$tr(e^{\beta H})$

$$tr(e^{A}) = tr(e^{\lambda_i}) = \sum_{i=1}^{N} e^{\lambda_i}$$

$\Rightarrow$ ~~det~~ ~~Log tr(e^A)~~ $=$

$$\det(e^{A}) = \prod_{i=1}^{N} e^{\lambda_i} = e^{tr(A)}$$

$$\Rightarrow \quad tr(A) = \mathrm{Log}(\det(e^{A}))$$

$\Uparrow e^{B}$

$\Downarrow$

$$= \mathrm{Log}\det(e^{e^{B}})$$

- Solution: 1 Bose-Einstein condensa

$$tr(e^{\beta H}) = {}_1\langle x| e^{\beta H} |x\rangle_1$$

$$+ {}_2\langle x| e^{\beta H} |x_2\rangle$$

$${}_{\alpha_0}\langle x_0 | e^{\beta H} | x_N \rangle_{\alpha_N} = {}_{\alpha_0}\langle x_0 | e^{\beta H \cdot \Delta \tau} | x_1 \rangle_{\alpha_1}$$

$${}_{\alpha_2}\langle x_\phi | e^{\beta H \cdot \Delta \tau} | x_2 \rangle_{\alpha_2} \cdots \; {}_{\alpha_{N-1}}\langle x_{N-1} | e^{\beta H \cdot \Delta \tau} | x_N \rangle_{\alpha_N}$$

- $${}_{\alpha_n}\langle x_n | 1 + \beta H \cdot \Delta t | x_n \rangle_{\alpha_{N-1}}$$

$$= {}_{\alpha_n}\langle x_n | x_n \rangle_{\alpha_{n-1}} + {}_{\alpha_n}\langle x_n | x_{n-1} \rangle_{n-1}$$

$$+ {}_{\alpha_n}\langle x_n | x_n \rangle_{\alpha_n} + \langle x_n | \beta H \cdot \Delta t | x_\alpha \rangle_{n-1}$$

$$= {}_{\alpha_n}\langle x_n | \partial_t | x_n \rangle_{\alpha_1} + e^{\beta H (x_\alpha (x_1))}$$

$$\Rightarrow e^{\int_0^\beta {}_{\alpha}\langle x | \partial_\tau | x \rangle_1 d\tau} + \underline{H (\beta \ldots}$$

$$\Downarrow$$

$$\underline{\left(\oint_c F^a_{\mu\nu} \sigma_a \, dx_\mu \wedge dx_\nu \right)_{11}}$$

$$\Downarrow \quad \underline{e^{i\gamma_{11}}}$$

$$\mathrm{tr}(e^{\gamma_{11}}$$

• SU(4) Heisenberg model:

$$|\alpha\rangle = \sum_{M=-J}^{J} \phi_{J,M}(\alpha)\ |J,M\rangle$$

(u,v)

$$(\hat{r}\cdot J) = \begin{pmatrix} x_3 & x_1 - x_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial u} \end{pmatrix}$$

$$= u\frac{\partial}{\partial u} + v\cdot\frac{\partial}{\partial v}$$

$$\langle\alpha\ |(\hat{r}\cdot J)|\alpha\rangle \equiv \hbar S\ |\alpha\rangle$$

$$\Rightarrow \quad \vec{J}\,|\alpha\rangle \equiv \hbar S\cdot\hat{r}\ |\alpha\rangle = \hbar\vec{S}'|\alpha\rangle$$

$\Downarrow$ J只有在这个方向分量:

$$\boxed{S_a\,|\alpha\rangle \equiv}$$

$\otimes$ S_a

$$\boxed{x_a\cdot L_{ba} = I}$$

可以推广为 SU(4) matrix

$$\boxed{(x_a\cdot L_{ba})\,\phi = I\,\phi}$$

$$\Rightarrow \quad \cancel{L_{ba}} = I$$

• SU(4) Heisenberg model:

$$★★★ \left(\frac{1+\vec{S}_{\alpha\beta}\cdot\vec{S}_{\alpha\beta}}{4}\right)^{2I} \Rightarrow \equiv$$

measurement

$\Downarrow$

搞清楚 Fierz identity

$$H = J\,\vec{S}_{\alpha\beta}(i)\cdot\vec{S}_{\alpha\beta}(j)$$

$\Downarrow$

由于: $\vec{S}_{\alpha\beta}$

$$= J\,\vec{J}_{\alpha b}(i)\cdot\vec{J}_{\alpha b}(j) + J\sum_{SO(5)} J^{(i)}_{\alpha\beta}\cdot J^{(j)}_{\alpha\beta})$$

$$\left(\frac{1+\vec{n}\cdot\vec{n}'}{2}\right)^{2s}$$

$\Downarrow$

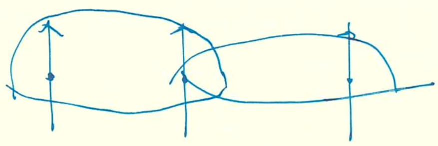
How to minimize this

$$[(J^{(i)}_{\alpha\beta}+J^{(j)}_{\alpha\beta}))^2 - \underline{J^{(i)2}_{\alpha\beta}} - \underline{J^{(j)}_{\alpha\beta}}]/2$$

$\boxed{\text{singlet}}$

$$\left(\frac{1+\vec{S}_{\alpha\beta}{}'\cdot\vec{S}_{\alpha\beta}}{4}\right)^{I}$$

