An Elementary Approach to the Weight Multiplicities for Sp(4)

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1 Introduction

Complex semisimple Lie algebras have the complete reducibility property. Each complex finite-dimensional irreducible representation of a complex semisimple Lie algebra is parameterized by a highest weight. Each finite-dimensional irreducible representation also has a unique weight diagram including a specific multiplicity for each weight; the multiplicity of a weight is the dimension of the corresponding weight space in the representation. These multiplicities have been the topic of many research efforts. Several formulas for computing these multiplicities have been developed by Freudenthal [3], Kostant [7], Lusztig [10], Littelmann [9], and Sahi [12]. Many of these formulas are general and recursive.

In this note, we will focus on the weight multiplicities of finite-dimensional representations of the classical rank two Lie algebra $\mathfrak{sp}(4, \mathbb{C})$ corresponding to the Lie group Sp(4). These multiplicities are surprisingly difficult to obtain considering there is a nice formula for the weight multiplicities for another classical rank two Lie algebra, $\mathfrak{sl}(3, \mathbb{C})$. In 2004 in [2], Cagliero and Tirao gave an explicit closed formula for the weight multiplicities of any irreducible representation of this Lie algebra, and to the best of our knowledge, this was the first paper to do so. The method of proof in [2] employed a Howe duality theorem and the explicit decomposition of tensor products of exterior powers of fundamental representations of Sp(4). In this note, we will provide an alternate, elementary approach to finding an explicit closed formula for the weight multiplicities of any irreducible representation of $\mathfrak{sp}(4, \mathbb{C})$.

We first present a useful identity between finite-dimensional representations of the rank 2 symplectic Lie algebra. In Section 3, using a basic approach, we develop this first identity. It is based on a general result involving multilinear algebra for symmetric tensors; see Proposition 2.1 and Corollary 2.2 from Section 2. While these are certainly well known to experts, we have included proofs for completeness. Proposition 3.1 (and subsequently Corollary 3.2), follows from this together with the explicit determination of certain highest weight vectors occurring in a tensor product of symmetric powers of the standard representation of $\mathfrak{sp}(4, \mathbb{C})$. Corollary 3.2 then shows how an irreducible representation can be expressed as a linear combination of tensor products of symmetric powers of the standard representation. These results can also be found using Littelmann's paper [8] and Young tableaux or using a formula involving characters from Section 24.2 in [4].

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In Section 4, we determine the weight multiplicities of any dominant weight in a tensor product of symmetric powers of the standard representation. In Section 5, we use the results of Sections 3 and 4 to create an explicit closed formula for the weight multiplicities of the dominant weights in any irreducible representation of $\mathfrak{sp}(4,\mathbb{C})$.

2 A result on symmetric tensors

For a positive integer n, let S_n be the symmetric group on n letters. For this section, let V be a finite-dimensional vector space over a field with characteristic zero, F. S_n acts linearly on $V^{\otimes n}$ by $\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$. Let $\mathrm{sym} : V^{\otimes n} \to V^{\otimes n}$ be the usual symmetrization map, i.e., $\mathrm{sym}(v) = \sum_{\sigma \in S_n} \sigma(v)$. The kernel of this map is spanned by all elements of the form $v - \sigma(v)$ for $v \in V^{\otimes n}$ and $\sigma \in S_n$. We denote by $\mathrm{Sym}^n(V)$ the image of sym or equivalently the quotient of $V^{\otimes n}$ by the kernel of sym.

Let V have the basis $\{v_1, v_2, \ldots, v_k\}$, and let V^* be the dual space with corresponding dual basis $\{f_1, f_2, \ldots, f_k\}$. Let W be another finite-dimensional vector space over F with the basis $\{w_1, w_2, \ldots, w_l\}$. Using the given bases of V and W, we identify the standard basis elements of $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m W$ with pairs of k- and l-tuples $(c_1, \ldots, c_k) \times (d_1, \ldots, d_l)$ such that, for a particular basis element, c_i equals the number of times v_i appears in that basis element and d_j equals the number of times w_j appears in that basis element. When $W = V^*$, $l = k$ and the standard basis elements of $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^*$ are identified with pairs of k-tuples such that c_i is as before and d_j equals the number of times f_j appears in the basis element. The standard basis for $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m W$ is then given by

$$\{(c_1, \ldots, c_k) \times (d_1, \ldots, d_l) \mid c_i \in \mathbb{Z}_{\geq 0}, d_j \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{k} c_i = n, \sum_{j=1}^{l} d_j = m\}.$$

For $n, m \geq 1$, consider the linear map $\rho : \mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V^* \to \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^*$ with the property

$$\begin{aligned} &\mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_{n-1}) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_{m-1}) \\ &\qquad \longmapsto \sum_{i=1}^{k} \mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_{n-1} \otimes v_i) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_{m-1} \otimes f_i). \end{aligned}$$

This is the map defined as multiplication by the element $\sum_{i=1}^{k} v_i \otimes f_i$, which generates the trivial representation in $V \otimes V^*$. The following shows ρ is an injective intertwining map.

