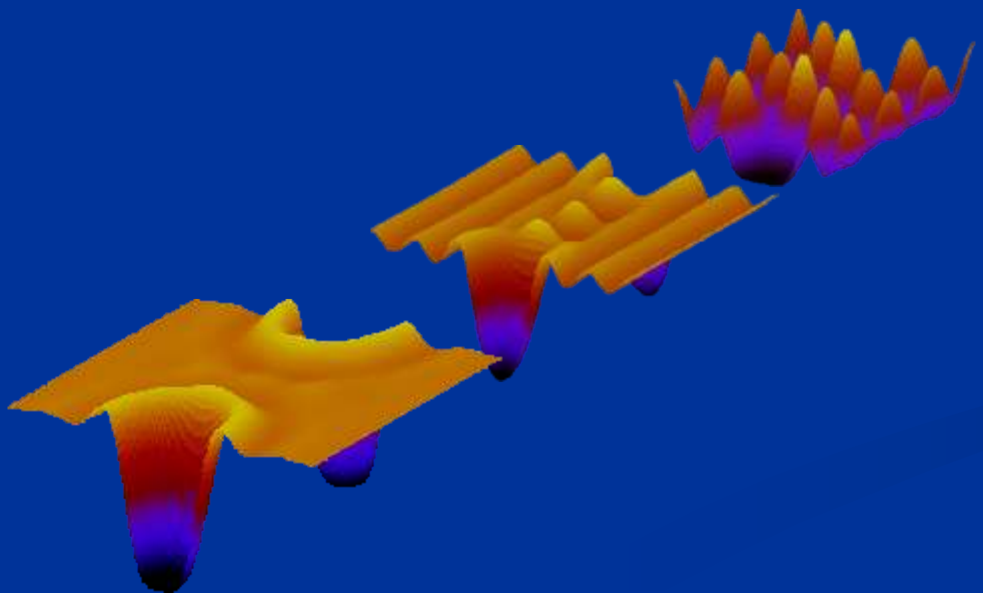
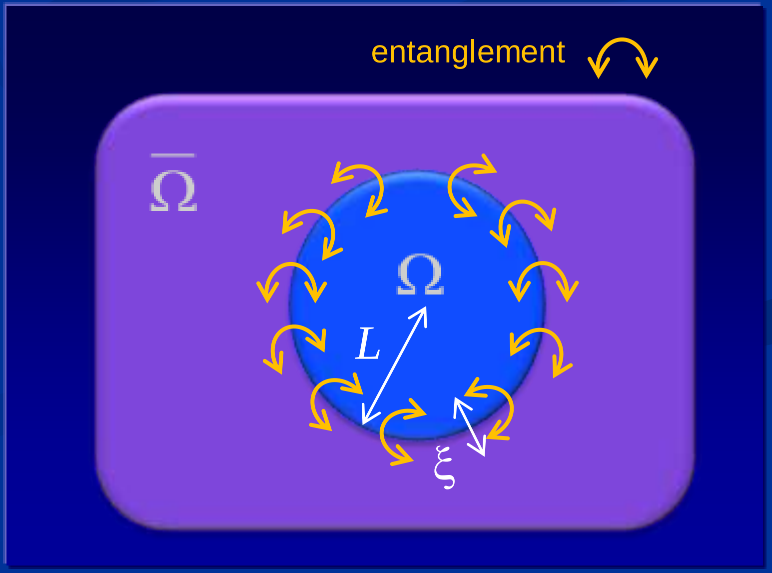
Quantum Hall States and Entanglement Entropy

Tohoku Univ. N. Shibata

- Entanglement entropy
- Density matrix renormalization group
- Topological entanglement entropy of Quantum Hall states
- Relation between 2D quantum Hall systems and 1D systems

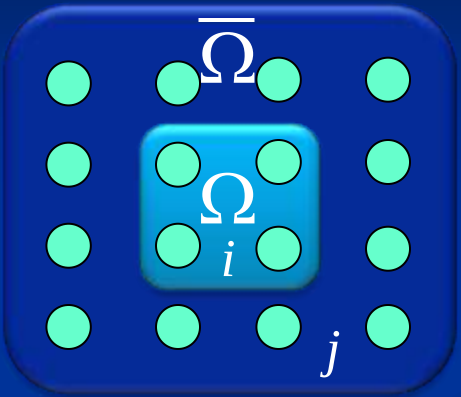




Entanglement entropy

Basis states : $|i\ j\rangle = |i\rangle_{\Omega} |j\rangle_{\bar{\Omega}}$

Wave function : $|\Psi\rangle = \sum_{ij} \Psi_{ij} |i\rangle_{\Omega} |j\rangle_{\bar{\Omega}}$



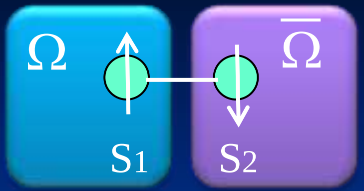
Reduced density matrix : $\rho_{\Omega} = Tr_{\bar{\Omega}} |\Psi\rangle\langle\Psi|$ $\quad (\rho_{\Omega})_{ii'} = \sum_{j} \Psi^{*}_{ij} \Psi_{i'j}$

Entanglement entropy : $S_{\Omega} = -Tr\, \rho_{\Omega} \ln \rho_{\Omega}$

Measure of entanglement between two regions

Entanglement entropy

S=1/2 2-Spin system



$S_1 = \uparrow$ or $\downarrow$

$S_2 = \uparrow$ or $\downarrow$

Wave function

$$|\Psi> = \sum_{S_1 S_2} \Psi_{S_1 S_2} \, |S_1> |S_2>$$

$$\Psi_{S_1 S_2} : \begin{pmatrix} \Psi_{\uparrow\uparrow} & \Psi_{\uparrow\downarrow} \\ \Psi_{\downarrow\uparrow} & \Psi_{\downarrow\downarrow} \end{pmatrix} = \Psi$$

Reduced density matrix

$$\rho_\Omega = \Psi^* \Psi^t$$

$$(\rho_\Omega)_{ii'} = \sum_j \Psi_{ij}^* \Psi_{i'j}$$

Entanglement entropy

$$S_\Omega = -Tr \, \rho_\Omega \ln \rho_\Omega$$

S_1 is independent of S_2

- Not correlated -

$$\frac{1}{\sqrt{2}}(|\uparrow\uparrow> + |\uparrow\downarrow>) \Rightarrow \Psi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

(S1S2)

$$= \frac{1}{\sqrt{2}} |\uparrow>_{S1} \otimes (|\uparrow> + |\downarrow>)_{S2}$$

$$\rho_\Omega = \overbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}}^{\Psi^*} \overbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}}^{\Psi^t} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S_\Omega = -\ln 1 = 0$$

disentangled

S_1 depends on S_2

- correlated -

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>) \Rightarrow \Psi = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Singlet state

