UNIVERSITY OF
Southampton
School of Physics
and Astronomy

Group Theory

Day 2: Continuous Groups
G3: Introduction to Lie Groups
G4: SU(N) groups
G5: SO(N) groups

Steve King, Dresden,
Germany 29th-30th August,
2016

G3: Introduction to Lie Groups

- Overview of group axioms
- Introduction to SU(N) and SO(N)
- SU(3) and subgroups
- U(1) as limit of Z_N
- U(1) is isomorphic to SO(2)
- Lie groups

Group axioms

- A group is a set of elements a,b,... which can be combined together with ab inside the set
- $(ab)c=a(bc)$
- One element e satisfies $ae=ea=a$ for all a
- For each element a there is an element a^{-1} which satisfies $aa^{-1}=a^{-1}a=e$
- e.g. special orthogonal or unitary matrices form groups under matrix multiplication

Orthogonal (real) matrix O(N)

$$O^T O = I$$

NxN

Unitary (complex) matrix U(N)

$$U^\dagger U = I$$

implies $U^\dagger = U^{-1}$ inverse

where $U^\dagger = (U^*)^T$

SU(N) and SO(N) form groups

SU(N) = ***S***pecial ***U***nitary NxN matrices

Unit determinant

$$\det U = 1$$

Unitary

Unit matrix

$$U^\dagger U = I$$

SO(N) = ***S***pecial ***O***rthogonal NxN matrices

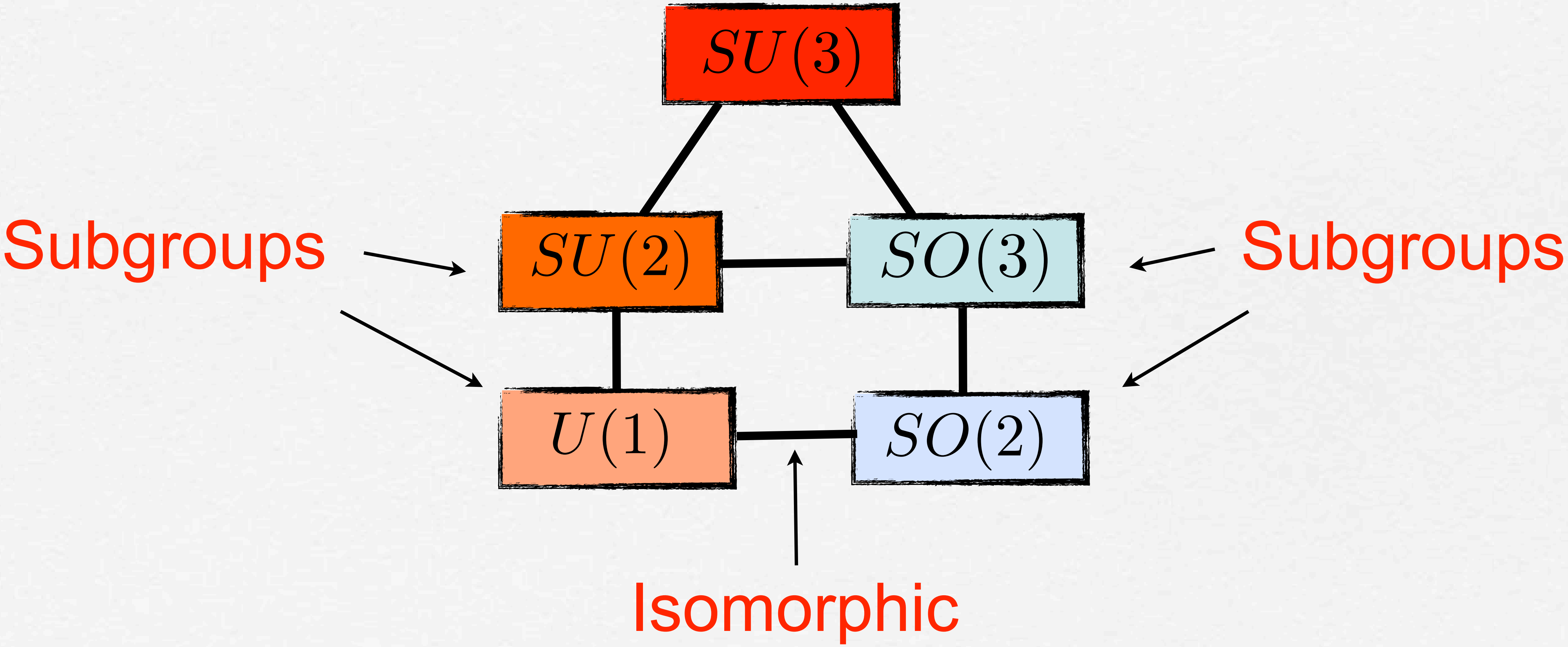
Unit determinant

$$\det O = 1$$

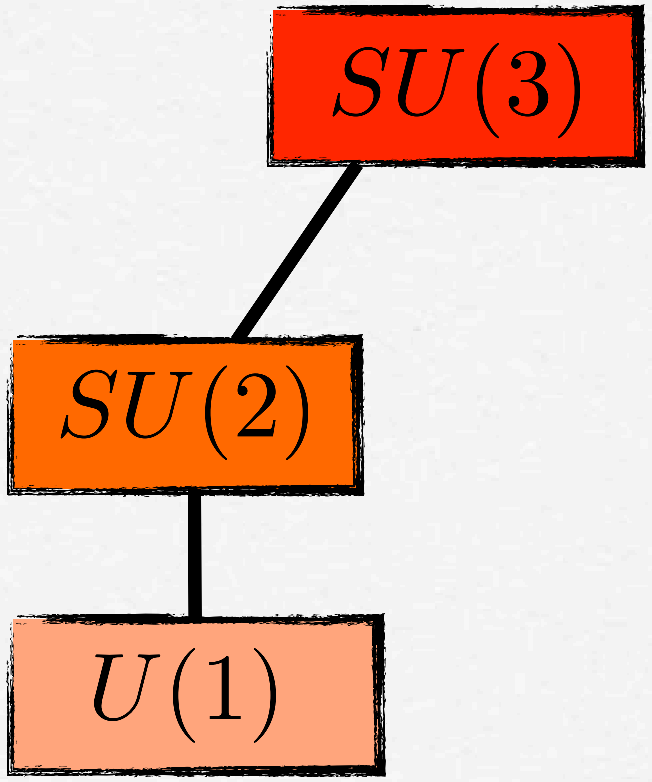
Orthogonal

$$O^T O = I$$

SU(3) and a few of its subgroups



SU(3) and a few of its subgroups



Unitary 3x3
matrices with
unit determinant

Unitary 2x2
matrices with
unit determinant

Unitary 1x1
matrices are
complex numbers

Special

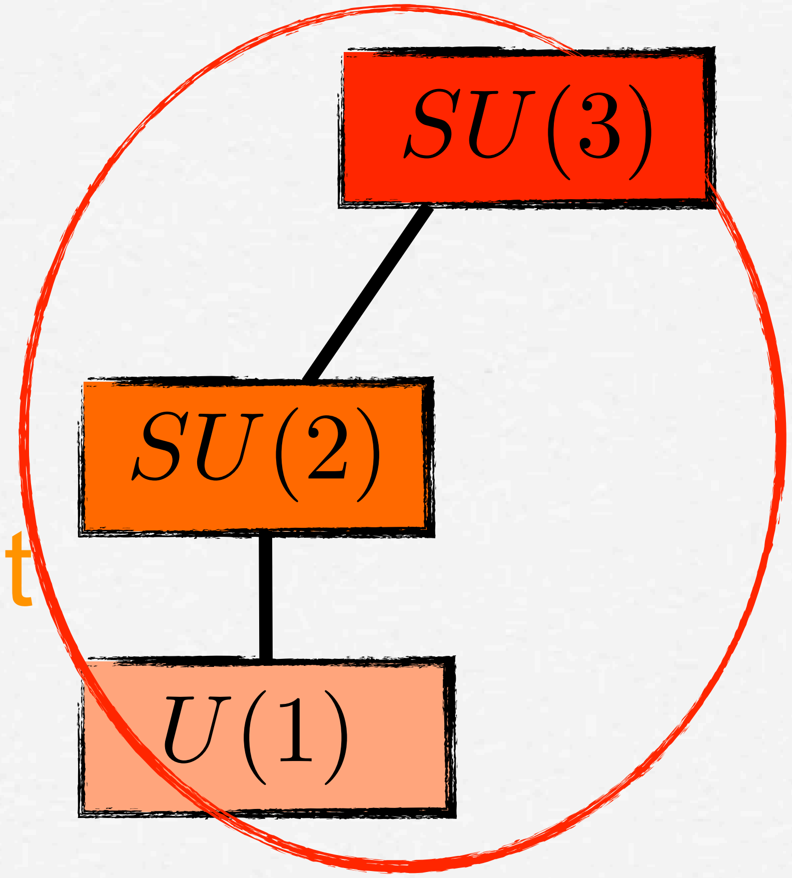
$$\det U = 1$$

$$U^\dagger U = I$$

Unitary

$$e^{i\theta}, \qquad \det e^{i\theta} = e^{i\theta} \neq 1$$ so U(1) not SU(1)

SU(3) and a few of its subgroups



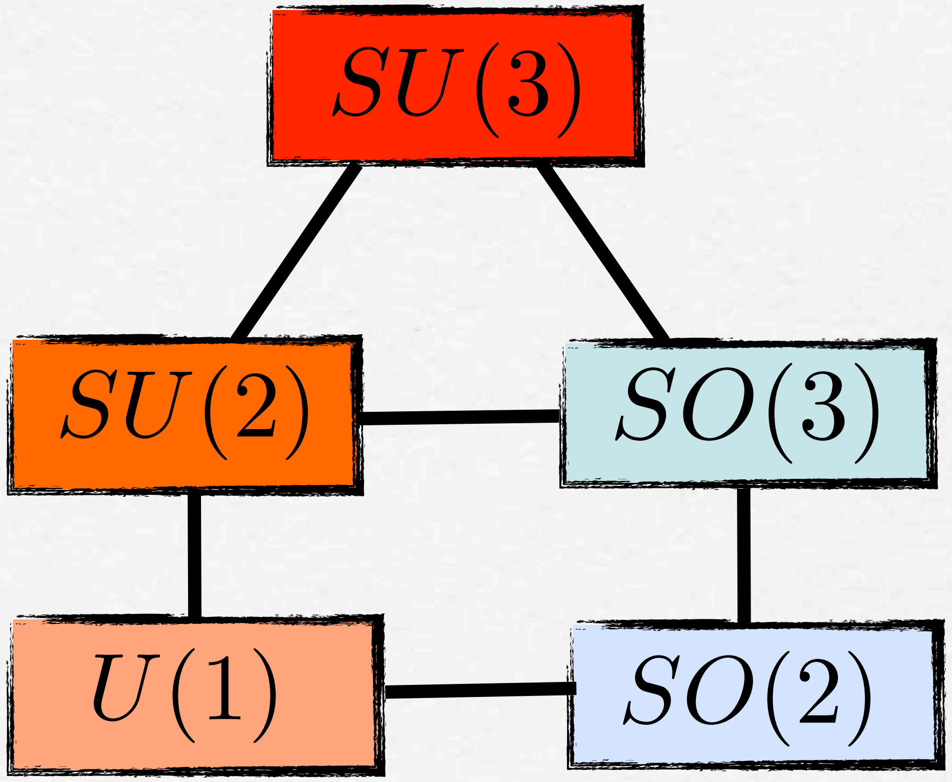
Unitary 3x3
matrices with
unit determinant

