The SU(N) Heisenberg model of quantum permutations on a lattice

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Scope

- Introduction: SU(N) models in condensed matter and cold atoms
- SU(3) on triangular and square lattice
  → 3-sublattice color order
- SU(4) on square lattice
  → dimerization and Néel order
- Probing color order with cold atoms
  → multiple occupancy
- Conclusions

Quantum permutations

- Objects with N flavours on a lattice
- Hilbert space = {| $\sigma_1\ \sigma_2 \ldots \sigma_L$ >}

  σ_i = 1,2, …,N or σ_i = A, B, C,… or 

$$\mathcal{H} = \sum_{\langle i,j \rangle} P_{ij}$$

$$P_{ij} |\sigma_1 \ldots \sigma_i \ldots \sigma_j \ldots \sigma_N\rangle = |\sigma_1 \ldots \sigma_j \ldots \sigma_i \ldots \sigma_N\rangle$$

SU(N) formulation

$$H = \sum_{\langle i,j \rangle} S_m^n(i) S_n^m(j)$$

$$S_m^n |\mu\rangle = \delta_{n,\mu} |m\rangle$$

$$[S_m^n, S_k^l] = \delta_{n,k} S_m^l - \delta_{m,l} S_k^n$$

S_m^n → generators of SU(N)

At each site: fundamental N-dimensional representation

Physical realizations I

Magnetic insulators

- N=2 → spin-1/2 Heisenberg $P_{ij} = 2\vec{S}_i \cdot \vec{S}_j + 1/2$
- N=3 → S=1 biquadratic $P_{ij} = \vec{S}_i \cdot \vec{S}_j + (\vec{S}_i \cdot \vec{S}_j)^2 - 1$
- N=4 → symmetric Kugel-Khomskii model

$$H = \sum_{ij} J_{ij}\left(2\vec{s}_i \cdot \vec{s}_j + \frac{1}{2}\right)\left(2\vec{\tau}_i \cdot \vec{\tau}_j + \frac{1}{2}\right)$$

Physical realizations II

N-flavour fermions in optical lattice (^{40}K, ^{87}Sr,…)

N-flavour Hubbard model

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} \sum_{\alpha=1}^{N} (c_{i,\alpha}^{\dagger} c_{j,\alpha} + h.c.) + U \sum_{i} \sum_{(\alpha,\beta)} n_{i,\alpha} n_{i,\beta}$$

1/N filling ↓ (1 fermion per site)

$$\mathcal{H} = \frac{2t^2}{U} \sum_{\langle i,j \rangle} P_{ij}$$

General properties

- Soluble in 1D with Bethe Ansatz
  → algebraic correlations with periodicity $2\pi/N$
  Sutherland, 1974

- Equivalent of SU(2) dimer singlet: N sites

  $| S > = (1/\sqrt{N!}) \sum_P (-1)^P | \sigma_{P(1)} \sigma_{P(2)} \dots \sigma_{P(N)} >$

  with $\{\sigma_1 \sigma_2 \dots \sigma_N\}=\{1\ 2 \dots N\}$

  Li, Ma, Shi, Zhang, PRL'98

Hartree approximation

$$|\psi\rangle = \prod_i |\varphi_i\rangle$$

$$\langle\varphi_1\varphi_2|P_{12}|\varphi_1\varphi_2\rangle = \langle\varphi_1\varphi_2|\varphi_2\varphi_1\rangle = |\langle\varphi_1|\varphi_2\rangle|^2$$

→ on 2 sites, energy minimal if $\langle\varphi_1|\varphi_2\rangle = 0$

→ on a lattice, Hartree energy minimal as soon as colors on nearest neighbors are different