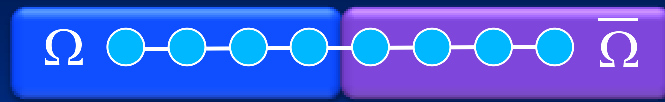
$$\rho_\Omega = \overbrace{\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}}^{\Psi^*} \overbrace{\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}}^{\Psi^t} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$S_\Omega = 2\left(\frac{1}{2}\ln 2\right) = \ln 2$$

entangled

Entanglement entropy

Many electron system



$|\,i>$ $\qquad$ $|\,j>$

Basis states : $|\,i\;j> \;=\; |\,i>|\,j>$ $\qquad$ $i \rightarrow \{1,2,3,4,\ldots m\}$, $\; j \rightarrow \{1,2,3,4,\ldots m\}$ $\qquad$ $|\,\Psi> \;=\; \sum_{ij} \Psi_{ij}\,|\,i>|\,j>$

Ω is independent of $\overline{\Omega}$

$$\frac{1}{\sqrt{m}}(\,|\,1\;1> + |\,1\;2> + |\,1\;3> + \ldots + \;|\,1\;m>\,) \;\Rightarrow\; \Psi = \begin{pmatrix} \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{m}}\underbrace{|\,1\;>}_{i} \otimes \underbrace{(\,|\,1> + |\,2> + \ldots + |\,m>\,)}_{j}$$

$$\rho_{ii'} = \sum_{j} \Psi^{*}_{ij}\Psi_{i'j}$$

$$\rho = \begin{pmatrix} \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{m}} & \cdots & 0 \\ \frac{1}{\sqrt{m}} & \cdots & 0 \\ \vdots & 0 & \vdots \\ \frac{1}{\sqrt{m}} & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \rightarrow$$

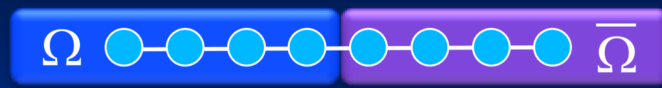
Entanglement entropy

$$S = -Tr\,\rho \ln \rho$$
$$= -\ln 1 = 0$$

disentangled

Entanglement entropy

Basis states :



$|i\ j> \; = \; |i>|j>$ $\qquad\qquad$ $|i>$ $\qquad\qquad$ $|j>$

Ω and $\overline{\Omega}$ are correlated

$$\frac{1}{\sqrt{m}}\left(|1\ 1>+|2\ 2>+|3\ 3>+\ldots+\ |m\ m>\right) \Rightarrow \Psi=\begin{pmatrix} \frac{1}{\sqrt{m}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m}} & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{m}} \end{pmatrix}$$

$$\rho_{ii'}=\sum_j \Psi^*_{ij}\Psi_{i'j}$$

$$\rho=\begin{pmatrix} \frac{1}{\sqrt{m}} & 0 & & 0 \\ 0 & \frac{1}{\sqrt{m}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{m}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{m}} & 0 & & 0 \\ 0 & \frac{1}{\sqrt{m}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{m}} \end{pmatrix}=\begin{pmatrix} \frac{1}{m} & 0 & & 0 \\ 0 & \frac{1}{m} & & \\ & & \ddots & \\ 0 & & & \frac{1}{m} \end{pmatrix}$$

Entanglement entropy

$$S=-Tr\,\rho\ln\rho$$

$$=m\left(\frac{1}{m}\ln m\right)=\ln m$$

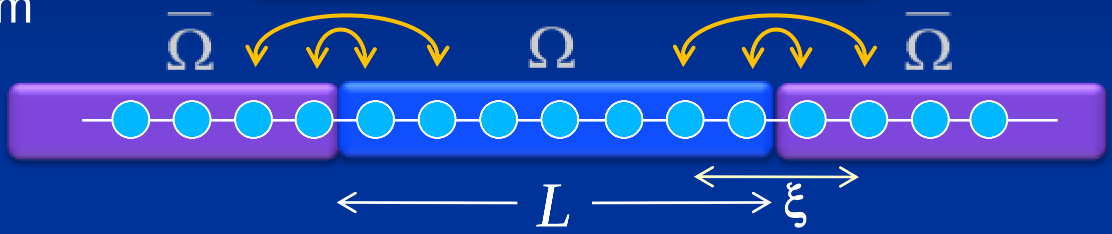
Maximally entangled

Scaling of entanglement entropy

Relation between correlations and scaling of entanglement entropy

$$S_{\Omega(L)} = -Tr\, \rho_{\Omega(L)} \ln \rho_{\Omega(L)}$$

1D system



Short range correlation :

$$S_{\Omega(L)} \approx const.$$

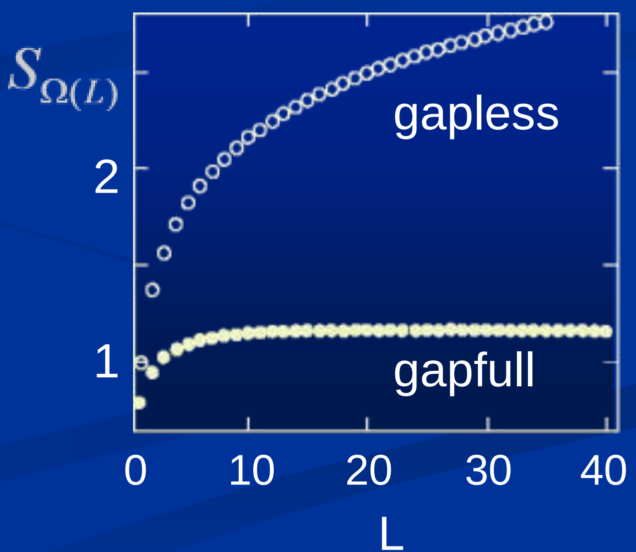
$(L >> \text{correlation length } \xi)$

Power law correlation : (1D critical system)

c: central charge

$$S_{\Omega(L)} \approx \frac{c}{6} \ln L + s_0$$

1D-XXZ model



Vidal, Latorre, Rico, and Kitaev, PRL, 2003

Scaling of entanglement entropy

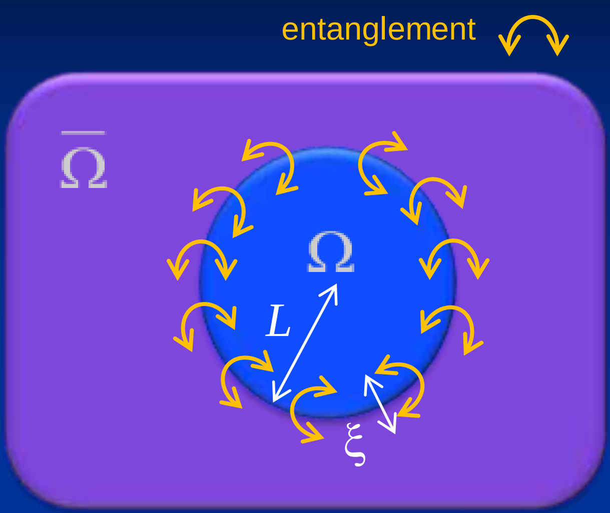
Short range correlation $(L >> \xi)$

D-dimensional system

Area law $S_{\Omega(L)} \propto L^{D-1}$

L^{D-1} : boundary size (length)