Unitary 2x2
matrices with
unit determinant

Unitary 1x1
matrix

Standard
Model

SU(3) and a few of its subgroups



$SU(3)$

$SU(2)$ $SO(3)$

$U(1)$ $SO(2)$

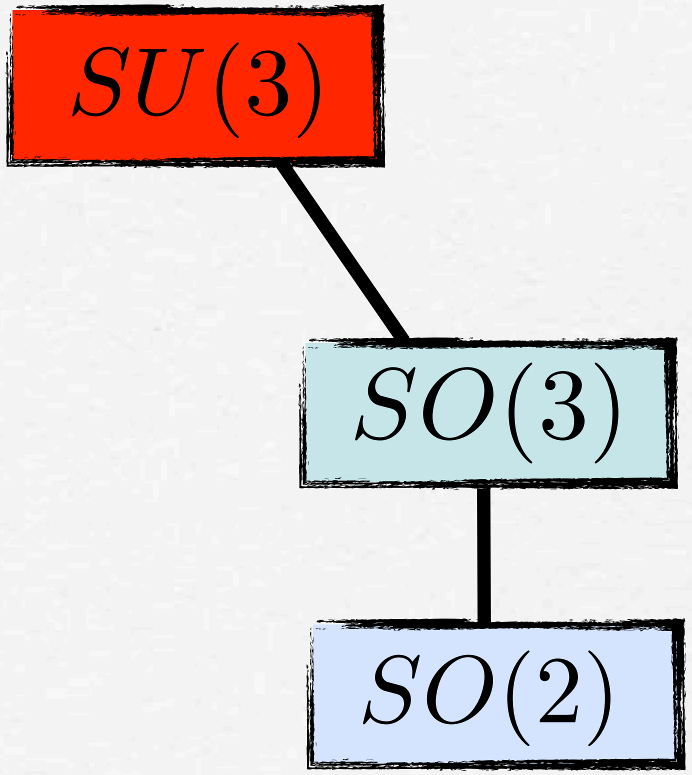
SU(3) and a few of its subgroups

Special

$$\det O = 1$$

$$O^T O = I$$

Orthogonal

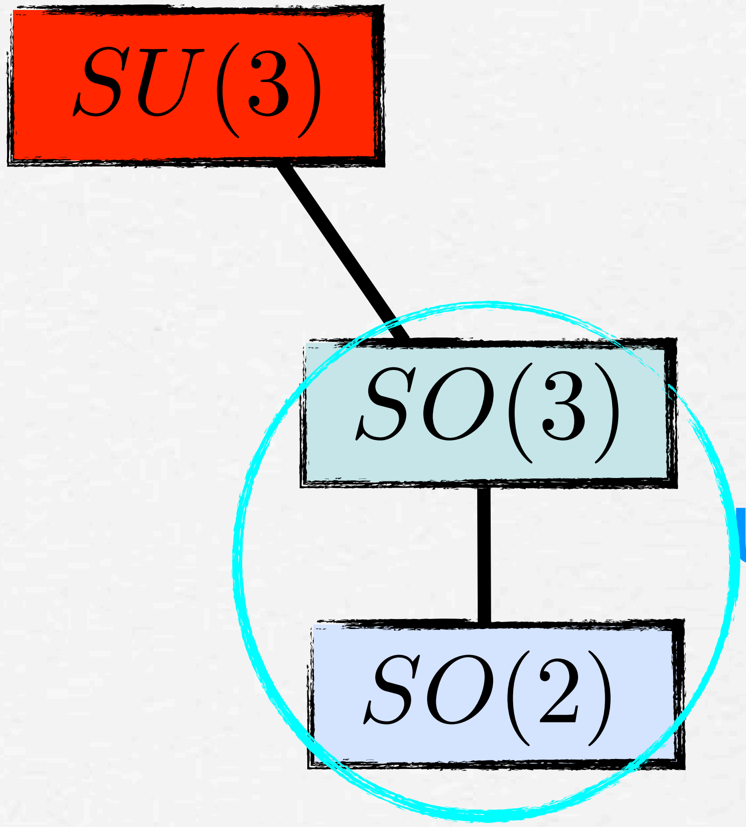


Orthogonal 3x3 matrices with unit determinant

Orthogonal 2x2 matrices with unit determinant