NB: For SU(2), Hartree ⇔ classical

→ fundamental representation: S=1/2

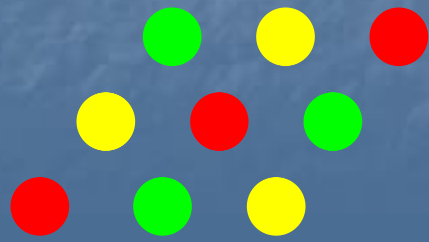
→ S=1/2: all states are magnetic

SU(3) on triangular lattice

- Unique ‘classical’ (Hartree) state
  → 3-sublattice covering of triangular lattice
  → The equivalent of Néel on square lattice
- Schwinger bosons → Flavour wave theory

$$\mathcal{P}_{ij} = \sum_{\mu,\nu \in \{A,B,C\}} a^{\dagger}_{\mu,i} a^{\dagger}_{\nu,j} a_{\nu,i} a_{\mu,j}$$

$$\tilde{a}^{\dagger}_{A,i}, \tilde{a}_{A,i} \rightarrow \sqrt{M - \tilde{a}^{\dagger}_{B,i}\tilde{a}_{B,i} - \tilde{a}^{\dagger}_{C,i}\tilde{a}_{C,i}}$$



3-sublattice order stable

Tsunetsugu, Arikawa, JPSJ 2006
A. Läuchli, FM, K. Penc, PRL 2006

SU(3) on square lattice

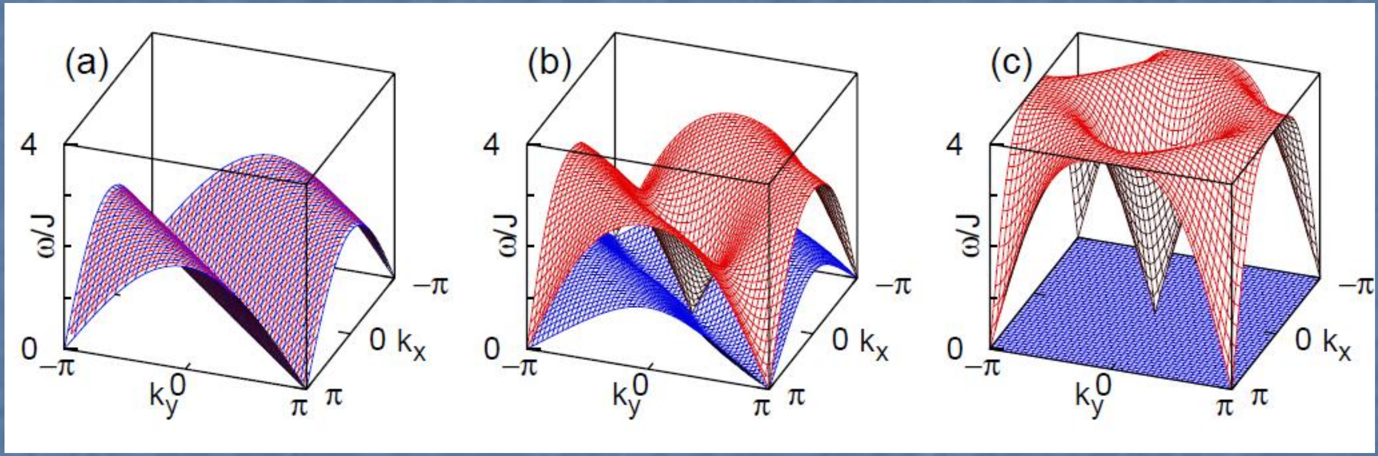
- Infinite number of 'Hartree' ground states

A B A B
B A B A   A or B → C at any site
A B A B

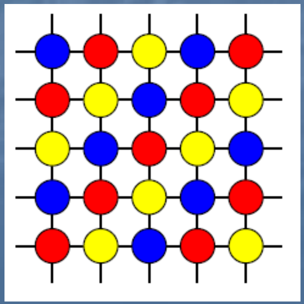
- Quantum fluctuations: order by disorder?
  → Flavour-wave theory
  → Zero-point energy

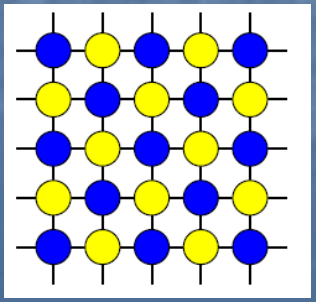
T. Toth, A. Läuchli, FM, K. Penc, PRL 2010

