Appendix B

Representations of the algebra sp(2,R)

ABSTRACT: Finite-dimensional representations of the Lie algebra *sp(2,ℜ)* are used in geometric aberration optics, and the self-adjoint ones are relevant for wave optics, especially in the paraxial approximation. In this appendix we gather some information about this Lie algebra, its self-adjoint, indecomposable, and finite-dimensional representations.

B.1 The Lie algebra *sp(2,ℜ)* = *su(1,1)* = *so(2,1)*

We dedicate this section to obtain the generators of the Lie group *Sp(2,ℜ)* summarized in Appendix A; these constitute a basis for the Lie algebra *sp(2,ℜ)*, isomorphic to *sl(2,ℜ)*, *su(1,1)*, and *so(2,1)*, the Lie algebras of the corresponding homomorphic groups. It helps intuition to work with the 'relativistic' *so(2,1)*, justifying the notation and relating its structure to that of the compact rotation algebra *so(3)* more readily, since the properties and conventions of the latter are well known and established.

B.1.1 The cartesian basis

From equations $(A.14a,b,c)$[1] we may find the basic matrix representatives of the one-parameter subgroup generators J_k, $k=1,2,0$, $g_k(\tau)=\exp(i\tau J_k)$. We indicate their correspondence by "$\leftrightarrow$":

$$\begin{array}{ccccccc} & & sp(2,\Re) & & su(1,1) & & so(2,1) \\ J_1 & \leftrightarrow & \frac{-1}{2i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} & \leftrightarrow & \frac{1}{2i}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} & \leftrightarrow & \frac{1}{i}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \end{array} \qquad (1a)$$

$$\begin{array}{ccccccc} J_2 & \leftrightarrow & \frac{-1}{2i}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} & \leftrightarrow & \frac{-1}{2i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} & \leftrightarrow & \frac{1}{i}\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \end{array} \qquad (1b)$$

$$\begin{array}{ccccccc} J_0 & \leftrightarrow & \frac{-1}{2i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} & \leftrightarrow & \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} & \leftrightarrow & \frac{1}{i}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}. \end{array} \qquad (1c)$$

These operators satisfy the commutation relations

$$[J_0,J_1]=iJ_2, \qquad [J_1,J_2]=-iJ_0, \qquad [J_2,J_0]=iJ_1. \qquad (2a,b,c)$$

[1] That is, equations (14*a*), (14*b*), and (14*c*) of Appendix A.

The minus sign in (2*b*) distinguishes *so(2,1)* from *so(3)*, the compact rotation algebra.

We may now abstract the above commutation relations for any particular representation of the quantities $\{J_i\}_{i=0}^{2}$.

Most of the representation theory of semisimple Lie algebras (and groups) deals with representations that are self-adjoint (or unitary) and irreducible. These provide a deep understanding of many of the properties such as muliplet classification by the compact subgroup generator [1]. For noncompact groups, there are also integral transforms on the continuous spectrum of the parabolic [2] or the hyperbolic generator [3]. Generally one requires the existence of a complex Hilbert space $\mathcal{H}$ with a sesquilinear inner product $(\cdot,\cdot)$. Analytic continuation of the expressions from the self-adjoint representations provides the conventions for the finite dimensional, *non*-self-adjoint representations.

B.1.2 Raising, lowering, and Casimir operators

We follow Bargmann [1] in the definition of the raising and lowering operators as the *complex*[2] linear combinations of the *sp(2,ℜ)* generators J_k, $k = 0, 1, 2$:

$$J_\uparrow := J_1 + iJ_2, \qquad J_\downarrow := J_1 - iJ_2. \tag{3a, b}$$

Their commutation relations with the *weight* operator J_0 are

$$[J_0, J_{\uparrow\downarrow}] = \pm J_{\uparrow\downarrow}, \qquad [J_\uparrow, J_\downarrow] = -2J_0. \tag{5}$$

