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Solutions of the Maxwell and Yang–Mills Equations Associated with Hopf Fibrings

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Abstract

It is shown that the magnetic pole of lowest strength and the pseudoparticle solution of the Yang–Mills equations correspond to natural connections defined on the principal bundles $U(2)/U(1) = S_3 \to S_2$ and $Sp(2)/Sp(1) = S_7 \to S_4$, respectively. This observation leads to a general method of constructing new, topologically nontrivial solutions of the Maxwell and Yang–Mills equations. Among them is an "electromagnetic instanton" defined over the two-dimensional complex projective space endowed with the Fubini-Study metric.

Recent theoretical work on the properties of magnetic poles (Nambu, 1974; Parker, 1975; Goldhaber, 1976; Wu and Yang, 1976; many references are given by Goldhaber and Smith, 1975) and on the Yang–Mills instanton (Belavin et al., 1975; Hooft, 1976a, b; Jackiw and Rebbi, 1976a; Callan et al., 1976) encouraged me to consider the geometrical models that can be associated with the corresponding classical gauge fields. It is known that electromagnetism and the Yang–Mills theory admit an interpretation in terms of connections and curvatures on principal bundles with the structure groups $U(1)$ and $SU(2)$, respectively (Yang and Mills, 1954; Lubkin, 1963; Trautman, 1970). Clearly, the $U(1)$ bundle carrying a connection corresponding to a magnetic pole is nontrivial (Wu and Yang, 1975; Ezawa and Tze, 1976). Consider a magnetic pole at rest relative to an inertial frame in Minkowski space-time R^4; the manifold R^4 with the worldline of the pole removed is diffeomorphic to $R^2 \times S_2$. One is thus led to consider circle bundles over S_2; they are all known. The "simplest," nontrivial among them was described by Hopf (1931) in the same year Dirac (1931) published his paper on magnetic poles.

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Let z_0, z_1 be two complex numbers, then $\bar{z}_0 z_0 + \bar{z}_1 z_1 = 1$ defines a three-dimensional sphere S_3. The group $U(1)$ acts on S_3 by $(z_0, z_1)u = (z_0 u, z_1 u)$, where $u \in U(1)$, i.e., $\bar{u}u = 1$. The orbits (fibers) of $U(1)$ in S_3 are circles and the quotient of S_3 by this action is S_2. The projection $S_3 \to S_2$ is given by a composition of $(z_0, z_1) \mapsto z_1/z_0$ with the stereographic map $C \to S_2$. This Hopf fiber bundle admits a natural connection, which may be conveniently expressed in terms of the Euler angles: Set

$$z_0 = [\exp \tfrac{1}{2} i (\chi + \phi)] \cos \tfrac{1}{2}\theta, \qquad z_1 = [\exp \tfrac{1}{2} i (\chi - \phi)] \sin \tfrac{1}{2}\theta$$

and compute the Riemannian line element of S_3,

$$4(d\bar{z}_0 dz_0 + d\bar{z}_1 dz_1) = d\theta^2 + \sin^2 \theta \, d\phi^2 + (d\chi + \cos\theta \, d\phi)^2$$

The form $\alpha = \frac{1}{2}(d\chi + \cos\theta \, d\phi)$ defines a connection on S_3 considered as a circle bundle over S_2. Its curvature $F = \frac{1}{2} \sin\theta \, d\phi \wedge d\theta$, extended to Minkowski space-time is the electromagnetic field of a magnetic pole of strength $g = \frac{1}{2}$. (The units are such that the charge of the electron is equal to the fine-structure constant). The form α is smooth and invariant under the transitive action of $U(2)$ on S_3. The singularities of the potentials of the magnetic pole are due to the nontrivial character of the bundle $S_3 \to S_2$. The map s, sending S_2, with the north pole ($\theta = 0$) removed, into S_3, and defined by $s(\theta, \phi) = (z_0 = e^{i\phi} \cos \frac{1}{2}\theta, z_1 = \sin \frac{1}{2}\theta)$ is smooth, but it cannot be extended throughout S_2. Therefore, s is only a local section and the potential A in the gauge s, $A = s^*(\alpha) = \frac{1}{2}(1 + \cos\theta) d\phi$, is singular at $\theta = 0$ because its essential component with respect to an orthonormal frame is $A_\phi = (1 + \cos\theta)/2r \sin\theta$.