Flavour wave spectrum



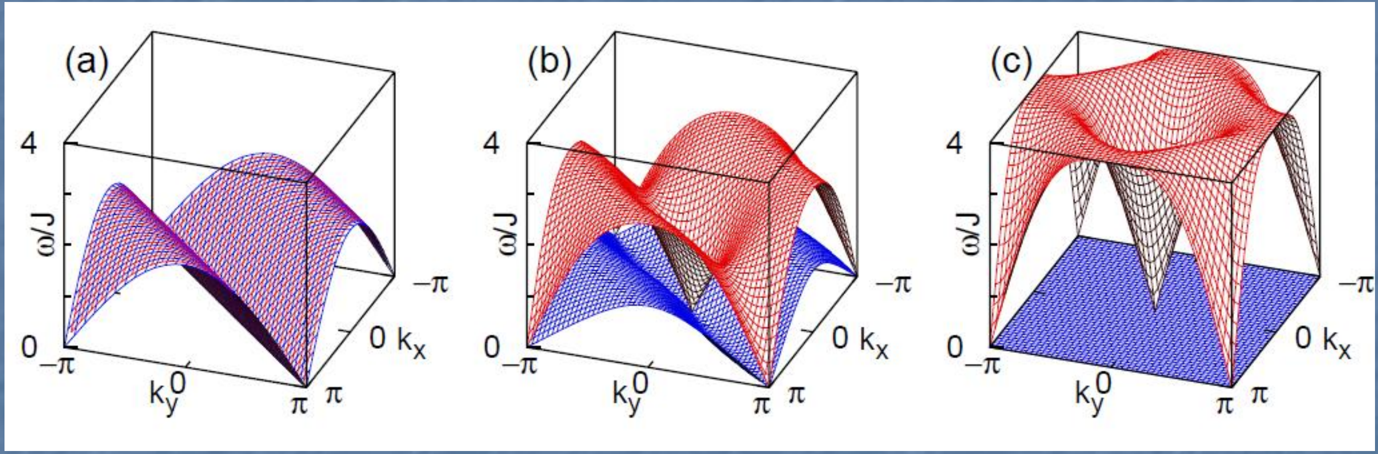
3-sublattice helical 2-sublattice





Order by disorder

3-sublattice helical 2-sublattice



- Quantum fluctuations:
  minimize $\sum\omega_q$ → 3-sublattice order
- Thermal fluctuations:
  maximize # zero modes → 2-sublattice order

Flavour-wave theory

$$\mathcal{H} = \sum_{\substack{\alpha,\beta = A,B,C\ldots \\ \alpha \neq \beta}} \mathcal{H}_{\alpha\beta}$$

$$\mathcal{H}_{\alpha\beta} = \sum_{\substack{\text{disconnected} \\ \text{clusters } \mathcal{C}}} \sum_{\substack{\langle i,j\rangle \in \mathcal{C} \\ i \in \alpha, j \in \beta}} \mathcal{H}_{\alpha\beta}(i,j)$$

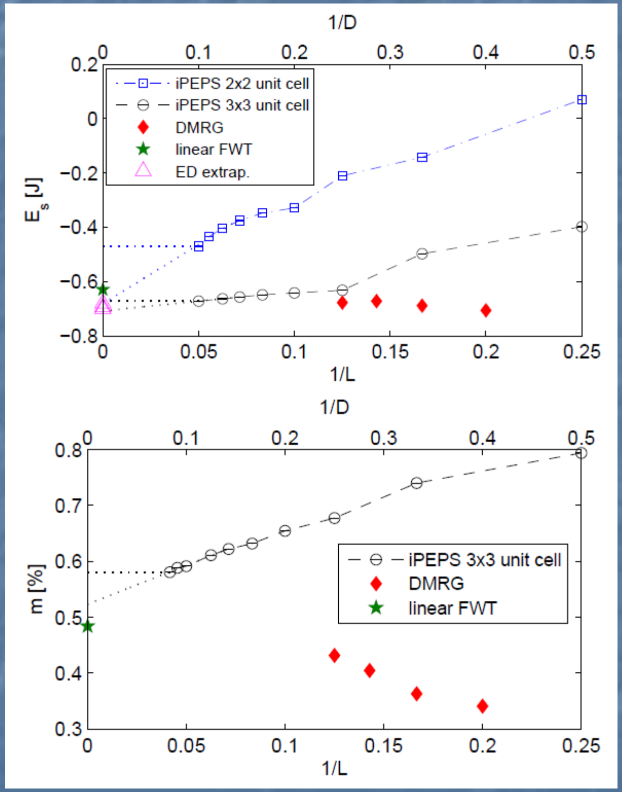
$$\mathcal{H}_{\alpha\beta}(i,j) = (\alpha_i^\dagger + \beta_j)(\alpha_i + \beta_j^\dagger) - 1$$

