Lect 9: Symmetry dictates Yang-Mills theory

1. Weyl idea of gauge theory

2. U(1) gauge theory — quantization, Anderson-Higgs

3. Yang-Mills-Faddeev-Popov

4. Glashow-Weinberg-Salam model

5. Nonabelian Berry phase:

Weyl gauge theory (1918-1919)

	point 1	points 1, neighborhood
coordinate	x^μ	$x^\mu + dx^\mu$
field	f	$f + (\partial_\mu f) dx^\mu$
scale	1	$1 + S_\mu dx^\mu \leftarrow$ metric
scalar field	f	$(f + (\partial_\mu + S_\mu) f) dx^\mu$

Weyl: $f(x+\Delta x) = e^{S_\mu \Delta x^\mu} f(x)$

就是放缩后会出现：$(\partial_\mu + S_\mu) f$

- Yang 先生认为规范原理是第一位的

- 1. Yang-Mills 场如何 Quantization
  2. Qantization - Perturbation
  3. Yang-Mills field mass

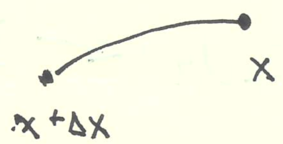
人过了十年. Fock: 在量子力学中把 EM Field 放进去:

$$\pi_\mu = p_\mu - \frac{e}{c}A_\mu = -i\hbar\left(\partial_\mu - \frac{ie}{\hbar c}A_\mu\right)$$

$\Psi(x+\Delta x)$ ———— $\Psi(x)$

$$\Psi(x+\Delta x) - e^{-i\frac{e}{\hbar c}A_\mu \Delta x^\mu}\,\Psi(x)$$

$\downarrow$ non-integrable phase



- U(1) gauge theory:

gauge symmetry:

$$\Psi(x) - e^{i\alpha}\,\Psi(x)$$

$\uparrow$ Global

Global $\xrightarrow{\text{break}}$ Local

$$\Psi(x) \longrightarrow \Psi(x-a)$$

必须通过平移：

$$\Psi(x) \longrightarrow u(x,x-a)\Psi(x-a)$$

$$u(x,x-a) = e^{-i\frac{e}{\hbar c}\int_{x-a}^{x}\vec{A}\cdot d\vec{l}} = 1 - i\frac{e}{\hbar c}\vec{A}\cdot d\vec{l}$$

$$\cdot\ \Psi'(x') - u'(x,x-a)\Psi'(x-a) = e^{i\alpha(x)}\big(\Psi(x) - u(x,x-a)\Psi(x-a)\big)$$

$$\Rightarrow e^{i\alpha(x)}\Psi(x) - u(x,x-a)\,e^{i\alpha(x-a)}\Psi(x-a)$$

$$= e^{i\alpha(x)}\big(\Psi(x) - u(x,x-a)\Psi(x-a)\big)$$

$$\Rightarrow u(x,x-a) = e^{-i\alpha(x)}\,u'(x,x-a)\,e^{i\alpha(x)}$$

$$\Rightarrow \exp\!\left(-i\frac{e}{\hbar c}A'_x - i\partial_x\alpha\right)\cdot a\,) = e^{-\frac{ie}{\hbar c}A_x a}$$

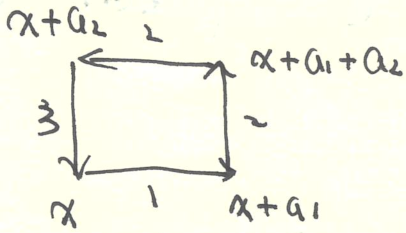
$$\Rightarrow A'_x = A_x - \frac{\hbar c}{e}\partial_x\alpha \leftarrow$$ covariant derivative

• $\lim_{a\to 0}\frac{1}{a}\left(\Psi(x) - u(x,x-a)\Psi(x-a)\right)$

$= \frac{1}{a}\left(\Psi(x) - \Psi(x-a) + \frac{ie}{\hbar c}A_x\, a\Psi(x)\right)$