The Casimir operator —commuting with all J_k— is

$$\begin{aligned} C := {} & J_1^2 + J_2^2 - J_0^2 \\ = {} & J_\uparrow J_\downarrow - J_0^2 + J_0 \\ = {} & J_\downarrow J_\uparrow - J_0^2 - J_0. \end{aligned} \tag{5}$$

B.1.3 A Hilbert space and a basis

A *representation* of *sp(2,ℜ)* is a mapping of *sp(2,ℜ)* onto a linear space of operators on $\mathcal{H}$, with a vector basis $\mathcal{J}_k$, $k = 1, 2, 0$, whose commutators in $\mathcal{H}$ follows equations (2). We shall require the $\mathcal{J}_k$ to be *self-adjoint* in $\mathcal{H}$, *i.e.*,

$$(\mathcal{J}_k^\dagger f, g) := (f, \mathcal{J}_k g) = (\mathcal{J}_k f, g), \qquad f, g \in \mathcal{H}, \quad k = 0, 1, 2; \tag{6}$$

the domain of $\mathcal{J}_k$ and of $\mathcal{J}_k^\dagger$ are assumed to be the same. The raising and lowering operators (3) will then be one the adjoint of the other:

$$(\mathcal{J}_\uparrow f, g) = (f, \mathcal{J}_\downarrow g), \qquad (\mathcal{J}_\downarrow f, g) = (f, \mathcal{J}_\uparrow g). \tag{7a, b}$$

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[2] These should **not** be confused with the *real* linear combinations $J_\pm := J_0 \pm J_1$ that generate the parabolic subgroups of *sp(2,ℜ)*.

The operator $\mathbb{C}$ representing the Casimir operator through (C.3) will thus be also self-adjoint in $\mathcal{H}$.

We consider a complete basis for $\mathcal{H}$ given by the simultaneous eigenfunctions of $\mathbb{J}_0$ and $\mathbb{C}$ with (real) eigenvalues μ and q, respectively:

$$\mathbb{C} f^q_\mu = q f^q_\mu, \qquad q \in \Sigma(\mathbb{C}, \mathcal{H}), \tag{8a}$$
$$\mathbb{J}_0 f^q_\mu = \mu f^q_\mu, \qquad \mu \in \Sigma(\mathbb{J}_0, \mathcal{H}), \tag{8b}$$

where we use the eigenvalues as eigenfunction labels, and denote by $\Sigma(\mathbb{A}, \mathcal{H})$ the spectrum of $\mathbb{A}$ in $\mathcal{H}$.

B.1.4 Normalization coefficients

When we appply $\mathbb{J}_\uparrow$ and $\mathbb{J}_\downarrow$ to the above pair of equations and use (4a), we see that $\mathbb{J}_\uparrow f^q_\mu$ and $\mathbb{J}_\downarrow f^q_\mu$, if not null, are eigenfunctions of $\mathbb{C}$ with the same eigenvalue q, and of $\mathbb{J}_0$ with eigenvalues $\mu+1$ and $\mu-1$, respectively. The spectrum of $\mathbb{J}_0$ in $\mathcal{H}$ must be thus a collection of *equally spaced* points. If the classification (8) resolves the eigenfunctions uniquely, then

$$\mathbb{J}_\uparrow f^q_\mu = c^q_{\uparrow\mu} f^q_{\mu+1}, \tag{9a}$$
$$\mathbb{J}_\downarrow f^q_\mu = c^q_{\downarrow\mu} f^q_{\mu-1}. \tag{9b}$$

The constant $c^q_\uparrow\mu$ will be zero if $\mu+1$ is *not* a point in the spectrum of $\mathbb{J}_0$ in $\mathcal{H}$, and analogously for $c^q_\downarrow\mu$, that is, $\mu \pm 1 \notin \Sigma(\mathbb{J}_0, \mathcal{H}) \Rightarrow c^q_{\uparrow\downarrow\mu} = 0$. The eigenfunctions may be normalized to unity: $(f^q_\mu, f^q_\mu) = 1$ for all $\mu \in \Sigma(\mathbb{J}_0, \mathcal{H})$. That inner product of (9) with itself may be written