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

SU(3) and a few of its subgroups



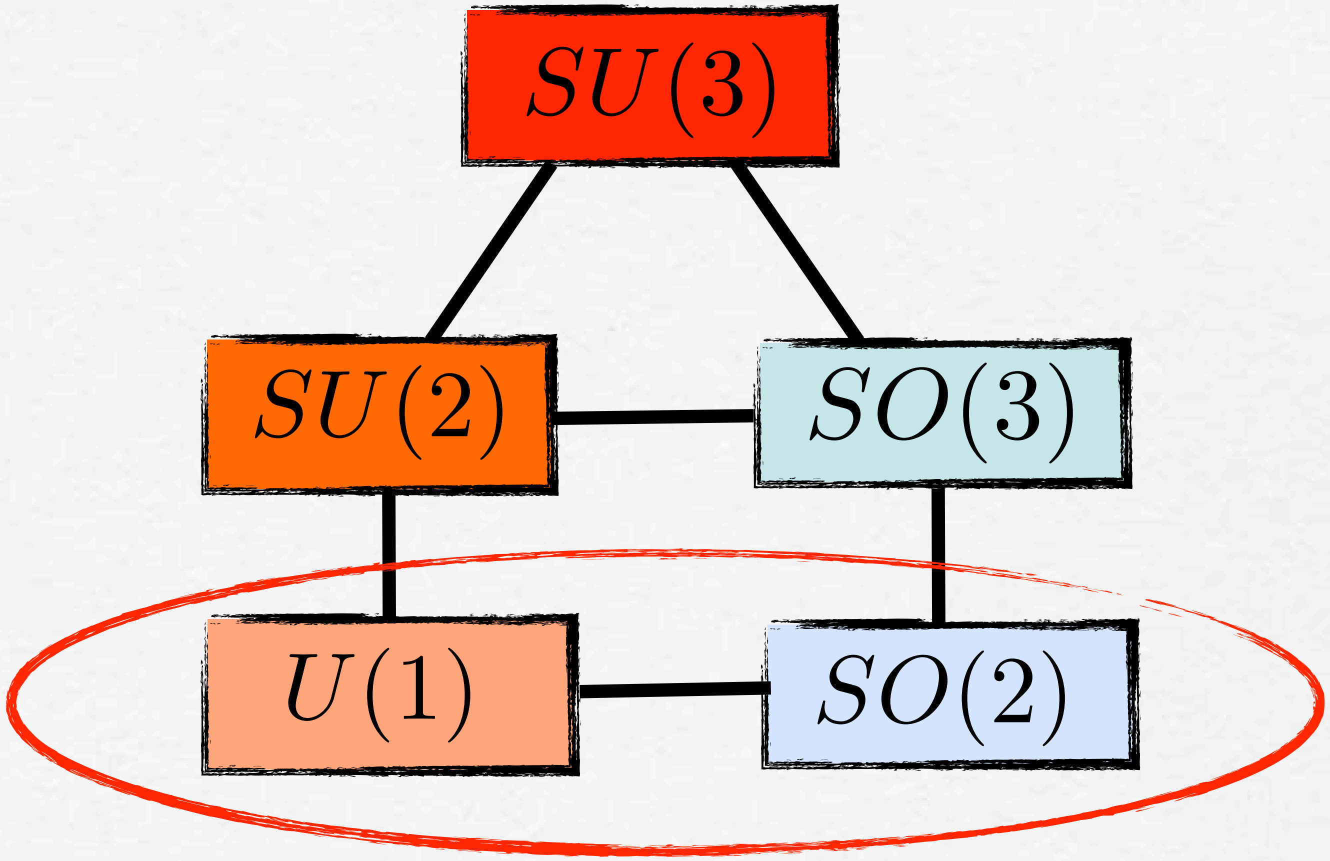
Rotation Groups

$$O^T O = I$$

Orthogonal 3x3 matrices with unit determinant

Orthogonal 2x2 matrices with unit determinant

SU(3) and a few of its subgroups



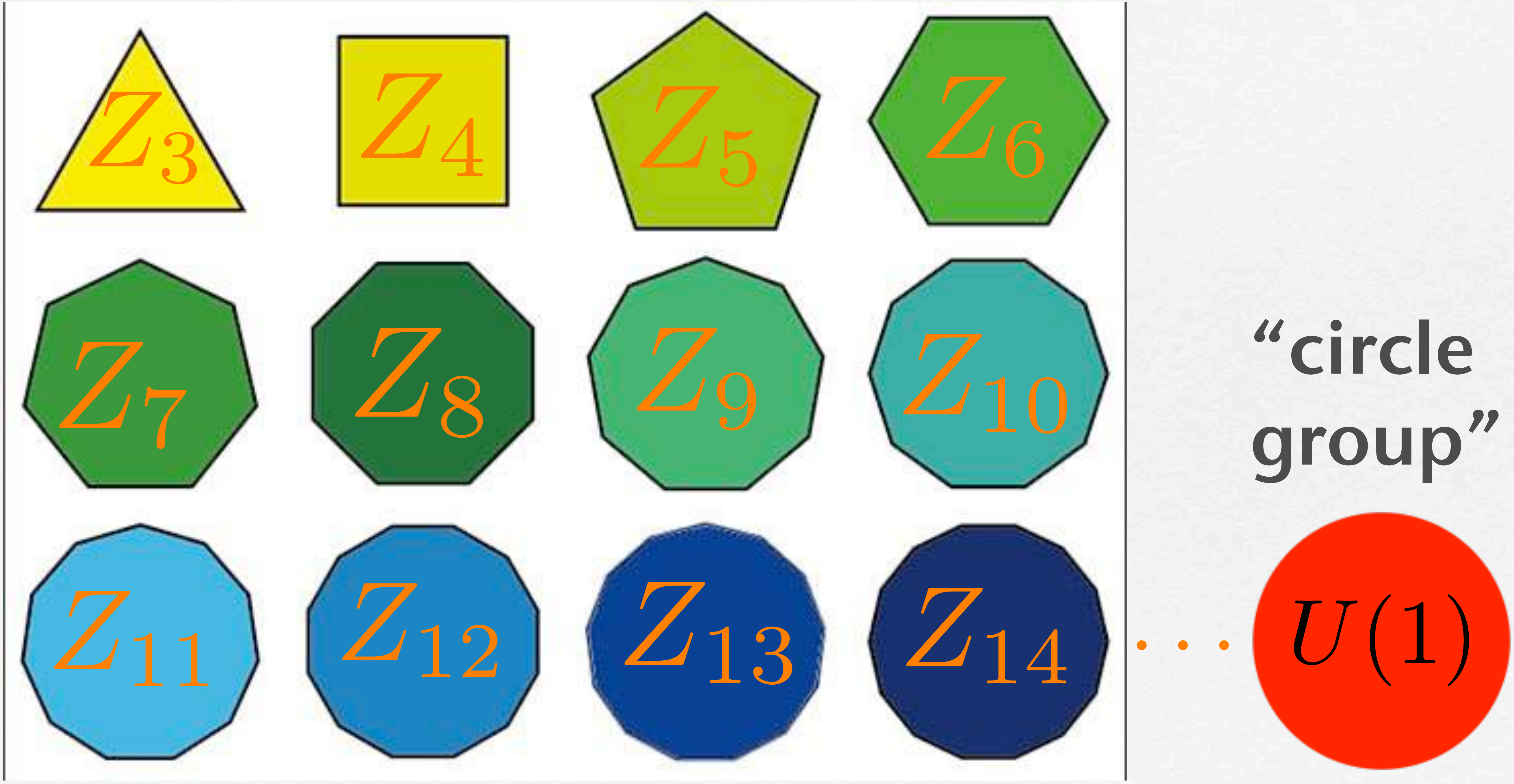
We start with U(1) and SO(2) where U(1) is limiting case of Z_N

