum_{a_1 a_2} \varepsilon_{a_1 a_2} T_{a_1 a_2 \ldots a_n}.$$

The trace operation gives a tensor of rank $n-2$. The subspace of such trace contraction tensors becomes an invariant space under the transformation in $SP(4)$. There are $n(n-1)$ traces for a n-th rank tensor. So we divide the space of tensors of rank n into two subspaces, one with all traceless tensors, the other with the trace contraction tensors. a general nth rank tensor of $SP(4)$ can be written as

$$T_{a_1 a_2 \ldots a_n} = T^0_{a_1 a_2 \ldots a_n} + F_{a_1 a_2 \ldots a_n}$$

where $T^0_{a_1 a_2 \ldots a_n}$ is the traceless part and $F_{a_1 a_2 \ldots a_n}$ is the trace contraction part. Generally we can use traceless condition to get the formation of T^0. The irreducible representation spaces can be obtained by use of the Young pattern. However, the rows of the Young pattern to mark the irreducible representations $SP(4)$ should be not larger than two. The reason is that a space of nth traceless tensors is exactly equal to duality space of $(N-n)th$ traceless tensors by use of anti-symmetry tensor ϵ given in Appendix (**??**). Therefore, if the boxes of a column in a Young pattern are three, we can use its duality column with boxes equal to $4-3=1$. The difference on $SP(4)$ from that discussed on $SU(N)$ is that since the tensor is traceless, we can directly combine the duality column with the left part of origin Young pattern together. The representations with respect to the origin and later Young pattern are equivalent. Each irreducible representation of $SP(4)$ can be marked by Young pattern $[p, q]$, $p, q =$ the boxes in first and second rows. There is an easy way to calculate the dimension of the irreducible representation of $SP(4)$ with respect to that of $SU(4)$,

$$D(p, q) = d(p, q) - d(p-1, q-1) \tag{6}$$

where the d(p,q) is the dimension of irreps of $SU(4)$ labeled by the same Young pattern $[p, q]$. The result counts on the following reason: Since the tensors in the representation space for $SP(4)$ are traceless, we divide the representation tensor space for $SU(4)$ into two subspace, one with trace contraction tensors and the other with traceless tensors. The later subspace should be equal to the representation space for $SP(4)$. The dimension for the former space is exactly equal to the dimension of representation of $SU(4)$ labeled by the Young pattern $[p-1, q-1]$. Utilizing the result for $SU(4)$ given in Appendix (**??**), we obtain

$$D(p, q) = \frac{1}{6}(p+2)(q+1)(p-q+1)(p+q+3). \tag{7}$$

It is very tedious and tiresome work to obtain the generator irreps matrix based on the invariant traceless tensor space. However, the Young pattern is rather helpful to discuss other aspects.

The first issue can be readily handled here is to decompose an irreps of $SU(4)$ into irrepses of $SP(4)$. The reason why an irreps of $SU(4)$ can be decomposed into $SP(4)$ is the irreducible tensor space for $SU(4)$ can be divided into two subspaces as that discussed above. The trace contraction tensors can be reconsidered as the lower rank tensors, and can be divided successively into traceless tensors and sub-trace contraction tensors again. This division ultimately leads to decompose the original tensor space for $SU(4)$ into the irreducible representation tensor spaces for $SP(4)$. The hint to solve this problem has been shown in (**??**). The result is straightforwardly derived , which is

$$[p, q]_{SU(4)} = [p, q]_{SP(4)} \bigoplus [p-1, q-1]_{SP(4)} \ldots \bigoplus [p-q, 0]_{SP(4)}. \tag{8}$$

However, this result is not complete. The Young pattern labeling the irreps of $SU(4)$ can be more than two rows. In particular in case of $SU(4)$, we need only to handle the Young diagram having three rows. We developed a general way to decompose such Young diagram. The method includes the following steps:

1. Re-write Young diagram more than two rows as the equivalent mixed sign $[s]^*/[p,q]$, as the way shown in Appendix (**??**).

2. Decompose $[p,q]$ according to Eq.(**??**).

3. Apply the Littlewood-Richardson rule to decompose all of the products of one decomposed component Young patterns, which comes from $[p,q]$, with $[s]^*$.

4. Disregard the Young patterns which the total rows are more than two.

The decomposition result is the sum of all of the Young patterns which satisfy above condition. There is an example to illustrate above procedure.

example: $[3,3,2]_{SU(4)}$

(1). $[3,3,2]_{SU(4)} \simeq [2]^*/[1,1]$

(2). $[1,1]_{SU(4)} \simeq [1,1] + [0]$

(3). $[2] \otimes ([1,1] + [0]) \Longrightarrow [3,1] + [2,1,1] + [2]$

(4). Dropping off [2, 1 ,1], the final result is:

$$\begin{array}{rcccccc} & [3,3,2]_{SU(4)} & \simeq & [3,1]_{SP(4)} & + & [2]_{SP(4)} \\ Dimension: & 45 & & 35 & & 10 \end{array}$$

The second easily solvable problem is to obtain all irrepses including an product of two irrepses of $SP(4)$, which means

$$[p_1,q_1] \bigotimes [p_2,q_2] = ? \bigoplus ?... \bigoplus ?.$$

An approach is developed in the following. It is also connected with our above results.

1. Use Equation **??** to rewrite the product into

$$[p_1,q_1] \bigotimes [p_2,q_2] \simeq \{[p_1,q_1] - [p_1-1,q_1-1]\} \bigotimes \{[p_2,q_2] - [p_2-1,q_2-1]\},$$

where the Young pattern in the right side of equation denotes the irreps of $SU(4)$.

2. Apply Littlewood-Richardson rule to revolve above products into the sum of the irreducible representations of $SU(4)$ on the right part of above equation and keep the sign.

3. Use the method shown above to decompose each obtained irreps of $SU(4)$ into irrepses of $SP(4)$.

The results is the sum of the total irrepses we obtained.

Two examples are illustrated in following by using above method:

(1) The product of spinor irreps $[1]$(dimension 4) with a general irreps $[p,q]$, i.e. $[1]\otimes[p,q]$.

$$[1] \otimes [p,q] \simeq [p+1,q] + [p,q+1] + [p,q-1] + [p-1,q] \tag{9}$$

Results of several low dimension irreps products :

$$\begin{array}{lccccccccccc}
 & [1] & \otimes & [1] & \simeq & [2] & + & [1,1] & + & [0] & & \\
Dimension: & 4 & \times & 4 & = & 10 & + & 5 & + & 1 & & \\
 & [1] & \otimes & [2] & \simeq & [3] & + & [2,1] & + & [4] & & \\
Dimension: & 4 & \times & 10 & = & 20 & + & 16 & + & 4 & & \\
 & [1] & \otimes & [2,1] & \simeq & [3,1] & + & [2,2] & + & [2] & + & [1,1] \\
Dimension: & 4 & \times & 16 & = & 35 & + & 14 & + & 10 & + & 5
\end{array}$$

(2) The product of symmetric tensor irreps $[1,1]$(dimension 5) with general irreps $[p,q]$, i.e. $[1,1]\otimes[p,q]$.

$$[1,1] \otimes [p,q] \simeq \begin{cases} [p+1,q+1] + [p,q] + [p+1,q-1] + [p-1,q+1] + [p-1,q-1], & p \neq q \\ [p+1,q+1] + [p+1,q-1] + [p-1,q-1], & p = q \end{cases} \tag{10}$$

Results of several low dimension irreps products:

$$
\begin{array}{rccccccccc}
 & [1,1] & \otimes & [1] & \simeq & [2,1] & + & [1] & & \\
Dimension: & 5 & \times & 4 & = & 16 & + & 4 & & \\
 & [1,1] & \otimes & [1,1] & \simeq & [2,2] & + & [2] & + & [0] \\
Dimension: & 5 & \times & 5 & = & 14 & + & 10 & + & 1 \\
 & [1,1] & \otimes & [2,0] & \simeq & [3,1] & + & [2] & + & [1,1] \\
Dimension: & 5 & \times & 10 & = & 35 & + & 10 & + & 5 \\
 & [1,1] & \otimes & [2,2] & \simeq & [3,3] & + & [3,1] & + & [1,1] \\
Dimension: & 5 & \times & 14 & = & 30 & + & 35 & + & 5
\end{array}
$$

B. Highest weight representation of $SO(5)$

The general method to construct irreps of an semi-simple Lie Group is to analyze its Lie algebra. The irreps is so-called highest weights representation. There are many concepts with respect to this general representation method. In this section, we just simplify this method to apply for $SO(5)$. The detailed proof is omitted.