$$\begin{aligned}
|c^q_{\uparrow\downarrow\mu}|^2 (f^q_{\mu\pm1}, f^q_{\mu\pm1}) &= (\mathbb{J}_{\uparrow\downarrow} f^q_\mu, \mathbb{J}_{\uparrow\downarrow} f^q_\mu) && \text{[by (9)]}\\
&= (f^q_\mu, \mathbb{J}_{\downarrow\uparrow}\mathbb{J}_{\uparrow\downarrow} f^q_\mu) && \text{[by (7)]}\\
&= \big(f^q_\mu, [\mathbb{C} + \mathbb{J}_0^2 \pm \mathbb{J}_0] f^q_\mu\big) && \text{[by (5)]}\\
&= (q + \mu^2 \pm \mu)\big(f^q_\mu, f^q_\mu\big). && \text{[by (8)]}
\end{aligned} \tag{10}$$

Hence, $|c^q_{\uparrow\downarrow\mu}|^2 = 0$ when $q + \mu^2 \pm \mu = 0$, and otherwise they must be *positive*. Any eigenvalue pair q, μ must therefore satisfy

$$|c^q_{\uparrow\downarrow\mu}|^2 = q + \mu^2 \pm \mu \geq 0. \tag{11}$$

The regions of positivity of $|c^q_{\uparrow\mu}|^2$ and of $|c^q_{\downarrow\mu}|^2$ are depicted in Figure B.1, next page.

B.2 The self-adjoint irreducible representations

B.2.1 Bounds on multiplets

Consider an eigenfunction f^q_μ such that q and μ *do* satisfy (11). They will determine a point *outside* the striped regions of Figure B.1, or on the boundaries. Through successive application of $\mathbb{J}_\uparrow$ and $\mathbb{J}_\downarrow$ we can produce the sequence of eigenfunctions $f^q_{\mu\pm1}, f^q_{\mu\pm2}, \ldots$ which should also fall outside the striped regions. For fixed q, the ***multiplet*** of eigenvalues $\{\mu\}$ forms thus a vertical lattice of points. If any of these points falls on the *forbidden* regions, (11) is violated and $\mu \notin \Sigma(\mathbb{J}_0, \mathcal{H})$ for that value of q, since further application of $\mathbb{J}_\downarrow$ or of $\mathbb{J}_\uparrow$ will yield zero; if the point falls *on* the boundary, this value of μ will be a *bound* —lower or upper— of the multiplet. In this regard we have the following distinct intervals for q:

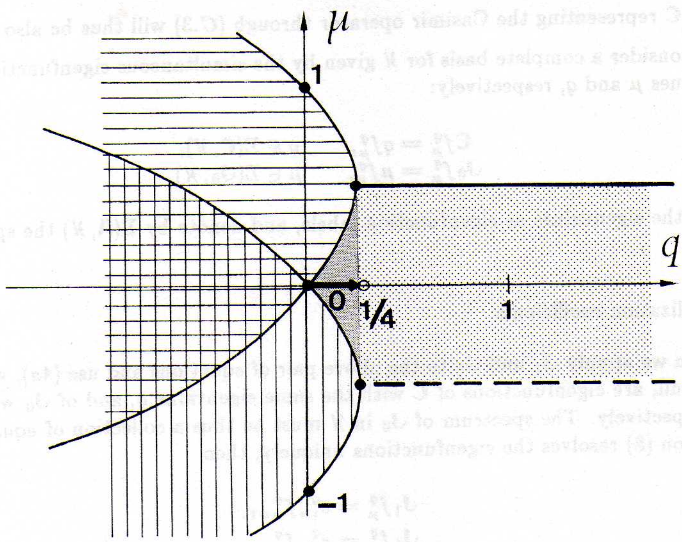

Figure B.1 The *forbidden* regions $q + \mu^2 \mp \mu < 0$ are marked with horizontal and vertical stripes, respectively. No eigenfunction f^q_μ of $\mathbb{C}$ and $\mathcal{J}_0$ within a self-adjoint representation space may correspond to points q, μ inside these striped regions.