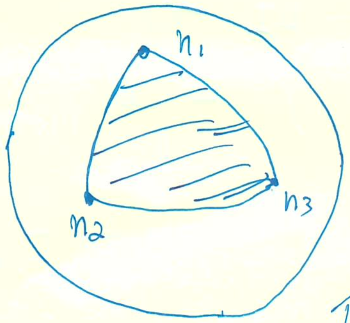
SO(5) Casimir element



$$\sum_{m=-J}^{J}\phi^{J*}_{m}(\alpha)\,\phi^{J}_{m}(\alpha') = \sum_{m=-J}^{J}\frac{(2s)!}{(J-m)!(J+m)!}$$

$$(\bar{u}'u)^{J-m}(\bar{v}'v)^{J+m} = (\bar{u}'u+\bar{v}'v)^{2s}$$

• Nonabelian Berry phase



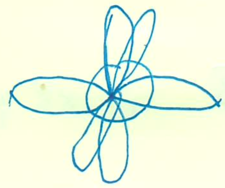
$\Uparrow$ Bell entanglement

$\Longrightarrow$ $(|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$

<u>Path integral to study Bell inequality</u>

• Quaternion coherent state

d_{z^2}



super-exchange

Cuprate :

① many body coherent state

②: Hopf term

③: Gauge field and ~~hop~~ string theory

Lie Group:

1. Weyl integration formula

2. Weyl character formula

3. Young tableaux branching rule

Heavy fermion superconductivity

$$\Gamma^1 = (S_x S_y + S_y S_x)/\sqrt{3}$$

$$\Gamma^2 = (S_x S_z + S_z S_x)/\sqrt{3}$$

$$\Gamma^3 = (S_z S_y + S_y S_z)/\sqrt{3}$$

$$\Gamma^4 = S_z^2 - 5/4$$

$$\Gamma^5 = \frac{1}{\sqrt{3}}(S_x^2 - S_y^2)$$

⇒ 对应 Qua

- Qudrapole matrix

怎么去做 Four mode 的

$$S_z^2 = \begin{pmatrix} 9/4 & & & \\ & 1/4 & & \\ & & 1/4 & \\ & & & 9/4 \end{pmatrix} - 5/4 = \begin{pmatrix} 2 & & & \\ & -2 & & \\ & & -2 & \\ & & & 2 \end{pmatrix}$$

$$(S_x + iS_y)^2 = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 & 2\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2n}\left((S_x + iS_y)^2 + (S_x - iS_y)^2\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ & & & \\ & & & \end{pmatrix}$$

$\sum_\nu (C\; x_\delta \hbar_\delta + \hbar_\delta x_\delta) = 1$

例:

$$[S_x S_y + S_y S_x ,\quad S_x S_z + S_z S_x]$$

term1: $[S_x S_y ,\ S_x S_z] = S_x [S_y,\ S_x] S_z$

$+ S_x [S_x,\ S_z] S_y = -i\hbar$

最越峰

$RNiO_2$: 把 $SrTiO_3$ 中的顶角氧拿掉

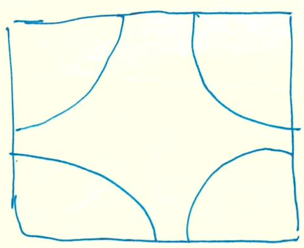
(李丹枫) $RNiO_3 \xrightarrow{CaH_2} RNiO_2$

- Cuprate: magnetic ordering: AFM

  Nicolate: ?

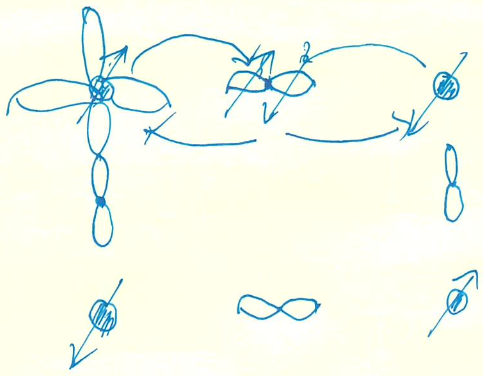
Nicolate: 多带

Cuprate:

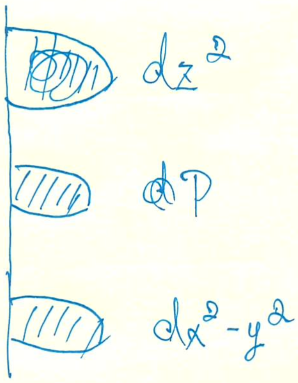


single band

$RNiO_2$ 需要用 CaH_2 将 $RNiO_3$ 中顶角氧拿掉



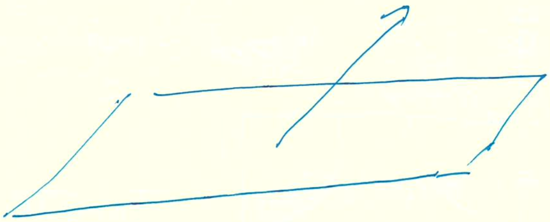
Symmetry and Polynomials:

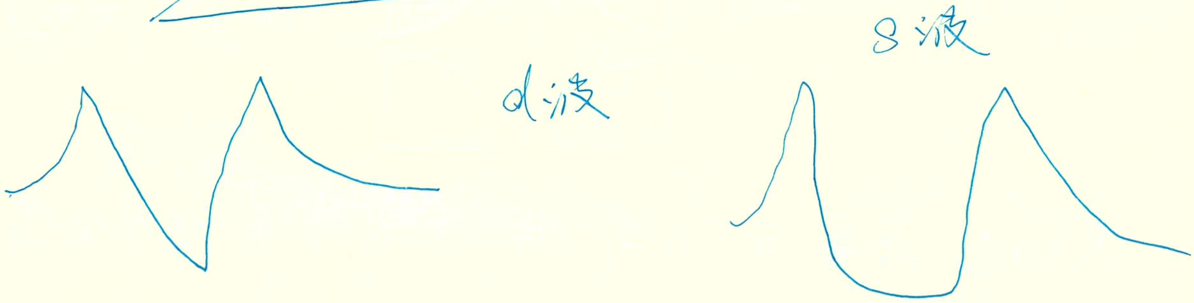


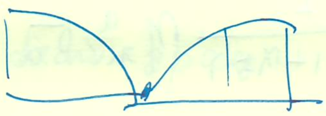
空穴掺杂先从氧的p轨道等电子

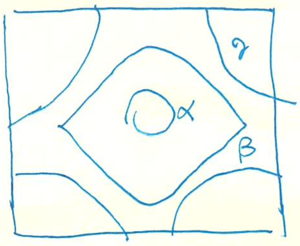
主要是从氧的2p轨道头跳起交换

· <u>Zhang-Rice Singlet: 自旋单态 model</u>





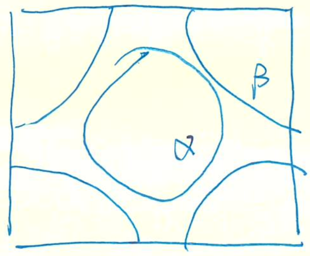




三带的模型

用 strain 实现
空间依赖的 SOC

掺杂的时候



$$\Psi = \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix} 1+x_5 \\ x_4 + i\sigma_i x_i \end{pmatrix}$$