At first, we explain several concepts with respect to semi-simple algebra.

1. *Lie algebra*,

$$[L_A, L_B] = iC_{AB}^{C} L_C,$$

where L_A is infinitesimal generators and C_{AB}^{C} is called structure constants.

2. *Killing* matrix:

$$K_{AB} = \sum_{PQ} C_{AP}^{Q} C_{BQ}^{P}.$$

Lie algebra is semi-simple if $detK \neq 0$. With the generators of $SO(5)$ shown as before, the *Killing* matrix K for $SO(5)$ is

$$K_{AB} = -6\delta_{AB}.$$

3. *Cartan subalgebra*:

if H is an element of Lie algebra and its eigenvector E is defined as

$$[H, E] = \alpha E,$$

then the vanishing eigenvalue is the unique degenerate eigenvalue. The degeneracy l is called the *rank* of Lie algebra. All of the eigenvectors with eigenvalue zero form an abelian subalgebra called *Cartan subalgebra*. In the case of $SO(5)$, the *rank* of Lie algebra is two. We can choose

$$H_1 = L_{12}, H_2 = L_{34},$$

to form its *Cartan subalgebra*.

4. *Root*, *simple root*, and *root diagram*:

The left generators can consist of the eigenvectors of *Cartan subalgebra* with nonvanishing eigenvalue. Each eigenvalue, we call *root*, can be considered as a "vector" in a l-dimensional space. This space is called root space. Th e graphical representation of the root vectors in root space is called *root diagram*. it is easy to prove that if α is a nonvanishing root, then $-\alpha$ is also a root and it is not degenerate. If the first component in the *root* vector is positive, we call this root is positive root. A positive root is a *simple root* if it cannot be decomposed into the sum of two positive roots.

In case of $SO(5)$, there are eight *roots* and the root space is a two dimensional space.:

$$[H_i, E_\alpha] = \alpha_i E_\alpha, i = 1, 2, \alpha = (\alpha_1, \alpha_2)$$

Based on the *Cartan subalgebra* and the eigenvectors, the *Killing* matrix changes to be

$$K = \begin{pmatrix} g & & & & \\ & -\sigma_1 & & & \\ & & -\sigma_1 & & \\ & & & -\sigma_1 & \\ & & & & -\sigma_1 \end{pmatrix}$$

where the matrix $g_{ij} = -6\delta_{ij}, i, j = 1, 2$ and σ_1 is the first Pauli matrix, and the Lie algebra of $SO(5)$ is modified to

$$[H_1, H_2] = 0$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \alpha+\beta \text{ is a root} \\ \sum_i \alpha^i H_i & \alpha = -\beta \\ 0 & \text{other} \end{cases} \tag{11}$$

where $\alpha^i = -\sum_j g^{ij}\alpha_i = \frac{1}{6}\alpha_i$, $\sum_k g^{ik} g_{kj} = \delta_{ij}$, and $N_{\alpha\beta}$ depends on our choice of eigenvectors and it satisfies

$$N_{\alpha\beta} = -N_{-\alpha-\beta} = N_{-\beta-\alpha}.$$

The *root vectors* and *roots* in $SO(5)$ are,

Eigenvalues	*Eigenvector*
$\alpha_1 = [1,-1]$	$E_1 = \frac{1}{\sqrt{24}}[L_{13} + L_{24} + i(L_{23} - L_{14})]$
$\alpha_{-1} = [-1,1]$	$E_{-1} = \frac{1}{\sqrt{24}}[L_{13} + L_{24} - i(L_{23} - L_{14})]$
$\alpha_2 = [0,1]$	$E_2 = \frac{1}{\sqrt{12}}[L_{35} + iL_{45}]$
$\alpha_{-2} = [0,-1]$	$E_{-2} = \frac{1}{\sqrt{12}}[L_{35} - iL_{45}]$
$\alpha_3 = [1,1]$	$E_3 = \frac{1}{\sqrt{24}}[L_{13} - L_{24} + i(L_{23} + L_{14})]$
$\alpha_{31} = [-1,-1]$	$E_{-3} = \frac{1}{\sqrt{24}}[L_{13} - L_{24} - i(L_{23} + L_{14})]$
$\alpha_4 = [1,0]$	$E_4 = \frac{1}{\sqrt{12}}[L_{25} - iL_{15}]$
$\alpha_{-4} = [-1,0]$	$E_{-4} = \frac{1}{\sqrt{12}}[L_{25} + iL_{15}]$

There are four positive roots. The first two positive roots $\alpha_1 = [1,-1], \alpha_2 = [0,1]$ are the simple roots. The *root diagram* of $SO(5)$is shown in Picture **??**.

FIG. 1. Root diagram of $SO(5)$

5. *Dynkin diagram* and *Cardan matrix*:

In Lie algebra theory, there is an alternative ingenious scheme to draw the root diagrams for any rank Lie algebra. It is called *Dynkin* diagram. It just tells the information of the simple roots of Lie algebra. The classic Lie group can be totally defined according to it's *Dynkin* diagram. In this definition, $SO(5)$is called B_2 and $SP(4)$ is called C_2. The Dynkin diagrams of $SO(5)$ and $SP(4)$ are shown in Picture **??** where the empty hole represents the longer amptitute simple root $[1,-1]$, and filled hole represents the shorter roots $[0,1]$. The two lines between two roots are corresponding to the angle between them is equal to $\frac{3\pi}{4}$.

FIG. 2. *Dynkin* diagrams of $SO(5)$and $SP(4)$

There is also a matrix, called *Cartan* matrix to label the simple roots of Lie algebras. It is defined as

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{\alpha_i, \alpha_i},$$

where (α_i, α_j) denote the scalar product, and α_i, α_j are simple roots. *Cortan* matrix of $SO(5)$ is

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

6. *weight*, *weight* diagram and *highest weight* representation

Since *Cartan* algebra is an abelian algebra, we can choose the space generated by the common eigenvectors of the elements H_i as the representation spaces. These kinds of spaces can be divided into the irreps spaces. In case of $SO(5)$we denote an eigenvectors of its *Cartan* subalgebra as $|m_1, m_2>$,

$$H_i|m_1, m_2 >= m_i|m_1, m_2 >, i = 1, 2.$$

The two dimensional vector $M = [m_1, m_2]$ is called *weight* and the 2-dimensional vector space extended by the set of weights is called weight space. If the first nonvanishing component of a weight is positive, this weight is called *positive* weight. A weight $[m_1, m_2]$ is said to be higher than $[w_1, w_2]$ if $[m_1 - w_1, m_2 - w_2]$ is positive. In an irreducible representation space, a weight is said to be *simple* if it belongs to only one eigenvector.

Since the Lie algebra is totally determined by its simple roots, we can guess the irreducible representations of Lie algebra must depend on simple roots. A set of weights M_1, M_2, called *fundamental* weights, is defined as

$$2(M_i, \alpha_i) = (\alpha_i, \alpha_i)\delta_{ij}, i, j = 1, 2$$

where α_i is a *simple* root. The fundamental weights of $SO(5)$ are

$$M_1 = [1, 0], M_2 = [\frac{1}{2}, \frac{1}{2}].$$

There is an obvious geometrical relation between *simples* roots and *fundamental* weights:

if the simple roots are a group of basic vectors in root lattice space, the fundamental weights are just the basic vectors in the reverse lattice space.

