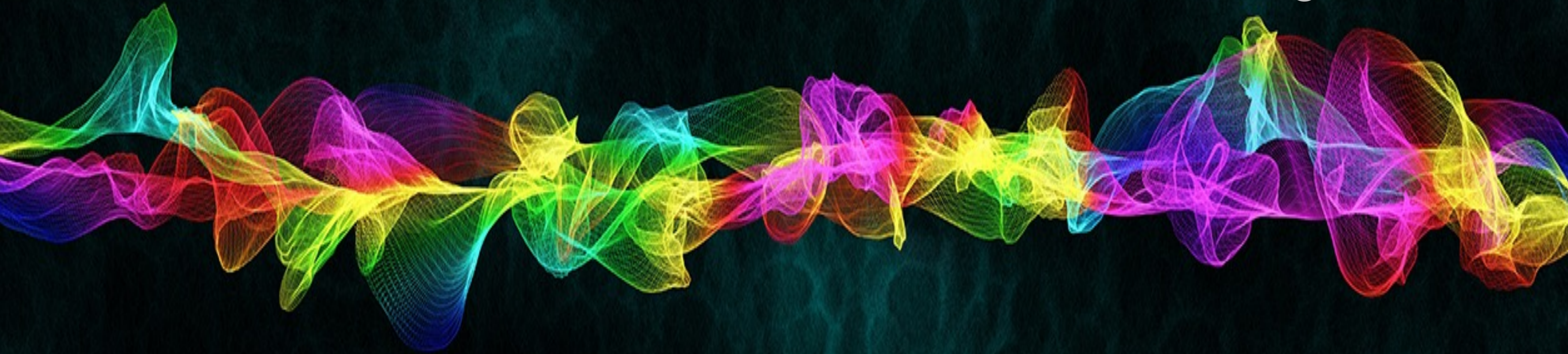
Generalized Symmetries in Quantum Field Theory



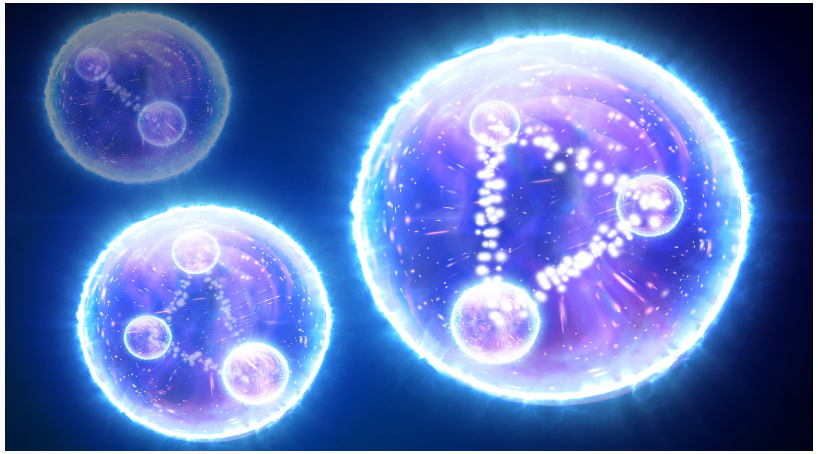
Clay Córdova

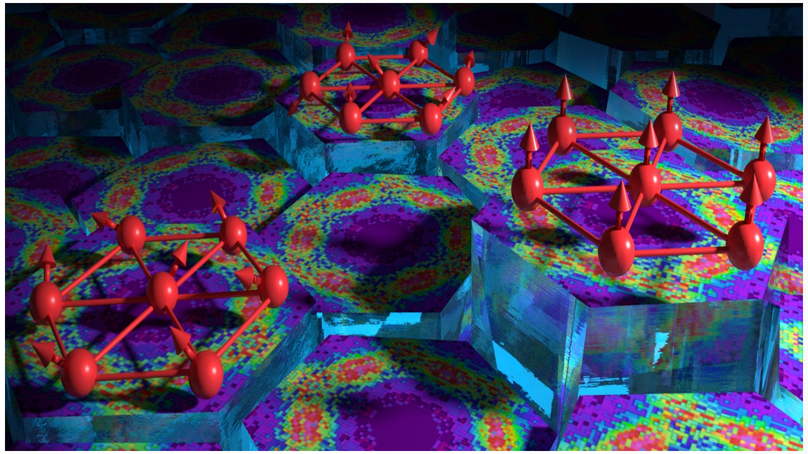
April 27th 2022

Microscopics & Macroscopics

Unifying problem: Given interacting microscopic constituents determine macroscopic emergent behavior