Proposition 2.1. *Let V and W be finite-dimensional representations of a Lie algebra L, such that there is a trivial representation contained in $V \otimes W$. For integers $n, m \geq 1$, let $\phi : \mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} W \to \mathrm{Sym}^n V \otimes \mathrm{Sym}^m W$ be the linear map defined as multiplication by some fixed generator of the trivial representation. Then ϕ is an injective intertwining map.*

Proof. Given V and W are representations with the previously-defined bases, let $\sum_{i,j} a_{ij} v_i \otimes w_j$, for some coefficients a_{ij}, generate a trivial representation in $V \otimes W$, and assume without loss of generality $a_{11} \neq 0$. Then ϕ becomes the linear map defined by the property

$$\begin{aligned} &\mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_{n-1}) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_{m-1}) \\ &\qquad \longmapsto \sum_{i=1}^{k} \sum_{j=1}^{l} a_{ij} \mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_{n-1} \otimes v_i) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_{m-1} \otimes w_j). \end{aligned}$$

To prove injectivity directly, consider an element of the kernel of ϕ written as a linear combination of basis vectors of $\mathrm{Sym}^{n-1}V \otimes \mathrm{Sym}^{m-1}W$ such that each basis vector has the form $(n-r_1, c_2, \ldots, c_k) \times (m-r_2, d_2, \ldots, d_l)$ for some r_1, $1 \leq r_1 \leq n$, and some r_2, $1 \leq r_2 \leq m$, with $c_i, d_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=2}^{k} c_i = r_1 - 1$, and $\sum_{i=2}^{l} d_i = r_2 - 1$. Using a similar form for the basis of $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m W$, it is then straightforward to prove by induction on $s = r_1 + r_2$ that each of the coefficients in this linear combination equals zero, thus proving $\ker(\phi) = \{0\}$.

To prove injectivity an alternate way, let $\mathrm{Sym}(V)$ be the algebra $\bigoplus_{x=0}^{\infty} \mathrm{Sym}^x V$. Then

$$\mathrm{Sym}(V) \otimes \mathrm{Sym}(W) = \mathrm{Sym}(V+W)$$

is isomorphic to the algebra of polynomials on $(V^* + W^*)$ over F, which has no zero divisors. This implies ϕ is injective because in this setting, ϕ is equivalent to multiplying certain homogeneous degree $n-1+m-1$ polynomials by a fixed homogeneous degree 2 polynomial.

The intertwining property of ϕ is easy to verify using the fact that $\sum_{i,j} a_{ij} v_i \otimes w_j$ generates a trivial representation in $V \otimes W$. This concludes the proof. $\square$

The dual map to ρ (with n and m interchanged) is the linear map $\rho^* : \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^* \to \mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V^*$ with the property

$$\begin{aligned} &\mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_n) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_m) \\ &\quad \longmapsto \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j(\alpha_i) \mathrm{sym}(\alpha_1 \otimes \ldots \otimes \hat{\alpha_i} \otimes \ldots \otimes \alpha_n) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \hat{\beta}_j \otimes \ldots \otimes \beta_m). \end{aligned}$$

ρ^* is a surjective intertwining map.

We obtain the following result from Proposition 2.1.

Corollary 2.2. *Let V be a finite-dimensional representation of a Lie algebra. Then there exists an invariant subspace*

$$\mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V^* \subset \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^* \textit{ for all integers } n, m \geq 1.$$

3 The case of $\mathfrak{sp}(4, \mathbb{C})$

We will apply the above result of Corollary 2.2 to representations of the Lie algebra $\mathfrak{sp}(4, \mathbb{C})$, where

$\mathfrak{sp}(4, \mathbb{C}) = \{A \in \mathfrak{gl}(4, \mathbb{C}) \mid A^t J + JA = 0\}$ and $J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$.

Evidently, $\mathfrak{sp}(4, \mathbb{C})$ is 10-dimensional and has the following basis,

$$H_1 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{bmatrix}$$

$$X_{\alpha_1} = \begin{bmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&0&0 \end{bmatrix} \qquad X_{2\alpha_1+\alpha_2} = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \qquad X_{\alpha_1+\alpha_2} = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \qquad X_{\alpha_2} = \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

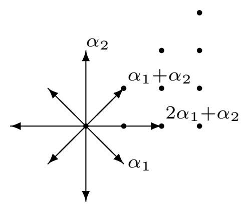
$$Y_{\alpha_1} = \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&-1&0 \end{bmatrix} \qquad Y_{2\alpha_1+\alpha_2} = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{bmatrix} \qquad Y_{\alpha_1+\alpha_2} = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix} \qquad Y_{\alpha_2} = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{bmatrix}.$$

In this basis, the simple roots are α_1 and α_2, the Cartan subalgebra is $\mathfrak{h} = \langle H_1, H_2 \rangle$, and for each root α,

$$\mathfrak{s}^\alpha = \operatorname{span}\{X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]\} \cong \mathfrak{sl}(2, \mathbb{C}).$$

Any weight (w_1, w_2) can be thought of as the pair of eigenvalues associated to H_1 and H_2, respectively, for the corresponding weight vector. The dominant Weyl chamber is $\{(n, m) \in \mathbb{Z} \times \mathbb{Z} : n \geq m \geq 0\}$. Let $V(n, m)$ be the irreducible representation with highest weight (n, m), $n \geq m$.

The following displays the root system and the first few weights of the dominant Weyl chamber.