$= \left(\partial_x + i\frac{e}{\hbar c}\right)\Psi(x) = D_x\Psi(x)$

$D_\mu\Psi(x) \to e^{i\alpha(x)}D_\mu\Psi(x)$

• 

绕一圈之后

$u_1u_2u_3u_4 \equiv \exp\left(-i\frac{e}{\hbar c}a\left(-A_2(x+\frac{a}{2}\hat{i})\right.\right.$

$\left.\left.+ -A_1(x+a_2+\frac{a}{2}) + A_2(x+\hat{a}+\frac{a}{2}) + A_1(x+\frac{a}{2}\hat{1}\right)\right)$

$= \exp\left(-i\frac{e}{\hbar c}a^2 F_{12}\right) \longrightarrow$ 正比于面积

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$[D_\mu, D_\nu] = i\frac{e}{\hbar c}F_{\mu\nu}$

• 与 action:

$$\mathcal{L} = \bar{\Psi} \not{D} \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - m\bar{\Psi}\Psi$$

费米子与电磁场耦合

Dirac 性质 mass:

topological terms:

Witten

$$\underbrace{\frac{i\theta e^2}{32\pi^2\hbar c} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}}$$

当 $\theta = \pi$，保持时间反演

topological term 破坏了时间反演和 inversion

• 电磁场的量子化

正则量子化：A_0 没有动力学 (Prob)

注意：μ, ν 有4个分量，

$$\mathcal{L} = -\cancel{\Phi}\mathcal{L} - \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial_\mu A_\nu$$

$$\Pi_\mu = \frac{\partial L}{\partial(\partial \dot{A}_\mu)}$$

ϕ 不能定义 Π_μ

- Gauge transformation:

$$A_\mu \to A^\mu + \partial_\mu \alpha$$

$\downarrow$

纵向分量

$$A_\mu'(k) = A_\mu(k) + k^\mu \alpha$$

$\Downarrow$

α 与 k^μ 平行

ϕ 没有自由度，规范 α 有不当作用，实际自由度只有两个

$$\pi^i = \partial^0 A^i - \partial^i A^0 = -E^i$$

Demand: $[E^j(r,t), A^j(r',t)] = i\delta_{jk}\delta(r-r')$

$\nabla\cdot\vec{A} = 0$ $\longleftrightarrow$ inconsistent

$$\nabla^k [E^j(r,t), A^j(r',t)] = 0$$

所以 δ_{jk} 不是真实三维空间中 δ_{jk}

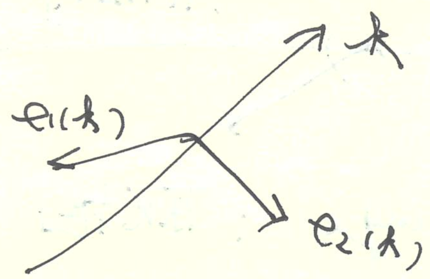
$$\delta_{jk}(\vec{r}) = \int \frac{dk^3}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \left(\delta_{ij} - \frac{k_i k_j}{|k^2|} \right)$$

$\Downarrow$ 即把 k 投影掉 $\qquad \Downarrow$ 扔.

$$A^j(k,0) \equiv \sum_{k} \frac{1}{V} \frac{1}{\sqrt{2\omega_k}} \sum_{s=1}^{2} e_s^j(k)$$

$$\left(a_s(k)\, e^{i\vec{k}\cdot\vec{r}} + a_s^{+}(k)\, e^{-i\vec{k}\cdot\vec{r}} \right)$$

$\dot{A}$



· propagator:

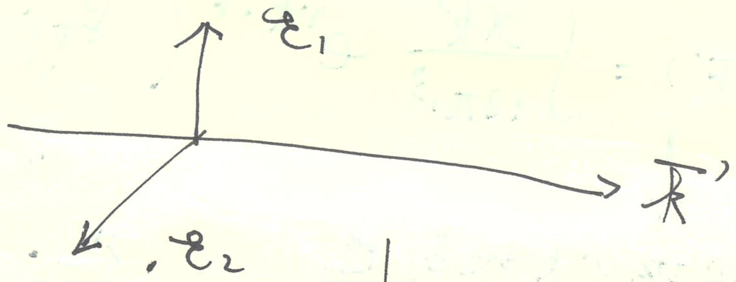
$$D^{ij}(x) = \int \frac{dk^4}{(2\pi)^4} \frac{e^{i\vec{k}\cdot\vec{r}}}{k_0^2 - k^2 + i\eta} \left(\delta^{ij} - \frac{k_i k_j}{k^2} \right)$$