$q > \frac{1}{4}$ Multiplets are unbounded.

$q \le 0$ Multiplets are upper- or lower-bound; to describe them it is convenient to introduce the *Bargmann index* k, related to q:

$$q(k) := k(1-k),$$
$$k(q) := \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - q}, \qquad \operatorname{Re} k \ge \tfrac{1}{2},\ \operatorname{Im} k \ge 0. \tag{12}$$

Then:

\+ If a point q, μ falls on the *upper* branch of the *upper* parabola, $\mu_{\min} := k$ is the *lower* bound of its multiplet.

– If a point q, μ falls on the *lower* branch of the *lower* parabola, $\mu_{\max} := -k$ is the *upper* bound of the multiplet.

0 For $q = 0$ we have the ***sp(2,ℜ) trivial*** representation by zero: $\mathcal{J}_n f^0_0 = 0$ for $n = 0, \uparrow, \downarrow, 1, 2$.

$0 < q \le \frac{1}{4}$ Both half-bounded *and* unbounded multiplets coexist in this ***exceptional*** interval: the former ones happen when some μ falls *on* the parabolæ, the latter when the unit spacing of the μ's allow them to *jump over* the forbidden regions. (+) Lower-bound multiplets are obtained when ***either*** $\mu_{\min} = k$ as before (falling on the upper branch of the upper parabola), *or* when $\mu_{\min} = 1 - k$ (falling on the lower branch of the upper parabola). (–) Upper-bounded multiplets occur when *either* $\mu_{\max} = -k$ (falling on the lower branch of the lower parabola) *or* $\mu_{\max} = k-1$ (falling on the upper branch of the lower parabola). When $q = \frac{1}{4}$, then $k = \frac{1}{2}$, and the two branches coalesce on the same point $\mu_{\min} = \frac{1}{2}$ and $\mu_{\max} = -\frac{1}{2}$. Unbounded multiplets occur when $\mu = \epsilon + n$, $|\epsilon| < k$, $n \in \mathfrak{Z}$ (the set of integers).

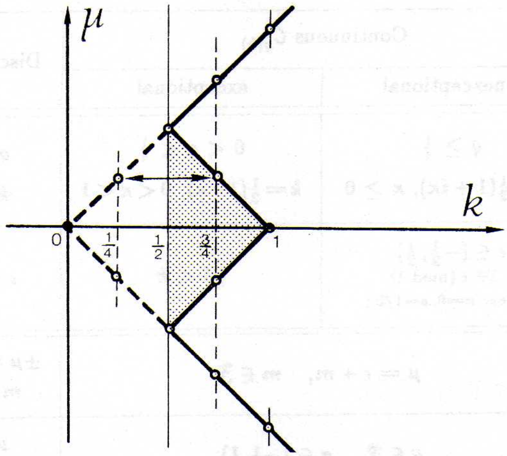

Figure B.2 The Bargmann index k is used as coordinate axis to show the *sp(2,ℜ)* representation structure in the exceptional interval. The bold lines correspond to the lower- and upper-bound representations. The open gray region indicates the exceptional continous representations. The dotted lines prolong the index k to the origin, unfolding thereby the discrete series doubling due to $q(k) = q(1-k)$.

B.2.2 Resolution of irreducible subspaces

In contradistinction to the familiar *su(2)* case, where the Casimir operator eigenvalue $j(j+1)$ uniquely specifies the spectrum of the compact generator, the *sp(2,ℜ)* Casimir operator eigenvalue does not. This is so because, as we have seen in the preceding subsection, $\mathcal{H}$ is *not* irreducible under the action of the algebra. In addition to the direct integral decomposition $\mathcal{H} = \int_{\Re}^{\oplus} \mathcal{H}^{(q)}$ into eigenspaces of the Casimir operator $\mathbb{C}$, within each $\mathcal{H}^{(q)}$, J_0 exhibits *more than one self-adjoint extension* (*two* for $q < 0$, *three* for $q = 0$, and a *one-parameter family* for $q > 0$). Since we have identified the multiplets themselves, the reduction of $\mathcal{H}^{(q)}$ to its irreducible components proceeds as follows: *(i)* we build the *linear span* of all functions $\{f_\mu^q\}$ belonging to a given multiplet (q fixed), and *(ii)* the *completion* of this space with respect to the original inner product $(\cdot,\cdot)$ will *define* the Hilbert space $\mathcal{H}_E^{(q)}$ —labelled by E— which is irreducible under *sp(2,ℜ)* and where $\Sigma(J_0, \mathcal{H}_E^{(q)})$ is *unique*. We shall now specify what E is. See Figure B.2, above.