The Weyl dimension formula, tailored to our situation, appears in [5], Section 7.6.3. It states that

$$\dim V(n, m) = \frac{1}{6}(n - m + 1)(m + 1)(n + 2)(n + m + 3).$$

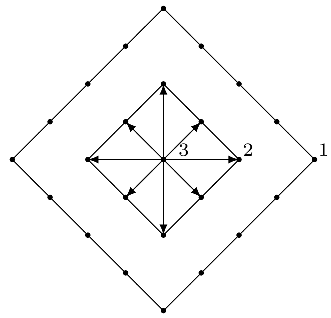
The following table displays $V(1, 0)$, the standard representation of $\mathfrak{sp}(4, \mathbb{C})$, for this previously defined basis of $\mathfrak{sp}(4, \mathbb{C})$ and the standard basis of $\mathbb{C}^4$ and its dual representation with corresponding basis $\{f_1, f_2, f_3, f_4\}$. These representations are isomorphic via $f_1 \mapsto -e_4$, $f_2 \mapsto -e_3$, $f_3 \mapsto e_2$, $f_4 \mapsto e_1$, but the different formulas for both of them will be used in subsequent calculations.

	e_1	e_2	e_3	e_4	f_1	f_2	f_3	f_4
H_1	e_1	0	0	$-e_4$	$-f_1$	0	0	f_4
H_2	0	e_2	$-e_3$	0	0	$-f_2$	f_3	0
X_{α_1}	0	e_1	0	$-e_3$	$-f_2$	0	f_4	0
$X_{2\alpha_1+\alpha_2}$	0	0	0	e_1	$-f_4$	0	0	0
$X_{\alpha_1+\alpha_2}$	0	0	e_1	e_2	$-f_3$	$-f_4$	0	0
X_{α_2}	0	0	e_2	0	0	$-f_3$	0	0
Y_{α_1}	e_2	0	$-e_4$	0	0	$-f_1$	0	f_3
$Y_{2\alpha_1+\alpha_2}$	e_4	0	0	0	0	0	0	$-f_1$
$Y_{\alpha_1+\alpha_2}$	e_3	e_4	0	0	0	0	$-f_1$	$-f_2$
Y_{α_2}	0	e_3	0	0	0	0	$-f_2$	0

The weights of $V(1, 0)$ are $\{(1, 0), (0, 1), (0, -1), (-1, 0)\}$, and e_1 is a highest weight vector.

It can be easily shown that $V(n, 0) = \operatorname{Sym}^n V(1, 0)$. First, there is a highest weight vector, $\operatorname{sym}(e_1 \otimes \ldots \otimes e_1)$, in $\operatorname{Sym}^n V(1, 0)$ with weight $(n, 0)$, and therefore $V(n, 0) \subset \operatorname{Sym}^n V(1, 0)$. Then using the Weyl dimension formula, $V(n, 0)$ has the same dimension as $\operatorname{Sym}^n V(1, 0)$ and thus $V(n, 0) = \operatorname{Sym}^n V(1, 0)$.

The weight diagram for $V(n, 0) = \operatorname{Sym}^n V(1, 0)$ is a series of nested diamonds with leading weights $(n - 2i, 0)$ and with multiplicities $i + 1$ along the diamonds, $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. The following is the weight diagram for $V(4, 0)$.



Proposition 3.1. *For $\mathfrak{sp}(4,\mathbb{C})$ and its standard representation $V = V(1,0)$,*

$$\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V = (\mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V) \oplus \bigoplus_{p=0}^{m} V(n+m-p,p)$$

for integers $n \geq m \geq 1$.

Proof. Given $n \geq m$ and using the previously described basis, we define for all integers p such that $0 \leq p \leq m$ the following vector in $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^*$,

$$\begin{aligned} v_p &= \sum_{i=0}^{p} \binom{p}{i} (-1)^i (n-p+i, p-i, 0, 0) \times (0,0,i,m-i) \\ &= \sum_{i=0}^{p} \binom{p}{i} (-1)^i \mathrm{sym}(\underbrace{e_1 \otimes \ldots \otimes e_1}_{n-p+i} \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{p-i}) \\ &\qquad\qquad \otimes \mathrm{sym}(\underbrace{f_3 \otimes \ldots \otimes f_3}_{i} \otimes \underbrace{f_4 \otimes \ldots \otimes f_4}_{m-i}). \end{aligned}$$

This vector is in the kernel of the map ρ^* defined in Section 2 because $(n-p+i, p-i, 0, 0) \times (0,0,i,m-i) \mapsto 0 + \ldots + 0 = 0$. Also, this vector is a highest weight vector with weight

$$(n-p+i)(1,0) + (p-i)(0,1) + i(0,1) + (m-i)(1,0) = (n+m-p,p).$$

To see v_p is a highest weight vector, it is enough to show that it is in the kernel of X_α for any α.

First, the only relevant calculations are $X_\alpha.e_1$, $X_\alpha.e_2$, $X_\alpha.f_3$, and $X_\alpha.f_4$. These will all be equal to zero except for $\alpha = \alpha_1$. Therefore, we only need to show v_p is in the kernel of X_{α_1}. $X_{\alpha_1}.e_1 = X_{\alpha_1}.f_4 = 0$, $X_{\alpha_1}.e_2 = e_1$, and $X_{\alpha_1}.f_3 = f_4$. By definition,

$$\begin{aligned} &X_{\alpha_1}.(n-p+i, p-i, 0, 0) \times (0,0,i,m-i) \\ &\quad = X_{\alpha_1}.\mathrm{sym}(\underbrace{e_1 \otimes \ldots \otimes e_1}_{n-p+i} \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{p-i}) \otimes \mathrm{sym}(\underbrace{f_3 \otimes \ldots \otimes f_3}_{i} \otimes \underbrace{f_4 \otimes \ldots \otimes f_4}_{m-i}). \end{aligned}$$