$\downarrow$

Fermi

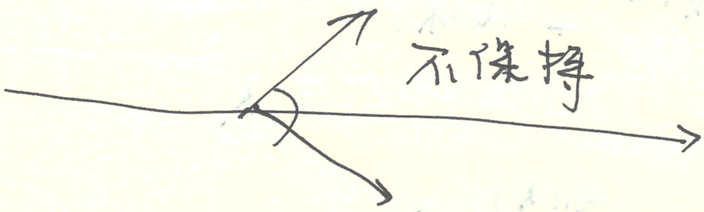
$= \frac{1}{2}\int d^4k \, A_\mu(k)(-k^2 g^{\mu\nu} + k^\mu k^\nu) A_\nu(k)$

· Columb gauge 不满足于 Lorentz covariance



$\vec{A}'(k) = \hat{\varepsilon}_1(k) A(k)$

LZ transformation



· Feyman gauge ：就把 $k^\mu k^\nu$ 去掉

· Path integral Quantization

$\int D[A]\, e^{iS[A]}$ $\qquad S = \int dx^4 \left(-\frac{1}{4} F_{\mu\nu}^2\right)$

$= -\frac{1}{2}\int dx^4 \, A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu) A_\nu$

kernel: $\longleftarrow$ $(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)$

. Coulomb gauge 不满足 Lorentz covariance

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) A_\nu(k)$$

$$\det\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) = 0$$

$$\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) k_\nu \alpha(k) = 0$$

$$\Rightarrow \boxed{k^\mu \alpha_\mu(k)} = 0$$

非物理自由度
困难

- Faddeev-Popov：（把非物理自由度丢掉）

Gauge fixing: $G(A) = \partial_\mu A^\mu = 0$

$$A^\alpha_\mu = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

引入identity：　　怎么找到这个identity

$$1 = \int D[\alpha(x)]\, \delta\left(G(A^\alpha(x))\right) \cdot \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

$$G(A^\alpha) \equiv \partial^\mu A^\alpha_\mu = \partial^\mu A_\mu + \frac{1}{e}\partial^2 \alpha(x)$$

$e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2}$

$$\Rightarrow \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{1}{e}\partial^2\right)$$

$$\bullet \int DA\, e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int D\alpha \int DA\, e^{iS[A]}\, \delta(G(A'))$$

$$= \det\left(\frac{1}{e}\partial^2\right)\int D\alpha \underline{\int DA^\alpha e^{iS[A^\alpha]}\, \delta(G(A^\alpha))}$$

$\longrightarrow$ 继续处理

$$G[A] = \partial_\mu A^\mu - \omega(x) = 0$$

$$\int D[A]\, e^{iS[A]} \equiv \det\left(\frac{1}{e}\partial^2\right)\int D[\alpha]\, \underbrace{\int D\omega(x)\, e^{-i\int d^4x \frac{\omega^2(x)}{2\xi}}}_{\text{加权平均 对所有 Gauge-样的}} \int D[A]\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x))$$

$$\equiv N(\xi)\det\left(\frac{1}{e}\partial^2\right)\int D\alpha \int DA$$

$$e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2}$$

$$\Rightarrow \left(-k^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right) k_\mu k_\nu\right) D^\mu(k) = i\delta_\mu^\rho$$

$$\Rightarrow \tilde{D}^{\mu\nu}(k) = \frac{-i}{k_0^2 - k^2 + i\eta}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