B.2.2.1 The continuous series $\mathcal{C}_{q>0}^{\epsilon}$

For the case of *unbounded multiplets* in $q > 0$, the label E that fully specifies the self-adjoint irreducible representations is denoted ϵ, and is

$$\epsilon \equiv \mu \,(\mathrm{mod}\ 1) \in (-\tfrac{1}{2}, \tfrac{1}{2}]. \tag{13}$$

The spectrum of J_0 is $\mu = \epsilon + n$, $n \in \mathfrak{Z}$. There are two subintervals of interest:

Nonexceptional In $q > \frac{1}{4}$, $[k = \frac{1}{2}(1 + i\kappa)$, $\kappa \in \Re^+]$, the full range $(-\frac{1}{2}, \frac{1}{2}]$ is available for ϵ. This is the *continuous nonexceptional* series of representations.

Exceptional In $0 < q \leq \frac{1}{4}$, $[\frac{1}{2} \leq k < 1]$, ϵ is constrained by $|\epsilon| < 1 - k$. This is the *continuous exceptional* series of representations. See the Table in the following page.

Representation series:	Continuous $C^{\epsilon}_{q(k)}$ nonexceptional	Continuous $C^{\epsilon}_{q(k)}$ exceptional	Discrete $D^{\pm}_{k}$
$q \in \Sigma(\mathbb{C})$ Bargmann label k: $q = k(1-k)$	$q \geq \frac{1}{4}$ $k = \frac{1}{2}(1+i\kappa),\ \kappa \geq 0$	$0 < q < \frac{1}{4}$ $k=\frac{1}{2}(1+\kappa),\ 0<\kappa<1$	$q \leq \frac{1}{4}$ $k > 0$
'multivaluation' index: ϵ	$\epsilon \in (-\frac{1}{2}, \frac{1}{2}]$ $\epsilon \equiv \epsilon\,(\mathrm{mod}\ 1)$ (except $\kappa=0, \epsilon=1/2$)	$\lvert\epsilon\rvert < k$	$\epsilon = \pm k$
$\mu \in \Sigma(\mathsf{J}_0)$	$\mu = \epsilon + m, \quad m \in \mathfrak{Z}$		$\pm\mu = \epsilon + m$ $m \in \mathfrak{Z}^{0+}$
$\nu \in \Sigma(\mathsf{J}_1)$ $\sigma \in \Sigma(\mathsf{A})$	$\nu \in \Re, \quad \sigma \in \{-1, 1\}$		$\nu \in \Re$ $\pm\sigma = 1$
$\xi \in \Sigma(\mathsf{J}_-)$	$\xi \in \Re$		$\pm\xi \in \Re^+$

Table. Casimir operator eigenvalues and spectra of the elliptic (J_0), hyperbolic (J_1), and parabolic (J_-) subalgebra representatives for all representation series. (The outer algebra automorphism A is described in reference [3].)