Srednicki, PRL, 1993 Wolf,Verstraete,Hastings,Cirac,PRL,2008



Topological order in 2D

$$S_{\Omega(L)} = \alpha L - \ln D$$

Boundary term

Non-trivial universal correction

Topological term

Fractional quantum Hall state

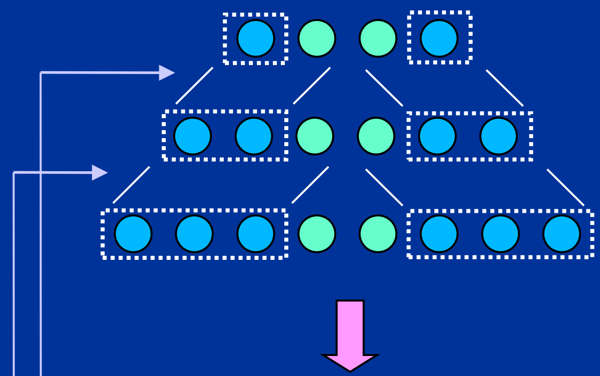
$\nu = 1/m$ Laughlin state : $D = m^{1/2}$

Kitaev & Preskill; Levin & Wen, PRL,2006

Application of DMRG to quantum Hall systems

To confirm the prediction of boundary term and topological term

- Real space renormalization

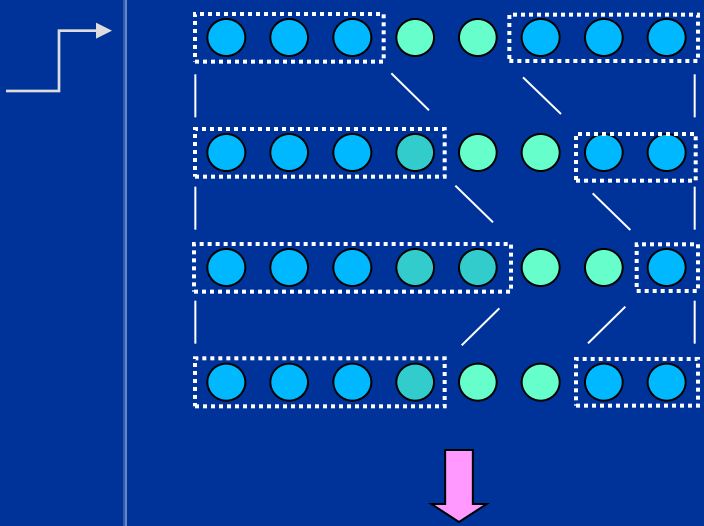


- Large size of systems

Extend the blocks
with the restriction of basis states

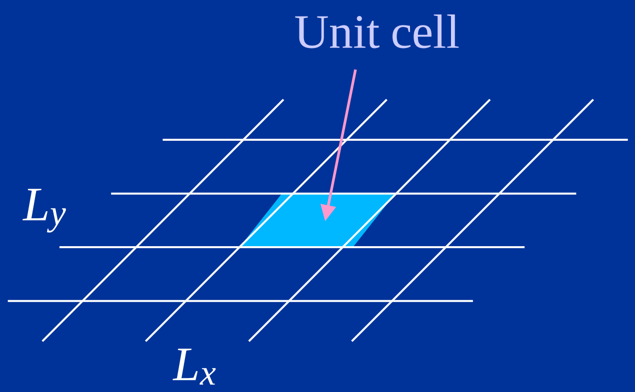
Steven White (1992)

- Variational method



- Controlled accuracy

Application of DMRG



Periodic boundary conditions
for both x and y directions

$$k_y = 2\pi n/L_y = X_n / l^2$$

Initial basis states (Landau gauge)

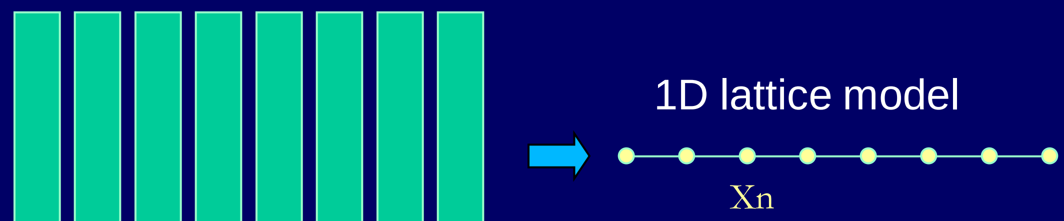
$$\varphi_{XN}(\mathbf{r}) = \exp\left[\, i \frac{X_n y}{l^2} - \frac{(x - X_n)^2}{2l^2} \right] H_N\left(\frac{x - X_n}{l}\right)$$

H_N : Hermite polynomials

One particle states are uniquely
specified by X_n and N

X_n: guiding center
N : Landau level index

Mapping on to effective 1D lattice model



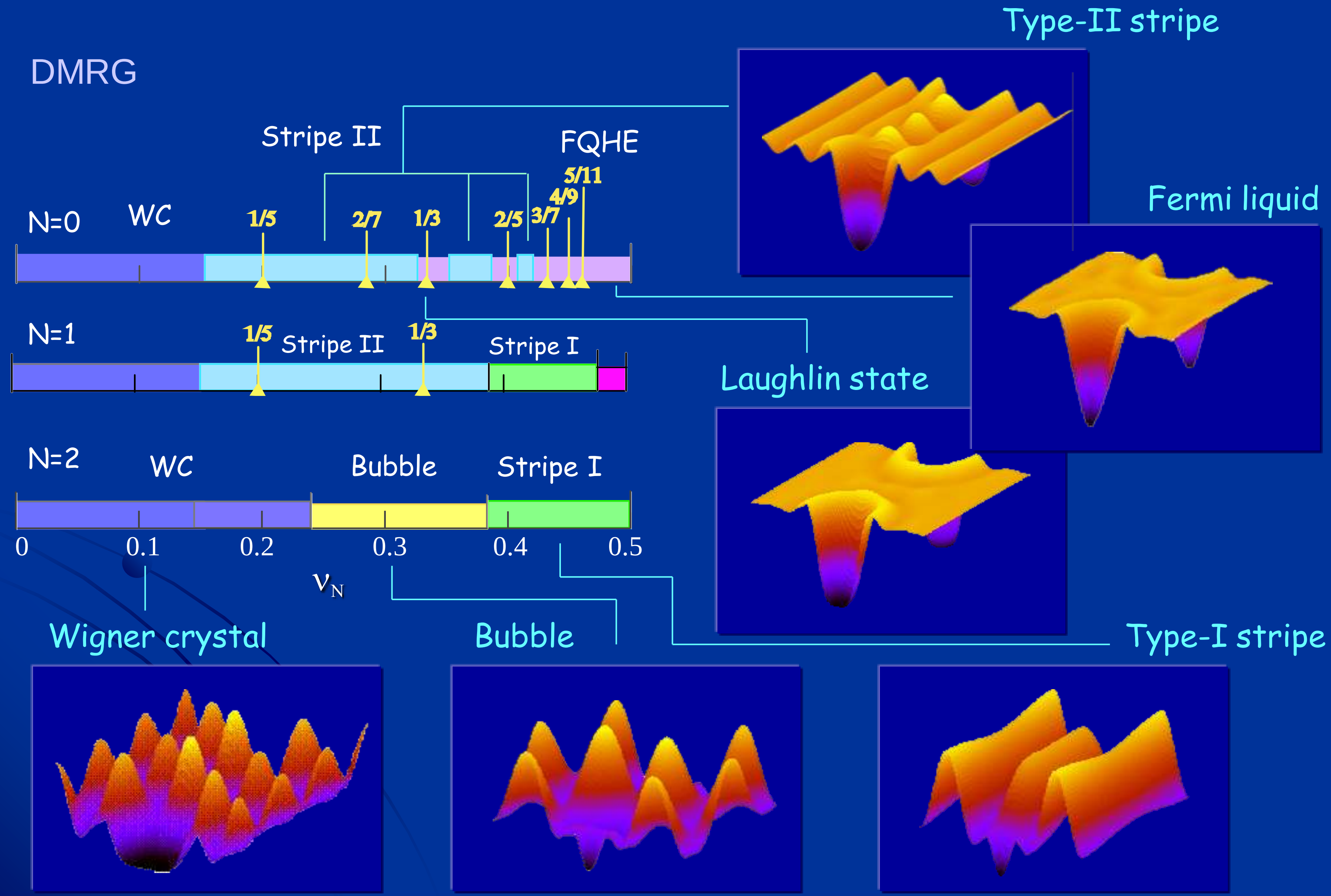
Ground state energy (Ne=10)