This becomes

$$\begin{aligned} &(n-p+i)\mathrm{sym}(X_{\alpha_1}.e_1 \otimes e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2) \\ &\qquad \otimes \mathrm{sym}(f_3 \otimes \ldots \otimes f_3 \otimes f_4 \otimes \ldots \otimes f_4) \\ &+ (p-i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes X_{\alpha_1}.e_2 \otimes e_2 \otimes \ldots \otimes e_2) \\ &\qquad \otimes \mathrm{sym}(f_3 \otimes \ldots \otimes f_3 \otimes f_4 \otimes \ldots \otimes f_4) \\ &+ (i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2) \\ &\qquad \otimes \mathrm{sym}(X_{\alpha_1}.f_3 \otimes f_3 \otimes \ldots \otimes f_3 \otimes f_4 \otimes \ldots \otimes f_4) \\ &+ (m-i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2) \\ &\qquad \otimes \mathrm{sym}(f_3 \otimes \ldots \otimes f_3 \otimes X_{\alpha_1}.f_4 \otimes f_4 \otimes \ldots \otimes f_4). \end{aligned}$$

This is equal to $(n-p+i)(0)+(p-i)(n-p+i+1,p-i-1,0,0)\times(0,0,i,m-i)+(i)(n-p+i,p-i,0,0)\times(0,0,i-1,m-i+1)+(m-i)(0)$ (with the understanding that when $i=p$ there is no second term and when $i=0$ there is no third term here). From here $X_{\alpha_1}.v_p=0$ is a straightforward calculation.

For each of the highest weight vectors, v_p, with weight $(n+m-p,p)$ and in the kernel of ρ^*, there is an irreducible representation $V(n+m-p,p)$ contained in the kernel. Since all of the weights $\{(n+m-p,p) : 0\le p\le m\}$, are distinct,

$$\bigoplus_{p=0}^{m} V(n+m-p,p)\subset \ker(\rho^*).$$

It follows from semisimplicity and the surjectivity of ρ^* that

$$\begin{aligned}(\mathrm{Sym}^{n-1}V\otimes \mathrm{Sym}^{m-1}V^*)\oplus \bigoplus_{p=0}^{m} V(n+m-p,p)\\ \subset (\mathrm{Sym}^{n-1}V\otimes \mathrm{Sym}^{m-1}V^*)\oplus \ker(\rho^*)\\ = \mathrm{Sym}^{n}V\otimes \mathrm{Sym}^{m}V^*\end{aligned}$$

for $n\ge m\ge 1$. The Weyl dimension formula shows that this inclusion is actually an equality. Note that V^* can be replaced by V since this representation is self-dual. □

Note that all of the highest weight vectors in $\mathrm{Sym}^nV\otimes\mathrm{Sym}^mV$, $V=V(1,0)$, can be determined using the proof of Proposition 3.1, the map ρ from Section 2, and the isomorphism between the standard representation and its dual.

In [8], Littelmann provides a generalization of the Littlewood-Richardson rule in the cases of all simple, simply connected algebraic groups of type A_m, B_m, C_m, D_m, G_2, E_6, and partial results for F_4, E_7, and E_8. This generalization provides an algorithm for decomposing tensor products of irreducible representations using Young tableaux and can be utilized to produce the result of Proposition 3.1.

Corollary 3.2. *For integers $n\ge m=1$,*

$$V(n,0)\otimes V(1,0)=V(n+1,0)\oplus V(n,1)\oplus V(n-1,0)$$

For $n\ge m\ge 2$,

$$\begin{aligned}&(V(n,0)\otimes V(m,0))\oplus(V(n,0)\otimes V(m-2,0))\\ &\quad=(V(n+1,0)\otimes V(m-1,0))\oplus V(n,m)\oplus(V(n-1,0)\otimes V(m-1,0))\end{aligned}$$

Proof. Recall $\mathrm{Sym}^nV(1,0)=V(n,0)$. The first assertion is the special case of Proposition 3.1 where $m=1$. Using Proposition 3.1, when $n\ge m\ge 2$,

$$V(n,0)\otimes V(m,0)=(V(n-1,0)\otimes V(m-1,0))\oplus\bigoplus_{p=0}^{m}V(n+m-p,p)$$

and

$$V(n+1,0)\otimes V(m-1,0)=(V(n,0)\otimes V(m-2,0))\oplus\bigoplus_{p=0}^{m-1}V(n+m-p,p).$$

Combining these equations yields the assertion. □

In the Grothendieck group of all representations of $\mathfrak{sp}(4,\mathbb{C})$, for $V = V(1,0)$, we get

$$\begin{aligned}
V(n,0) &= \mathrm{Sym}^n V && n \geq 0\\
V(n,1) &= \mathrm{Sym}^n V \otimes V - \mathrm{Sym}^{n+1} V - \mathrm{Sym}^{n-1} V && n \geq 1\\
V(n,m) &= \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V + \mathrm{Sym}^n V \otimes \mathrm{Sym}^{m-2} V && n \geq m \geq 2\\
&\quad - \mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V - \mathrm{Sym}^{n+1} V \otimes \mathrm{Sym}^{m-1} V.
\end{aligned}$$

This result can also be derived in a less elementary way from a proposition in Section 24.2 in [4], which gives a formula for the character of an irreducible representation of a simplectic Lie algebra in terms of the characters of symmetric powers of the standard representation.

4 Weight multiplicities in $V(n,0) \otimes V(m,0)$

Since any irreducible representation of $\mathfrak{sp}(4,\mathbb{C})$ can be written as a formal combination of tensor products of symmetric powers of the standard representation, the problem of determining weight multiplicities in an irreducible representation is reduced to the problem of determining weight multiplicities in $V(n,0) \otimes V(m,0)$. We will now begin a combinatorics argument, which will produce an explicit formula for the weight multiplicities of $V(n,0) \otimes V(m,0)$.