gluons
quarks
↓
confinement





spins
↓
liquid

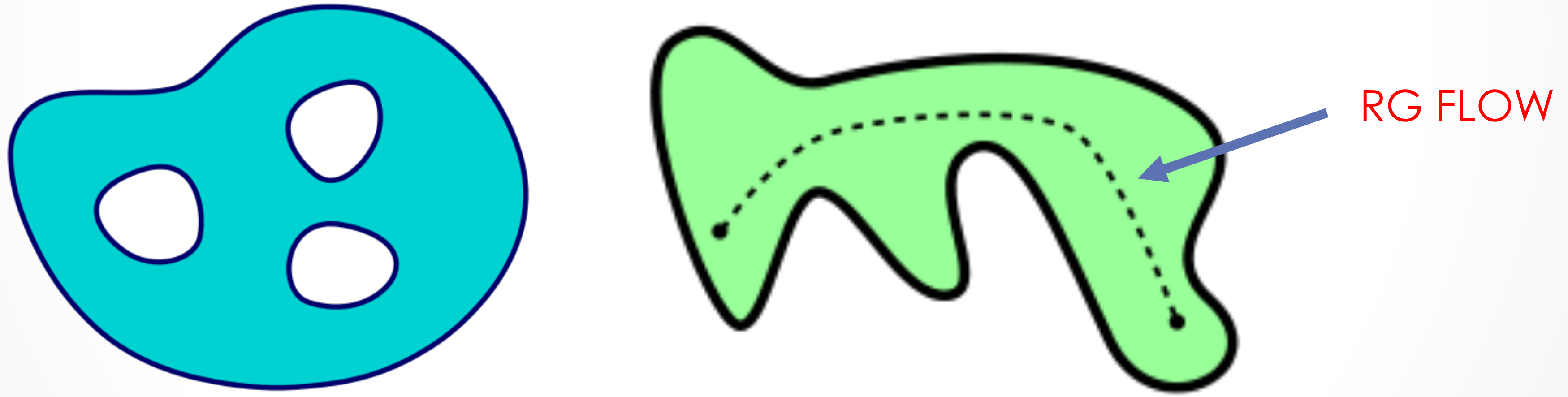
Renormalization Group Flow paradigm organizes physics by energy scale

Symmetry defines "constants of motion" for this flow in the space of QFTs

Aspirational Questions

Which field theories can be connected by continuous deformations?
(tuning couplings/masses, integrating in and out massive fields, RG flows, …)

Abstractly: in the space of all QFTs what are the connected components?



Modern viewpoint, connected component labelled (at least) by symmetries and associated 't Hooft anomalies (see e.g. [Seiberg Strings 2018]:)

Aspirational Questions

Can we organize phases of QFTs, i.e. long distance physics, by patterns of symmetry realization in the IR? (e.g. spontaneously broken or not …)

Landau paradigm **(delusional fantasy?!)** of phases via local order parameters

Challenged by exotic phase transitions and the richness of infrared physics

- Confinement/deconfinement transition in Yang-Mills refers to behavior of Wilson loops (area law vs perimeter law), not a local operator condensate
- General IR can be a topological quantum field theory: gapped with long range correlations, but not classified by any traditional notion of symmetry

Landau paradigm can only work with more expansive concept of symmetry

Symmetry $\Rightarrow$ Topological Operators

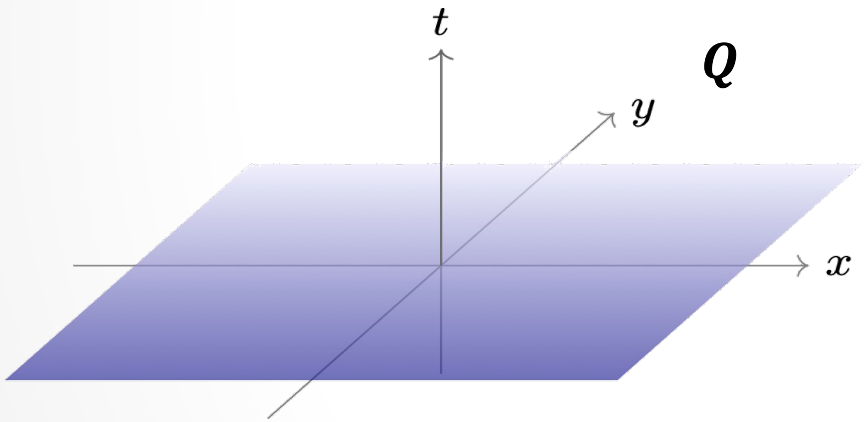
Currents, Locality, Deformations

Modern notion of symmetry in quantum field theory due to Noether 1915

Symmetry is local. Continuous symmetries encoded in current operator J^m

Probe theory by (source) background gauge field A :