B.2.2.2 The discrete series $D^{\pm}_{k}$

The lower- and upper-bound multiplets belong to the so-called *discrete*[3] representation series. There, it is $\mu_{\min}$ or $\mu_{\max}$ which becomes the label E specifying the representation bound. The main division concerns the direction in which the multiplet extends, while the bound itself is given in terms of Bargmann's label. We thus take E to be $+$ or $-$, and we write, following the established convention:

Positive $\quad D^{+}_{k}: \quad k > 0, \quad \mu = k + n,\ n \in \{0, 1, 2, \ldots\} = \mathfrak{Z}^{0+}.$

Negative $\quad D^{-}_{k}: \quad k > 0, \quad \mu = -k + n,\ n \in \{0, -1, -2, \ldots\} = \mathfrak{Z}^{0-}.$

We may uphold the choice of the parameter ϵ, simply setting $\epsilon = \pm k$ and abandon the modulo 1 condition since now it is a lower bound for $\{\mu\}$. It is also known [3] that by means of an (outer) automorphism A of the algebra, $\mathsf{A} : \{J_+, J_0, J_-\} = \{J_-, -J_0, J_+\}$ we may intertwine the positive and the negative discrete series.

As we can see in Figures B.1 and B.2, the number of discrete-series multiplets corresponding to a given value of the Casimir eigenvalue q is *two* for $q < 0$ ($k > 1$ *i.e.*, D^{+}_{k} and D^{-}_{k}), *three* for $q = 0$ ($k = 1$ *i.e.*, D^{+}_{1}, D^{-}_{1} and the trivial D^{0}), *four* for $0 < q < \frac{1}{4}$ ($\frac{1}{2} < k < 1$ *i.e.*, D^{+}_{k}, D^{+}_{1-k}, D^{-}_{k}, and D^{-}_{1-k})[4]

[3] The name *discrete* for these series was given by V. Bargmann [1], who considered the single-valued *group* representations, rather than the algebra representations as here. In that case, the spectrum $\{\mu\}$ is restricted to *integers* for *SO(2,1)*, and the half-integers for *Sp(2,R)*. For the group, the allowed values of Bargmann's label k are *discrete*. For the algebra, they are continuous.

[4] The well-known *oscillator* representation falls on $q = 3/16$; there we have $k = 3/4$, and so $D^{+}_{3/4}$ and $D^{+}_{1/4}$ are the irreducible representations spanned by the odd and even states. The $D^{-}_{3/4}$ and $D^{-}_{1/4}$ representation spaces would contain negative unbounded energies. They are disregarded as unphysical.

and finally *two* again for $q = \frac{1}{4}$ ($k = \frac{1}{2}$ *i.e.*, $D^+_{1/2}$ and $D^-_{1/2}$). In all but the first case, these representations coexist with continuous-series ones. The plethora of cases is conveniently reduced by disregarding the Casimir operator eigenvalue q in favor of Bargmann's label $k > 0$ and $\pm$ for the discrete series, and q —or equally well k— and ϵ for the continuous series[5] In the table of last page we abstract this information.

For all self-adjoint representations series, using (11) and the Bargmann label (12), we may write

$$J_\uparrow f^k_\mu = \sqrt{(\mu + k)(\mu - k + 1)}\, f^k_{\mu+1}, \tag{14a}$$

$$J_\downarrow f^k_\mu = \sqrt{(\mu - k)(\mu + k - 1)}\, f^k_{\mu-1}, \tag{14b}$$

where the radicands are positive.

B.3 Indecomposable and finite-dimensional representations

We now relax the condition (7) that the raising operator be the adjoint of the lowering operator. We should keep the condition of self-adjointness for J_0 and $\mathbb{C}$, however, if in (8) we still want real eigenvalue labels μ and q for the multiplet members. This means that both $J_\uparrow J_\downarrow$ and $J_\downarrow J_\uparrow$ are self-adjoint, as may be seen from (5). We may thus allow the matrix elements of $J_{\uparrow\downarrow}$ to keep the absolute values of (14), but to differ by conjugate *phases* from those of $(J_{\downarrow\uparrow})^\dagger$, so that $c^k_{\uparrow\mu} c^k_{\downarrow\mu+1}$ be real, *i.e.*,

$$c^{q(k)}_{\uparrow\ \mu} = e^{i\phi(k,\mu)} \sqrt{|(\mu + k)(\mu - k + 1)|}, \tag{15a}$$

$$c^{q(k)}_{\downarrow\ \mu} = e^{-i\phi(k,\mu-1)} \sqrt{|(\mu - k)(\mu + k - 1)|}. \tag{15b}$$

B.3.1 Indecomposable representations

The Bargmann label k will prove to describe the class of indecomposable representations better than the Casimir operator eigenvalue q.