The irreducible representations of Lie group are established by the following important theorem:

Theorem: The irreducible representation is uniquely determined by its highest weight M^*, the highest weight is a simple weight and can be written as an linear combination of fundamental weights, $M^* = \sum_i \mu_i M_i, \mu_i, i = 1, 2$, is non-negative integer.

A highest weight of $SO(5)$can be written as

$$M^* = [\mu_1 + \frac{\mu_2}{2}, \frac{\mu_2}{2}]. \tag{12}$$

Similar to the roots diagram, A *weight* diagram can be drawn to represent an irreps in weight space.

The other powerful theorem is very useful:

Theorem : For any weight M and root α, the quantity $\frac{2(M,\alpha)}{(\alpha,\alpha)}$ is an integer, and the weight $M' = M - \frac{2(M,\alpha)}{(\alpha,\alpha)}\alpha$ is also a weight and has the same degeneracy as M.

The geometrical meaning of above Theorem is very obvious in the weight space . M' is corresponding to the vector reflecting M through a hyperplane perpendicular to the root α.

Therefore, generally we can define an group S which includes the total reflection operations related to the planes perpendicular to the roots and the operation products.

The group S for $SO(5)$ includes eight elements. An weight $[m_1, m_2]$ by operating transformations in S can be changed to $[\pm m_1, \pm m_2] and [\pm m_2, \pm m_1]$. $Weyl$ has derived the general formula to calculate the dimension of any irreps of any semi-simple Lie algebra. Given the highest weight M^*, the dimension of the irreps is

$$d(M_*) = \prod_{\alpha}[1 + \frac{M^* \dot{\alpha}}{R\dot{\alpha}}],$$

where $R = \frac{1}{2}\sum_{\alpha} \alpha$ and the sum here is over all positive roots.

In case of $SO(5)$, $R = [\frac{3}{2}, \frac{1}{2}]$, and for a given highest weight M^*. above dimension equation becomes (see Eq.(**??**))

$$d(M^*) = (1 + u_1)(1 + u_2)(1 + \frac{u_1}{2} + \frac{u_2}{2})(1 + \frac{2u_1}{3} + \frac{U_2}{3}),$$

Specifically, it is tensor irreps when u_2 is an even integer, spinor irreps when $u_2 = 1, u_1 = 0$, and spinor tensor irreps in the other situations.

7. details in spinor irreps (d=4) and vector irreps (d=5) of $SO(5)$

(1) spinor irreps $M^* = [\frac{1}{2}, \frac{1}{2}]$:

$$\begin{aligned}
H_1 &= \tfrac{1}{2}(|1><1| + |2><2| - |3><3| - |4><4|) \\
H_2 &= \tfrac{1}{2}(|1><1| - |2><2| - |3><3| + |4><4|) \\
E_1 &= \tfrac{1}{\sqrt{6}}|2><4| \\
E_2 &= \tfrac{1}{\sqrt{12}}(|1><2| + |4><3|) \\
E_3 &= -\tfrac{1}{\sqrt{6}}|1><3| \\
E_4 &= \tfrac{1}{\sqrt{12}}(-|1><4| + |2><3|)
\end{aligned}$$

FIG. 3. weight diagram of spinor irreducible representation of $SO(5)$

(2) vector irreps $M^* = [1, 0]$:

$$\begin{aligned}
H_1 &= |1><1| - |4><4| \\
H_2 &= |5><5| - |2><2| \\
E_1 &= \tfrac{1}{\sqrt{6}}(|1><5| + |2><4|) \\
E_2 &= \tfrac{1}{\sqrt{6}}(|3><2| + |5><3|) \\
E_3 &= \tfrac{1}{\sqrt{6}}(|1><2| + |5><4|) \\
E_4 &= \tfrac{1}{\sqrt{6}}(-|2><4| + |1><3|)
\end{aligned}$$

FIG. 4. weight diagram of vector irreducible representation of $SO(5)$

8. The second order Casimir operator

Casimir operators are invariant operators in carrier spaces of an irrepses. We only discuss the second order Casimir operator here, which is defined as

$$C = \sum_{a<b} L_{ab}^2. \tag{13}$$

In the case of $SO(5)$, we can easily calculate it for a given irreps. By using of *Cartan subalgba* and *eigenvectors* defined before, Casimir operator becomes,

$$C = H_1^2 + H_2^2 + 6 \sum_{\alpha, positive\ root} E_\alpha E_{1\alpha}.$$

The value of Casimir operator acting on the state in the carrier spaces of a given irreps $[m_1 = u_1 + \frac{u_2}{2}, m_2 = \frac{u_2}{2}]$ is

$$C = m_1^2 + m_2^2 + 3m_1 + m_2 \tag{14}$$

$$= u_1^2 + \frac{u_2^2}{2} + u_1 u_2 + 3u_1 + 2u_2. \tag{15}$$

C. Irreducible representation of $SO(5)$ based on $SO(4)$

For practical application, it is convenient and necessary to realize irreps in an explicit way. In quantum mechanics, we label a state with quantum numbers. In an irreps of specific group, we can also offer the weights with certain physical meaning. In case of $SO(5)$, there is a natural way to identify the vectors of weights: indicating the irreps of $SO(5)$with respect to $SO(4)\cong SO(3)\times SO(3)$. There are ten generators in $SO(5)$. Six of them can be used to construct those of the subgroup $SO(4)$. We make use of the following six generators to set up the Lie algebra of $SO(4)$:

$$J_1 = \tfrac{1}{2}(L_{14} + L_{23}), \quad J_2 = \tfrac{1}{2}(L_{24} + L_{31}), \quad J_3 = \tfrac{1}{2}(L_{12} + L_{34})$$
$$; K_1 = \tfrac{1}{2}(L_{23} - L_{14}), \; K_2 = \tfrac{1}{2}(L_{31} - L_{24}), \; K_3 = \tfrac{1}{2}(L_{12} - L_{34}).$$

The remaining four generators, $L_{15}, L_{25}, L_{35}, L_{45}$, can be considered to construct irreducible tensor operators of $SO(4)$. We use the following definition :

$$T^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} = \tfrac{1}{\sqrt{2}}(L_{25} - iL_{15}), \quad T^{\frac{1}{2},\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} = \tfrac{1}{\sqrt{2}}(L_{25} + iL_{15})$$
$$T^{\frac{1}{2},\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} = \tfrac{1}{\sqrt{2}}(L_{45} + iL_{35}), \; T^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} = \tfrac{-1}{\sqrt{2}}(L_{45} - iL_{35})$$

$T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\beta}, \alpha, \beta = \pm\frac{1}{2}$, denote the rank of irreducible tensor of $SO(4)$ (also see Table.(**??**). The commutation relations between $J_i, K_i, T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\beta}$ are

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [K_i, K_j] = i\epsilon_{ijk} K_k, \quad [J_i, K_j] = 0;$$
$$[T^{\frac{1}{2},\frac{1}{2}}_{\pm\frac{1}{2},\beta}, J_\pm] = 0, \quad [T^{\frac{1}{2},\frac{1}{2}}_{\pm\frac{1}{2},\beta}, J_\mp] = T^{\frac{1}{2},\frac{1}{2}}_{\pm\frac{1}{2},\beta}$$
$$[T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\pm\frac{1}{2}}, K_\pm] = 0, \quad [T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\pm\frac{1}{2}}, K_\mp] = T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\pm\frac{1}{2}}$$
$$[T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\beta}, J_3] = \alpha T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\beta}, \quad [T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\beta}, K_3] = \beta T^{\frac{1}{2},\frac{1}{2}}_{\alpha,\beta} \tag{16}$$

We have five operators: C (Casimir operator), J, J_3, K and K_3, which are Hermitian and mutually commute each other. The states in an irreps can be labeled by above quantum numbers. Therefore, for a given irreps labeled by the highest weight $M^* = [m_1 = u_1 + \frac{u_2}{2}, m_2 = \frac{u_2}{2})$, We denote an common eigenstate of above five operators as $|J_M K_M; JJ_m KK_m>$, where

$$J_M = \frac{1}{2}(m_1 + m_2) = \frac{1}{2}(u_1 + u_2), \quad K_M = \frac{1}{2}u_1.$$

One of convenient treatment in this representation is to obtain the irreps matrixes of the infinitesimal operators [?]. The matrix elements follow from the explicit expression for the states $|J_M K_M; JJ_m, KK_m>$. The matrix elements of J_i, K_i are the general angular momentum matrix elements. To calculate matrix elements of the remaining four infinitesimal operators, we slightly repeat the method given in Ref. [?] in the following.