$$\langle(\alpha_i^\dagger + \beta_j)(\alpha_i + \beta_j^\dagger)\rangle \geq 0 \quad \Rightarrow \quad \langle\mathcal{H}_{\alpha\beta}(i,j)\rangle \geq -1$$

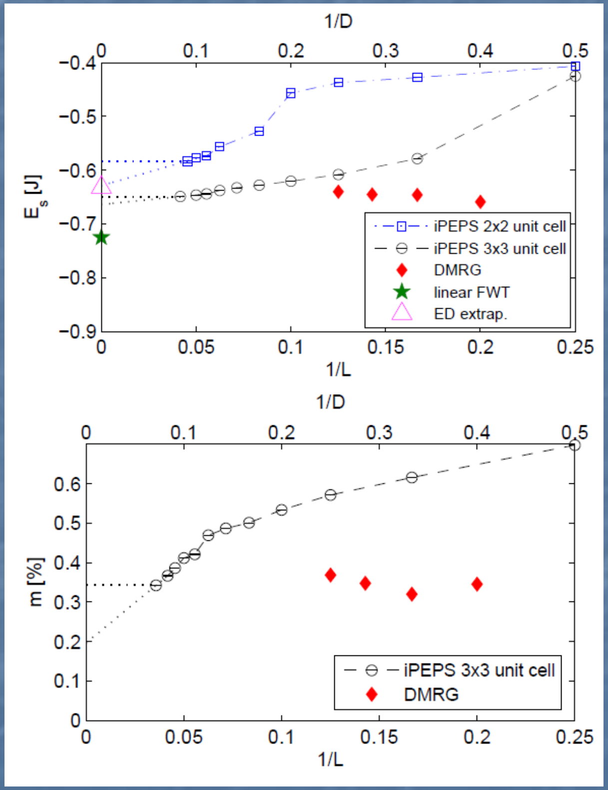
- Lower bound saturated for two sites
- Make clusters as small as possible

DMRG and iPEPS for SU(3)