Z_N, rotation group of regular N-polygon

- Z_4 is square, Z_5 is pentagon, Z_6 hexagon, etc.
- Z_N generators given by 2pi/N rotation
- Order = N group elements $\{e,a,a^2,...,a^{N-1}\}$
- We write e.g. a=ρ where

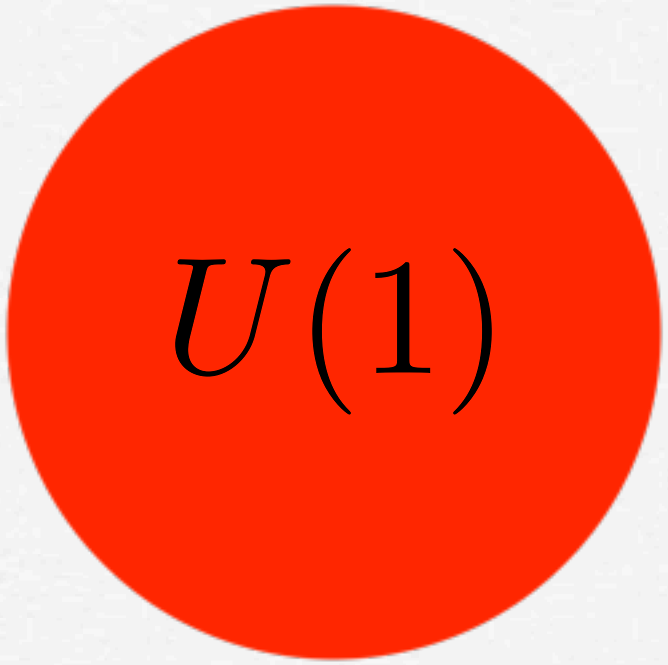
$$\rho = e^{i2\pi/N}, \quad \rho^N = 1$$

Now take limit: $N \to \infty$



“circle group”

In the limit N→∞ discrete group Z_N becomes continuous group U(1) parameterised by the real angle θ



“circle group”

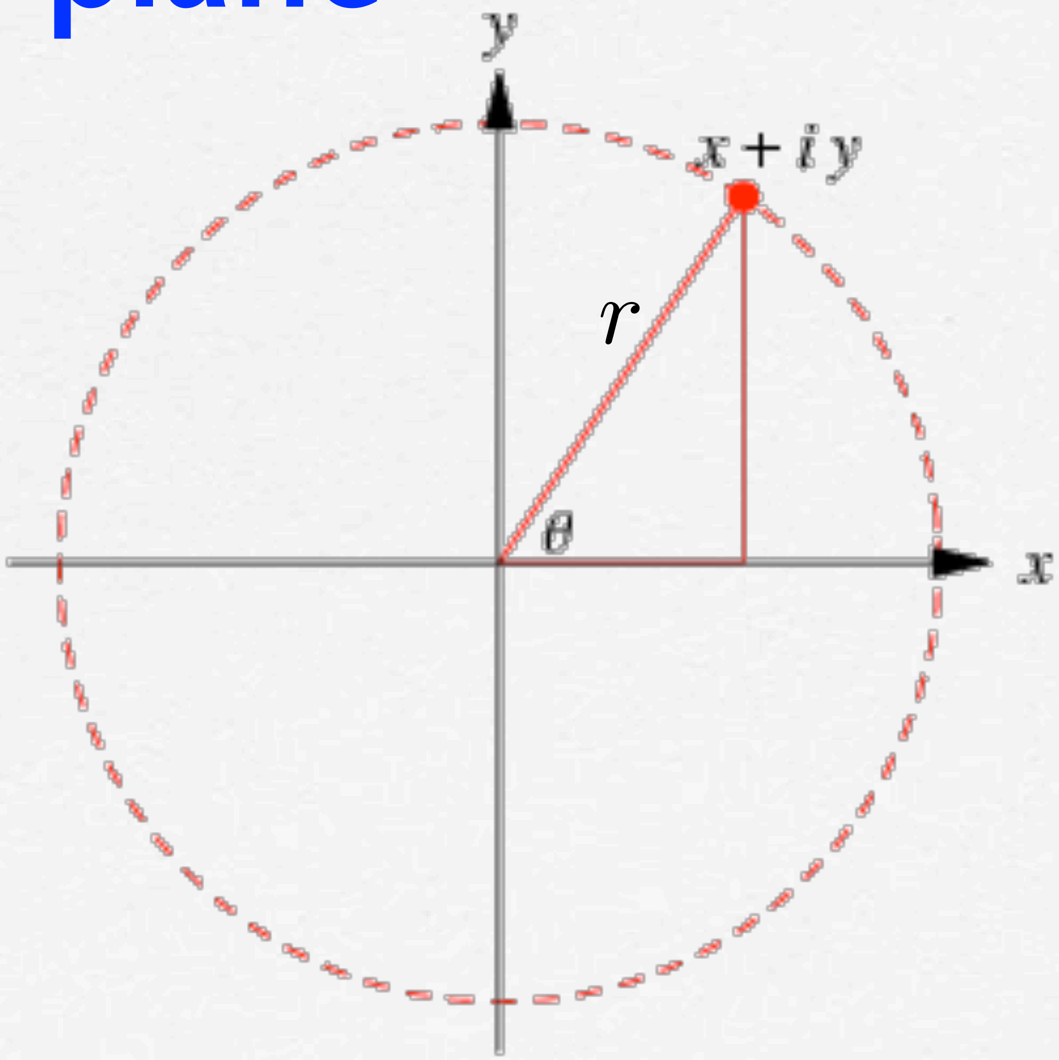
group element

$$\rho^n = e^{i2\pi n/N} \xrightarrow{N \to \infty} e^{i\theta}, \qquad \theta \equiv 2\pi n/N$$

U(1) is isomorphic to SO(2)

Argand plane

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$



U(1) transformation

$$z \to e^{i\theta} z$$

equivalent to rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

SO(2) = orthogonal 2x2 matrices with det = 1

Lie Groups

A Lie group is a group whose elements are labelled by a set of continuous parameters with a multiplication law that depends smoothly on the parameters