Hecht Ref. [?] introduced two operators, $O{-+}$ and O_{--} , which are defined as

$$O_{-+} = -\sqrt{2}[(J_- T^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} + T^{-\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(2J_3 + 1)]$$
$$O_{--} = -K_- O_{-+} + \sqrt{2}[-J_- T^{\frac{1}{2},-\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} + T^{-\frac{1}{2},-\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(2J_3)](2K_3 + 1). \tag{17}$$

These two operators have the following important properties,

$$O_{-+}|J_M K_M; JJ, KK> \; = c|J_M K_M; J - \frac{1}{2}J - \frac{1}{2}, K + \frac{1}{2}K + \frac{1}{2}>,$$
$$O_{--}|J_M K_M; JJ, KK> \; = c^{'}|J_M K_M; J - \frac{1}{2}J - \frac{1}{2}, K - \frac{1}{2}K - \frac{1}{2}> . \tag{18}$$

where c and $c^{'}$ are constants. Moreover, these two operators commute with each other, i.e. $[O_{-+}, O_{--}] = 0$. This property allows us to obtain the general states included in an irreps disregard of the order of operation. The general state $|J_M K_M; JJ_m KK_m>$ can be written as

$$|J_M K_M; J J_m K K_m >= N(J_M, K_M, n, m, x, y) J_-^x K_y O_{--}^m O_{-+}^n |J_M K_M; J_M J_M K_M K_M > . \quad (19)$$

with

$$\begin{aligned} J &= J_M - \tfrac{1}{2}n - \tfrac{1}{2}m, \quad 0 \le n \le 2(J_M - K_M), \\ K &= K_M + \tfrac{1}{2}n - \tfrac{1}{2}m, \quad 0 \le n \le 2K_M, \\ J_m &= J - x, \qquad 0 \le x \le 2J, \\ K_m &= K - y, \qquad 0 \le x \le 2K, \end{aligned}$$

where N is normalization constant and can be written as $N = f(n,m)g(J,x)g(K,y)$, with

$$f(n,m) = (n!)^{-2}(m!)^{-4}(C_{m+n}^n C_{2J_M+1}^n C_{2J_M+1}^m C_{2J_M+1}^{n+m} C_{2J_M-2K_M}^n C_{2J_M+2K_M+2}^m C_{2K_M}^m C_{2K_M+n+1}^m)^{-\frac{1}{2}},$$

and

$$g(z_1, z_2) = (z_2!)^{-1}(C_{2z_1}^{z_2})^{-\frac{1}{2}},$$

where $C_{n_1}^{n_2} = \frac{n_1!}{(n_1-n_2)!n_2!}$.

From above results, we can also get the dimension of irreps $[J_M, K_M]$, which is exactly identical to that derived (Eg.**??**) in previous section.

The matrix elements of operators $T_{\alpha,\beta}^{\frac{1}{2},\frac{1}{2}}$ can be obtained directly by operating on the state $|J_M K_M; Jj, Kk>$. However, we need not to respectively calculate matrix elements for each operator because of the well known Wigner-Eckart theorem. Since operators $T_{\alpha,\beta}^{\frac{1}{2},\frac{1}{2}}$ are irreducible tensors of $SO(4)$, the Wigner-Eckart gives us the following result,

$$< J_M K_M; J'j', K'k'|T_{\alpha,\beta}^{\frac{1}{2},\frac{1}{2}}|J_M K_M; Jj, Kk >= C_{Jj,\frac{1}{2}\alpha}^{J'j'} C_{Kk,\frac{1}{2}\beta}^{K'k'} < J_M K_M; J'K'||T^{\frac{1}{2}\frac{1}{2}}|||J_M K_M; JK >, \quad (20)$$

where C is the C-G coefficients of $SO(3)$ and $< J_M K_M; J'K'||T^{\frac{1}{2}\frac{1}{2}}|||J_M K_M; JK >$ is independent of α, β, j, and k. The C-G coeffients are shown in Table(**??**). For a given state $|J_M K_M; Jj, Kk >$, there are four nonvanishing matrix elements. The results are

$$< J_M K_M; J - \frac{1}{2} K - \frac{1}{2}||T^{\frac{1}{2}\frac{1}{2}}|||J_M K_M; JK > \; = \frac{1}{2}[\frac{f(J+K)}{2JK}]^{\frac{1}{2}},$$

$$< J_M K_M; J+\frac{1}{2} K+\frac{1}{2} || T^{\frac{1}{2}\frac{1}{2}} ||| J_M K_M; JK > \; = -\frac{1}{2}[\frac{f(J+K+1)}{2(J+1)(K+1)}]^{\frac{1}{2}},$$

$$< J_M K_M; J-\frac{1}{2} K+\frac{1}{2} || T^{\frac{1}{2}\frac{1}{2}} ||| J_M K_M; JK > \; = -\frac{1}{2}[\frac{g(J-K)}{2J(K+1)}]^{\frac{1}{2}},$$

$$< J_M K_M; J+\frac{1}{2} K-\frac{1}{2} || T^{\frac{1}{2}\frac{1}{2}} ||| J_M K_M; JK > \; = -\frac{1}{2}[\frac{g(J-K-1)}{2K(J+1)}]^{\frac{1}{2}}, \quad (21)$$

$$(22)$$

where

$$g(t) = (J_M - K_M + t)(J_M - K_M - t + 1)(J_M + K_M + t + 1)(J_M + K_M - t + 2)$$

$$f(s) = (J_M + K_M + s + 2)(J_M - K_M + s + 1)(J_M + K_M - s + 1)(K_M - J_M + s).$$

	$s = \frac{1}{2}$	$s = -\frac{1}{2}$
$J_2 = J_1 + \frac{1}{2}$	$(\frac{J_1+j+1}{2J_1+1})^{\frac{1}{2}}$	$(\frac{J_1-j+1}{2J_1+1})^{\frac{1}{2}}$
$J_2 = J_1 - \frac{1}{2}$	$-(\frac{J_1-j}{2J_1+1})^{\frac{1}{2}}$	$(\frac{J_1+j}{2J_1+1})^{\frac{1}{2}}$

TABLE II. C-G coefficients $C^{J_2 j+s}_{J_1 j, \frac{1}{2} s}$

II. HARMONICS AND MONOPOLE HARMONICS OF $SO(5)$

In many physical problem, we deal with functions over the homogeneous or symmetric spaces, in particular, on group spaces. These functions can be decomposed over a set of eigenfunctions of Casimir operators. Such decomposition is extremely useful and has a clear physical interpretation. For example, in case of $SO(3)$, Casimir operator is total angular momentum and eigenfuncitons are spherical harmonics $Y_m^j(\theta,\phi)$.

To derive the $SO(5)$harmonics functions, we consider a S^4 sphere, and choose a specific coordinate system, so-called biharmonic coordinate system, which will lead us to express $SO(5)$harmonics solely in terms of $SO(3)$ well known $d^j_{m,n}(\theta)$-functions.

$$\begin{aligned}
x_5 &= cos\theta_2, \\
x_4 &= sin\theta_2 cos\theta_1 sin\phi_1, \\
x_3 &= sin\theta_2 cos\theta_1 con\phi_1, \\
x_2 &= sin\theta_2 sin\theta_1 sin\phi_2, \\
x_1 &= sin\theta_2 sin\theta_1 con\phi_2,
\end{aligned} \tag{23}$$

where $\theta_1 \in [0,\pi/2)$, $\theta_2 \in [0,\pi)$, $\phi_1 \in [0,2\pi)$, and $\phi_2 \in [0,2\pi)$.