Triangular lattice

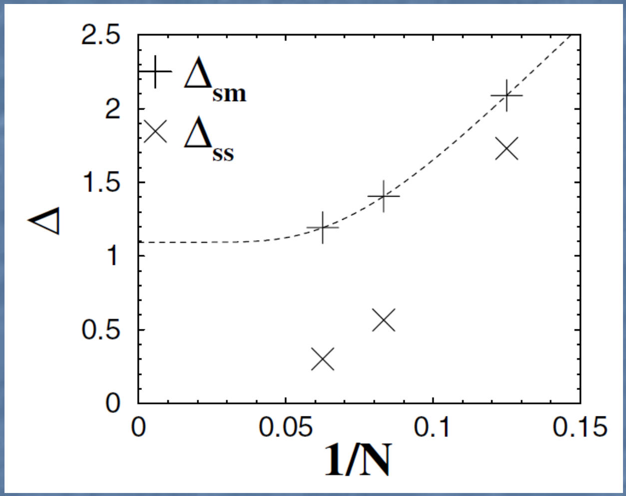


Square lattice

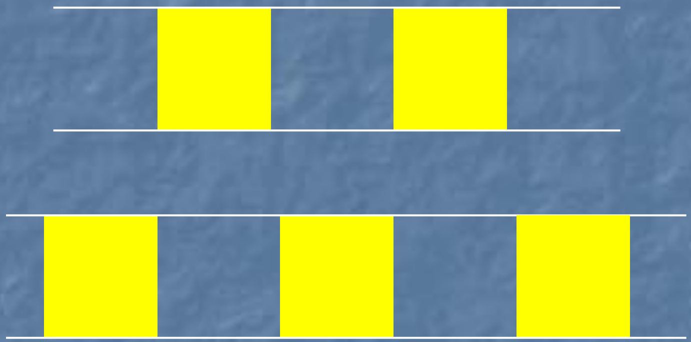


B. Bauer, P. Corboz, A. Laeuchli, L. Messio, K. Penc, M. Troyer, FM, unpublished

SU(4) ladder



2-fold degenerate GS



Spontaneous SU(4) plaquette singlet formation

Confirmed by field theory in weak and strong rung limits

M. Van Den Bossche, P. Azaria, P. Lecheminant, FM, PRL 2001

SU(4) on square lattice: early results

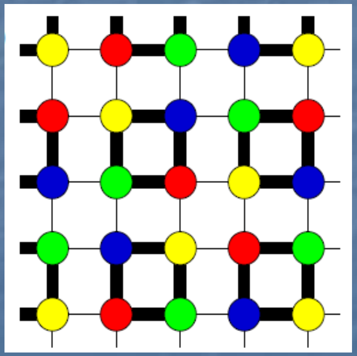
- Low-lying SU(4) singlets: plaquette coverings?
  M. Van den Bossche, F.-C. Zhang, FM, EPJB 2001
- Plaquette long-range order
  H.-H. Hung, Y. Wang, and C. Wu, Modern Phys Lett 2006
- Liquid with emergent nodal fermions
  Fang, Vishwanath, PRB 2009
- Chiral spin liquid ground state with topological order for N>4
  Hermele et al, PRL 2009
- Stripe color order?

SU(4) on square lattice

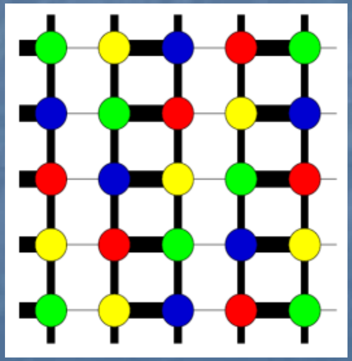
- Hartree: infinite number of coverings
- Flavor-wave theory

  → small clusters favored (2 and 4 sites)