The above construction may be generalized by considering multidimensional spaces and allowing the coordinates $z_\alpha \in K$ to be either complex ($K = C$) or quaternionic (Finkelstein et al., 1973) ($K = H$). The equation

$$\bar{z}_0 z_0 + \bar{z}_1 z_1 + \cdots + \bar{z}_n z_n = 1 \tag{1}$$

defines an S_{2n+1} or an S_{4n+3}, depending on whether $K = C$ or H. The group $G(n+1)$ of linear, K-valued transformations acting on the z's on the left and preserving the quadratic form (1) is $U(n+1)$ in the first, and $Sp(n+1)$ in the second case (Steenrod, 1951; Husemoller, 1966). The group $Sp(1)$ of unit quaternions is isomorphic to $SU(2)$. In either case, the group $G(1)$ acts freely on the sphere (1) by $(z_0, \ldots, z_n)u = (z_0 u, \ldots, z_n u)$, $u \in G(1)$. The quotient of (1) by this action is the projective space in n dimensions over K. There are thus two sequences of Hopf principal fiber bundles:

$$S_{2n+1} \to CP_n \qquad \text{with group } U(1)$$

$$S_{4n+3} \to HP_n \qquad \text{with group } Sp(1) = SU(2)$$

Assuming $z_0 \neq 0$ one can introduce a local trivialization of the sphere (1) by writing $z_0 = \rho u$ and $z_a = \zeta_a z_0$, where $\rho = |z_0| > 0$ and $a = 1, \cdots, n$. It follows from these definitions that $u \in G(1)$ and $\rho^{-2} = 1 + \Sigma_a \bar{\zeta}_a \zeta_a$. The

ζ's constitute a local coordinate system on the projective space. The Riemannian line element on the sphere is

$$dl^2 = \sum_{\alpha=0}^{n} d\bar{z}_\alpha dz_\alpha$$

and may be computed in terms of u and ζ_a:

$$dl^2 = ds^2 - \omega^2$$

where

$$\omega = u^{-1}du + \tfrac{1}{2}\rho^2 u^{-1} \sum_a [\bar{\zeta}_a d\zeta_a - (d\bar{\zeta}_a)\zeta_a] u$$

and ds^2 is the symmetric part of the positive definite Hermitean form

$$\sum_{a,\,b} d\bar{\zeta}_a h_{ab} d\zeta_b$$

with $\bar{h}_{ab} = h_{ba}$ given by

$$\Omega = d\omega + \omega \wedge \omega = u^{-1} \sum_{a,\,b} (d\bar{\zeta}_a \wedge h_{ab} d\zeta_b) u \tag{2}$$

The forms $u^{-1}du$, ω and Ω have values in the Lie algebra of $G(1)$, i.e., in the pure imaginary subspace of K. Therefore, the quadratic form $-\omega^2$ is positive definite. Since both the latter form and dl^2 are invariant under the action of $G(1)$, so is ds^2 and it defines a Riemannian metric on the projective space. In the complex case, $\omega \wedge \omega = 0$, and, if one writes $\omega = i\alpha$, $\Omega = iF$, then both α and F are real, and F is the Hodge form (Weil, 1958; Chern, 1967; Morrow and Kodaira, 1971) of CP_n.