DMRG	m=100	-3.239340
	m=200	-3.239686
	m=300	-3.239981
	m=400	-3.239993
	Exact	-3.239995

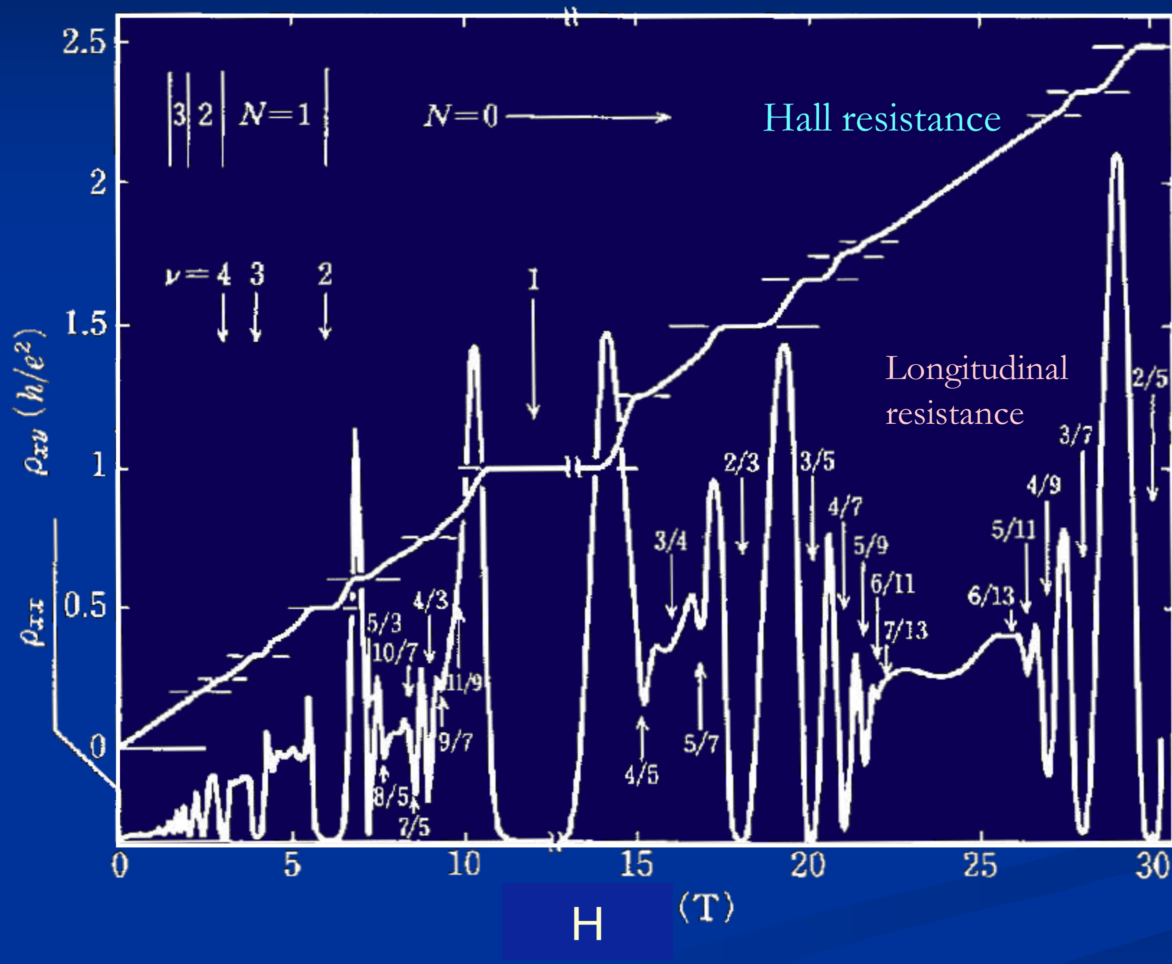
N=2
$\nu=1/2$

Ground state of Quantum Hall systems

DMRG

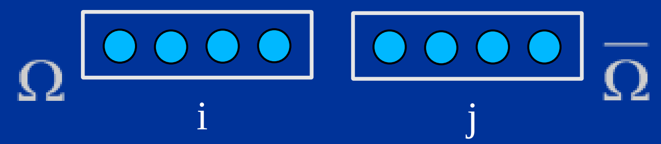


Fractional quantum Hall effect



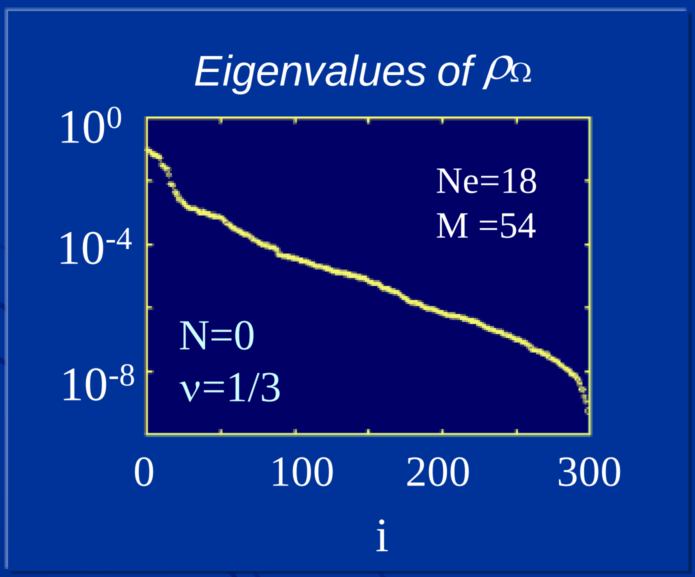
R. Willett *et al* (1987)