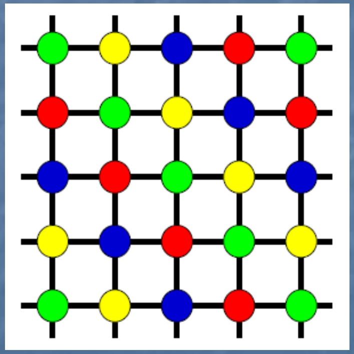
  → stripe order not stabilized



E/J=-1.5



E/J=-1.29



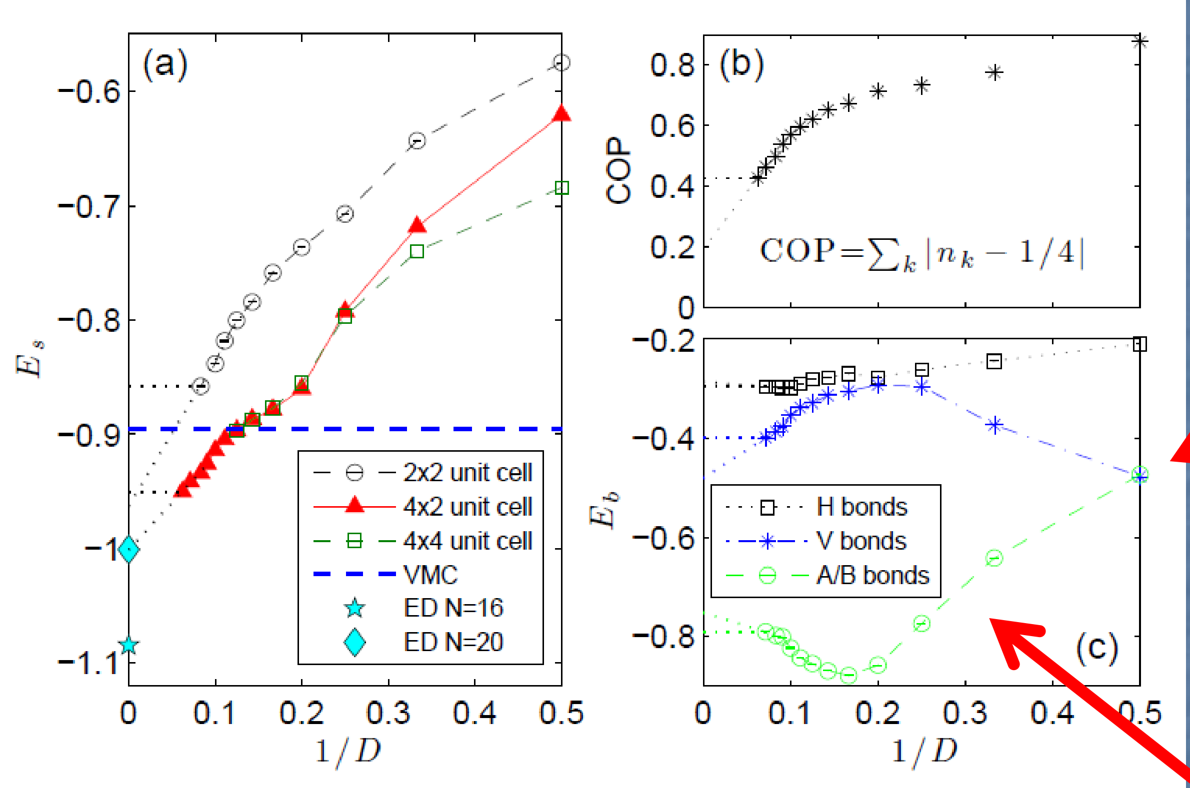
E/J=-0.73

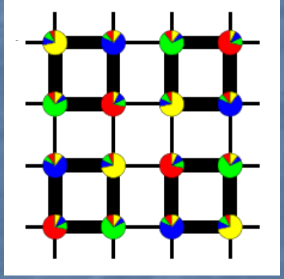
iPEPS

- iPEPS = infinite Projected Entangled Pair States
- Variational method based on a tensor product wave-function
- Becomes exact if the dimension D of the tensors → infinity
- Can be seen as a generalization of DMRG