$$S \supseteq \int A_m J^m$$



$$Q = \int_{space} J^0$$

Conservation allows us to change time slice where symmetry acts

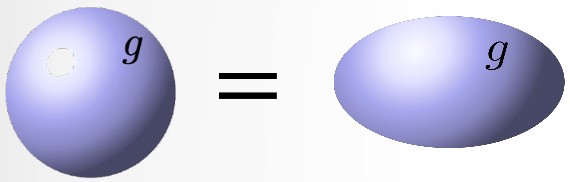
This is a topological deformation of the charge operator

Symmetry & Topology

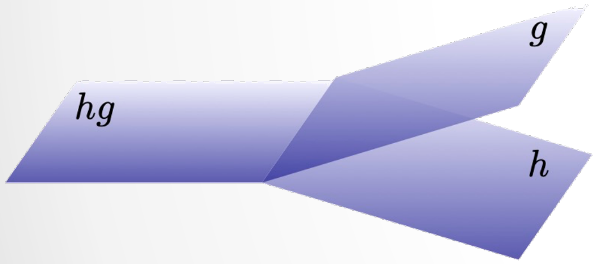
Abstractly, the charge is an extended operator (defect) of codimension one

(notation: $\boldsymbol{U_g(\Sigma)}$, $g \in G$, Σ codimension 1)

Key Properties:

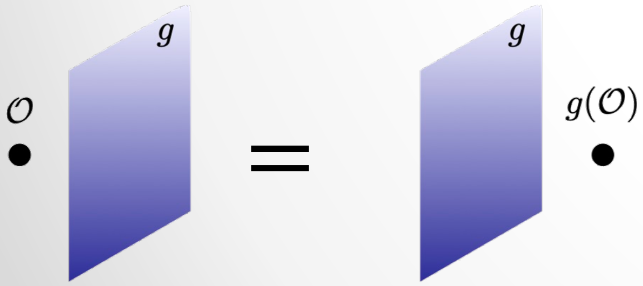


Topological: small changes in shape of Σ do not modify correlators of $U_g(\Sigma)$



Fusion algebra: yields group multiplication law

$$U_h(\Sigma) \times U_g(\Sigma) = U_{hg}(\Sigma)$$



Linking rule: crossing the defect with a local point operator leads to symmetry action

Higher Symmetry

Consider topological operators obeying same axioms but now of larger codimension. Define higher-form global symmetry [Gaiotto-Kapustin-Seiberg-Willett]

p-form global symmetry acts on extended operators of dimension p. Generated by symmetry defects $U_g(\Sigma)$ where support Σ is dimension d-p-1

Higher symmetry operators $U_g(\Sigma)$ act on extended operators L in QFT by linking

<u>Key Modifications for p>0:</u>

Symmetry group G necessarily abelian. (higher homotopy groups abelian)

Sources higher-form fields. p-form symmetry ↔ $A^{(p+1)}$, p+1 form gauge field

Confinement & Higher Symmetry

$SU(N)$ YM has $\mathbb{Z}_N$ 1-form symmetry from topological surface operators

Wilson loops charged under this symmetry. The charge of a Wilson loop is the N-ality of the representation. Cannot be screened by dynamical gluons

In a confined (deconfined) phase this 1-form symmetry is preserved (spontaneously broken) by the vacuum [Gaiotto-Kapustin-Seiberg-Willett]

Brings phases of gauge theories under the umbrella of the Landau paradigm (order parameter now a loop operator as opposed to local operator)

Also illustrates the inevitability of higher symmetry perspective: extended operators are ubiquitous in QFT. Natural to organize them using topology

Higher Symmetries & Anomalies

For any global symmetry we can ask if it can be gauged, i.e. if we can promote the background fields to dynamical fields (integrate over them)

't Hooft anomaly is any obstruction to gauging. It is scale invariant, so constant along RG flow or more general symmetry preserving deformation

In the case of continuous symmetries these can be extracted from a characteristic scale invariant structure function in the current correlators