For Lie groups U(1) or SO(2) the continuous Lie parameter is just angle θ
The Lie group is compact since θ=[0..2π]

Quantum Mechanics

Physical states represented by state vectors,

$$|v\rangle$$

Physical transformations on physical states represented by Unitary operators,

$$U|v\rangle = |v'\rangle$$

Unitary operators as matrices

Consider U acting on some orthonormal basis vectors,

$$|i\rangle, |j\rangle, \cdots$$

Then U may be represented by the Unitary matrix

$$U_{ij} = \langle i|U|j\rangle$$

Lie Groups

Any representation of compact Lie group is equivalent to a representation by Unitary operators U

So Lie groups correspond to unitary transformations in quantum mechanics

Lie group ⇔ U ⇔ quantum mechanics

Lie Groups

Any group element which can be obtained from the identity by continuous changes in parameters can be written as:

$$U = e^{i\alpha_a X_a} = e^{i(\alpha_1 X_1 + \cdots \alpha_N X_N)}$$

where α_a are real Lie parameters, and X_a are linearly independent Hermitian operators.

Lie Groups

For infinitesimal transformations

$$U = e^{i\alpha_a X_a} \approx 1 + i\alpha_a X_a \longleftarrow$$

generators of the Lie group

Their commutation relations determine the full structure of the group

$$[X_a, X_b] = i f_{abc} X_c$$

"Lie algebra"

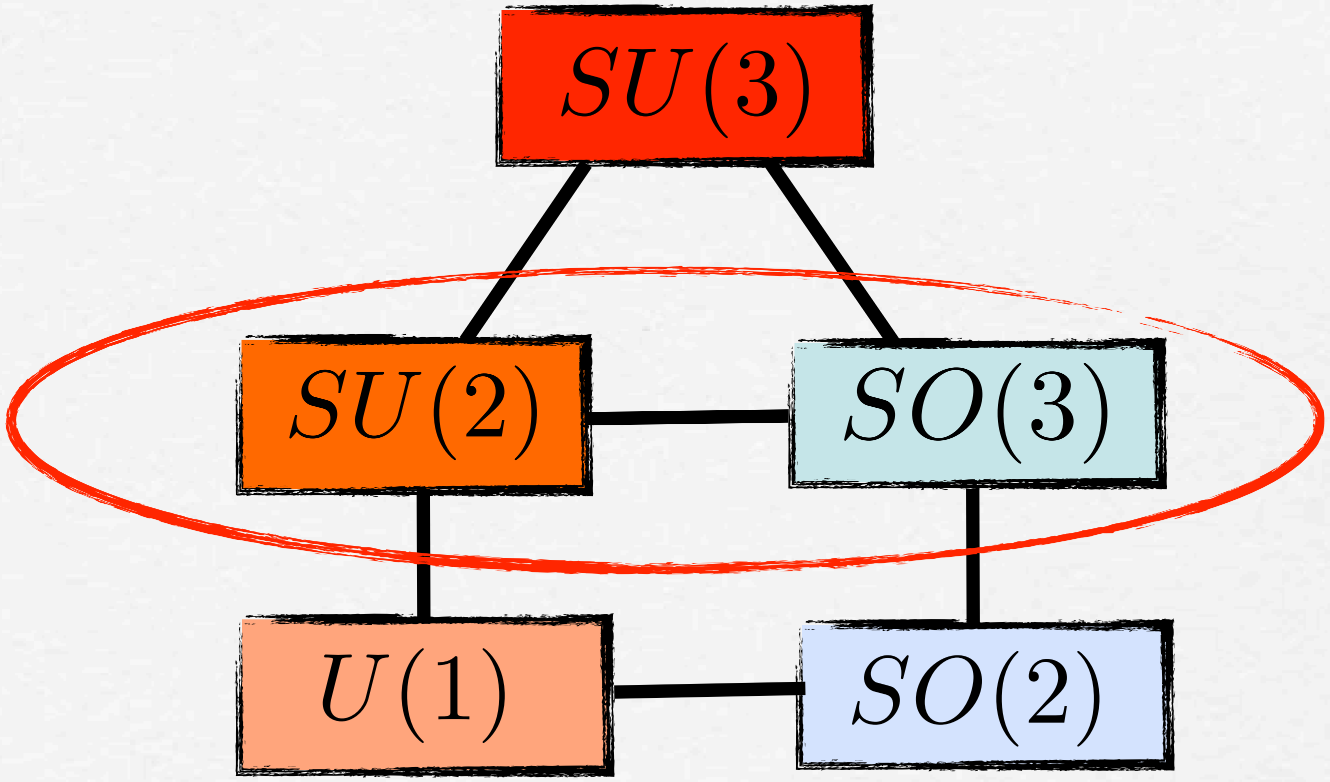
$$\uparrow$$

"structure constants"

G4: SU(N) groups

- SU(2) and angular momentum
- U(2), subalgebras, simple groups
- SU(2)xSU(2) as a semi-simple group
- SU(2) representations
- SU(2) ~ SO(3)
- U(3) and its subgroups
- SU(3)
- SU(N)

SU(3) and a few of its subgroups



We now consider SU(2) ~ SO(3) as examples of Lie groups

SU(2)

Lie algebra is just algebra of the angular momentum operators J_1, J_2, J_3

$$U = e^{i\theta_a J_a} \quad [J_a, J_b] = i\varepsilon_{abc} J_c$$

SU(2) group element

SU(2) generator

totally antisymmetric Levi-Civita tensor

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

Angular momentum eigenstates

$$J_3|j,m\rangle = m|j,m\rangle \qquad (J_a J^a)|j,m\rangle = j(j+1)|j,m\rangle$$

give matrix representation of Lie algebra

$$\langle j,m'|J^a|j,m\rangle$$

e.g. spin 1/2 $j = 1/2, \quad m, m' = \pm 1/2$

$$\langle m'|J^a|m\rangle = \langle \pm|J^a|\pm\rangle = \frac{1}{2}\sigma^a$$

Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{array}{c} \begin{array}{cc} |+\rangle & |-\rangle \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$U(2) \sim SU(2) \otimes U(1)$

In the fundamental representation the generators of $U(2)$ group can be written as

$U(1)$

$SU(2)$

$$T^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T^a = \frac{1}{2}\sigma^a, \qquad a = 1,\, 2,\, 3$$

$$Tr(T^a T^b) = \frac{1}{2}\delta^{ab}$$

where σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Generators T^a form a subalgebra of $U(2)$ because

$$\left[T^a, T^b\right] = i\varepsilon_{abc}T^c\,, \qquad a,\, b,\, c = 1,\, 2,\, 3\,.$$

The set of generators T^1, T^2, T^3 represents the Lie algebra of $SU(2)$ which elements satisfy the conditions

$$UU^\dagger = 1\,, \qquad det\,U = 1\,.$$

Thus in the fundamental representation the elements of $SU(2)$ group are Special ($det\,U = 1$), Unitary, 2×2 matrices.

An invariant subalgebra is a set of generators, X^a, which when commuted with any of the generators of the Lie group either gives zero or another generator in the set, X^a.

- In the $U(2)$ group T^a and T^0 form two invariant subalgebras corresponding to $SU(2)$ and $U(1)$ groups.

Groups which do not possess invariant subalgebras are called simple groups.

- $SU(2)$ is an example of a simple group while $U(2)$ is not simple.