Using previous notation, the set

$$\{(c_1,c_2,c_3,c_4) \times (d_1,d_2,d_3,d_4) | c_i, d_j \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{4} c_i = n, \sum_{j=1}^{4} d_j = m\}$$

is a basis of weight vectors for $V(n,0) \otimes V(m,0)$. The weight of $(c_1,c_2,c_3,c_4) \times (d_1,d_2,d_3,d_4)$ is $((c_1+d_1)-(c_4+d_4), (c_2+d_2)-(c_3+d_3))$. The only dominant weights of $V(n,0) \otimes V(m,0)$ with a nonzero multiplicity are of the form $(n+m-2i-j, j)$ for $0 \leq i \leq \lfloor \frac{n+m}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{n+m}{2} \rfloor - i$. To determine the multiplicity of a dominant weight, we need only count the number of distinct vectors of the form $(c_1,c_2,c_3,c_4) \times (d_1,d_2,d_3,d_4)$ with that weight.

In other words, the multiplicity of the dominant weight $(n+m-2i-j, j)$ is equal to the number of distinct vectors $(c_1,c_2,c_3,c_4) \times (d_1,d_2,d_3,d_4)$ such that $(c_1+d_1)-(c_4+d_4) = n+m-2i-j$ and $(c_2+d_2)-(c_3+d_3) = j$. Let $x_r = c_r + d_r$. Solving the system

$$\begin{aligned}
x_1 - x_4 &= n+m-2i-j\\
x_2 - x_3 &= j
\end{aligned}$$

yields the solution set satisfying

$$\begin{aligned}
x_1 &= n+m-2i-j+x_4\\
x_2 &= i+j-x_4\\
x_3 &= i-x_4
\end{aligned}$$

for $x_4 \in \mathbb{Z}$ and $0 \leq x_4 \leq i$.

For a fixed $x = x_4$, the number of distinct vectors $(c_1,c_2,c_3,c_4) \times (d_1,d_2,d_3,d_4)$ such that $(c_1+d_1)-(c_4+d_4) = n+m-2i-j$ and $(c_2+d_2)-(c_3+d_3) = j$ is equivalent to the number of distinct ways to find (d_1,d_2,d_3,d_4) such that $\sum_{r=1}^{4} d_r = m$ and $0 \leq d_r \leq x_r$ for any r. Since $x_3 + x_4 = i$ and $x_1 + x_2 = n+m-i$, we can fix an integer k such that $0 \leq k \leq \min(m,i)$, and the number of distinct ways to find (d_1,d_2,d_3,d_4) with the desired conditions is equivalent to

$\sum_{k=0}^{\min(m,i)} f(x,k) * g(x,k)$, where $f(x,k)$ is the number of distinct ways to find (d_1, d_2) such that $d_1 + d_2 = m - k$ and $0 \le d_r \le x_r$ for $r = 1, 2$ and where $g(x,k)$ is the number of distinct ways to find (d_3, d_4) such that $d_3 + d_4 = k$ and $0 \le d_r \le x_r$ for $r = 3, 4$. With these definitions,

$$\begin{aligned} f(x,k) &= \min(n+m-2i-j+x+1, i+j-x+1, n+k-i, m-k+1) \\ g(x,k) &= \min(x+1, i-x+1, k+1, i-k+1). \end{aligned}$$

The multiplicity of $(n+m-2i-j, j)$ in $V(n,0) \otimes V(m,0)$, for $0 \le i \le \lfloor \frac{n+m}{2} \rfloor$ and $0 \le j \le \lfloor \frac{n+m}{2} \rfloor - i$, is

$$M(n+m-2i-j, j) = \sum_{x=0}^{i} \sum_{k=0}^{\min(m,i)} f(x,k) * g(x,k).$$

Theorem 4.1. *The multiplicity of the dominant weight* $(n+m-2i-j, j)$, $0 \le i \le \lfloor \frac{n+m}{2} \rfloor$ *and* $0 \le j \le \lfloor \frac{n+m}{2} \rfloor - i$, *in the representation* $V(n,0) \otimes V(m,0)$ *of* $\mathfrak{sp}(4,\mathbb{C})$, $n \ge m \ge 0$, *is given in the following table. The conditions on* n, m, i, *and* j *are in the first two columns, and the third column is the corresponding multiplicity.*

$n \ge 2i+j$	$m \ge 2i+j$	$\frac{1}{12}(i+1)(i+2)(i+3)(i+2j+2)$
	$i+j \le m \le 2i+j$	$\frac{1}{12}(i+1)(i+2)(i+3)(i+2j+2) - R(\beta)$
	$j \le m \le i+j, m \ge i$	$\frac{1}{12}(i+1)(i+2)(i+3)(2m-i+2) - R(\gamma)$
	$m \le j, m \ge i$	$\frac{1}{12}(i+1)(i+2)(i+3)(2m-i+2)$
	$j \le m \le i+j, m \le i$	$\frac{1}{12}(m+1)(m+2)(m+3)(2i-m+2) - R(\gamma)$
	$m \le j, m \le i$	$\frac{1}{12}(m+1)(m+2)(m+3)(2i-m+2)$
$n \le 2i+j$	$i+j \le m \le 2i+j$	$\frac{1}{12}(i+1)(i+2)(i+3)(i+2j+2) - R(\alpha) - R(\beta)$
	$j \le m \le i+j, m \ge i$	$\frac{1}{12}(i+1)(i+2)(i+3)(2m-i+2) - R(\alpha) - R(\gamma)$
	$j \le m \le i+j, m \le i$	$\frac{1}{12}(m+1)(m+2)(m+3)(2i-m+2) - R(\alpha) - R(\gamma)$