In momentum space, along locus where $p_i^2 = p^2$, important term is (exactly!):
[Frishman-Schwimmer-Banks-Yankielowicz , Coleman-Grossman]

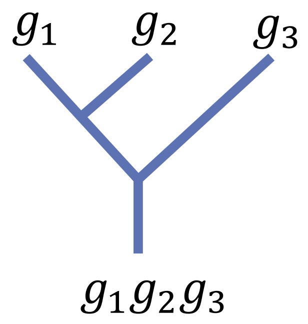
$$< J^a(p_1) J^b(p_2) J^c(p_3) > \supseteq \kappa \frac{\varepsilon^{abmn}}{P^2} p_{1m} p_{2n}\, p_3^c + perms$$

The constant κ is the scale invariant anomaly coefficient

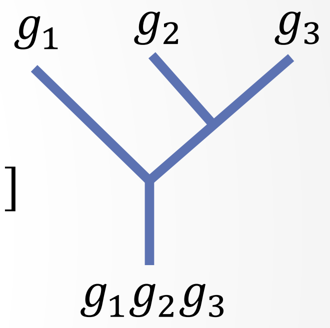
Higher Symmetries & Anomalies

For discrete symmetries characterized using topological defects, alternatively view anomaly as characteristic correlator of the defects $U_g(\Sigma)$

example of phase encountered when moving topological defect lines in 2d. This phase is an anomaly (captured by group cohomology)



$$= \exp[i\,\phi(g_1, g_2, g_3)]$$



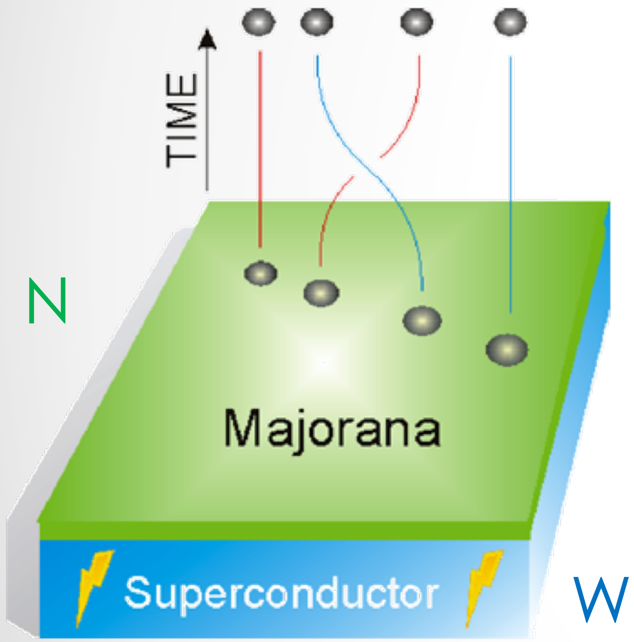
This perspective naturally generalizes to higher-form global symmetry:

- continuous p-form symmetry→p+1-form currents. Extract pieces of correlators
- discrete higher-form symmetry → directly investigate correlators of $U_g(\Sigma)$

In particular, **anomalies of generalized symmetries are invariant under RG flows**

Anomalies via Inflow

Anomaly defined by inflow. Dynamical QFT on N, anomaly theory on W
[Callan-Harvey, Freed-Hopkins-Teleman]



N

W

Classic examples involve Chern-Simons terms

In condensed matter physics this setup comes to life. Bulk W supports a material with trivial local dynamics

Boundary N has edge modes carrying anomaly

Mathematically, anomaly theory is invertible QFT:

$\mathcal{Z}(W) \in \mathbb{C}^*$ for any closed d+1-manifold W. (Equivalently single state in Hilbert space on any spatial slice.) Explosive progress in classification.
[Kapustin, Kapustin-Thorngren, Wang-Wan, Gaiotto-Johnson-Freyd, Lee-Ohmori-Tachikawa, ...]