DMRG and Entanglement entropy



Ground state $|\Psi\rangle = \Psi_{ij}\,|i\rangle\,|j\rangle$

Density matrix $(\rho_\Omega)_{ii'} = \sum_j \Psi_{i'j}\,\Psi_{ij}$



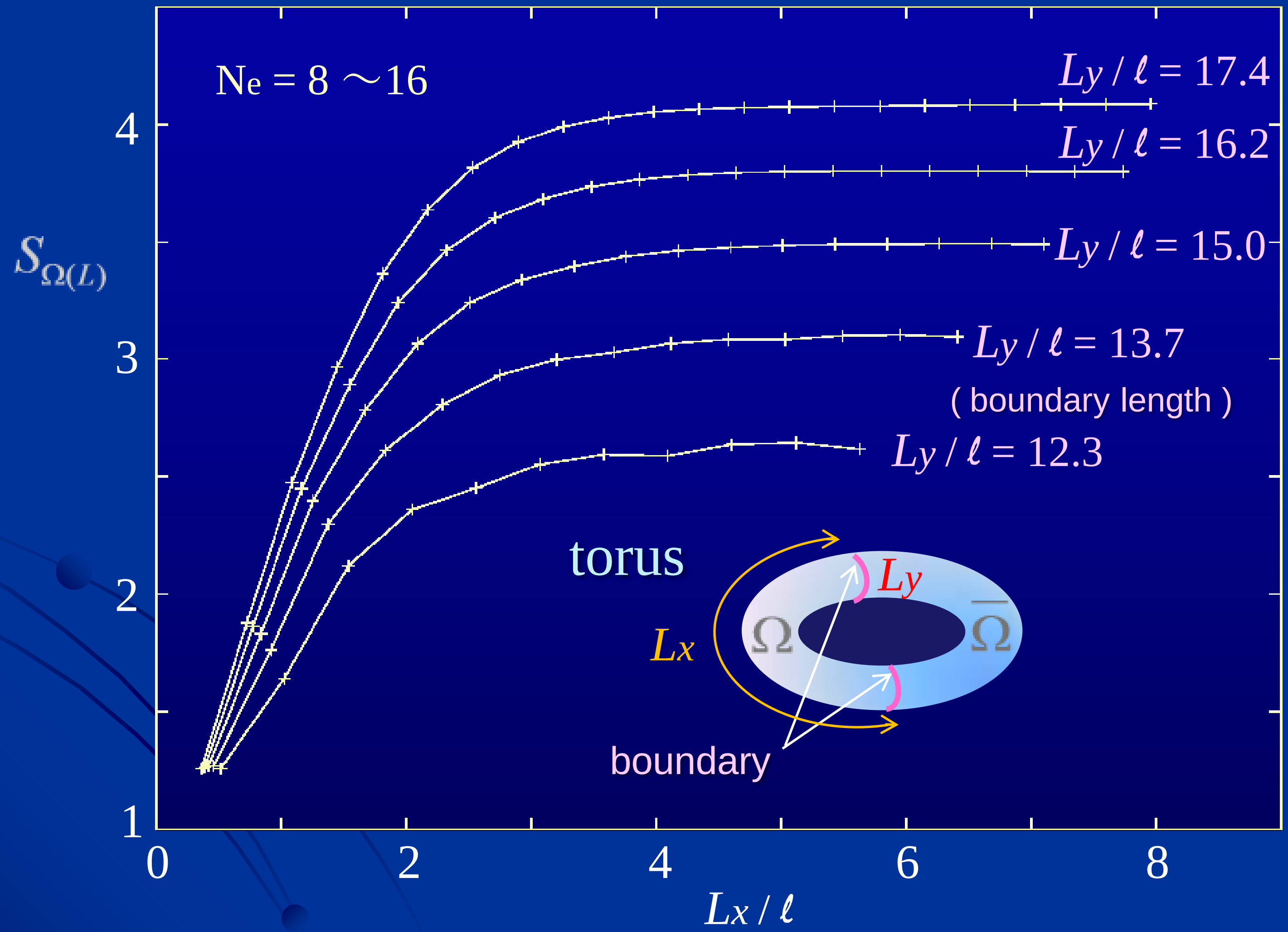
$$S_\Omega = -Tr\,\rho_\Omega \ln \rho_\Omega$$

Entanglement entropy is calculated from eigenvalues of ρ_Ω

Entanglement entropy

Torus geometry

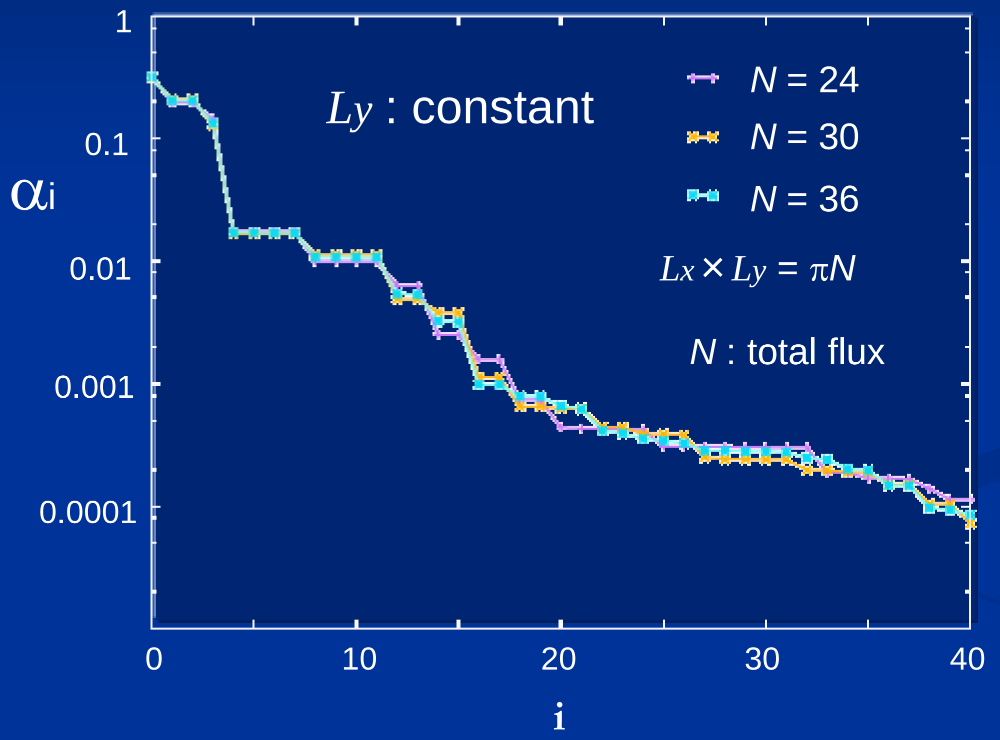
Fractional quantum Hall state $\nu = 1/3$