$$A = -i\langle \Psi | d\Psi \rangle = -i\frac{1}{2(1+x_5)}\begin{pmatrix} 1+x_5 & x_4 - i\sigma_i x_i \end{pmatrix}\left[\begin{pmatrix} dx_5 \\ dx_4 + i\,dx_i\,\sigma_i \end{pmatrix} - \frac{dx_5}{2(1+x_5)}\begin{pmatrix} 1+x_5 \\ x_4 - i\sigma_i x_i \end{pmatrix}\right]$$

$$\Rightarrow (1+x_5)\,dx_5 + (x_4 - i\sigma_i x_i)(dx_4 + i\sigma_i dx_i)$$

$$= -\frac{1}{2(1+x_5)}\,\eta^a_{\mu\nu}\,\sigma_a\, x_\mu\, dx_\nu \quad (☆)$$

· gauge potential:

$$A = \frac{1}{-2(1+x_5)} \eta^a_{\mu\nu} \sigma_a x_\mu dx_\nu$$

$$\Rightarrow F = dA + i A \wedge A$$

$$\left.\begin{aligned} &= -\frac{1}{2(1+x_5)} \eta^a_{\mu\nu} \sigma_a dx_\nu \wedge dx_\mu \\ &+ \frac{1}{2(1+x_5)^2} \eta^a_{\mu\nu} \sigma_a x_\mu dx_\nu \wedge dx_5 \end{aligned}\right] \quad (2)$$

$$+ i \frac{1}{4(1+x_5)} \eta^a_{\mu\nu} \eta^b_{\nu\mu} \sigma^a \sigma^b x_\mu x_\nu \underline{dx_\mu \wedge dx_\nu}$$

根据(2)我们可以确定这个 $F^a_{\mu\nu}$

$$\boxed{\int F = 0}$$

· Second Chern number

$$C_2 = \frac{1}{2!}\left(\frac{1}{2\pi}\right)^2 \int F \wedge F = \frac{1}{32\pi^2} \int$$

$$\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

我们怎么能够表达这个 Nonabelian Berry phase!

这个物理意义是啥?

· An soliton is classical solition of Yang-Mills field in Euclidean space:

孤子是经典Yang-Mills场在欧氏空间中的解

$$\frac{1}{4}\varepsilon_{ijkl}\,\mathrm{tr}(F_{ij}F_{kl}) = \partial_i\left(\varepsilon_{ijkl}\,\mathrm{tr}\left(A_j\partial_k A_l - \frac{2i}{3}A_jA_kA_l\right)\right)$$

$$\varepsilon_{ijkl}(\partial_i A_j - iA_iA_j)(\partial_k A_l - iA_kA_l)$$

$$= \varepsilon_{ijkl}(\partial_i A_j\partial_k A_l - 2iA_iA_jA_kA_l)$$

1. $\varepsilon_{ijkl}\partial_i A_j\partial_k A_l = \varepsilon_{ijkl}\partial_k(\partial_i A_j A_k)$

2. $\varepsilon_{ijkl}\,\mathrm{tr}(A_iA_j\partial_k A_l) = \frac{1}{3}\varepsilon_{ijkl}\partial_k\,\mathrm{tr}(A_iA_jA_k)$

$$\Rightarrow \partial_i\left(\varepsilon_{ijkl}A_j\partial_k A_l - \frac{2i}{3}A_jA_k\cdot A_l\right)$$

• Chern - Simons term:

$$S_{CS} = \frac{k}{4\pi}\left(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A\right)$$

$$\boxed{A_k \to i U^{+} \partial_k U}$$

topological current densital

$$J_i = \varepsilon_{ijkl} \, tr(\,.$$

$\boxed{\text{Chern - Simons term}}$

$\boxed{U^{+} \partial_k}$

$$C_2 = \frac{1}{4\pi^2} \int d\theta_1 d\theta_2 d\theta_3 \, tr\left(\underline{U^{+} \partial_1 U \, U^{+} \partial_2 U \, \partial \ldots} \right)$$

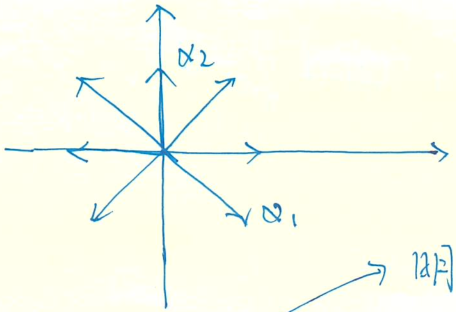
$$U^{+} = e^{\frac{i}{2} \sigma_1 \sigma_3} \, e^{\frac{i}{2} \theta_2 \sigma_2} \, e^{\frac{i}{2} \theta_3 \sigma_3}$$

Electromagnetism response of surface states:

- Cartan matrix:

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_j)} \qquad |\alpha_1| = 1, \quad |\alpha_2| = \sqrt{2}$$



明确什么是 ρ

$$\rho = (\vec{\alpha_1} + \vec{\alpha_2})/2 = \left(\frac{3}{2}, \frac{1}{2}\right)$$