Due to (12), $q(k) = q(1 - k)$, but k itself, allowed to range over $k > 0$ above, may be extended here to the real line[6] As for the discrete series, we see from (14b) that $\mu_{\min} = k$ is a lower bound for multiplets extending up, and from (14a) that $\mu_{\max} = -k$ is an upper bound for multiplets extending down, since $J_\downarrow f^k_{\mu_{\min}} = 0$ and $J_\uparrow f^k_{\mu_{\max}} = 0$ (here we have switched from q to k to denote the representation). Yet, these are *one-way* barriers, because we may arrive at the lower bound from *below*, *i.e.*, $J_\uparrow f^k_{\mu_{\min}-1} = e^{i\phi}\sqrt{2k} f^k_{\mu_{\min}}$ and correspondingly $J_\downarrow f^k_{\mu_{\max}+1} = e^{-i\phi'}\sqrt{2k} f^k_{\mu_{\max}}$. See Figure B.3, next page.

Suppose now we have k integer and we start with $f^k_{\mu=0}$. In Figure B.3 this falls within the 'non-self-adjoint' region to the right of the origin. We may now raise μ with $J_\uparrow$ past the $\mu_{\min} = k$ barrier, *into* the D^+_k region, or we may lower it with $J_\downarrow$ into the D^-_k region. Once there, however, we cannot go back to f^k_0 because of the one-way barriers. It follows that f^k_μ, $\mu \in \mathfrak{Z}$ is a basis for an *indecomposable* representation of *sp(2,*$\mathfrak{R}$*)* [and of *Sp(2,*$\mathfrak{R}$*)*], with two irreducible, self-adjoint pieces D^+_k and D^-_k. The block form of the algebra (and group) representation is thus $\begin{pmatrix} D^+ & \times & 0 \\ 0 & \times & 0 \\ 0 & \times & D^- \end{pmatrix}$.

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[5]Further study of the *exceptional interval* $0 < k < 1$ in terms of quantum mechanical eigenfunctions of a harmonic oscillator with a *weakly* attractive or repulsive x^{-2}-core, is undertaken in reference [4].

[6]The Bargmann label is, in fact, complex for the continuous representation series: $k = \frac{1}{2}(1 + i\kappa)$, $\kappa \geq 0$.

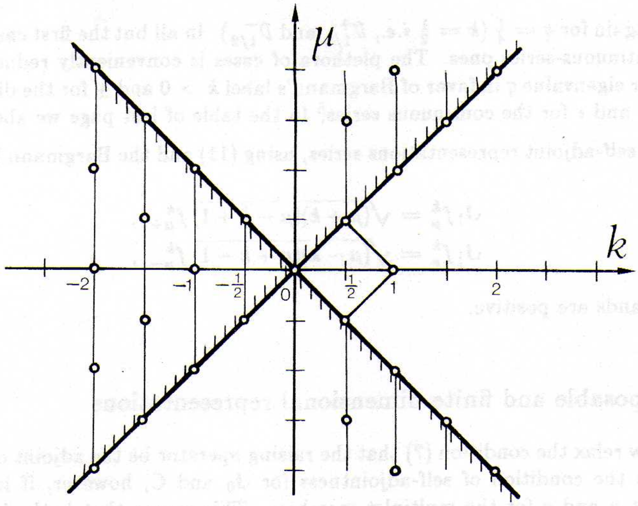

Figure B.3 The Bargmann index k is prolongued to negative values, showing the one-way barriers that hold the irreducible spaces within the indecomposable ones. The finite-dimensional representation multiplets f^j_μ are to the left.

For noninteger k, when $\mu \equiv k \pmod 1$ we obtain upper-triangular indecomposable representations of 2×2 block form, with $\mathcal{D}^+$ in the 1–1 position. Similarly, when $\mu \equiv 1 - k \pmod 1$, the lower-triangular indecomposable representations contain $\mathcal{D}^-$ in the 2–2 position.