$\xi = 1$

$\Downarrow$

它指了很多 zero mode

- Anderson - Higgs mode $U(1)$ SC

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^2 - V(\phi)$$

$\downarrow$

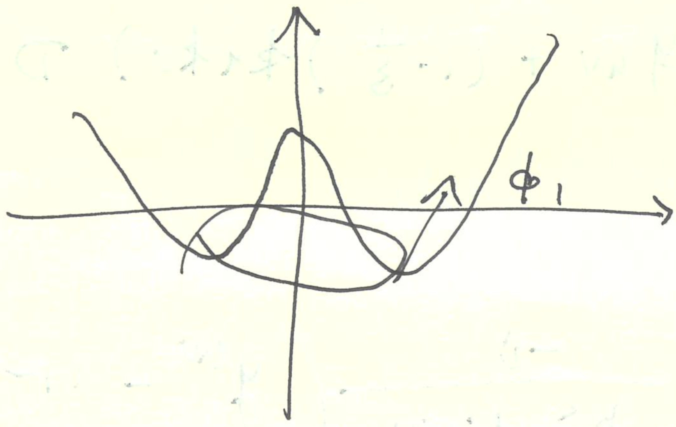
ϕ : complex

$$\phi(x) \longrightarrow e^{i\alpha(x)}\phi(x)$$

$$A'_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\alpha$$

$$V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4$$

$$\langle \phi \rangle = \phi_0 = \left(\frac{\mu^2}{\lambda}\right)^{0.5}$$



$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right)$$

$$V(\phi^2) = -\frac{1}{2\lambda}\phi^4 + \frac{1}{2}(2\mu)^2\phi_1^2$$

Higgs:

$$|D_\mu\phi|^2 = \frac{1}{2}\left((\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + e^2\phi_0^2 A_\mu^2 + \sqrt{2}A_\mu e\phi_0\partial^\mu\phi_2 + \cdots\right)$$

$$\frac{1}{2}(\partial_\mu\phi_1)^2 + \left(\frac{1}{\sqrt{2}}\partial_\mu\phi_2 + e\phi_0 A_\mu\right)^2 + \cdots$$

$\Downarrow$

Goldstein mode 被 gauge field 吃掉，变成纵向分量

$\left(\frac{\ }{\sqrt{2}}\right) = \phi = \langle \phi \rangle$

零质量 Gauge field 与 Goldstone mode 耦合時，然后变成 Gauge field 的纵向

- Yang - Mills field

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}$$

$$\Psi'(x) = e^{i\alpha(x)\cdot\frac{\vec{\sigma}}{2}} \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}$$

$$\Psi'(y) - U'(y,x)\Psi'(x) = V(y)\big(\Psi(y) - U(y,x)\Psi(x)\big)$$

$$= V(y)\big(\Psi(y) - V^{\dagger}(y)\,U'(y,x)\,V(x)\,\Psi(x)\big)$$

$$U'(y,x) = V(y)\,U(y,x)\,V^{\dagger}(x)$$

- $U(y,x) \equiv \exp\left(ig A^i_\mu \frac{\sigma^i}{2} \varepsilon^\mu\right)$

$$\equiv 1 + \frac{ig}{\hbar c}\vec{A_\mu}\cdot\frac{\vec{\sigma}}{2}\cdot\varepsilon_\mu$$

$$A'_\mu \cdot \frac{\vec{\sigma}}{2} = V A_\mu \frac{\vec{\sigma}}{2} V^+ + \frac{i}{g} V \partial_\mu V$$

$$\lim_{y \to x} \Psi(y) - u(y,x)\Psi(x) \equiv \varepsilon^\mu \left(\partial_\mu - i g \vec{A}_\mu \cdot \frac{\vec{\sigma}}{2} \right) \Psi(x)$$

$$\Rightarrow \quad D_\mu \Psi(x) \equiv \left(\partial_\mu - i g A_\mu \frac{\vec{\sigma}}{2} \right) \Psi(x)$$

$$D'_\mu \Psi(x) \equiv \left(\partial_\mu - i g A(x') \right) V(x) \Psi(x)$$

$$D'_\mu \Psi(x) \equiv V(x) \left(V^+(x) \partial_\mu V(x) - i g V^+ A'_\mu \frac{\vec{\sigma}}{2} V \right) \Psi(x)$$

$$V^+(x) \partial_\mu \left(V(x) \Psi(x) \right) = \partial_\mu \Psi(x) + V^+ \partial_\mu V$$