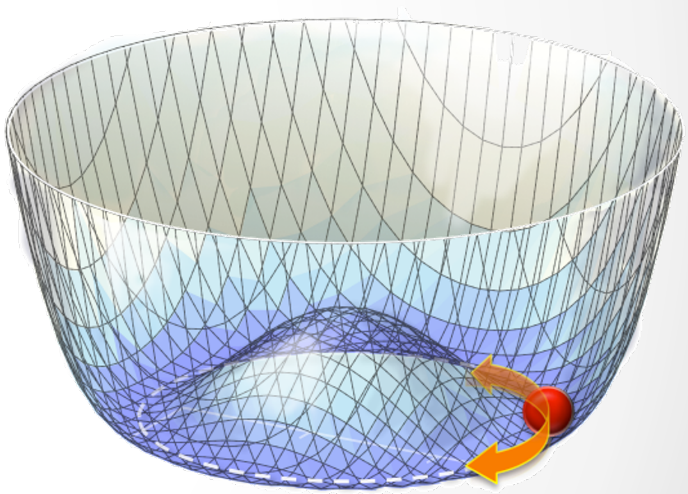
Dynamical Results From Anomalies

Anomalies and the Mass Gap

Given symmetry and anomaly fixed in UV. What IR behavior is possible?

Distinguish possibilities based on properties of vacuum:

Symmetry preserving phases. All charged excitations have vanishing expectation value



Spontaneous symmetry breaking phases.
For ordinary symmetry this means degenerate vacua

continuous symmetry breaking → goldstones bosons

Discrete symmetry breaking → domain walls

Anomalies and the Mass Gap

We can also ask about the mass gap in the theory. Two possibilities

- Gapless: IR described by a conformal field theory (sometimes simply a free theory of massless fields)

- Gapped: IR described by a topological quantum field theory

any anomaly can be carried by a gapless phase (free massless fermions)

any anomaly can be matched in spontaneous symmetry breaking by suitable Wess-Zumino terms or domain wall physics

Anomalies and the Mass Gap

Most constrained possibility is a symmetry preserving gapped phase. (TQFT)

Which anomalies can be carried by symmetry preserving gapped phases?

For instance returning to the anomalous current correlation function:

$$< J^a(p_1)J^b(p_2)J^c(p_3) > \supseteq \kappa \frac{\varepsilon^{abmn}}{p^2} p_{1m}p_{2n}\, p_3^c + perms$$

Power law behavior means that any theory with such an anomaly is always gapless. First applied by 't Hooft to constrain hypothetical models of preons

goal is to understand this interplay between the mass gap and symmetry
[Wang-Wen-Witten, Kobayashi-Ohmori-Tachikawa, Córdova-Ohmori, Thorngren]

A Topological Example

3d theories with time-reversal symmetry have a possible anomaly $\nu \in \mathbb{Z}_{16}$

N massless Majoranas have $\nu = N \; mod \; 16$ → symmetry preserving gapless phase

Chern-Simons TQFT $SO(N)_K$. Non-zero level breaks time-reversal

Level-rank duality: $SO(N)_K \cong SO(K)_{-N}$ → $SO(N)_N$ T invariant!

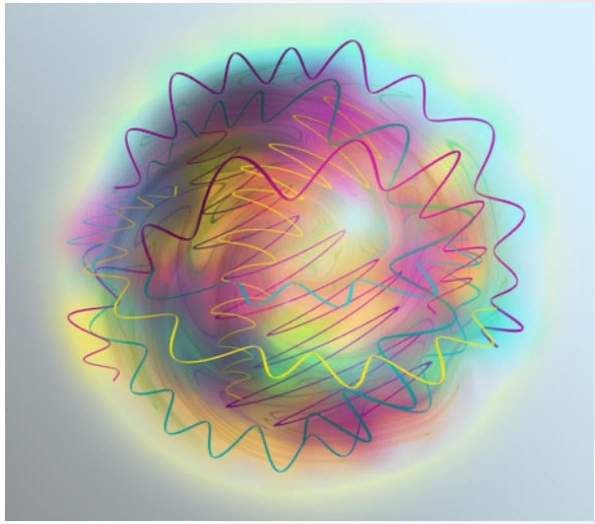
[see e.g. Hsin-Seiberg]

$SO(N)_N$ has $\nu = N \; mod \; 16$ → symmetry preserving gapped phase

[Cheng, Levin, Gomis-Komargodski-Seiberg]

Application: Yang-Mills Theory

Infrared believed to be gapped and color confining.
Numerical, analytical, and experimental evidence



glueball

Yang-Mills theory comes in a one-parameter family:

$$S \supseteq \frac{\theta}{8\pi^2}\int Tr(F \wedge F) \qquad \theta \in [0, 2\pi)$$

Special values $\theta = 0$ and $\theta = \pi$ are distinguished by time-reversal symmetry

Angle θ controls how instantons are weighted in the path integral. Lore about IR above concerns the theory at $\theta = 0$. **Does it hold for general θ?**