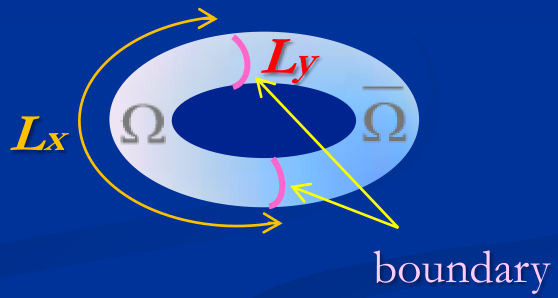
Fixed boundary length

Fractional quantum Hall state $\nu = 1/3$

density matrix eigenvalues α_i



Torus geometry



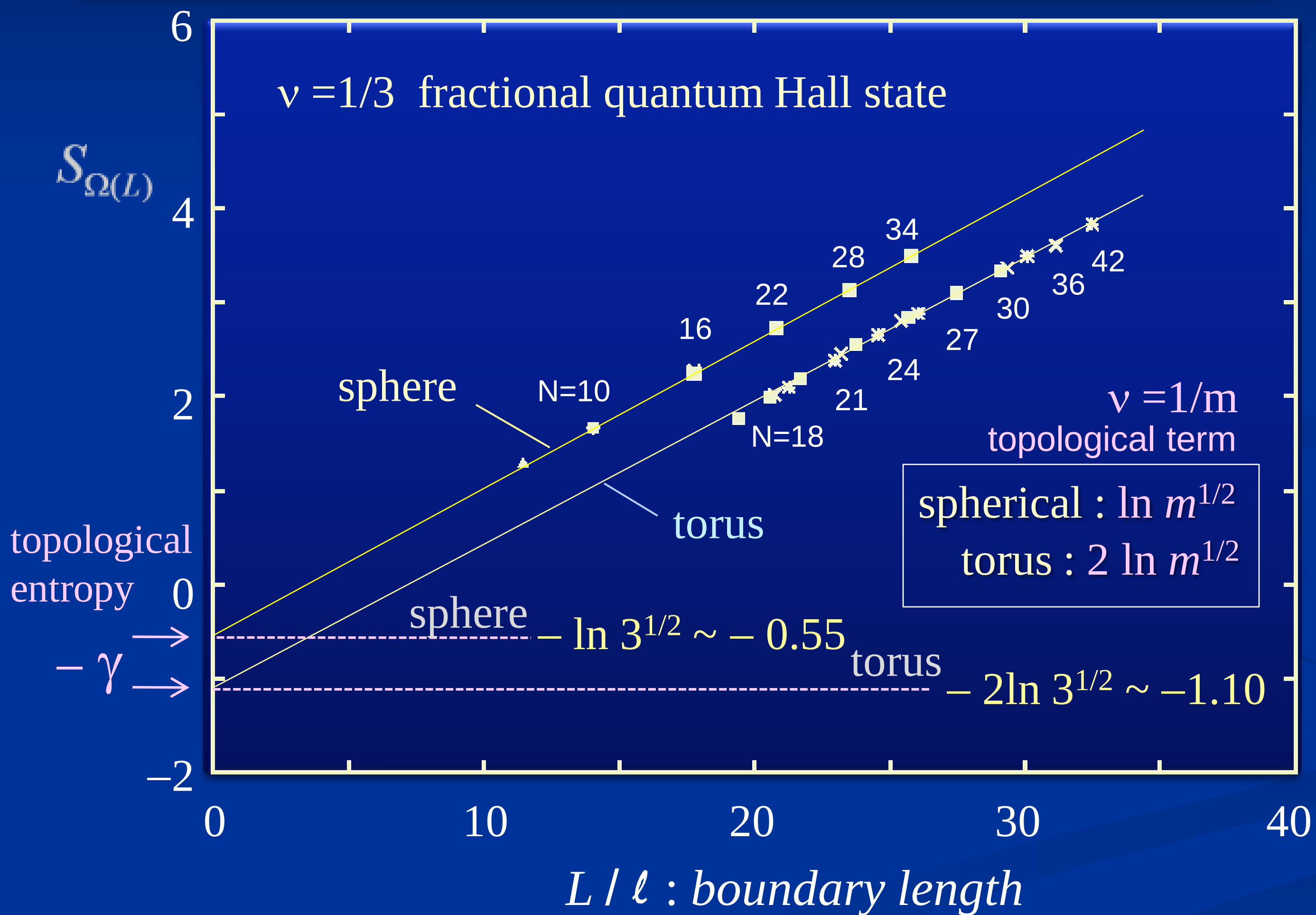
α_i is almost the same when the boundary length is the same

→ **Area law**

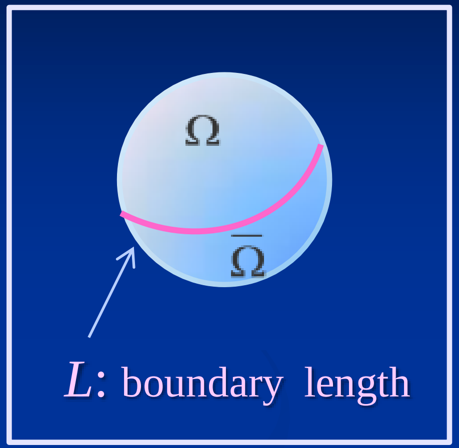
Entanglement entropy

entanglement entropy $S_{\Omega(L)} = \underline{\alpha L} - \underline{\gamma}$ → topological term

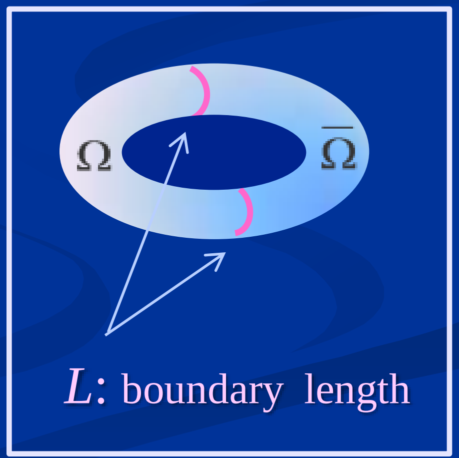
↓ boundary term



sphere



torus



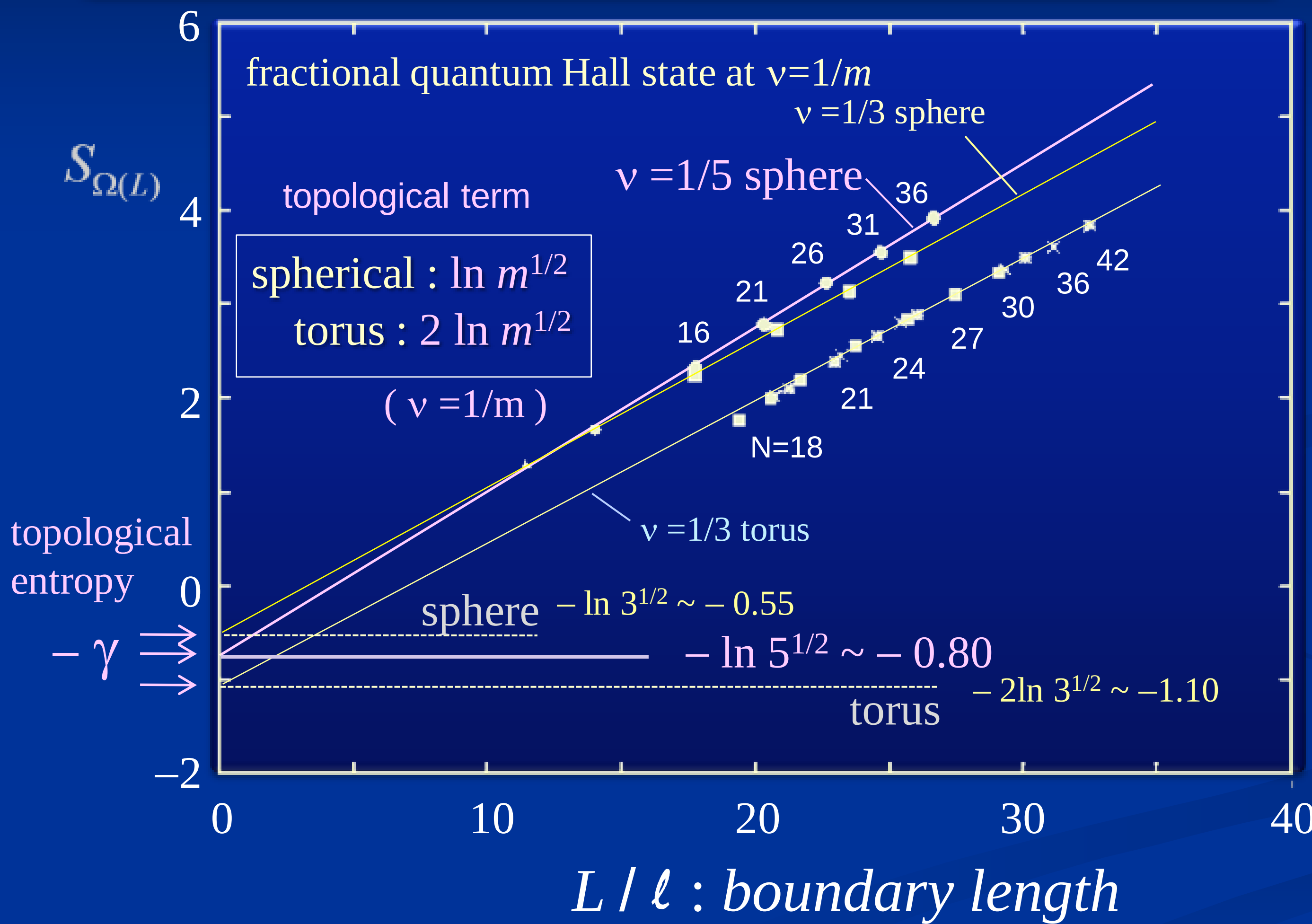
Disk geometry : Haque, Zozulya, and Schoutens, PRL, 2007