The metric tensor $g_{\alpha\beta}$ matrix in S^4 sphere in terms of above coordinate choices is

$$g = \begin{pmatrix} sin^2\theta_1 sin^2\theta_2 & 0 & 0 & 0 \\ 0 & cos^2\theta_1 sin^2\theta_2 & 0 & 0 \\ 0 & 0 & sin^2\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{24}$$

with the Jocobii term is $J = |det(g)|^{\frac{1}{2}} = |sin\theta_1 cos\theta_1 sin\theta_2^3|$.

Lapalace operator (Casimir operator) is

$$\begin{aligned}
-\hat{C} &= J^{-1}\partial_\alpha g^{\alpha\beta} J \partial_\beta \\
&= sin^{-3}\theta_2 \frac{\partial}{\partial\theta_2} sin^3\theta_2 \frac{\partial}{\partial\theta_2} + sin^{-2}\theta_2 \hat{P}(\theta_1,\phi_1,\phi_2),
\end{aligned} \tag{25}$$

where

$$\hat{P} = sin^{-1}\theta_1 cos^{-1}\theta_1 \frac{\partial}{\partial\theta_1}(sin\theta_1 cos\theta_1)\frac{\partial}{\partial\theta_1} + cos^{-2}\theta_1 \frac{\partial^2}{\partial\phi_1^2} + sin^{-2}\theta_1 \frac{\partial^2}{\partial\phi_2^2}. \tag{26}$$

The integral measure of S^4 in biharmonic coordinate system is

$$d\mu = Jd\theta_1 d\theta_2 d\phi_1 d\phi_2 = sin\theta_1 cos\theta_1 sin^3\theta_2 d\theta_1 d\theta_2 d\phi_1 d\phi_2. \tag{27}$$

To solve above eigenvalue equation of Lapalace operator, we separate variables i.e. the eigenvector function $\Psi^{l_1 l_2}_{m_1 m_2}(\theta_1,\theta_2,\phi_1,\phi_2) = \psi^{l_1}_{l_2}(\theta_2)\varphi^{l_1}_{m_1 m_2}(\theta_1)exp(im_1\phi_1 + im_2\phi_2)$ and obtain the following second-order ordinary differential equation (the eigenvalues of this Lapalace operator are $l_2(l_2+3)$, which can be derived after having solved the equations),

$$\hat{C}\psi^{l_1}_{l_2} = l_2(l_2+3)\psi^{l_1}_{l_2}, \tag{28}$$

$$\hat{P}\varphi^{l_1}_{m_1 m_2} = -l_1(l_1+2)\varphi^{l_1}_{m_1 m_2}. \tag{29}$$

The general solutions of above two equations are

$$\psi^{l_1}_{l_2}(\theta_2) = tan^{l_2}\theta_2 cos^{l_2}\theta_2 F_1[\frac{1}{2}(l_1-l_2), \frac{1}{2}(l_1-l_2+1), l_1+2; -tan^2\theta_2] \tag{30}$$

$$\varphi^{l_1}_{m_1 m_2}(\theta_1) = tan^{m_2}\theta_1 cos^{l_1}\theta_1 F_1[\frac{1}{2}(m_1+m_2-l_1), \frac{1}{2}(|m_2|-m_1+l_1), m_2+1, -tan^2\theta_1], \tag{31}$$

where F_1 is the standard hypergeometric function. The restriction relations among the eigenvalues m_1, m_2, l_1 and l_2 are

$$|l_1| = l_2 - n, \quad l_2 \geq 0, \quad n = 0,1,...,l_2. \tag{32}$$

$$|m_1| + |m_2| = l_1 - 2k, \qquad k = 0,1,...,[\frac{1}{2}l_1], \tag{33}$$

Above solutions can be written in terms of $d^J_{m,m'}$ - functions of the ordinary rotation group $SO(3)$. The result is

$$\Psi^{l_1 l_2}_{m_1 m_2}(\theta_1,\theta_2,\phi_1,\phi_2) = N^{-\frac{1}{2}} sin^{-1}\theta_2 d^{l_2+1}_{l_1+1,0}(\theta_2) d^{\frac{1}{2}l_1}_{\frac{1}{2}(m_1+m_2),\frac{1}{2}(m_1-m_2)}(2\theta_1)exp(im_1\phi_1 + im_2\phi_2), \tag{34}$$

where the normalizations constant $N = 4\pi^2(2l_2+3)^{-1}(l_1+1)^{-1}$, and the orthogonality relation is

$$\int_0^\pi sin^3\theta_2 d\theta_2 \int_0^{\frac{\pi}{2}} sin\theta_1 cos\theta_1 d\theta_1 \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \bar{\Psi}^{l_1' l_2'}_{m_1' m_2'} \Psi^{l_1 l_2}_{m_1 m_2} = \delta_{l_1 l_1'} \delta_{l_2 l_2'} \delta_{m_1 m_1'} \delta_{m_2 m_2'}. \quad (35)$$

These harmonics only give the basis for the symmetric tensor irreps of $SO(5)$. In the next section, we will extend them to monopole harmonics, which includes the basis for spinor and spinor tensor representations. However, it is still necessary to express the infinitesimal generators in above biharmonic coordinate system. Moreover, it appears there are apparent connections between the eigenvalues characterizing the harmonics and the angular momentum discussed in Sec.(**??**). The following equations gives the detailed representations of the operators defined in Sec.(**??**):

$$\begin{aligned}
J_3 &= -\frac{i}{2}\frac{\partial}{\partial\phi_J}, \qquad K_3 = \frac{i}{2}\frac{\partial}{\partial\phi_K}, \\
J_+ = J_1 + iJ_2 &= \frac{i}{2}(tan^{-1}2\theta_1 \frac{\partial}{\partial\phi_J} - sin^{-1}2\theta_1 \frac{\partial}{\partial\phi_K} - i\frac{\partial}{\partial\theta_1}), \\
J_- = J_1 - iJ_2 &= \frac{i}{2}(tan^{-1}2\theta_1 \frac{\partial}{\partial\phi_J} - sin^{-1}2\theta_1 \frac{\partial}{\partial\phi_K} + i\frac{\partial}{\partial\theta_1}), \\
K_+ = J_1 + iJ_2 &= \frac{i}{2}(sin^{-1}2\theta_1 \frac{\partial}{\partial\phi_J} - tan^{-1}2\theta_1 \frac{\partial}{\partial\phi_K} - i\frac{\partial}{\partial\theta_1}), \\
K_- = J_1 - iJ_2 &= \frac{i}{2}(sin^{-1}2\theta_1 \frac{\partial}{\partial\phi_J} - tan^{-1}2\theta_1 \frac{\partial}{\partial\phi_K} + i\frac{\partial}{\partial\theta_1}), \\
T^{-\frac{1}{2}-\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} &= \frac{-1}{\sqrt{2}} e^{-i(\phi_J-\phi_K)}[cos\theta_1 tan^{-1}\theta_2 \frac{\partial}{\partial\theta_1} - \frac{i}{2} sin^{-1}\theta_1 tan^{-1}\theta_2 (\frac{\partial}{\partial\phi_J} - \frac{\partial}{\partial\phi_K}) + sin\theta_1 \frac{\partial}{\partial\theta_2}], \\
T^{\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} &= \frac{1}{\sqrt{2}} e^{i(\phi_J-\phi_K)}[cos\theta_1 tan^{-1}\theta_2 \frac{\partial}{\partial\theta_1} + \frac{i}{2} sin^{-1}\theta_1 tan^{-1}\theta_2 (\frac{\partial}{\partial\phi_J} - \frac{\partial}{\partial\phi_K}) + sin\theta_1 \frac{\partial}{\partial\theta_2}], \\
T^{-\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} &= \frac{1}{\sqrt{2}} e^{-i(\phi_J+\phi_K)}[sin\theta_1 tan^{-1}\theta_2 \frac{\partial}{\partial\theta_1} + \frac{i}{2} sin^{-1}\theta_1 tan^{-1}\theta_2 (\frac{\partial}{\partial\phi_J} + \frac{\partial}{\partial\phi_K}) - cos\theta_1 \frac{\partial}{\partial\theta_2}], \\
T^{-\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} &= \frac{1}{\sqrt{2}} e^{i(\phi_J+\phi_K)}[sin\theta_1 tan^{-1}\theta_2 \frac{\partial}{\partial\theta_1} - \frac{i}{2} sin^{-1}\theta_1 tan^{-1}\theta_2 (\frac{\partial}{\partial\phi_J} + \frac{\partial}{\partial\phi_K}) - cos\theta_1 \frac{\partial}{\partial\theta_2}], \quad (36)
\end{aligned}$$