$$\Rightarrow \quad - i g A_\mu = - i g V^+ A'_\mu \frac{\vec{\sigma}}{2} V + V^+ \partial_\mu V$$

$$A_\mu^{\prime}\cdot\frac{\vec{\sigma}}{2} = V A_\mu\cdot\frac{\vec{\sigma}}{2} V^{+} + \frac{i}{g} V^{+}\partial_\mu V$$

$$[D_\mu', D_\nu']\Psi(x) \equiv V(x)[D_\mu, D_\nu]\Psi(x)$$

$$[D_\mu, D_\nu] \equiv \left[\partial_\mu - ig\vec{A}\cdot\frac{\vec{\sigma}}{2},\ \partial_\nu - igA_\nu\frac{\vec{\sigma}}{2}\right]$$

$$\equiv ig\left[\partial_\mu A_\nu^{i} - \partial_\nu A_\mu^{i} - ig\,\epsilon_{ijk}A_\mu^{j}A_\nu^{k}\right]\cdot\frac{\sigma^{i}}{2}$$

$$\equiv -igF_{\mu\nu}^{i}\left(\frac{\sigma^{i}}{2}\right)$$ — Wilson Loop

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$$F_{\mu\nu}^{i} \longrightarrow \underline{V(x)\,F_\mu\,V^{+}}$$

$\downarrow$

它是么变的.

- invariant: $\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$

- $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$

• 弱电统一理论 : $SU_L(2) \otimes U(1)$

Yang-Mills Lagrangian

$$\mathcal{L} \equiv \bar{\Psi} \not{D} \Psi \qquad \Psi^i_\alpha \begin{array}{l} - \text{doublet} \\ - \text{spacetime} \end{array}$$

$$- \frac{1}{2} \mathrm{tr} \left(F^i_{\mu\nu} \frac{\sigma^i}{2} \right)^2 - m \bar{\Psi} \Psi$$

• $\dfrac{\partial L}{\partial A^{i\mu}} \equiv ① + ②$

① ⇒ $g \bar{\Psi} \gamma_\mu \frac{\sigma^i}{2} \Psi$

Hopf term

$$\partial^\mu F^i_{\mu\nu} + g \varepsilon^{ijk} A^{\mu j} F^k_{\mu\nu}$$

$$= g \bar{\Psi} \gamma_\nu \frac{\sigma^i}{2} \Psi$$

⇓

Hopf term

• 弱电理论： $SU_L(2) \otimes U(1)$

$$\phi \rightarrow \phi' = e^{i\alpha^a \cdot \frac{\sigma^a}{2}} e^{i\frac{\beta}{2}} \phi$$

$$V(\phi) \equiv -\mu^2 \phi^+ \phi + \frac{\lambda}{2}(\phi^+ \phi)^2$$

$$\downarrow$$

$$\langle \phi \rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \longleftarrow \text{取下面}$$

$$\alpha^3 = \beta, \quad \alpha^1 = \alpha^2 = 0$$

$U(1)$和 σ_3 combine — preserve vaccum

$$D_\mu \phi \equiv \left(\partial_\mu - ig A_\mu^a \frac{T^a}{2} - i\frac{g'}{2} \underbrace{B_\mu}_{U(1)}\right)\phi$$

$$L = (D^\mu \phi^*)(D_\mu \phi) - V(\phi)$$

Gauge boson mass:

真空 $\longrightarrow L$

$$\Delta L \equiv \frac{1}{2}(0, v)\underbrace{\left(g A_u^a T^a + \frac{g'}{2} B_u\right)}_{A}\left(\begin{matrix}0\\v\end{matrix}\right)$$

$$A\left(\begin{matrix}0\\v\end{matrix}\right) \equiv \left(g^2 A_u^a A^{bu} \tau^a \tau^b + \frac{1}{4}\right.$$

$$\left. g'^2 B_u B^u + g g' A_u^a \tau^a B^u\right)$$

$$\frac{g^2}{4}\left[A_u^1 A^{1u} + A_u^2 A^{2u}\right]^2$$

$$\Delta L \equiv \frac{v^2}{4}\left(g^2 W_u^+ W_u^- -\right.$$

$A_u =$