Groups that do not possess an Abelian invariant subalgebra are called semi–simple Lie groups.

SU(2)xSU(2): semi-simple group

Group $SU(2) \otimes SU(2)$ has six generators which in the fundamental representation can be written in the block diagonal form

$$T^a = \frac{1}{2}\begin{pmatrix} \sigma^a & 0 \\ 0 & 0 \end{pmatrix}, \qquad T^{b+3} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma^b \end{pmatrix}, \qquad a,\, b = 1,\, 2,\, 3.$$

The first three generators of this group form an invariant $SU(2)$ subalgebra.

Therefore $SU(2) \otimes SU(2)$ group is not simple.

SU(2) algebra representations

spin 1 representation $J_3|j,m\rangle = m|j,m\rangle$

$$j = 1, \quad m, m' = +1, 0, -1$$

$$\langle m'|J^a|m\rangle = T^a$$

matrix representation of the algebra

$$T^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, T^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$ normalisation of generators

SU(2) algebra representations

Adjoint rep of algebra is defined as

$$(T^a)_{bc} = -i\varepsilon_{abc}$$

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

equivalent to spin 1 rep

$$T^a \to W T^a W^{-1}$$

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

SU(2) is rotation group in QM

In QM the action of rotating a spin j particle through angle θ_3 about 3-axis is given by

$$|j\rangle \to R_3(\theta_3)|j\rangle = e^{i\theta_3 J_3}|j\rangle$$

In general a rotation through angle θ about unit axis $\boldsymbol{n}=n_1\boldsymbol{i}+n_2\boldsymbol{j}+n_3\boldsymbol{k}$ is given by

$$|j\rangle \to R_{\mathbf{n}}(\theta)|j\rangle = e^{i\theta\mathbf{J}.\mathbf{n}}|j\rangle = \boxed{e^{i\theta_a J^a}}|j\rangle$$

SU(2) group element

where $\theta_1 = \theta n_1$, $\theta_2 = \theta n_2$, $\theta_3 = \theta n_3$

SU(2) group representations

spin 1/2 representation of group

$$U_{ij} = [R_{\mathbf{n}}(\theta)]_{ij} = e^{i\theta_a \frac{1}{2}\sigma^a_{ij}}$$

$\nearrow$

Special Unitary 2x2 matrices with unit determinant:

Proof

$$\det U = e^{Tr(i\theta_a \frac{1}{2}\sigma^a)} = e^0 = 1$$

$$U^\dagger U = e^{-i\theta_a J^a} e^{i\theta_b J^b} = I$$

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

Baker-Campbell-Hausdorff (BCH)

SU(2) group representations

spin 1/2 representation of group

$$U_{ij} = [R_{\mathbf{n}}(\theta)]_{ij} = e^{i\theta_a \frac{1}{2}\sigma^a_{ij}}$$

For rotations about the 2-axis

$$[R_{\mathbf{2}}(\theta)]_{ij} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

For rotations about the 3-axis

$$[R_{\mathbf{3}}(\theta)]_{ij} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$$

These are subgroups: SU(2) is the group of rotations about all axes

SU(2) ~ SO(3)

spin 1 adjoint representation of group $O_{ij} = [R_{\mathbf{n}}(\theta)]_{ij} = e^{i\theta_a T^a_{ij}}$

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

O_{ij} are special orthogonal 3x3 matrices, real with unit determinant, i.e. SO(3)

$$O^T O = e^{i\theta_a (T^a)^T} e^{i\theta_b T^b} = e^{-i\theta_a T^a} e^{i\theta_b T^b} = I$$

SU(2) ~ SO(3)

spin 1 adjoint representation of group

$$O_{ij} = [R_{\mathbf{n}}(\theta)]_{ij} = e^{i\theta_a T^a_{ij}}$$

$$R_{\mathbf{1}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad R_{\mathbf{2}}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_{\mathbf{3}}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Ex. check:

$$R_{\mathbf{3}}(\theta) = e^{i\theta T_3} \to T_3 = \frac{1}{i}\frac{dR_{\mathbf{3}}(\theta)}{d\theta}|_{\theta=0}$$

special orthogonal 3x3

SU(2) ~ SO(3)

spin 1 adjoint representation of group

$$O_{ij} = [R_{\mathbf{n}}(\theta)]_{ij} = e^{i\theta_a T^a_{ij}}$$

$$R_{\mathbf{1}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad R_{\mathbf{2}}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_{\mathbf{3}}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

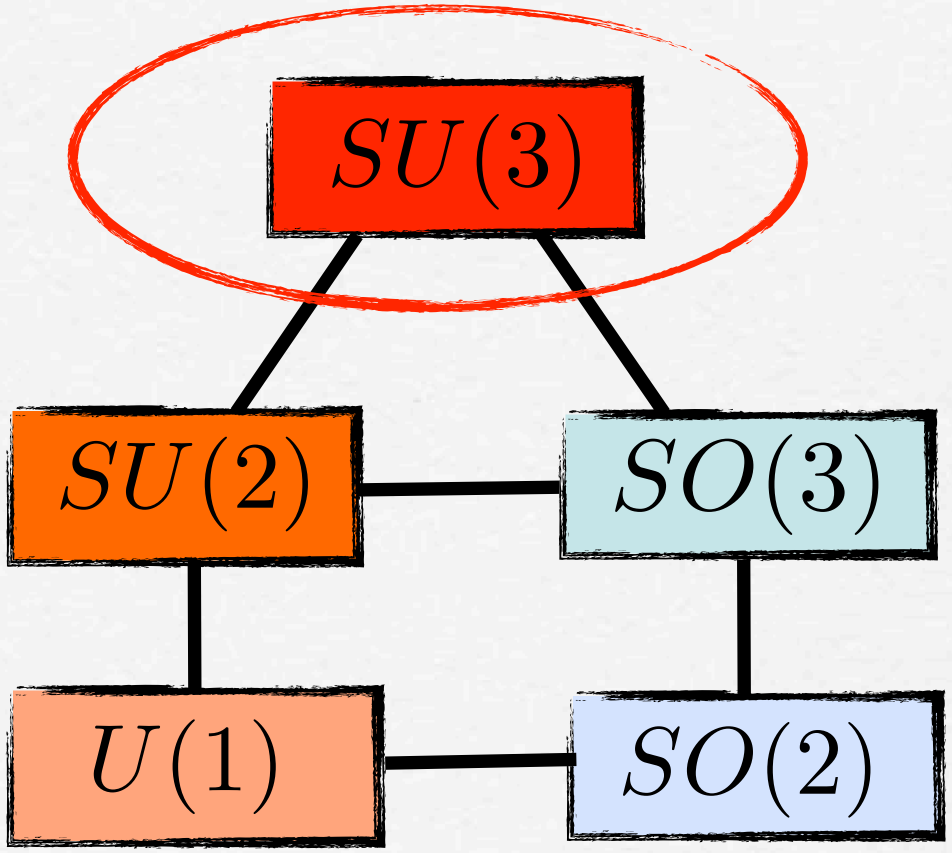
Ex. using BCH show:

$$R_{\mathbf{1}}(\theta_{23})R_{\mathbf{2}}(\theta_{13})R_{\mathbf{3}}(\theta_{12}) = e^{i\theta_a T^a_{ij}}$$

special orthogonal 3x3

where: $R_{\mathbf{1}}(\theta_{23}) = e^{i\theta_{23} T^1_{ij}}$

SU(3) and a few of its subgroups



We first consider SU(3) then SU(N) and SO(N)