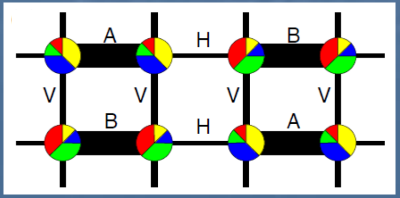
Verstraete and Cirac, 2004

iPEPS: SU(4) on square lattice





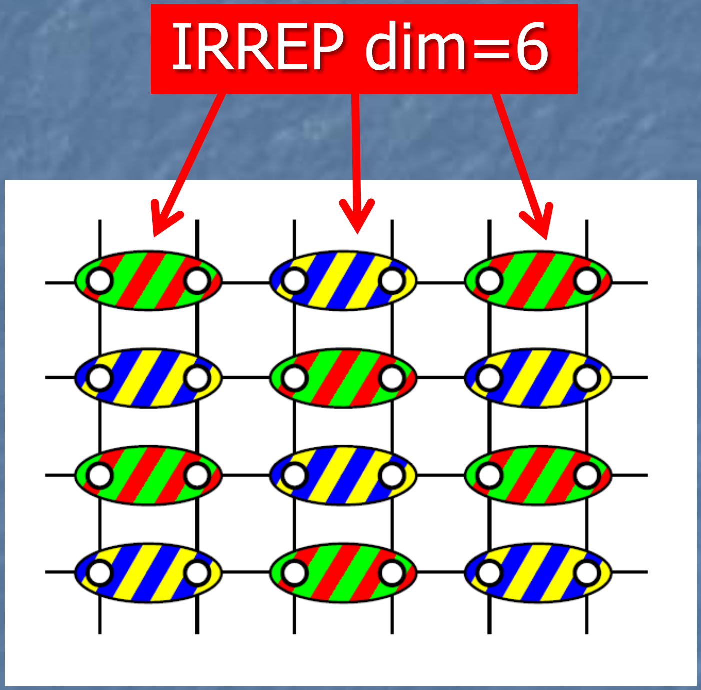
D=2



D>2

SU(4) on square lattice

IPEPS, ED, Hartree + flavour-wave theory,…



Dimerized ground state

+ Néel order

P. Corboz, A.Läuchli, K. Penc,
M. Troyer, F. Mila, PRL 2011

IRREP 6 on square lattice: Algebraic order? Assaad 2005
Long-range order? Paramekanti and Marston, 2007

SU(4) spin-orbital model

$$A = |\uparrow, a\rangle, \quad B = |\downarrow, a\rangle, \quad C = |\uparrow, b\rangle, \quad D = |\downarrow, b\rangle$$

IRREP6

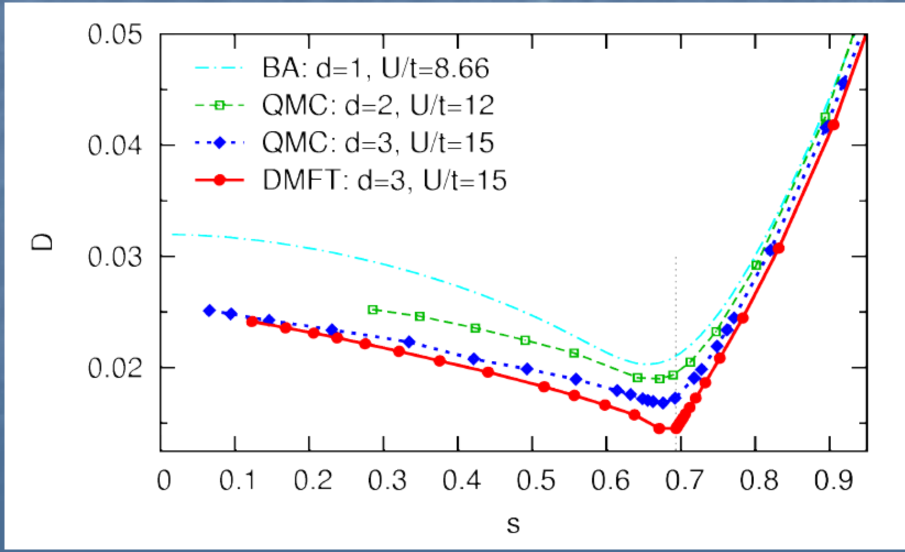
$$(AB - BA)/\sqrt{2} \rightarrow |\text{spin singlet}\rangle \otimes |a, a\rangle$$
$$(CD - DC)/\sqrt{2} \rightarrow |\text{spin singlet}\rangle \otimes |b, b\rangle$$

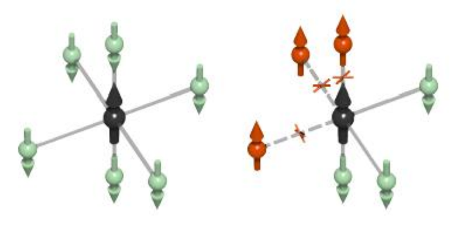
$$(AC - CA)/\sqrt{2} \rightarrow |\uparrow, \uparrow\rangle \otimes |\text{orbital singlet}\rangle$$
$$(BD - DB)/\sqrt{2} \rightarrow |\downarrow, \downarrow\rangle \otimes |\text{orbital singlet}\rangle$$

$$\begin{matrix}(AD - DA)/\sqrt{2} \\ (BC - CB)/\sqrt{2}\end{matrix} \rightarrow \begin{matrix}|\text{spin singlet}\rangle \otimes |\text{orbital } T_0\rangle \\ |\text{spin } T_0\rangle \otimes |\text{orbital singlet}\rangle\end{matrix}$$