Application: Yang-Mills Theory

For all values of θ, $SU(N)$ Yang-Mills theory has a $\mathbb{Z}_N^{(1)}$ 1-form global symmetry generated by topological surface operators. (Wilson lines are charged)

At $\theta = \pi$: anomaly involving the 1-form symmetry and time-reversal

scale invariance → IR must be non-trivial

[Gaiotto-Kapustin-Komargodski-Seiberg, Kikuchi-Tanizaki, Córdova-Freed-Lam-Seiberg]

[Córdova-Ohmori] proved that vacuum at $\theta = \pi$ must be :

T-breaking or deconfined or gapless

Lore: T-symmetry spontaneously broken at $\theta = \pi$ (stable domain walls)

Rigorous new results about foundational QFT core to modern physics.

Higher Group Global Symmetry

Symmetry of 4d QED

When a QFT has higher symmetry of different degrees, mixture is possible
[Kapustin-Thorngren, Tachikawa, Córdova-Dumitrescu-Intriligator, Benini-Córdova-Hsin]

Consider e.g. $U(1)$ gauge theory with N_f massless fermions of charge Q:

- There is $SU(N_f)_L \times SU(N_f)_R$ ordinary symmetry acting on the fermions
- There is a $U(1)$ 1-form symmetry with conserved current $J_{ab} = (*F)_{ab}$ (conserved by Bianchi). Charged objects are 't Hooft lines

Mixing between 0-form and 1-form symmetry encoded in $< J_a \, J_b \, J_{cd} >$

(Analog: structure constants for non-abelian ordinary global symmetry are encoded in 3-point functions of J_a. Thus we are discovering an algebra)

2-Group Ward Identities

Ward identities relating $< J_a J_b J_{cd} >$ and $< J_{ab} J_{cd} >$. On the locus in momentum space $p^2 = q^2 = (p+q)^2$ with M a mass scale:

$$< J_{ab}(p) J_{cd}(-p) > = \frac{1}{p^2} f\left(\frac{p^2}{M^2}\right) tensor_{abcd}$$

$$< J_a^k(p) J_b^l(q) J_{cd}(-p-q) > = \alpha \, \frac{\delta^{kl}}{p^2} f\left(\frac{p^2}{M^2}\right) tensor'_{abcd}$$

4d QED realizes these identities with $\alpha = Q$, J_a^k either chiral $SU(N_f)$, $J_{ab} = (* F)_{ab}$

[Córdova-Dumitrescu-Intriligator]

Alternatively in OPE: $\partial^a J_a(x) J_b(0) \sim \alpha \, \partial^c \delta(x) J_{cd}(0)$ (compare α to f^{klm})

2-Group Defect Fusion Algebra

Ward identities and OPEs can be encoded in background fields

Say both 0-form and 1-form are $U(1)$, appropriate background fields locally

1-form gauge field $A^{(1)}$, 2-form gauge field $B^{(2)}$

Under general gauge transformations these now mix as:

$$A^{(1)} \to A^{(1)} + d\Lambda^{(0)}, \qquad B^{(2)} \to B^{(2)} + d\Lambda^{(1)} + \frac{\alpha}{2\pi}\Lambda^{(0)} dA^{(1)}$$

This is a Green-Schwarz transformation for background fields. [Kapustin-Thorngren, Córdova-Dumitrescu-Intriligator, Benini-Córdova-Hsin]