Weight vector: $\vec{\alpha_i} \cdot \vec{\omega_j} = \delta_{ij}$

$$\det = 2: \quad A^{-1} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} / 2 = \begin{pmatrix} 1 & 1 \\ 1/2 & 1 \end{pmatrix}$$

$$\begin{cases} \vec{\omega_1} = \vec{\alpha_1} + \vec{\alpha_2} = (1, 0) \\ \vec{\omega_2} = 1/2\,\vec{\alpha_1} + \vec{\alpha_2} = \left(\frac{1}{2}, \frac{1}{2}\right) \end{cases}$$

SO(5), Algebra

$$\vec{\mu} = \left(\lambda_1 + \frac{\lambda_2}{2}, \frac{\lambda_2}{2}\right) : \ SO(5)$$

• 学习一下 SO(2N), SO(2N+1) 的 rep (一直都没搞)

$$A_\mu(r) = i f(r)\, U(\vec{x})\, \partial_\mu U(x)$$

$$= i f(r)\, \bar{q}\, dq$$

$$= i f(r)\, \eta^a_{\mu\nu}\, e_a\, \cancel{x_\mu d}\, q_\mu\, dq_\nu$$

$$\cancel{F_{\mu\nu} = i \partial f \; i \partial}$$

$$\cancel{\partial_\mu A_\nu = i \partial_\nu f(r)\, \eta^a_{\rho\mu}\, e_a\, d\rho\, q_\rho}\, dq_\mu \wedge dq_\nu$$

$$\partial_\nu \cancel{A_\mu = i \partial}$$

$$F = dA + i A \wedge A = i\, df(r)\, \eta^a_{\mu\nu}\, e_a\, q_a\, dq_\nu$$

$$+ i f(r)\, d(\eta^a_{\mu\nu}\, e_a\, q_\mu\, dq_\nu)$$

$$+ i f(r)^2\, \eta^a_{\mu\nu}\, e_a\, \eta^b_{\rho\sigma}\, e_b\, q_\mu\, q_\rho\, dq_\nu\, dq_\sigma$$

$$q_a = \frac{x_a}{r} \longrightarrow dq_a = \frac{dx_a}{r} - \frac{x_a}{r^3} \cdot x_a\, dx_a$$

$$= \frac{(r^2 - x_a^2)\, dx_a}{r^3}$$

$$\Rightarrow \eta^a_{\mu\nu} \frac{x_\mu}{r} \cdot \frac{(r^2 - x_\nu^2)}{r^3} dx_{\cancel{a}\nu}$$

$$\Rightarrow A_\mu = i f(r)\, \eta^a_{\mu\nu} \frac{x_\mu}{r} \cdot \frac{dx_\nu}{r} e_a$$

$$\Rightarrow \cancel{d} F = dA + i A \wedge A$$

$$= i\, df(r)\, \eta^a_{\mu\nu} \frac{x_\mu}{r} \frac{dx_\nu}{r} e_a + i f(r)\, \eta^a_{\mu\nu} \frac{1}{r^2} dx_\nu \wedge dx_\mu + i f(r^2)\, \underline{\eta^a_{\mu\nu}\, \eta^b_{\rho\sigma}}\, \cancel{\sigma^a \sigma^b}\, e^a e^b\, x_\mu x_\rho\, dx_\nu \wedge dx_\sigma$$

① 求解instanton的过程 Polyakov (1975)

看[illegible]的书把Instanton这些彻底搞懂

Effective action for one-dimensional

Quantum antifferomagnets:

1. 先写下 many body 的 action

$$S_M[\vec{n}] = S\sum_{j=1}^{N} S_{WZ}[n(j)] - \int_0^T dx_0 \sum_{j=1}^{n} \vec{n}(j, x_0)\,\vec{n}(j+1, x_0)$$

antifferomagnets:

$$S_M[\vec{n}] = S\sum_{j=1}^{N} (-1)^j S_{WZ}[n(j)] - \frac{JS^2}{2}$$

$$\int_0^T dx_0 \sum (n(j, x_0) - n(j+1, x_0))^2$$

$$n(j) \equiv m(j) + (-1)^j a_0 l(j)$$

$$S\sum_{r=1}^{N/2} \left(S(n(2r)) - S(2r-1) \right)$$

$$= m(2r) - m(2r-1) + a_0(l(2r) + l(2r-1))$$

$$= a_0(\partial_1 m(2r) + 2l(2r)) + O(a_0^2)$$

· 的 topological term 写为

$$\cong S\sum_{r=1}^{N/2}\int_0^T dx_0\; a_0\left(\partial_1\vec{m}(2r,x_0)+2\vec{l}(2r,x_0)\right)$$

$$\times\left(\vec{m}(2r,x_0)\times\partial_0\vec{m}(2r,x_0)\right)$$

take continuum limits:

$$\lim S_{WZ}=\frac{S}{2}\int d^2x\;\vec{m}\cdot\left(\partial_0\vec{m}\times\partial_1\vec{m}\right)$$

$$+\;S\int d^2x\cdot\vec{l}\left(\vec{m}\times\partial_0\vec{m}\right)$$

$$\Rightarrow\; S_{topological}=\frac{\theta}{8\pi}\varepsilon_{\mu\nu}\,\vec{m}\times\left(\partial_\mu\vec{m}\times\partial_\nu\vec{m}\right)$$