In the table, $\alpha = 2i+j-n, \beta = 2i+j-m, \gamma = m-j$, *and* $R(z)$ *is defined as*

$$R(z) = \begin{cases} \frac{1}{48} z(z+2)^2(z+4) & z \text{ even} \\ \frac{1}{48}(z+1)(z+3)(z^2+4z+1) & z \text{ odd.} \end{cases}$$

Proof. This is a direct result of

$$M(n+m-2i-j, j) = \sum_{x=0}^{i} \sum_{k=0}^{\min(m,i)} f(x,k) * g(x,k).$$

To use this definition to compute the multiplicity, consider

$$M(n+m-2i-j, j) = \sum_{k=0}^{\min(m,i)} S(k)$$

for $S(k) = \sum_{x=0}^{i} f(x,k) * g(x,k)$. The definitions of f and g then produce different cases. We will show one case as an example.

For $n \geq 2i+j$ and $m \geq 2i+j$,

$$\min(n+m-2i-j+x+1, i+j-x+1, n+k-i, m-k+1) = i+j-x+1,$$

and $S(k) = \sum_{x=0}^{i}(i+j-x+1) * g(x,k)$. To determine $S(k)$, we wish to sum over $x = 0, \ldots, i$ given a particular k. We will separately consider the cases when $k < \frac{i}{2}$, $k = \frac{i}{2}$, and $k > \frac{i}{2}$.

Assume $k < \frac{i}{2}$. When $x \leq k$, $g(k,x) = x+1$. When $k \leq x \leq i-k$, $g(k,x) = k+1$. When $x \geq i-k$, $g(k,x) = i-x+1$.

$$\begin{aligned} S(k) = &\sum_{x=0}^{k}(i+j-x+1)(x+1) + \sum_{x=k}^{i-k}(i+j-x+1)(k+1) \\ &+ \sum_{x=i-k}^{i}(i+j-x+1)(i-x+1) \\ &- (i+j-k+1)(k+1) - (i+j-(i-k)+1)(k+1) \\ = &\frac{1}{2}(k+1)(i-k+1)(i+2j+2). \end{aligned}$$

Assume $k = \frac{i}{2}$. When $x \leq k$, $g(k,x) = x+1$. When $x \geq k$, $g(k,x) = i-x+1$.

$$\begin{aligned} S(k) &= \sum_{x=0}^{k}(i+j-x+1)(x+1) + \sum_{x=k}^{i}(i+j-x+1)(i-x+1) - (i+j-k+1)(k+1) \\ &= \frac{1}{2}(k+1)(i-k+1)(i+2j+2). \end{aligned}$$

Finally, assume $k > \frac{i}{2}$. When $x \leq i-k$, $g(k,x) = x+1$. When $i-k \leq x \leq k$, $g(k,x) = i-k+1$. When $x \geq k$, $g(k,x) = i-x+1$.

$$\begin{aligned} S(k) = &\sum_{x=0}^{i-k}(i+j-x+1)(x+1) + \sum_{x=i-k}^{k}(i+j-x+1)(i-k+1) \\ &+ \sum_{x=k}^{i}(i+j-x+1)(i-x+1) \\ &- (i+j-(i-k)+1)(i-k+1) - (i+j-k+1)(i-k+1) \\ = &\frac{1}{2}(k+1)(i-k+1)(i+2j+2). \end{aligned}$$

Since $S(k) = \frac{1}{2}(k+1)(i-k+1)(i+2j+2)$ for any $k = 0, \ldots, i$,

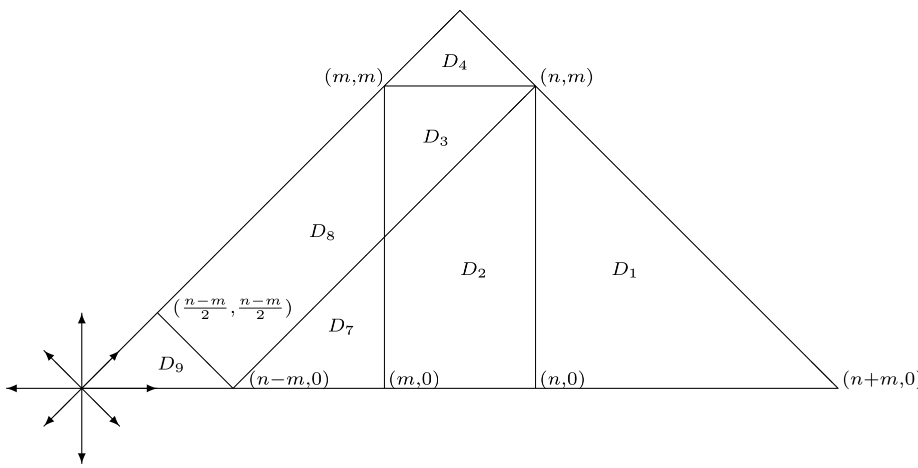
$$M(n+m-2i-j, j) = \sum_{k=0}^{i}\frac{1}{2}(k+1)(i-k+1)(i+2j+2) = \frac{1}{12}(i+1)(i+2)(i+3)(i+2j+2).$$