Such a pair $(A^{(1)}, B^{(2)})$ defines a 2-connection on a 2-group bundle

Defect Fusion Algebra

Background field formalism has an analog via symmetry defect fusion

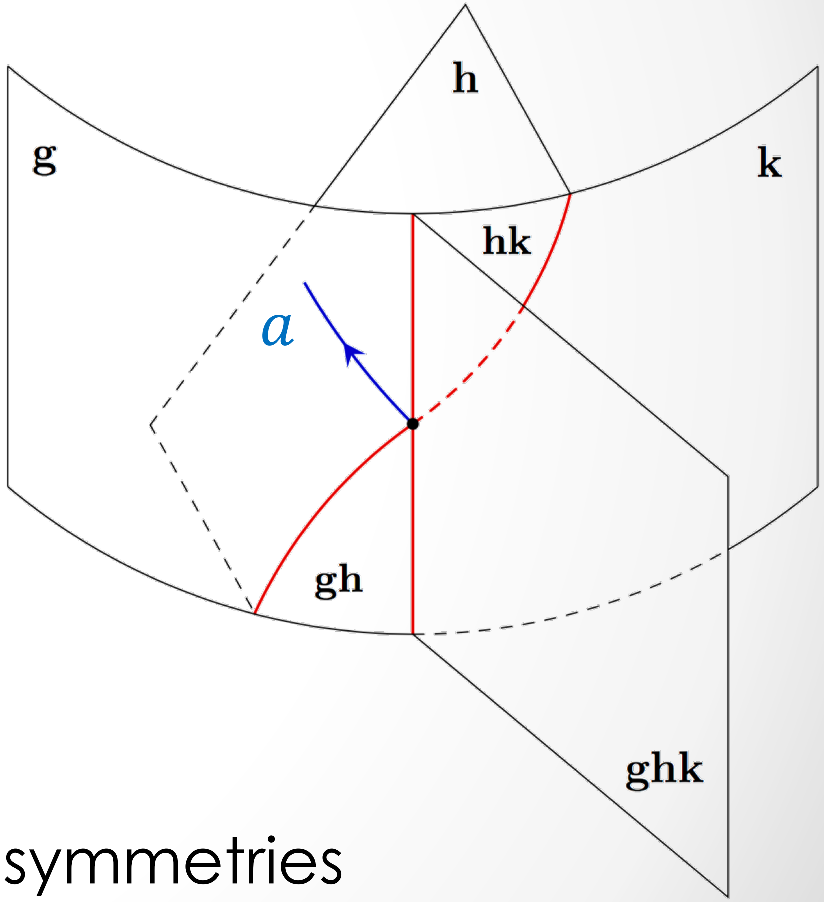
0-form symmetry G, 1-form symmetry $\mathcal{A}$

In $d \geq 3$, triple fusion of G defects generic

At intersection 1-form symmetry defect $a \in \mathcal{A}$ can appear (shown in blue)

fusion defined by cohomology class $\alpha \in H^3(G, \mathcal{A})$

[Kapustin-Thorngren, Tachikawa, Benini-Córdova-Hsin]



wide variety of constructions in setting of discrete symmetries

[Benini-Córdova-Hsin]

Higher Group Bonanza

mixing between higher-form symmetries of different degrees is everywhere!

- 3d TQFTs and Chern-Simons (ordinary symmetry and abelian anyons) [Barkeshli-Bonderson-Cheng-Wang, Benini-Cordova-Hsin]
- 4d gauge theories particularly with SO-type gauge groups [Hsin-Lam, Lee-Ohmori-Tachikawa]
- axion physics: axion string symmetry and flavor symmetry in a 3-group [Hidaka-Nitta-Yokokura, Córdova-Brennan]
- little string theories (natural due to string current) [del-Zotto-Ohmori, Córdova-Dumitrescu-Intriligator]
- 5d and 6d susy theories: ubiquoutous, matched across dualities [Bhardwaj-Schafer-Nameki, Appruzi-Bhardwaj-Gould-Oh-Schafer-Nameki]

Topological Operators ⇒ Symmetry

Non-Invertible Symmetries

As a final generalization of the concept of symmetry in QFT consider OPE

$$U_h(\Sigma)\times U_g(\Sigma) = U_{hg}(\Sigma)$$

More general fusion rule allows for the possibility of more terms on RHS:

$$U_i(\Sigma)\times U_j(\Sigma) = \sum_k c_{ij}^k \times U_k(\Sigma)$$

invertible (group-like) defects, defined by existence of inverse under fusion

topological operator, with no inverse is **non-invertible**

Symmetry?!

Like other concepts of symmetry, non-invertible operators are topological

(working) modern definitions:

symmetry of QFT = subset of topological operators (any codim, any fusion)

Solvable (and mathematically rigorous) subsector of a field theory. Metric independence is manifest and reflects scale invariance

anomaly = obstruction to gauging (summing over) general symmetry

Universality classes labelled, at least, by generalized symmetry and anomaly

Fusion Category Symmetry in 2d

2d CFTs provide a useful playground to explore non-invertible symmetry

A rational theory has Verlinde lines L_i. (one for each chiral primary operator)

Action on a chiral primary state is $L_k \,|\, \mathcal{O}_{\mathrm{i}} > = \frac{S_{ki}}{S_{oi}} \,|\, \mathcal{O}_{\mathrm{i}} >$ (modular S matrix)

Fusion algebra follows from the Verlinde formula:

$$L_i \times L_j = \sum_k N_{ij}^k \; L_k$$

Recent Work

[Chang-Lin-Shao-Wang-Xin, Thorngren-Wang, Carqueville-Runkel, Brunner-Carqueville-Plencner, Bhardwaj-Tachikawa]