其中 $\theta=2\pi S$：其中我一直很想知道 SU(4) 的 topological term 是什么？

$$\pi_2(S_2)=\mathbb{Z}$$

2D : instanton

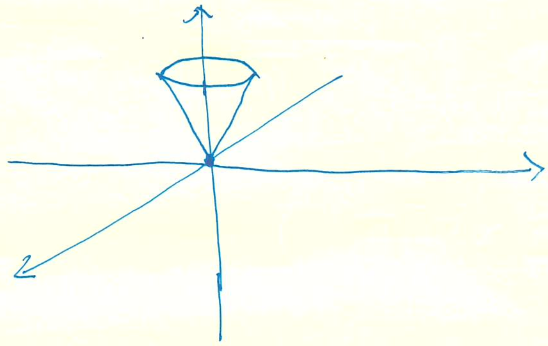
(2+1)D : Skymin

$$Q = \frac{1}{8\pi} \int d^2x\, \varepsilon_{\mu\nu}\, \vec{m} \times (\partial_\mu \vec{m} \times \partial_\nu \vec{m})$$

$$(-1)^{2\pi s Q}$$

leads to Haldane conjecture

§ Quantum Renormalization and the renormalization group:



m_3 . $m_1 = \sqrt{1-m_3^2}\cos\phi \qquad m_2 = \sqrt{1-m_3^2}\sin\phi$

$(\nabla_i m_3)^2$

$\nabla_i m_1 = \nabla_i m_1 \sqrt{1-m_3^2}\cos\phi = \sqrt{1-m_3^2} - \sin\phi\, \nabla_i\phi$

$- \frac{m_3}{\sqrt{1-m_3^2}} \nabla_i m_3 \cos\phi$

$$L_0^E = \frac{1}{2u a_0^{d-2}} \Big((\nabla_i m_3)^2 + (1-m_3^2)(\nabla_i \phi)^2 +$$

$$\frac{(m_3 \nabla_i m_3)^2}{1-m_3^2} \Big)$$

- Rescale field m_3: $\quad m_3 = (u a_0^{d-2})^{0.5}\, \psi$

$$L_0^E = \frac{1}{2}(\nabla_i \psi)^2 + \frac{1}{2u a_0^{d-2}} (1 - u a_0^{d-2}\psi^2)(\nabla_i \phi)^2$$

$$+ \frac{1}{2} \frac{u a_0^{d-2}}{1 - u a_0^{d-2}\psi^2} (\psi \nabla_i \psi)^2$$

$$\cong \frac{1}{2}(\nabla_i \psi)^2 + \frac{1}{2u a_0^{d-2}} (\nabla_i \phi)^2 - \frac{1}{2}\psi^2 (\nabla_i \phi)^2$$

$$+ \frac{1}{2} u a_0^{d-2} (\psi \nabla_i \psi)^2 + \frac{1}{2} u^2 (a_0)^{2(d-2)} \psi^2 (\psi \nabla_i \psi)^2$$

- Scale momentum space

$$\int_{b\Lambda < |\vec{p}'| < \Lambda} D\psi\, e^{-S_0^E[\psi, \phi]} = \int_{b\Lambda < |\vec{p}| < \Lambda} D\psi$$

$$\exp\Big(-\frac{1}{2} \int d^d x \big((\nabla_i \psi)^2 + \frac{1}{u a_0^{d-2}} (\nabla_i \phi)^2 - \psi^2 (\nabla_i \phi)^2 \big)$$

$$+ O(u) \Big)$$

φ是一个 slowly 变化的场

$$\int d^dx \, (\nabla_i \varphi)^2 + \frac{1}{\pi a_0^{d-2}} (\nabla_i \phi)^2 - \psi^2 \nabla_i(\phi)$$

$$= \int \frac{d^dp}{(2\pi)^d} \, \theta - p^2 \phi(p) + (\nabla_i \phi)^2 \phi(p)\phi(p)$$

$$= \prod_{b\Lambda < p < \Lambda} \left(\frac{2\pi}{p^2 - \nabla_i \phi^2} \right)^{0.5}$$

- $$\prod_{b\Lambda < p < \Lambda} \frac{2\pi}{p^2} \left(1 - \frac{1}{2} \frac{(\nabla_i \phi)^2}{p^2} \right)$$

$$\Rightarrow \exp\left(\frac{1}{2} \int_{b\Lambda<p<\Lambda} \frac{d^dp}{(2\pi)^d} \, \mathrm{Log}\left(\frac{2\pi}{p^2} \right) + \frac{1}{2} (\nabla_i \phi)^2 \int_{b\Lambda<p<\Lambda} \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} \right)$$

然后还有

- $$L_E = -\frac{1}{2} \int_{b\Lambda<p<\Lambda} \frac{d^dp}{(2\pi)^d} \, \mathrm{Log}\left(\frac{2\pi}{p^2} \right) + \frac{1}{2} (\nabla_i \phi)^2$$

$$+ \frac{1}{2} \left(\frac{1}{\pi a_0^{d-2}} - \int_{b\Lambda<p<\Lambda} \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} \right) (\nabla_i \phi)^2$$

$$- \frac{1}{2} \psi^2 (\nabla_i \phi)^2$$

例：

$$\frac{1}{u' a_0^{d-2}} = \frac{1}{u a_0^{d-2}} - \int_{b\Lambda<p<\Lambda} \frac{d^dp}{(2\pi)^d} \frac{1}{p^2}$$

$$\int_{b\Lambda<p<\Lambda} \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} = \int_{b\Lambda<p<\Lambda} \frac{S_d}{(2\pi)^d} p^{d-1} \cdot dp \cdot \frac{1}{p^2}$$

$$= \frac{S_d}{(2\pi)^d} \Lambda^{d-2} \int_b^1 x^{d-3} dx$$

$$= \frac{S_d}{(2\pi)^d} \Lambda^{d-2} \frac{1-b^{d-2}}{d-2}$$

$$\Rightarrow \frac{u'-u}{u u' a_0^{d-2}} = \frac{S_d}{(2\pi)^d} \Lambda^{d-2} \frac{1-b^{d-2}}{d-2}$$

$$\Rightarrow \frac{du}{u^2} \Big/ a_0^{d-2} = \frac{S_d}{2\pi^d} \Lambda^{d-2} \frac{1-b^{d-2}}{d-2}$$

$$\frac{b^{d-2}}{u' a_0^{d-2}} = \frac{b^{d-2}-1}{u' a_0^{d-2}} + \frac{1}{u' a_0^{d-2}}$$

$$\frac{u'-u}{u'u\, a_0^{d-2}} = \frac{S_d}{(2\pi)^d} \Lambda^{d-2} \frac{1-b^{d-2}}{d-2} + \frac{b^{d-2}-1}{u' a_0^{d-2}}$$

$$\frac{da_0}{a_0} \qquad \Uparrow \frac{da_0}{a_0}$$

$$-\operatorname{Log} b = \frac{d a_0}{a_0} \Rightarrow b^{d-2} = e^{(d-2)\operatorname{Log} b} = e^{(d-2)\cdot -\frac{da_0}{a_0}}$$

$$= 1 - (d-2)\cdot\frac{da_0}{a_0}$$

$$\Rightarrow \beta(u) = -\varepsilon u + \frac{u^2}{2\pi}$$

• Asymptotic Freedom and Haldane conjecture:

(1+1)-d non-linear sigma model:

$$\beta(u) = a_0\frac{du}{da_0} = \frac{u^2}{2\pi}$$

$$\Rightarrow \frac{du}{u^2} = \frac{da_0}{a_0}\cdot\frac{1}{2\pi}$$

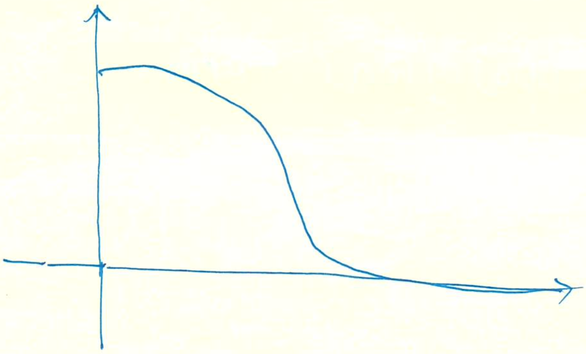
$$\Rightarrow \frac{1}{u(T)} - \frac{1}{u_0} = \frac{1}{2\pi}\operatorname{Log}\left(\frac{v_s a_0}{T}\right)$$

$$\Rightarrow u(T) = \frac{u_0}{1 + \frac{u_0}{2\pi}\operatorname{Log}\left(\frac{a_0 T}{v_s}\right)}$$

$$T_0 = \frac{v_s}{a_0} e^{-2\pi/u_0}$$

- $T < T_0$：耦合强度发散

- 从这一点上看 Non-linear sigma model 是 disordered 的



- Correlation length: ξ

$$\xi(u) = a_0 f(u)$$

RG invariant: $$a_0 \frac{d\xi(u)}{da_0} = 0$$

$$a_0 \frac{d\xi(u)}{da_0} = a_0 f(u) + a_0 \frac{df(u)}{da_0}$$

$$\Rightarrow \quad f(u) + \frac{df(u)}{du} \cdot \beta(u) = 0$$

$$f(u) = f(u') \exp(2\pi (\frac{1}{u} - \frac{1}{u'}))$$

- $$\frac{\xi(u_1)}{\xi(u_2)} = \exp(2\pi (\frac{1}{u_1} - \frac{1}{u_2}))$$

$\Rightarrow \quad \xi(u_0) = \xi(u_2)\, e^{\pi s} \approx a_0 e^{\pi s}$

整数自旋时，没有 topological term

- Hopf term:

推广到更高阶维度：hedgehog 构型

$$m^a = z_\alpha^* \sigma_{\alpha\beta}^a z_\beta$$

$$|D_\mu z|^2 = [(\partial_\mu - iA_\mu) z]^2$$

Hopf term 不改变运动方程，改变拓扑激发统计性质

$$\mathcal{L}_{CS} = \frac{\theta}{4\pi} \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}$$

$$D_\mu z = \partial_\mu z - i A_\mu z$$

$$(\partial_\mu z - i A_\mu z)(\partial_\mu z^* + i A_\mu z^*) \equiv |D_\mu z|^2$$

$$= \partial_\mu z \, \partial_\mu z^* + i (A_\mu z^*) \partial_\mu z - i (\partial_\mu z^*) A_\mu z$$

$$+ A_\mu^2 |z|^2$$

$$i A_\mu (z^* \partial_\mu z - z \partial_\mu z^*) = 2 A_\mu \cdot A_\mu$$

- $A_\mu = i (z_\alpha^* \partial_\mu z_\alpha - z_\alpha \partial_\mu z_\alpha^*) / 2$

- $\partial_\mu m^a = \partial_\mu (z_\alpha^* \sigma^a_{\alpha\beta} z_\beta) = \partial_\mu z_\alpha^* \sigma^a_{\alpha\beta} z_\beta$

$$+ z_\alpha^* \sigma^a_{\alpha\beta} \partial_\mu z_\beta$$

$$\Rightarrow \sigma^a_{\alpha\beta} \sigma^a_{\gamma\delta} \, ((\partial_\mu z_\alpha^* z_\beta \; \partial_\mu z_\gamma^* z_\delta) + \ldots)$$

$$z_\alpha^* \partial_\mu z_\beta \; \partial_\mu z_\gamma^* z_\delta + \partial_\mu z_\alpha^* z_\beta \; z_\gamma^* \partial_\mu z_\delta$$

$$+ (z_\alpha^* \partial_\mu z_\beta)(z_\gamma^* \partial_\mu z_\delta)$$

1. $\sigma^a_{\alpha\beta}\,\sigma^a_{\gamma\delta}\ \partial_\mu Z^*_\alpha Z_\beta\,(\partial_\mu Z^*_\gamma) Z_\delta$

$= (\partial_\mu Z^*_\alpha\ Z_\alpha)(\partial_\mu Z^*_\beta\ Z_\beta)$

$\quad - \ (\partial_\mu Z^*_\alpha\ Z_\alpha)(\partial_\mu Z^*_\beta\ Z_\beta)$

$= (\partial_\mu Z^*_\alpha\ Z_\alpha)(\partial_\mu Z^*_\beta\ Z_\beta)$

2. $\sigma^a_{\alpha\beta}\,\sigma^a_{\gamma\delta}\ (Z^*_\alpha\ \partial_\mu Z_\beta)(Z^*_\gamma\ \partial_\mu Z_\delta)$