where $\phi_J = \frac{1}{2}(\phi_1 + \phi_2)$ and $\phi_J = \frac{1}{2}(\phi_1 - \phi_2)$. Two total angular momentum operators, $\hat{J}^2$ and $\hat{K}^2$, are identical operators in the symmetric tensor irreps. They can be expressed with related to the above operator $\hat{P}$, by

$$\hat{J}^2 = \hat{K}^2 = -\frac{\hat{P}}{4}, \quad (37)$$

and the eigenvalues on the harmonics are

$$\hat{J}^2 \Psi^{l_1 l_2}_{m_1 m_2} = \hat{K}^2 \Psi^{l_1 l_2}_{m_1 m_2} = \frac{l_1}{2}(\frac{l_1}{2} + 1)\Psi^{l_1 l_2}_{m_1 m_2}, \quad (38)$$

$$\hat{J}_3 \Psi^{l_1 l_2}_{m_1 m_2} = \frac{1}{2}(m_1 + m_2)\Psi^{l_1 l_2}_{m_1 m_2}, \quad \hat{K}_3 \Psi^{l_1 l_2}_{m_1 m_2} = \frac{1}{2}(m_2 - m_1)\Psi^{l_1 l_2}_{m_1 m_2}. \tag{39}$$

The detailed formations of $SO(5)$harmonics with $l_2 = 0, 1$, and,2 are listed in the following:

1. $l_2 = 0 (d = 1)$:

$$\Psi^{00}_{00} = (\frac{4\pi^2}{3})^{-\frac{1}{2}} tan^{-1}\theta_2$$

2. $l_2 = 1 (d = 5)$:

$$\begin{aligned}
\Psi^{01}_{00} &= (\tfrac{8\pi^2}{15})^{-\frac{1}{2}} cos\theta_2, \\
\Psi^{11}_{\pm 10} &= (\tfrac{16\pi^2}{15})^{-\frac{1}{2}} sin\theta_2 cos\theta_1 exp(\pm i\phi_1), \\
\Psi^{11}_{0\pm 1} &= \mp(\tfrac{16\pi^2}{15})^{-\frac{1}{2}} sin\theta_2 sin\theta_1 exp(\pm i\phi_2)
\end{aligned}$$

3. $l_2 = 2 (d = 14)$:

$$\begin{aligned}
\Psi^{02}_{00} &= (\tfrac{64\pi^2}{21})^{-\frac{1}{2}} (1 - 5cos^2\theta_2), \\
\Psi^{12}_{\pm 10} &= (\tfrac{64\pi^2}{105})^{-\frac{1}{2}} sin2\theta_2 sin\theta_1 exp(\pm i\phi_1), \\
\Psi^{12}_{0\pm 1} &= \mp(\tfrac{64\pi^2}{105})^{-\frac{1}{2}} sin2\theta_2 sin\theta_1 exp(\pm i\phi_2), \\
\Psi^{22}_{00} &= (\tfrac{64\pi^2}{105})^{-\frac{1}{2}} sin^2\theta_2 \\
\Psi^{22}_{\pm 20} &= -(\tfrac{64\pi^2}{105})^{-\frac{1}{2}} sin^2\theta_2 cos^2\theta_1 exp(\pm i\phi_1), \\
\Psi^{22}_{0\pm 2} &= -(\tfrac{64\pi^2}{105})^{-\frac{1}{2}} sin^2\theta_2 sin^2\theta_1 exp(\pm i\phi_2), \\
\Psi^{22}_{\alpha\beta} &= -\alpha(\tfrac{128\pi^2}{105})^{-\frac{1}{2}} sin^2\theta_2 sin2\theta_1 exp[i(\alpha\phi_1 + \beta\phi_2)], \quad \alpha, \beta = \pm 1.
\end{aligned}$$

The concept of monopole harmonics was originally introduced by T.T. Wu and C.N.Yang [?] in 1970's. They derived the monopole harmonics in three dimension space and later C.N.Yang [?] generalized the idea and extended it to five dimension space which $SO(5)$ is concerned. Monopole harmonics, also called Dirac harmonics, are everywhere analytic and form a complete orthonormal set as the basis of expansion of any wave function around the monopole. In the 3-D space case, we can imagine it is a real physical model with a magnetic monopole and derive the monopole harmonics by solving the eigenvalue equation of total angular momentum. We briefly summarize the major results of monopole harmonics in 3-d space in the following:

1. Using spherical coordinates with a monopole of strength g at the origin , the vector potential can be chosen to be

$$\hat{A}_1^r = \hat{A}_1^\theta = 0, \quad \hat{A}_1^\phi = \frac{g}{sin\theta}(1 - cos\theta), \quad 0 \le \theta < \frac{\pi}{2} + \delta;$$
$$\hat{A}_2^r = \hat{A}_2^\theta = 0, \quad \hat{A}_2^\phi = \frac{-g}{sin\theta}(1 + cos\theta), \quad 0 \le \theta < \frac{\pi}{2} - \delta; \tag{40}$$

2. With a particle with charge Ze in this monopole model, the gauge transformation phase factor from A_1 to A_2 in the overlap area of above two chosen region is

$$S = exp(2iq\phi). \tag{41}$$

where $q = \frac{1}{2}DZ, \quad D = 2eg$, and $c = h = 1$. The transition can be written as

$$\hat{A}_1^i = \hat{A}_2^i + \frac{i}{Ze}S\frac{\partial S^{-1}}{\partial x^i}, \tag{42}$$

where $x^i, i = 1, 2, 3$ are the three local orthogonal coordinates.

3. The total angular momentum operator in above system can be written as

$$\hat{L} = \hat{r} \times (\hat{P} - Ze\hat{A}) - q\frac{\hat{r}}{r}. \tag{43}$$

4. The monopole harmonics is defined as

$$\hat{L}^2 Y_{l,m}^q = l(l+1)Y_{l,m}^q, \quad \hat{L}_3 Y_{l,m}^q = mY_{l,m}^q, \tag{44}$$

with $l = |q|, |q+1|, etc.$, and $\int_0^\pi sin\theta d\theta \int_0^{2\pi} |Y_{l,m}^q|^2 d\phi = 1$.