$\square$

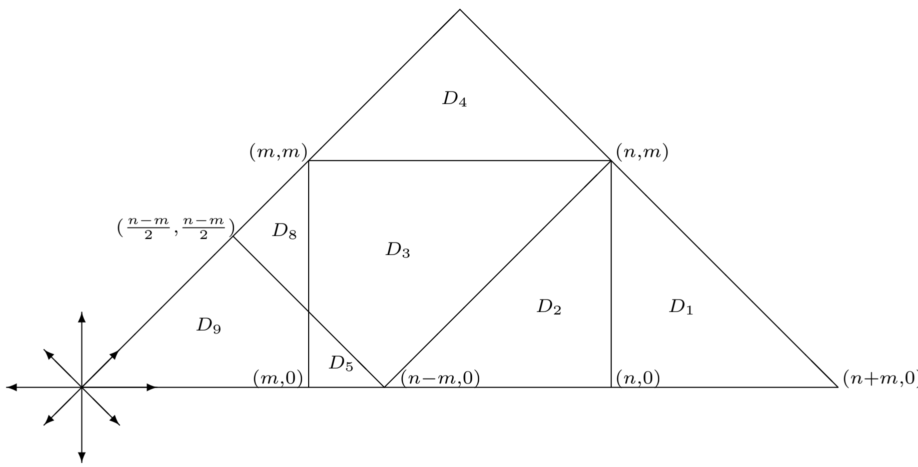
Index the conditions on n, m, i, and j as follows.

$n \geq 2i + j$	$m \geq 2i + j$	D_1
	$i + j \leq m \leq 2i + j$	D_2
	$j \leq m \leq i + j, m \geq i$	D_3
	$m \leq j, m \geq i$	D_4
	$j \leq m \leq i + j, m \leq i$	D_5
	$m \leq j, m \leq i$	D_6
$n \leq 2i + j$	$i + j \leq m \leq 2i + j$	D_7
	$j \leq m \leq i + j, m \geq i$	D_8
	$j \leq m \leq i + j, m \leq i$	D_9

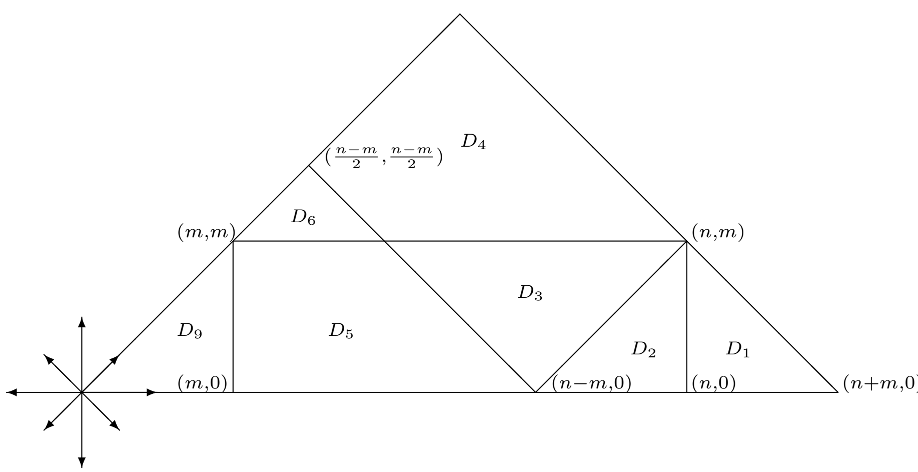
These conditions on the dominant weights of $V(n, 0) \otimes V(m, 0)$ create separate sections D_i. There are three main cases of this. The following diagrams display these cases. In the first case, $n \leq 2m$.



In the second case, $2m \leq n \leq 3m$.



In the third case, $n \geq 3m$.



The multiplicity of a weight lying on a line can be calculated using the formula for any section sharing that line as an edge.

In the boundary cases, such as when $m = 0$, $n = m$, $n = 2m$, or $n = 3m$, there will be fewer sections, but the sectioning of the triangle of dominant weights can still be derived from the main cases. For example, when $m = 0$ the triangle of dominant weights will only contain D_6, and when $n = m$ the triangle of dominant weights will be split down the middle into the sections D_7 and D_1.

5 Weight multiplicities in $V(n, m)$

Using the results of Section 3 and Section 4, the multiplicities of the weights in the representation $V(n, m)$ can be determined. Let $M(n + m - 2i - j, j)(V)$ be the multiplicity of the dominant weight $(n + m - 2i - j, j)$, $0 \leq i \leq \lfloor \frac{n+m}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{n+m}{2} \rfloor - i$, in the representation V. The results of Corollary 3.2 can be applied to weight multiplicities. The multiplicity of the weight $(n+m-2i-j,j)$ for $m=0$, $m=1$, and $m\geq 2$ can be found from the following identities, keeping in mind that $(n+m-2i-j,j)=((n+1)+(m-1)-2i-j,j)=(n+(m-2)-2(i-1)-j,j)=((n-1)+(m-1)-2(i-1)-j,j)$ and when $i=0$, any $M(n'+m'-2(i-1)-j,j)(V(n',0)\otimes V(m',0))=0$.

$$\begin{aligned}
M(n-2i-j,j)(V(n,0)) &= M(n-2i-j,j)(V(n,0)\otimes V(0,0))\\
M(n+1-2i-j,j)(V(n,1)) &= M(n+1-2i-j,j)(V(n,0)\otimes V(1,0))\\
&\quad - M(n+1-2i-j,j)(V(n+1,0)\otimes V(0,0))\\
&\quad - M(n-1-2(i-1)+j,j)(V(n-1,0)\otimes V(0,0))\\
M(n+m-2i-j,j)(V(n,m)) &= M(n+m-2i-j,j)(V(n,0)\otimes V(m,0))\\
&\quad + M(n+m-2-2(i-1)-j,j)(V(n,0)\otimes V(m-2,0))\\
&\quad - M(n+m-2-2(i-1)-j,j)(V(n-1,0)\otimes V(m-1,0))\\
&\quad - M(n+m-2i-j,j)(V(n+1,0)\otimes V(m-1,0)).
\end{aligned}$$

Combining these results with Theorem 4.1 gives a closed formula for the weight multiplicities of the dominant weights of $V(n,m)$.