$= (\partial_\mu Z_\alpha\ Z^*_\alpha)(\partial_\mu Z_\beta \cdot Z^*_\beta)$

3. $\sigma^a_{\alpha\beta}\,\sigma^a_{\gamma\delta} = (Z^*_\alpha\ \partial_\mu Z_\beta)(\partial_\mu Z^*_\gamma\ Z_\delta)$

$= 2(Z^*_\alpha\ Z_\alpha)(\partial_\mu Z_\beta \cdot \partial_\mu Z^*_\beta) - (Z^*_\alpha\ \partial_\mu Z_\alpha)$

$(\partial_\mu Z^*_\beta\ Z_\beta)$

4. $\sigma^a_{\alpha\beta}\,\sigma^a_{\gamma\delta}\ (\partial_\mu Z^*_\alpha\ Z_\beta)(Z^*_\gamma\ \partial_\mu Z_\delta)$

$= 2(Z^*_\beta\ Z_\beta)(\partial_\mu Z^*_\alpha\ \partial_\mu Z_\alpha) - (Z^*_\beta\ \partial_\mu Z_\beta)$

$(Z_\alpha\ \partial_\mu Z^*_\alpha)$

• $(\partial_\mu \vec{m})^2 = 4(\partial_\mu Z_\alpha)(\partial_\mu Z^*_\alpha)$

$+ (Z_\alpha \partial_\nu Z^*_\alpha - Z^*_\beta Z_\mu Z_\alpha)^2$

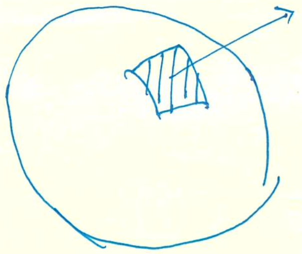
• $D_\mu Z = \partial_\mu Z - i A_\mu Z$

$(D_\mu Z)(D_\mu Z^*) = (\partial_\mu Z - i A_\mu Z)(\partial_\mu Z^* + i A_\mu Z^*)$

$= (\partial_\mu Z)(\partial_\mu Z^*) + i A_\mu (Z^* \partial_\mu Z - Z \partial_\mu Z^*) + A_\mu A_\mu$

$Z Z^*$

$= (\partial_\mu Z)(\partial_\mu Z^*) + A_\mu^2$



$F_{\mu\nu} = \vec{m} \cdot (\partial_\mu \vec{m} \times \partial_\nu \vec{m})$

Wess - Zumino - Witten model

- Nonabelian bosonization

SU(4) Heisenberg model – Nonabelian bosonization

#

Spin-liquid states:

- Spinons, holons, and valence-bond states:

$$\begin{cases} \vec{S}(x) = \frac{1}{2} a_\alpha^+(x)\, \sigma_{\alpha\beta}^+\, a_\beta(x) \\ \underline{a_\alpha^+(x)\, a_\alpha(x)} = 1 \end{cases}$$

$\Downarrow$ 表示什么意思：

$$S_z = \frac{1}{2}\left(a^+ a - b^+ b\right) = \pm\frac{1}{2} \quad \text{：即}$$

只能够表示 1/2，-1/2 两个态.

$\Downarrow$

Arovas – Auerbach 就是这个问题的升级

- Slave particles:

$$b^+(x)\, b(x) + f_\alpha^+(x)\, f_\alpha(x) = 1$$

$$|0\rangle,\quad |h\rangle,\quad \underbrace{|\uparrow\rangle,\ |\downarrow\rangle}_{\Downarrow\ \text{no charge}}$$

$$|h\rangle \equiv b^+|0\rangle \quad , \quad |\uparrow,\downarrow\rangle = f^+_{\uparrow,\downarrow}|0\rangle$$

$$C_\sigma^+(x) = b(x)\, f_\sigma^+(x)\, |0\rangle$$

$b^+(x)\, b(x)$ 和 $f_\sigma(x)\, f_\sigma(x)$ 是

- half filling 没有holon：即没有spinless的空穴
偏离半满的时候有Holon：

Mean-field theory (Affleck-Marston 1988)

超交换为：　　$\frac{1}{4}(n_+ - n_-)(n_+ - n_-)$

$$\vec{S}(x)\cdot\vec{S}(y)$$

$$= \cancel{S_{\alpha\beta}}\ C_\alpha^+(x)\, C_{\bar\alpha}(x)\, C_\beta^+(x)\, C_\beta(x)$$

$$\vec{S}(x)\cdot\vec{S}(y) = \frac{1}{4} C_\alpha^+(x)\, C_\beta(x)\, C_\gamma^+(y)\, C_\delta(y)$$

$$\sigma^a_{\alpha\beta}\,\sigma^a_{\gamma\delta} = \frac{1}{2} C_\alpha^+(x)\, C_\beta(x)\, C_\beta^+(x)\, C_\alpha(x) - \frac{1}{4} n(x)\, n(y)$$

- $$\begin{cases} H = \frac{J}{2} \sum C_\alpha^+(x)\, C_\beta(x)\, C_\beta^+(x+e_i)\, C_\alpha(x+e_i) \\ \vec{n}(x) = C_\alpha^+(x)\, C_\alpha(x) = 1 \end{cases}$$

$$\Rightarrow L(x,t) = \sum_{\vec{x}} C_\alpha^+(x,t)\,(\partial_\tau + u)\, C_\alpha(x,t)$$

$$+ \sum_{\vec{x}} \varphi(x,t)\,(C_\alpha^+(x,t)\, C_\alpha(x,t) - 1) - H$$

$\Downarrow$ HS Transformation

$\boxed{\text{怎么搞的}}$

$$C_{i\uparrow}^+ \cancel{C_{j\downarrow}\, C_{j\uparrow}^+}\, C_{j\downarrow}$$

$$= (C_{i\uparrow}^+ \cancel{C_{j\downarrow} - \chi_{ij}})\,(C$$

利用HS transformation得到effective的哈密顿量

$$L' = \sum_{\vec{x}} C_\alpha^+(x)(\partial_\tau + \mu) C_\alpha(x) + \sum_{\vec{x}} \varphi(x)(C_\alpha^+(x) C_\alpha(x) - 1) - \frac{2}{J}\sum_{\vec{x}, j} |x_j(\vec{x})|^2 + \sum_{\vec{x}, j} C_\alpha^+(x,t) X_{\alpha j}(x,t) C_\alpha(x+e_j, t) + C_\alpha^+(x+e_j), t) X_j(\vec{x}, t) C_\alpha(x, t)$$