We consider now the region $k < 0$ as continuation of the label k in (14) to negative values. Set $j := -k$, positive. Then, the Casimir eigenvalue is $q = k(1-k) = -j(j+1)$ [*cf.* Eq. (5)]; $\mu = -j$ remains a lower bound barrier and $\mu = j$ and upper bound barrier. Hence, for $j = \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots$, the set of vectors $\{f^j_\mu\}^j_{\mu=-j}$ forms a finite-dimensional basis for an irreducible *sp(2,ℜ)* representation [also valid for the group *Sp(2,ℜ)*], which we may call $\mathcal{D}^j$. The block form of the representation for all values $\mu \equiv j \pmod 1$ is $\begin{pmatrix} \times & 0 & 0 \\ \times & \mathcal{D}^j & \times \\ 0 & 0 & \times \end{pmatrix}$. If j is not in the above range of values but some μ falls on a boundary, the block form reduces to 2×2 block triangular cases.

B.3.2 The finite-dimensional representations of *sp(2,ℜ)*

We now consider specifically the finite dimensional (*non*-self-adjoint) representations $\mathcal{D}^j$, where $-k = j = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. The phase of the normalization constants in (14) may be chosen to be unity, so that

$$J_\uparrow f^j_\mu = \sqrt{(j-\mu)(j+\mu+1)}\, f^j_{\mu+1}, \tag{16a}$$

$$J_\downarrow f^j_\mu = \sqrt{(j+\mu)(j-\mu+1)}\, f^j_{\mu-1}. \tag{16b}$$

These equations have now exactly the same form as the familiar raising and lowering operator action of the rotation algebra *so(3)* [5, Eqs.(3.20)]. What we have done is part of the inverse Weyl trick: replacing $J_1 \mapsto iJ_1$, $J_2 \mapsto iJ_2$, whereby the minus sign in (2b) is now a plus, the Casimir operator C in (5) is now $-J^2$, and the i's have been brought into the radicands of (14) to yield those of (16). The other part of the trick, applied to groups, is the analytic continuation of the group parameters. This we need not do here; instead, we remain within *sp(2,ℜ)*.

B.3.3 The finite-dimensional representations of the group

The relation between the general element of the *sp(2,ℜ)* algebra and its exponentiation to $g\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2,\Re)$ is the following:

$$\exp i(\theta_0 J_0 + \theta_1 J_1 + \theta_2 J_2) = g\begin{pmatrix} \cos\frac{\theta}{2} - \frac{\theta_2}{\theta}\sin\frac{\theta}{2} & -\frac{\theta_0+\theta_1}{\theta}\sin\frac{\theta}{2} \\ \frac{\theta_0-\theta_1}{\theta}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + \frac{\theta_2}{\theta}\sin\frac{\theta}{2} \end{pmatrix}, \tag{17a}$$

where

$$\theta = \sqrt{\theta_0^2 - \theta_1^2 - \theta_2^2}. \tag{17b}$$

This can be verified to be consistent with (A.14) and (1) in the basic representation.

The self-adjoint representations of the algebra exponentiate to unitary representations of the group. These were found by Bargmann [1,§10] in 1947, and can be seen summarized for the symplectic group parameters $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in reference [2]. The finite-dimensional $\mathcal{D}^j$ representation matrix elements in the basis $\{f_\mu^j\}_{\mu=-j}^j$ were also given by Bargmann [1, §10g] and written in terms of hypergeometric functions. In polynomial form, we find

$$D^j_{m,m'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sqrt{\frac{(j-m')!\,(j+m')!}{(j-m)!\,(j+m)!}} \sum_n \binom{j-m}{j+m'-n}\binom{j+m}{n} a^n b^{j+m-n} c^{j+m'-n} d^{n-m-m'}. \tag{18}$$

This is a polynomial of degree $2j$ in the symplectic matrix group parameters.

References

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