$U(3) \sim SU(3) \otimes U(1)$

- In the fundamental representation the elements of the $U(3)$ group are 3×3 unitary matrices, i.e.

$$UU^{\dagger} = 1, \quad \Longrightarrow \quad U = \exp\left\{ i\omega^{\alpha} T^{\alpha} \right\}, \qquad T^{\alpha} = T^{\alpha\dagger} \, ,$$

$$T^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$U(3) \sim SU(3) \otimes U(1)$

- In the fundamental representation the elements of the $U(3)$ group are 3×3 unitary matrices, i.e.

$$UU^\dagger = 1, \quad \Longrightarrow \quad U = \exp\left\{ i\omega^\alpha T^\alpha \right\}, \qquad T^\alpha = T^{\alpha\dagger}\,,$$

$U(1)$

$$T^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$U(3) \sim SU(3) \otimes U(1)$

- In the fundamental representation the elements of the $U(3)$ group are 3×3 unitary matrices, i.e.

$$UU^\dagger = 1, \quad \Longrightarrow \quad U = \exp\left\{ i\omega^\alpha T^\alpha \right\}, \qquad T^\alpha = T^{\alpha\dagger}\,,$$

$SU(2)$

$$T^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$U(3) \sim SU(3) \otimes U(1)$

- In the fundamental representation the elements of the $U(3)$ group are 3×3 unitary matrices, i.e.

$$UU^\dagger = 1, \quad \Longrightarrow \quad U = \exp\left\{ i\omega^\alpha T^\alpha \right\}, \qquad T^\alpha = T^{\alpha\dagger}\,,$$

$SU(3)$

$$T^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

SU(3)

The set of generators T^a, where $a = 1, ...8$, form invariant subalgebra of $U(3)$ that corresponds to $SU(3)$

$\mathbf{3}$ $$\left[T^a, T^b\right] = i f_{abc} T^c$$

$\overline{\mathbf{3}}$ $(-T^a)^*$ also satisfy the algebra

Other reps include $\mathbf{1}, \mathbf{3}, \mathbf{6}, \mathbf{8}, \mathbf{15}, \ldots$

SU(N)

The elements of $SU(N)$ group obey the relations

$$UU^\dagger = 1\,, \qquad \det U = 1\,.$$

$SU(N-1)$,... , $SU(2)$ are subgroups of $SU(N)$.

But $SU(N)$ does not possess invariant subalgebras, i.e. $SU(N)$ is a simple group.

The quadratic Casimir operator $\sum(T^a)^2$ commutes with all generators of $SU(N)$ group.

The Cartan subalgebra of $SU(N)$ group involves $N-1$ traceless diagonal matrices

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \dots & \\ & & & & 0 \end{pmatrix}, \quad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & \dots & \\ & & & & 0 \end{pmatrix}, \dots$$

$$\dots H_{N-1} = \frac{1}{\sqrt{2N(N-1)}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \dots & \\ & & & & -(N-1) \end{pmatrix}.$$

The Cartan subalgebra of $SU(N)$ group involves $N-1$ traceless diagonal matrices

T_3 T_8

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \dots & \\ & & & & 0 \end{pmatrix}, \quad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & \dots & \\ & & & & 0 \end{pmatrix}, \dots$$

$SU(3)$ rank 2

$$\dots H_{N-1} = \frac{1}{\sqrt{2N(N-1)}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \dots & \\ & & & & -(N-1) \end{pmatrix}.$$

G5: SO(N) groups

- SO(N) and Clifford algebra
- SO(3) vector rep
- SO(2N+1) spinor rep
- SO(3) spinor rep
- SO(5) spinor rep
- SO(2N) vector and spinor reps
- SO(6) spinor rep
- SO(6) ~ SU(4) and SU(3) subgroup

SO(N) groups and Clifford algebra

$SO(N)$ is the group of rotations in N dimensions.

This group has $\frac{1}{2}(N^2 - N)$ generators $M_{ab} = -M_{ba}$, which represent rotations in the $a - b$ plane, i.e.

$$\Big(M_{ab}\Big)_{kl} = i\Big(\delta_{al}\delta_{bk} - \delta_{ak}\delta_{bl}\Big), \quad a, b, k, l = 1, \cdots, N$$

The generators of $SO(N)$ group obey algebra

$$\Big[M_{ab},\, M_{cd}\Big] = -i\Big(\delta_{bc}M_{ad} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac} + \delta_{ad}M_{bc}\Big)$$

SO(3) vector rep

e.g. SO(3) identify $T_1=M_{23}$, $T_2=M_{13}$, $T_3=M_{12}$

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, T^3 = \overset{\sigma_2}{\begin{pmatrix} \boxed{\begin{matrix} 0 & -i \\ i & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}}$$

The generators of the Cartan subalgebra may be written in 2×2 block form

$$M_{12} = \begin{pmatrix} \boxed{\sigma_2} & & & 0 \\ & 0 & & 0 \\ & & \dots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \dots \qquad M_{2N-1,2N} = \begin{pmatrix} 0 & & & 0 \\ & \dots & & 0 \\ & & \boxed{\sigma_2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

SO(2N+1) spinor rep

In order to find generators of $SO(2N+1)$ in the spinor representation we consider the Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\, I, \qquad a, b = 1, ..., (2N+1),$$

where Γ_a is a set of $(2N+1)$ matrices of size $2^N \times 2^N$.

In the spinor representation the generators of $SO(2N+1)$ group are given by

$$M_{ab} = -\frac{i}{4}\Big[\Gamma_a, \Gamma_b\Big].$$

SO(3) spinor rep $\mathbf{2}$

For the case of $SO(3)$ the matrices Γ_a are given by the three Pauli matrices

$$\{\sigma_a,\,\sigma_b\} = 2\,\delta_{ab}, \qquad M_{ab} = -\frac{i}{4}\Big[\sigma_a,\,\sigma_b\Big] = \frac{1}{2}\varepsilon_{abc}\sigma_c,$$

where $a, b, c = 1,\ 2,\ 3.$ $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2x2 dimensional spinor rep of SO(3) with generators M_{12}=-M_{21}, M_{13}=-M_{31}, M_{23}=-M_{32}

SU(2) double cover of SO(3) (same algebra and reps)

$$M_{12} = \frac{1}{2}\sigma_3 \qquad M_{13} = -\frac{1}{2}\sigma_2 \qquad M_{23} = \frac{1}{2}\sigma_1$$

SO(5) spinor rep

In the case of $SO(5)$ there are five 4×4 Γ matrices which may be written in block form as

$$\Gamma_a = \begin{pmatrix} 0 & i\sigma_a \\ -i\sigma_a & 0 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

The generators of $SO(5)$ in the spinor representation are given by

$$M_{ab} = \frac{\varepsilon_{abc}}{2} \begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{pmatrix}, \quad M_{a4} = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix},$$

$$M_{45} = \frac{1}{2} \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad M_{a5} = \frac{1}{2} \begin{pmatrix} 0 & -\sigma_a \\ -\sigma_a & 0 \end{pmatrix},$$

where a and b run from 1 to 3.