Probing color order

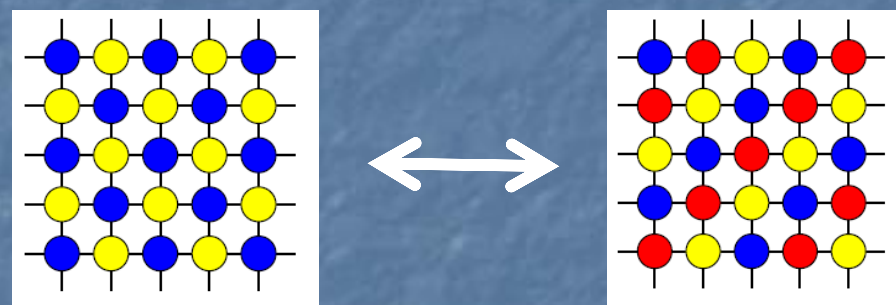
- Problem: long-range order sets in at low temperature (T=0 in 2D!)
- SU(2): probe short-range order through double-occupancy   Gorelik et al, PRL 2010





$$D = \frac{zt^2}{2U^2}\left(1 - \langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle\right)$$

How to probe local order for SU(3)?



Double occupancy? No

→ enhanced for both

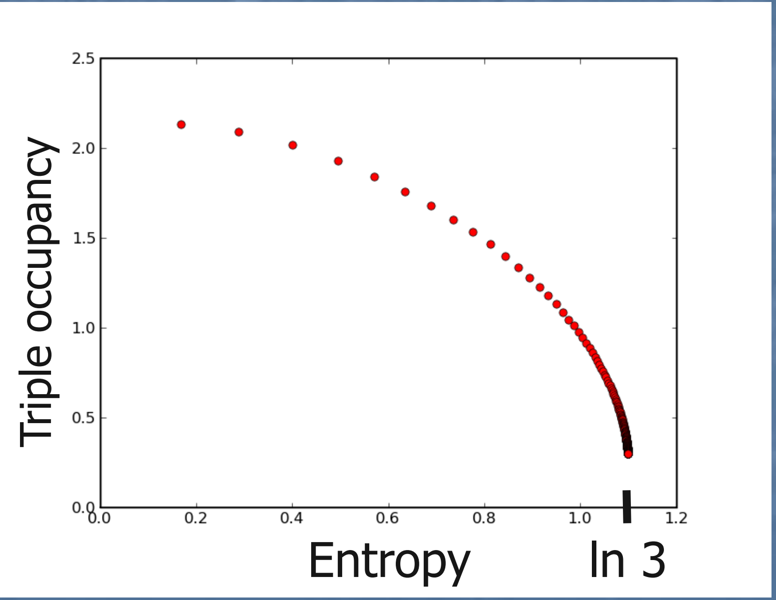
Triple occupancy? Yes!

→ suppressed for Néel order

→ enhanced for 3-sublattice order

Triple occupancy for SU(3) chain

$$n_{i,A}n_{i,B}n_{i,C} = \frac{4}{9}\frac{t^4}{U^4}\sum_{(j,k)}(1 - P_{ij} - P_{ik} - P_{jk} + P_{ijk} + P_{ijk}^{-1})$$



Quantum Monte Carlo

Strongly enhanced below S=ln3

L. Messio, HongYu Yang, FM, unpublished

Conclusions

- SU(3) on triangular lattice
  → canonical example of color order
- SU(3) on square lattice
  → 3-sublattice order at zero temperature
  → 2-sublattice correlations at large T?
- SU(4) on square lattice
  → dimerization + Néel order
- Probing color order
  → multiple occupancy