5. The explicit evaluation of the above equation is

$$[-sin\theta^{-1}\frac{\partial}{\partial\theta}sin\theta\frac{\partial}{\partial\theta} + sin^{-2}\theta(-i\frac{\partial}{\partial\phi} \mp q + qcos\theta)^2 + q^2]Y_{l,m}^q = l(l+1)Y_{l,m}^q \tag{45}$$

with the sign $\mp$ related to the two chosen region 1, 2. The explicit expression of $Y_{l,m}^q$ is

$$(Y_{l,m}^q)_1 = N(1 - cos\theta)^{\alpha/2}(1 + cos\theta)^{\beta/2}P_n^{\alpha,\beta}(cos\theta)exp[i(m+q)\phi], \tag{46}$$

$$(Y_{l,m}^q)_2 = (Y_{l,m}^q)_1 exp(2iq\phi), \tag{47}$$

where $\alpha = -q - m, \quad \beta = q - m, \quad n = l + m,$

$$N = 2^m[\frac{(2l+1)(l-m)!(l+m)!}{4\pi(l-q)!(l+q)!}^{\frac{1}{2}},$$

and $P_n^{\alpha,\beta}(cos\theta)$ are the Jacobi polynomials,

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}].$$

The construction of monopole harmonics in 5-d space (we call it $SO(5)$ monopole harmonics) is slightly complicated. The basic idea to realize the construction is to extend the above vector potential to a nonabelian $SU(2)$ gauge field. Since in this paper, we only focus on the monopole harmonics. Hence, we will construct the explicit expression of $SO(5)$ infinitesimal operators with a non-abelian $SU(2)$ gauge field in the biharmonic coordinate system , instead of repeating the abstract mathematical concepts and derivation in Ref. [**?**]. The basic idea is to express the generators of one of the subgroup $SO(3)$, $\hat{J}_i$ or $\hat{K}_i$, by adding the additional generators of $SU(2)$ gauge field, i.e,

$$\hat{J}_i = \hat{J}_i^0 + \hat{Y}_i, \quad \hat{K}_i = \hat{K}_i^0 \tag{48}$$

or

$$\hat{K}_i = \hat{K}_i^0 + \hat{Y}_i, \quad \hat{J}_i = \hat{J}_i^0. \tag{49}$$

Similar to $SO(3)$ monopole, the difference of above two choices results where the singularity locates when the gauge field is constructed. The results in the harmonics differs from a gauge transformation phase. We will make use of the first construction expression. The left job is to obtain the explicit expression of the remaining four $\hat{T}$ operators. By means of the commutators between the $SO(5)$ infinitesimal generators, the $\hat{T}$ operators can be written as

$$T^{\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} = T^{\frac{1}{2}\frac{1}{2}(0)}_{\frac{1}{2}\frac{1}{2}} + \sqrt{2}tan^{-1}\theta_2 cos\theta_1 exp(-i\phi_1)Y^+ - \sqrt{2}tan^{-1}\theta_2 sin\theta_1 exp(i\phi_2)Y^3 \tag{50}$$

$$T^{\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} = T^{\frac{1}{2}\frac{1}{2}(0)}_{\frac{1}{2}-\frac{1}{2}} + \sqrt{2}tan^{-1}\theta_2 sin\theta_1 exp(-i\phi_2)Y^+ + \sqrt{2}tan^{-1}\theta_2 con\theta_1 exp(i\phi_1)Y^3 \tag{51}$$

$$T^{\frac{1}{2}\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} = T^{\frac{1}{2}\frac{1}{2}(0)}_{-\frac{1}{2}-\frac{1}{2}} + \sqrt{2}tan^{-1}\theta_2 cos\theta_1 exp(i\phi_1)Y^- - \sqrt{2}tan^{-1}\theta_2 sin\theta_1 exp(-i\phi_2)Y^3 \tag{52}$$

$$T^{\frac{1}{2}\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} = T^{\frac{1}{2}\frac{1}{2}(0)}_{\frac{1}{2}-\frac{1}{2}} - \sqrt{2}tan^{-1}\theta_2 sin\theta_1 exp(i\phi_2)Y^- - \sqrt{2}tan^{-1}\theta_2 con\theta_1 exp(-i\phi_1)Y^3. \tag{53}$$

The Casimir operator can be calculated and be written as

$$\hat{C} = -sin^{-3}\theta_2\frac{\partial}{\partial\theta_2}sin^3\theta_2\frac{\partial}{\partial\theta_2} + \frac{4J^2}{sin^2\theta_2} + \frac{2(1-cos\theta_2)}{sin^2\theta_2}[\hat{J}^2 - \hat{K}^2] + \hat{Y}^2. \tag{54}$$

The $SO(5)$ monopole harmonics can be written as

$$\Psi^{J_M K_M}_{jm_j,km_k} = Nsin^{-1}\theta_2 F^{J_M K_M}_{jk}(\theta_2) G_{jm_j,km_k}(\theta_1,\phi_1,\phi_2), \tag{55}$$

where

$$\hat{J}^2 G_{jm_j,km_k}(\theta_1,\phi_1,\phi_2) = j(j+1)G_{jm_j,km_k}(\theta_1,\phi_1,\phi_2), \quad \hat{J}_3 G_{jm_j,km_k}(\theta_1,\phi_1,\phi_2) = m_j G_{jm_j,km_k}(\theta_1,\phi_1,\phi_2)$$

$$\hat{K}^2 G_{jm_j,km_k}(\theta_1,\phi_1,\phi_2) = k(k+1)G_{jm_j,km_k}(\theta_1,\phi_1,\phi_2), \quad \hat{K}_3 G_{jm_j,k\ m_k}(\theta_1,\phi_1,\phi_2) = m_k G_{jm_j,km_k}(\theta_1,\phi_1\ ,\phi_2)$$

and the N is the normalized constant. J_M and K_M label the given irreps (see eq.**??**). For a given irreps (J_M, K_M), the eigenvalue of Casimir operator ,

$$\hat{C}\psi^{J_M K_M}_{jm_j,km_k} = c\psi^{J_M K_M}_{jm_j,km_k} = 2(J_M^2 + K_M^2 + 2J_M + K_M)\psi^{J_M K_M}_{jm_j,km_k}.$$

Substituting the explicit expression of $\hat{C}$ into above equation, we obtain

$$[-sin^{-1}\theta_2 \frac{\partial}{\partial\theta_2} sin\theta_2 \frac{\partial}{\partial\theta_2} + \frac{\alpha^2+\beta^2+(\alpha^2-\beta^2)cos\theta_2}{2sin^2\theta_2}]F^{J_M K_M}_{jk}(\theta_2) = [c+2-Y(Y+1)]F^{J_M K_M}_{jk}(\theta_2). \tag{56}$$

where $\alpha = 2k+1$ and $\beta = -2j-1$. The above equation has the similar expression as Eq.(**??**). Thus we can get the allowable value of Y in order to obtain the allowable solution of above equation. Compared with Eq.(**??**), the value of Y should be not greater than $J_M - K_M$. However, since the angular momentum operator $\hat{J}$ is the sum of $\hat{K}$ and $\hat{Y}$, the value of Y should be not less than $|J_M - K_M|$. Therefore, the value of Y can only be equal to $J_M - K_M$ for a given irreps (J_M, K_M) and the right side of above equation becomes to be $(J_M + K_M + 1)(J_M + K_M + 2)]F^{J_M K_M}_{jk}(\theta_2)$. The explicit expression of function F can be written as

$$F^{J_M K_M}_{jk}(\theta_2) = (1-cos\theta_2)^{\alpha/2}(1+cos\theta_2)^{\beta/2}P^{\alpha,\beta}_n(cos\theta_2), \tag{57}$$

where $n = J_M + K_M + j - k + 1$.

We can construct the explicit expression of G_{jm_j,km_k} by means of C-G coefficients as

$$G_{jm_j,km_k}(\theta_1,\phi_1,\phi_2) = \sum_{m_k^{'}} C^{j,m_j}_{km_k^{'},Y(m_j-m_k^{'})}|km_k m_k^{'}, Y(m_j - m_k^{'})>, \tag{58}$$

where

$$|km_k m_k^{'}, Ym_Y >= \sqrt{\frac{2k+1}{2\pi^2}} d^k_{m_k^{'},m_k}(2\theta_1) exp[2i(m_k\phi_k + m_k^{'}\phi_j)]|Ym_Y > . \tag{59}$$

The renormalized constant N is

$$N = 2^{j-k}[\frac{(j+k+\frac{3}{2})(2k+1)!(2j+1)!}{(2j+2k+2)!}]^{\frac{1}{2}}. \tag{60}$$

The monopole harmonics have the same orthogonality relation as that in Eq.(**??**). The explicit forms of $SO(5)$ monopole harmonics of the spinor irreps(d=4), and adjoint representation are given in the following :