Torus geometry : Friedman and Levine, PRB, 2008

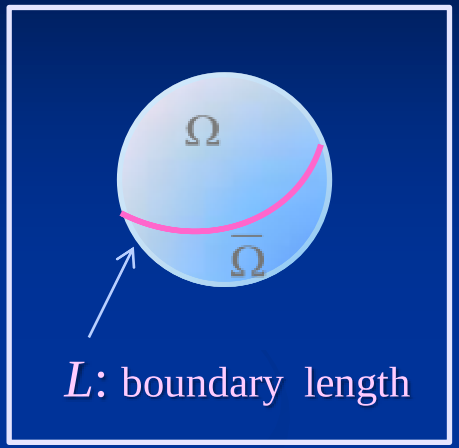
Entanglement entropy

entanglement entropy $S_{\Omega(L)} = \underline{\alpha L} - \underline{\gamma}$ → topological term

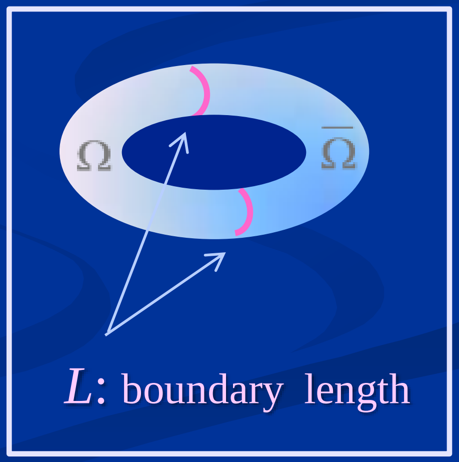
↓ boundary term



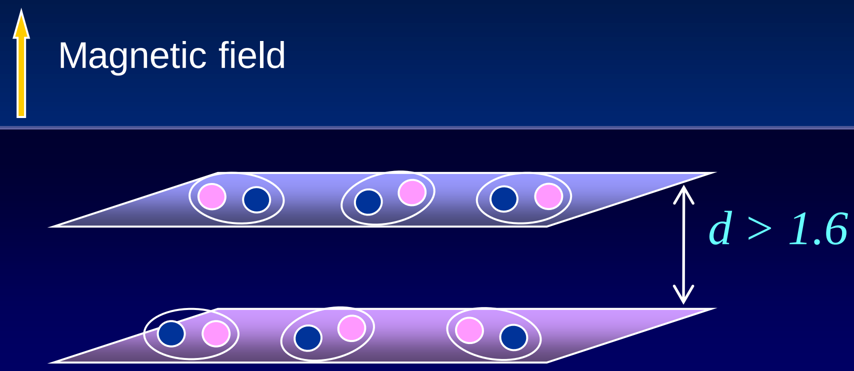
sphere



torus

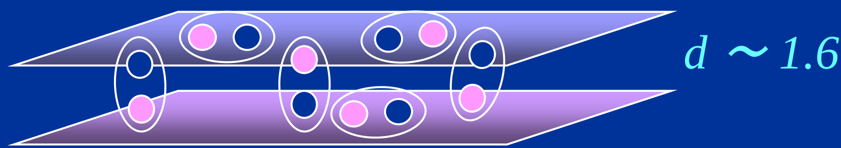


Bilayer quantum Hall system at ν=1



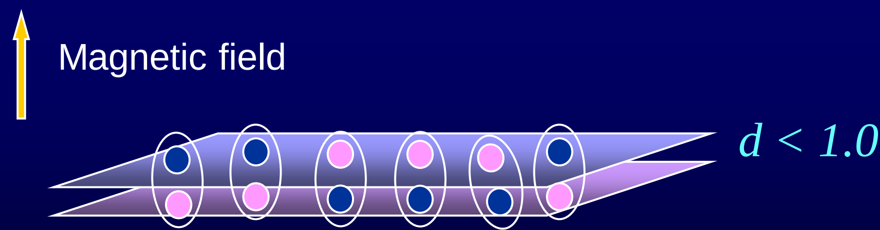
Composite fermion liquid

Gapless Fermi liquid



Brake down of composite fermion

Continuous transition



Excitonic state

Macroscopic coherence of excitons

Finite charge gap

Zero pseudo-spin gap

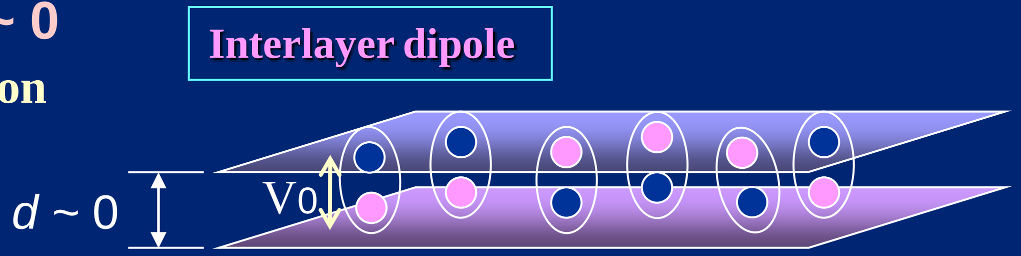
V_0-V_1 bilayer system at $\nu=1$

small interlayer distance $d \sim 0$