在Bose场x_j鞍点近似展开，其中x_j为complex场。$\rho_j(x)$和$A_j(x)$表示amplitude和phase部分

$$\begin{cases} A_j(x,t) = A_j'(x,t) + \Delta_j \phi(x,t) \\ \varphi(\vec{x},t) = \varphi'(x,t) + \partial_t \phi(x,t) \\ C_\alpha(x) = e^{i\phi(x)} C_\alpha'(X) \end{cases}$$

则$\varphi(\vec{x},t)$像一个$U(1)$的规范场变化

1. $|x_j(x)| = \rho_j(x)$ fluctuations
2. $F_{\mu\nu}^2$（动能项是没有的）

· 若将规范变换代入作用量：

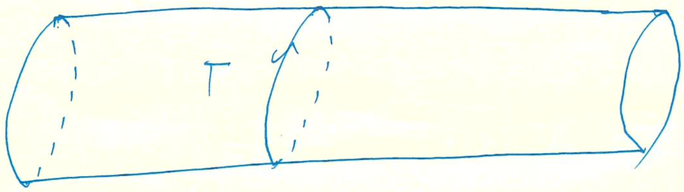
$$L \to L - \partial_t \phi$$

$$S \to S - \sum_x \int dt\, \partial_t \phi(x,t)$$

$$\varphi \equiv A_0$$

$$e^{-i\sum_x \int dt\, \varphi(x,t)} = \prod_x e^{-i\oint dt\, A_0(\vec{x},t)}$$

$$= \prod_{\vec{x}} e^{-i\oint_{\Gamma(\vec{x})} A^\mu dx_\mu}$$



Gauge invariace :

$$\oint_{\Gamma(x)} A^\mu(x)\, dx^\mu = \oint_{\Gamma(\vec{x})} dx_\mu A^{\mu\prime} + \oint_{\Gamma(\vec{x})} \partial^\mu \phi\, dx_\mu$$

规范不变性要求每个格点单占据

Affleck - Marston (1988, SU(N) Heisenberg model)

• Read and Sachdev (1989)

作用量则变为：

$$\mathcal{L}' = C^{+}_{\alpha a}(\vec{x},t)(i\partial_t + \mu) C_{\alpha a}(\vec{x},t)$$

$$+ \varphi_{ab}(\vec{x},t)\left(C^{+}_{\alpha a}(\vec{x},t) C_{\alpha b}(\vec{x},t) - \delta_{ab}\frac{N}{2} \right)$$

$$- \frac{N}{J}\left| \chi_j^{ab}(\vec{x},t) \right|^2 + C^{+}_{\alpha a}(x,t)\, \chi_j^{ab}(x,t)\, C_{\alpha b}(x+e_j,t)$$

$$+ C^{+}_{\alpha b}(\vec{x}+e_j,t)\, \chi_j^{ab\,*}(\vec{x},t)\, C_{\alpha a}(\vec{x},t)$$

$$\Downarrow$$

积掉费米子自由度，得到有效作用量

- SO(5) coherent states:

- SO(5) Quantum random walker

- Nonabelian Berry phases and path integral

$\Uparrow$ SU(4) AKLT

- SU(4) Heisenberg model: topological terms (Haldane chain & Generalization)

$\cancel{|\vec{n}\rangle = |\cos\frac{\theta}{2}, \sin\frac{\theta}{2}}\rangle$ $\quad T(g)|n\rangle$

$$\boxed{S^a_{\alpha} = \Psi^+_\alpha \Gamma^a_{\alpha\beta} \Psi_\beta} \quad = |g|n\rangle$$

$\langle \cancel{n| \vec{S} |\vec{n}\rangle = \hbar \vec{n} \cdot |\vec{h}\rangle}$

$$\boxed{\phi_{j_1 m_1; j_2 m_2} | j_1 m_1; j_2, m_2 \rangle}$$

$$\left(\frac{1 + \vec{S}_{\alpha\beta} \cdot S_{\alpha\beta}}{4}\right)^{DI}$$

$\cancel{T(g)} |n\rangle \otimes |3/2\rangle$

• saddle expansion:

$$\frac{\delta S_{tot}}{\delta \bar{\rho}_j(x,t)} = 0 ; \quad \sum_{plaquette} \bar{A}_j(x,t) = B$$

chiral spin liquid phases:

break translation and rotation invariance:

对应于 valence-bond crystals:

• 平均场

$$H = \frac{J}{2} \sum C_\alpha^+(x) C_\beta(x) C_\beta^+(x+e_j) C_\alpha(x+e_j)$$

$$\simeq -\frac{J}{2} \sum C_\alpha^+(x) C_\alpha(x+e_j) C_\beta^+(x+e_j) C_\beta(x)$$

$$\Downarrow$$

$$\Big(C_\alpha^+(x) C_\alpha(x+e_j) - \rho_j(x) e^{iA_j(x)}$$

$$+ \rho_j e^{iA_j(x)} \Big) \Big(C_\beta^+(x+e_j) C_\beta(x)$$

$$+ \rho_j \big(e^{iA_j(x)} + \rho_j e^{-iA_j(x)} \big)$$

$$= -\sum_{x,j} \rho_j(x) \Big(C_\alpha^+(x) C_\alpha(x+e_j) e^{iA_j(x)} + C_\alpha^+(x+e_j) C_\alpha(x) e^{-iA_j(x)} \Big) + \frac{N}{J} \sum \rho_j^2(x)$$