Theorem 5.1. *The multiplicity of the dominant weight $(n+m-2i-j,j)$, $0\leq i\leq\lfloor\frac{n+m}{2}\rfloor$ and $0\leq j\leq\lfloor\frac{n+m}{2}\rfloor-i$, in the representation $V(n,m)$ of $\mathfrak{sp}(4,\mathbb{C})$, $n\geq m\geq 0$, is given in the following table. The conditions on n,m,i, and j are in the first two columns, and the third column is the corresponding multiplicity.*

$n>2i+j$	$m>2i+j$	0
$n\geq 2i+j$	$m=2i+j$	1
	$i+j\leq m\leq 2i+j$	$P(\beta)$
	$j\leq m\leq i+j, m\geq i$	$\frac{1}{2}(i+1)(i+2)-Q(\gamma)$
	$m\leq j, m\geq i$	$\frac{1}{2}(i+1)(i+2)$
	$j\leq m\leq i+j, m\leq i$	$\frac{1}{2}(2i-m+2)(m+1)-Q(\gamma)$
	$m\leq j, m\leq i$	$\frac{1}{2}(2i-m+2)(m+1)$
$n\leq 2i+j$	$i+j\leq m\leq 2i+j$	$P(\beta)-Q(\alpha)$
	$j\leq m\leq i+j, m\geq i$	$\frac{1}{2}(i+1)(i+2)-Q(\alpha)-Q(\gamma)$
	$j\leq m\leq i+j, m\leq i$	$\frac{1}{2}(2i-m+2)(m+1)-Q(\alpha)-Q(\gamma)$

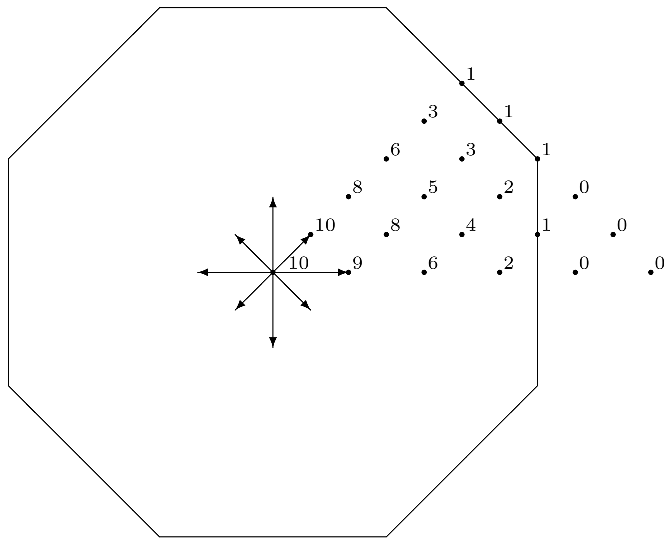
In the table, $\alpha=2i+j-n,\beta=2i+j-m,\gamma=m-j$, $P(z)$ and $Q(z)$ are defined as

$$P(z)=\begin{cases}\frac{1}{4}(z+2)^2 & z \text{ even}\\ \frac{1}{4}(z+1)(z+3) & z \text{ odd}\end{cases}$$

$$Q(z)=\begin{cases}\frac{1}{4}z(z+2) & z \text{ even}\\ \frac{1}{4}(z+1)^2 & z \text{ odd.}\end{cases}$$

The multiplicities of all other weights can be determined through reflections. It is also easy enough to check that these multiplicities coincide with the multiplicity formula found at the end of [2].

This picture is of the multiplicities of the dominant weights $(n+m-2i-j,j)$ in $V(7,3)$.



The following are a few examples of the calculations required to determine the multiplicities of these weights using Theorem 5.1.

The weight $(8,2) = (10-2(0)-2,2)$, where $i=0$ and $j=2$. Then $n=7>2=2i+j$ and $m=3>2=2i+j$. Therefore $M(8,2)=0$.

The weight $(6,0) = (10-2(2)-0,0)$, where $i=2$ and $j=0$. Then $n=7>4=2i+j$ and $2=i+j<m=3<2i+j=4$. Here, $\beta=2i+j-m=4-3=1$. Therefore $M(6,0)=\frac{1}{4}(1+1)(3+1)=2$.

The weight $(3,1) = (10-2(3)-1,1)$, where $i=3$ and $j=1$. Then $n=7=2i+j$, $1=j<m=3<i+j=4$, and $m=i=3$. Here, $\gamma=m-j=3-1=2$. Therefore $M(3,1)=\frac{1}{2}(3+1)(3+2)-\frac{1}{4}(2)(2+2)=8$.

The weight $(0,0) = (10-2(5)-0,0)$, where $i=5$ and $j=0$. Then $n=7<10=2i+j$, $0=j<m=3<i+j=5$, and $m<i=5$. Here, $\alpha=2i+j-n=3$ and $\gamma=m-j=3-0=3$. Therefore $M(0,0)=\frac{1}{2}(2*5-3+2)(3+1)-\frac{1}{4}(3+1)^2-\frac{1}{4}(3+1)^2=10$.

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