SO(5) spinor rep

Cartan generators are M_{12} and M_{34}

where

$$M_{12} = \frac{1}{2}\begin{pmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{pmatrix} \quad M_{34} = \frac{1}{2}\begin{pmatrix}\sigma_3 & 0 \\ 0 & -\sigma_3\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

in basis of two SO(3) spinors of M_{12} and M_{34}

$$|1\rangle, |2\rangle, |3\rangle, |4\rangle = \ |\underset{\overset{\uparrow}{M_{12}}}{+}\underset{\overset{\uparrow}{M_{34}}}{+}\rangle, |\underset{\overset{\uparrow}{M_{12}}}{-}\underset{\overset{\uparrow}{M_{34}}}{-}\rangle, |\underset{\overset{\uparrow}{M_{12}}}{+}\underset{\overset{\uparrow}{M_{34}}}{-}\rangle, |\underset{\overset{\uparrow}{M_{12}}}{-}\underset{\overset{\uparrow}{M_{34}}}{+}\rangle$$

$$\langle 1|M_{12}|1\rangle = +1/2 \qquad \langle 1|M_{34}|1\rangle = +1/2$$

$$\langle 2|M_{12}|2\rangle = -1/2 \qquad \langle 2|M_{34}|2\rangle = -1/2$$

$$\langle 3|M_{12}|3\rangle = +1/2 \qquad \langle 3|M_{34}|3\rangle = -1/2$$

$$\langle 4|M_{12}|4\rangle = -1/2 \qquad \langle 4|M_{34}|4\rangle = +1/2$$

SO(2N) vector rep

$$\Big(M_{ab}\Big)_{kl} = i\Big(\delta_{al}\delta_{bk} - \delta_{ak}\delta_{bl}\Big), \quad a, b, k, l = 1, \cdots, 2N$$

The Cartan subalgebra of $SO(2N)$ has N generators, $M_{12}, M_{34}, \ldots M_{2N-1,2N}$ which in $2N$ dimensional space can be written in 2×2 block form

$$M_{12} = \begin{pmatrix} \sigma_2 & & & \\ & 0 & & \\ & & \ldots & \\ & & & 0 \end{pmatrix}, \ldots \qquad M_{2N-1,2N} = \begin{pmatrix} 0 & & & \\ & \ldots & & \\ & & 0 & \\ & & & \sigma_2 \end{pmatrix}$$

SO(2N) spinor rep

The spinor representation of the generators of the $SO(2N)$ group are constructed from the $2^N \times 2^N$ Γ–matrices which satisfy the Clifford algebra so that

$$M_{ab} = -\frac{i}{4}\Big[\Gamma_a,\, \Gamma_b\Big], \qquad \{\Gamma_a,\, \Gamma_b\} = 2\delta_{ab}\, I, \qquad a, b = 1, ..., 2N.$$

The projection operators reduce 2^N spinor to the two irreducible spinors which have 2^{N-1} components

$$\Psi_L = P_L\,\Psi, \qquad \Psi_R = P_R\,\Psi. \quad P_{L,R} = \frac{1}{2}(I \pm \Gamma_{2N-1})$$

Thus the generators of $SO(2N)$ can be written as $2^{N-1} \times 2^{N-1}$ matrices.

SO(6) spinor reps $\mathbf{4}, \bar{\mathbf{4}}$

Therefore group $SO(6)$ has two four dimensional spinor representation.

The 15 generators of $SO(6)$ in the spinor representation can be presented in the following form:

$$\pm\frac{1}{2}\begin{pmatrix} 0 & i\sigma_a \\ -i\sigma_a & 0 \end{pmatrix}, \qquad \pm\frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \pm\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$$\frac{\varepsilon_{abc}}{2}\begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -\sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix},$$

where $a = 1, 2, 3$ and $\pm$ refers to the "left–handed" and "right–handed" representations. $\mathbf{4}, \bar{\mathbf{4}}$ (complex conjugates)

SO(6) spinor reps $\mathbf{4}, \overline{\mathbf{4}}$

Therefore group $SO(6)$ has two four dimensional spinor representation.

The 15 generators of $SO(6)$ in the spinor representation can be presented in the following form:

$$\Gamma_a \quad \pm\frac{1}{2}\begin{pmatrix} 0 & i\sigma_a \\ -i\sigma_a & 0 \end{pmatrix}, \qquad \pm\frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \pm\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$$M_{ab} \quad \frac{\varepsilon_{abc}}{2}\begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -\sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix},$$

where $a = 1, 2, 3$ and $\pm$ refers to the "left–handed" and "right–handed" representations. $\mathbf{4}, \overline{\mathbf{4}}$ (complex conjugates)

SO(6) spinor reps $\mathbf{4}, \overline{\mathbf{4}}$

As in SO(5), Cartan generators are M_{12}, M_{34}

where

$$M_{12} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad M_{34} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in basis of two SO(3) spinors of M_{12} and M_{34}

$$|++\rangle, |--\rangle, |+-\rangle, |-+\rangle$$

But SO(6) has further Cartan generators

$$\Gamma_5^+ = +\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma_5^- = -\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$ for $\mathbf{4}, \overline{\mathbf{4}}$

$|+\rangle, |+\rangle, |-\rangle, |-\rangle \qquad |-\rangle, |-\rangle, |+\rangle, |+\rangle$ spinors of Γ_5

SO(6) spinor rep $\mathbf{4} \oplus \overline{\mathbf{4}}$

The reducible $\mathbf{4} \oplus \overline{\mathbf{4}}$ can be written in basis of three SO(3) spinors of M_{12},M_{34} and Γ_5

$$\begin{array}{cccccccccccccccc}
|+ & + & +\rangle, & |- & - & +\rangle, & |+ & - & -\rangle, & |- & + & -\rangle & & \mathbf{4} \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \\
M_{12} & M_{34} & \Gamma_5 & M_{12} & M_{34} & \Gamma_5 & M_{12} & M_{34} & \Gamma_5 & M_{12} & M_{34} & \Gamma_5 & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
|+ & + & -\rangle, & |- & - & -\rangle, & |+ & - & +\rangle, & |- & + & +\rangle & & \overline{\mathbf{4}}
\end{array}$$

Note that the $\mathbf{4}$ has even number of - states

and that the $\overline{\mathbf{4}}$ has odd number of - states

SO(6) ~ SU(4)

$$|+++\rangle, |--+\rangle, |+--\rangle, |-+-\rangle \quad \mathbf{4}$$

identified as $\mathbf{4}$ of SU(4) with Cartan generators

$$H_1 = \frac{1}{2\sqrt{2}}(M_{12} + M_{34}) = \frac{1}{2} diag(1, -1, 0, 0)$$

$$H_2 = \frac{1}{\sqrt{12}}(-M_{12} + M_{34} + 2\Gamma_5^+) = \frac{1}{2\sqrt{3}} diag(1, 1, -2, 0)$$

$$H_3 = \frac{1}{\sqrt{6}}(M_{12} - M_{34} + \Gamma_5^+) = \frac{1}{\sqrt{24}} diag(1, 1, 1, -3)$$

SU(4) has SU(3) subgroup

$$|+++\rangle, |--+\rangle, |+--\rangle, |-+-\rangle$$

$\mathbf{3}$ states 1 state

Subgroup SU(3) involves Cartan generators H_1 and H_2 in the same basis as above

$$H_1 = \frac{1}{2\sqrt{2}}(M_{12} + M_{34}) = \frac{1}{2} diag(1, -1, 0, 0)$$

$$H_2 = \frac{1}{\sqrt{12}}(-M_{12} + M_{34} + 2\Gamma_5^+) = \frac{1}{2\sqrt{3}} diag(1, 1, -2, 0)$$