The fundamental result of this paper is that, for any n, Ω given by (2) is a solution of the source-free Maxwell ($K = C$) or Yang-Mills ($K = H$) equations, invariant under $SU(n + 1)$ or $Sp(n + 1)$, respectively. To prove this, we note that Ω satisfies the Bianchi identity,

$$D\Omega = d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0$$

and is invariant under $G(n + 1)$ by construction. The $2n$ form $F \wedge \cdots \wedge F$ (n factors) is a volume element on CP_n, whereas the $4n$ form $\Omega \wedge \cdots \wedge \Omega$ ($2n$ factors) plays a similar role on HP_n. These volume elements define orientations which, together with ds^2, determine the duals of differential forms. The dual $*\Omega$ of Ω is proportional to $\Omega \wedge \cdots \wedge \Omega$, where the exterior product contains $n - 1$ factors for CP_n and $2n - 1$ factors for HP_n. Therefore, the Bianchi identity implies that the gauge field Ω is source-free:

$$D{*\Omega} = 0$$

For example, the Belavin–Polyakov–Schwartz–Tyupkin solution corresponds to $K = H$ and $n = 1$: There is then one quaternion coordinate ζ, $\rho^{-2} = 1 + \bar{\zeta}\zeta$, and

$$ds^2 = \rho^4 d\bar{\zeta} d\zeta$$

is the line-element of a four-dimensional sphere of radius $\frac{1}{2}$. The local section $u = 1$ leads to the potential $\frac{1}{2}\rho^2 [\bar{\zeta} d\zeta - (d\bar{\zeta})\zeta]$ and the field $\rho^4\ d\bar{\zeta} \wedge d\zeta$. The action of $Sp(2)$ on S_7 projects to an action of $SO(5)$ on $HP_1 = S_4$ and the solution is invariant under the latter group (Jackiw and Rebbi, 1976b, Yang, 1977).

A new solution of Maxwell's equations is obtained for $K = C$ and $n = 2$. In local coordinates on CP_2 given by $\zeta_1 = e^{i\mu} \tan\theta \cos\phi$, $\zeta_2 = e^{i\nu} \tan\theta \sin\phi$, the electromagnetic field is

$$F = \sin 2\theta\, d\theta \wedge (\cos^2\phi\, d\mu + \sin^2\phi\, d\nu) - \sin^2\theta \sin 2\phi\, d\phi \wedge (d\mu - d\nu) \quad (3)$$

whereas the Fubini-Study metric assumes the form

$$ds^2 = d\theta^2 + \sin^2\theta\, [d\phi^2 + \cos^2\theta\, (\cos^2\phi\, d\mu + \sin^2\phi\, d\nu)^2 + \sin^2\phi \cos^2\phi\, (d\mu - d\nu)^2] \quad (4)$$

The field (3) is self-dual, $*F = F$, and its energy-momentum tensor vanishes. Therefore, equations (3) and (4) define a solution of Einstein's equations with a cosmological term. Following a suggestion by Eguchi and Freund (1976), this solution, which is invariant under $SU(3)$, could be called the gravitational and *electromagnetic instanton.* The integral $\int F \wedge F$ associated with the second Chern class is equal to $4\pi^2$.

If X is an analytic submanifold of CP_n, then the embedding $k : X \to CP_n$ may be used to pull the Hodge form F from CP_n back to X and to define thus a new solution of Maxwell's equations on X. For example, for any positive integer n there is an embedding $k_n : S_2 = CP_1 \to CP_n$ given in terms of the homogeneous coordinates (z_α) by

$$k_n(z_0, z_1) = (z_0{}^n, \tbinom{n}{1}^{1/2} z_0^{n-1} z_1, \ldots, \tbinom{n}{m}^{1/2} z_0^{n-m} z_1^m, \ldots, z_1{}^n)$$

An electromagnetic field pulled by k_n from CP_n to S_2 corresponds to a magnetic pole of strength $g = n/2$. Moreover, k_n induces over S_2 a circle bundle isomorphic to the lens space $L(n, 1)$ (Greenberg, 1967).

An interesting possibility, now under investigation, is to generalize the method described in this paper to spaces with an indefinite metric, by replacing the groups $U(n)$ and $Sp(n)$ by $U(p, q)$ and $Sp(p, q)$, respectively.

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