strong inter-layer Coulomb repulsion

⇓

Inter-layer interaction V_0

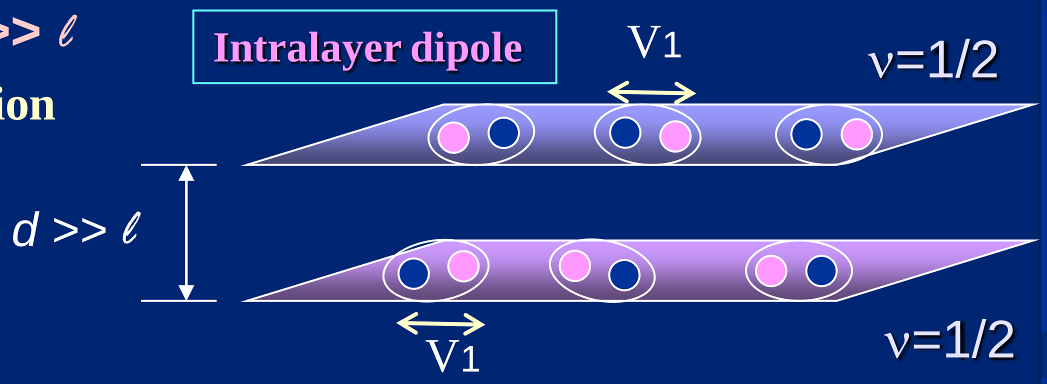


large interlayer distance $d \gg \ell$

strong intra-layer Coulomb repulsion

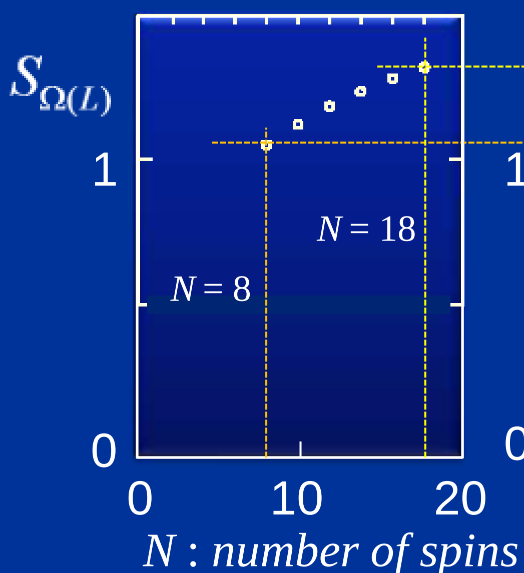
⇓

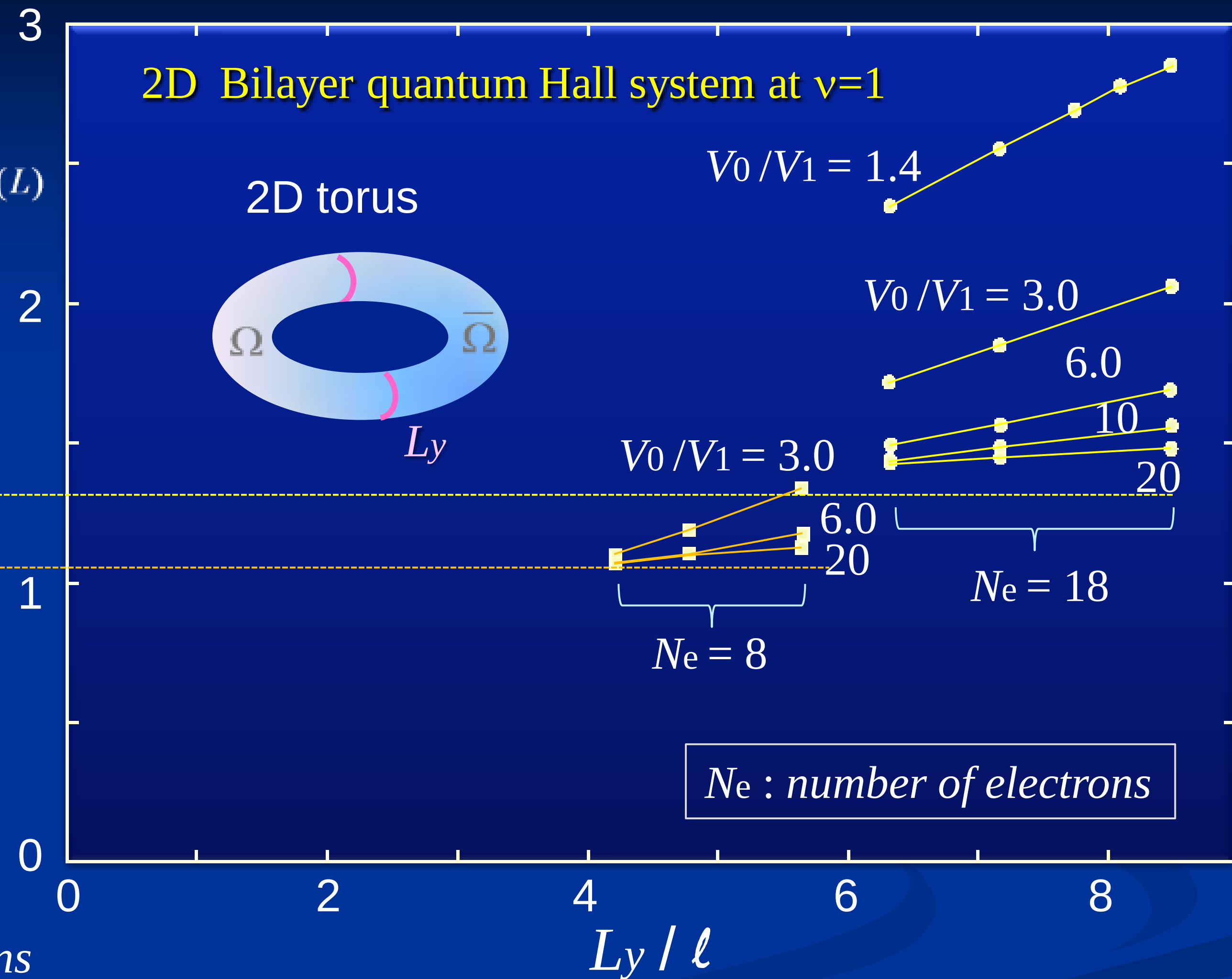
Intra-layer interaction V_1



Relation between 2D systems and 1D systems

1D Heisenberg model





2D topological charge entropy $= 2 \ln m^{1/2} = 0$ $(\nu = 1/m = 1)$

1D spin entropy = 2D pseudo-spin entropy

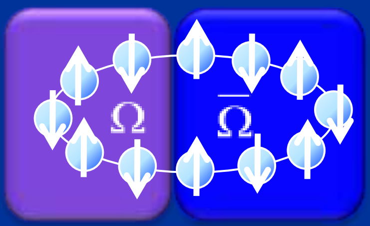
Summary

DMRG is applied to quantum Hall systems

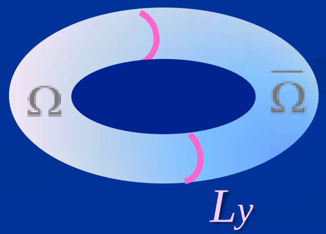
Entanglement entropy

Measure of
- Entanglement between two regions
- Correlation and topological order
- Similarity between different models

1D ring



2D torus



sphere