(1). spinor irreps, $d = 4$, $(J_M = \frac{1}{2}, K_M = 0)$, and $c = 2.5$:

$$\begin{aligned}
\Psi^{\frac{1}{2}0}_{\frac{1}{2}\pm\frac{1}{2},00} &= \sqrt{\tfrac{3}{8\pi^2}} sin\theta_2(1-cos\theta_2)^{-\frac{1}{2}}|\tfrac{1}{2}\pm\tfrac{1}{2}> \\
\Psi^{\frac{1}{2}0}_{00,\frac{1}{2}\frac{1}{2}} &= \sqrt{\tfrac{3}{8\pi^2}} sin\theta_2(1+cos\theta_2)^{-\frac{1}{2}} exp(i\phi_k)[cos\theta_1 exp(i\phi_j)|\tfrac{1}{2}-\tfrac{1}{2}> - sin\theta_1 exp(-i\phi_j)|\tfrac{1}{2}\tfrac{1}{2}>] \\
\Psi^{\frac{1}{2}0}_{00,\frac{1}{2}-\frac{1}{2}} &= \sqrt{\tfrac{3}{8\pi^2}} sin\theta_2(1+cos\theta_2)^{-\frac{1}{2}} exp(i\phi_k)[-sin\theta_1 exp(i\phi_j)|\tfrac{1}{2}-\tfrac{1}{2}> + cos\theta_1 exp(-i\phi_j)|\tfrac{1}{2}\tfrac{1}{2}>]
\end{aligned}$$

(2) adjoint irreps, $d = 10$, $(J_M = 1, K_M = 0)$, and $c = 6$:

$$\begin{aligned}
\Psi^{10}_{10,00} &= \sqrt{\tfrac{5}{8}}(1+cos\theta_2)|10> \\
\Psi^{10}_{1\pm1,00} &= \sqrt{\tfrac{5}{8}}(1+cos\theta_2)|1\pm1> \\
\Psi^{10}_{00,1\alpha} &= \sqrt{\tfrac{5}{8}}(1-cos\theta_2)G_{00,1\alpha}(\theta_1,\phi_k,\phi_j), \quad \alpha = 0, \pm1 \\
\Psi^{10}_{\frac{1}{2}\alpha,\frac{1}{2}\beta} &= \sqrt{\tfrac{15}{16}} sin\theta_2 G_{\frac{1}{2}\alpha,1\beta}(\theta_1,\phi_k,\phi_j), \quad \alpha = \pm\tfrac{1}{2}, \beta = \pm\tfrac{1}{2}
\end{aligned}$$

APPENDIX A: SUMMARY OF TENSOR REPRESENTATION OF $SU(N)$

1. The n rank tensor T of $SU(N)$ is defined as:

$$O_u T_{a_1a_2...a_n} = u_{a_1b_1}u_{a_2b_2}...u_{a_nb_n}T_{b_1b_2...b_n}, a_i, b_i = 1, 2, 3...N,$$

where, u is an element of $SU(N)$. Considering the linear space generated by all of components of tensor, we can get a representation of $SU(N)$. However, this representation is not irreducible. It is easy to see that the permutation symmetry in the index of tensor is kept under $SU(N)$. This means that set of the components of tensor with certain permutation symmetry is an invariable subspace under $SU(N)$. The mathematical technique to decompose a tensor space to its invariable subspace is developed by introducing Young Operator and Young Pattern with respect to $S(N)$ group (Readers who are not familiar with these concepts can find this related material in some standard books.)

2. The irreducible tensor representation of $SU(N)$ is labeled by a Young pattern $[Y = (y_1, y_2, ..., y_N)]$, whose raw is not greater than N. The dimension of irreducible representation given by a certain Young pattern is equal to the number of Young standard tableaux, which is defined in following: We label each box in Young pattern with a positive integer not greater than N. A labeled Young pattern is called Young standard tableux if in a raw, the number in a box is not greater than the number in its left boxes, and in a column, the number in a box is smaller than the number in the boxes below it (the numbers in the same column must be different).

3. We can also introduce nth rank tensor, which is defined as:

$$O_u T^{a_1 a_2 ... a_n} = u^*_{a_1 b_1} u_{a_2 b_2} ... u^*_{a_n b_n} T^{b_1 b_2 ... b_n}, a_i, b_i = 1, 2, 3...N,$$

Moreover, we can define general (m, n) rank mixed tensor,

$$O_u T^{a_1^{'} a_2^{'} ... a_m^{'}}_{a_1 a_2 ... a_n} = u_{b_1 b_1^{'}} u_{b_2 b_2^{'}} ... u_{b_n^{'} b_n} T^{b_1^{'} b_2^{'} ... b_m^{'}}_{b_1 b_2 ... b_n}, a_i, b_i, a^{'}, b^{'} = 1, 2, 3...N, .$$

The irreducible representation of $SU(N)$ can also be derived from tensor or mixed tensor invariant spaces labeled by Young patterns. We mark $[Y]^*$ with respect to the Young pattern related to invariant tensor space and $[Y]^*/[W]$ with respect to the young patterns related to invariant mixed tensor space . The invariant mixed tensor space should be satisfied with an additional traceless condition similar to what we discuss in $SP(4)$.

4. Defining a Nth rank antisymmetrical tensor,

$$\epsilon_{a_1 a_2 ... a_N} = \begin{cases} 1, & even \;\; permutation \\ -1, & odd \;\; permutation \\ 0, & others \end{cases} \tag{A1}$$

we can change an mth rank antisymmetrical tensor(tensor) to $(N-m)th$ antisymmetrical tensor(tensor) by

$$T_{a_1a_2...a_{N-m}} = \frac{1}{m!}\sum_b \epsilon_{a_1a_2...b_1b_2...b_m} T^{b_1b_2...b_m}. \tag{A2}$$

This relation leads to a general result about the equivalence of irreducible representations between invariant tensor and tensor space:

$$[y_1, y_2...y_N] \cong [y_1, y_1 - y_{N-1}, ..y_1 - y_2]^*.$$

5. Littlewood-Richardson rule:

Littlewood-Richardson rule determines what irreps $[Y]$ are included in the product of two irreps $[Y_1]$ and $[Y_2]$.

(1). Draw the Young diagrams of $[Y_1]$ and $[Y_2]$.

(2). Choose the simpler one between these two Young diagrams and fill up each of boxes with the line number it locates.

(3). Append each of boxes to the more complicated Young diagram with starting from the boxes filled with the lowest number in all possible ways subject to the rules(needed to be satisfied after adding each box):

I.resultant diagram are always regular. II. they are no two boxes filled with the same number are appeared in the same column III. the number of appended boxes filled with larger number counting from right to left and from top to bottom should not be greater than that of appended boxes filled with smaller number counting in the same way. (4). Disregard the Young diagrams $[Y]$ of more than N rows and delete the columns of length N in any diagram $[Y]$

6. Hook rule to calculate the dimension of irreps of $SU(N)$:

The formula to calculate the dimension of irreps of $SU(N)$ is:

$$d_{[Y]} = \prod_{1\leq i\leq j\leq N} \frac{Y_i - Y_j - i + j}{j - i}. \tag{A3}$$

There is a simple Hook rule related to above formula:

$$d_{[Y]} = \frac{d'_{[Y]}}{H_{[Y]}},$$

where the numerator $d'_{[Y]}$ is a product of integers filled in the boxes of Young diagram $[Y]$ as the following rule: filling all the diagonal boxes in the Young diagram with integer N, the subsequent boxes in the same rows with $N+1$, $N+2$ ..., and that in the same columns with $N-1$, $N-2$ etc,

and the denominator is the product of *hooklength* associated with each boxes in Young diagram. *hooklength* of a given box in a given Young diagram is defined as: the number of boxes in a *hook* which consists of the given box, those on its left side at the same row and below it at the same column.

7. Examples for $SU(4)$:

$$d_{[p,q]} = \frac{1}{12}(p+2)(p+3)(q+1)(q+2)(p-q+1),$$

and the Hook rule is shown in the Pic.**??**.

FIG. 5. Hook rule to calculate the dimension of irreps of $SO(4)$ with Yang pattern [p,q]