Generalization: **symmetry group** → **fusion category**

2d Adjoint QCD

Early dynamical application to 2d massless adjoint QCD

IR theory is deconfined, due to fractionalization of adjoint quarks [Kutasov-Schwimmer, Gross-Klebanov-Matytsin-Smilga, Cherman-Jacobson-Tanizaki-Unsal]

Modern viewpoint: algebra of non-invertible symmetry lines and one-form symmetry defects (points in 2d) obstructs a confined phase [Komargoski-Ohmori-Roumpedakis-Seifnashri]

implications for effective field theory. Consider quartic fermion operator

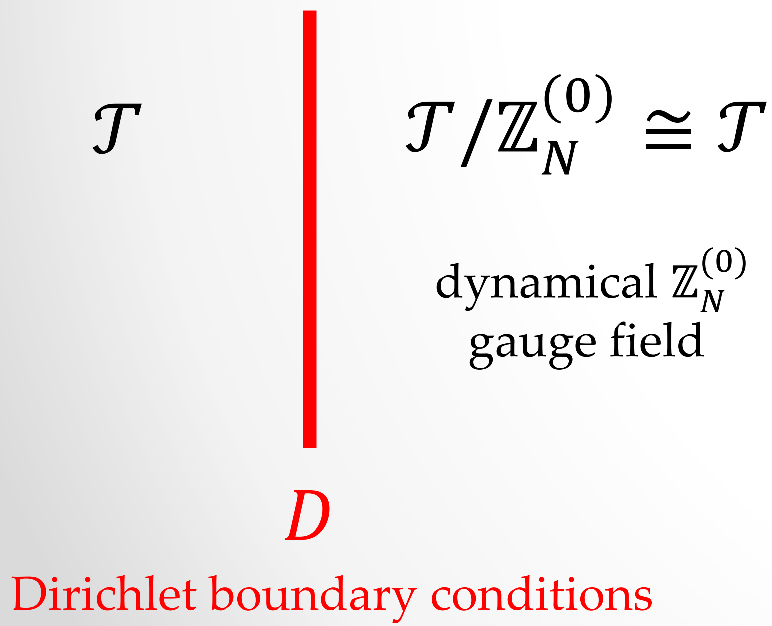
$$\mathcal{O} \sim Tr(\Psi_+\Psi_-)Tr(\Psi_+\Psi_-)$$

invertible symmetries allow $\mathcal{O}$ but non-invertible symmetries forbid $\mathcal{O}$

Kramers-Wannier Duality Defects

Consider 2d QFT $\mathcal{T}$ with $\mathbb{Z}_N^{(0)}$ ordinary symmetry. Gauging produces a new theory $\mathcal{T}/\mathbb{Z}_N^{(0)}$ which has dual $\mathbb{Z}_N^{(0)}$symmetry (orbifold). Define η to be $\mathbb{Z}_N^{(0)}$ line

If the theory invariant under gauging (i.e. self dual), $\mathcal{T} \cong \mathcal{T}/\mathbb{Z}_N^{(0)}$ construct new non-invertible duality (or Dirichlet) defect D with universal fusion rules



D is codimension one and non-invertible,

$\eta \times D = D \times \eta = D, \qquad \eta^N = 1, \qquad D \times D = \sum_{i=0}^{N-1} \eta^i$

example $N = 2$ realized by critical Ising model there, it is known as Kramers-Wannier duality

Duality Defects in 4d

Consider 4d QFT $\mathcal{T}$ with $\mathbb{Z}_N^{(1)}$ 1-form symmetry. Gauging produces theory $\mathcal{T}/\mathbb{Z}_N^{(1)}$ which has dual $\mathbb{Z}_N^{(1)}$ symmetry. Define η to be $\mathbb{Z}_N^{(1)}$ surface operator

[Kiadi-Ohmori-Zheng, Choi-Córdova-Lam-Hsin-Shao]

If the theory invariant under gauging (i.e. self dual), $\mathcal{T} \cong \mathcal{T}/\mathbb{Z}_N^{(1)}$ construct new non-invertible duality (or Dirichlet) defect D with universal fusion rules

$\mathcal{T}$ | $\mathcal{T}/\mathbb{Z}_N^{(1)} \cong \mathcal{T}$

dynamical $\mathbb{Z}_N^{(1)}$ gauge field

$D(M)$

Dirichlet boundary conditions

D is codimension one and non-invertible,

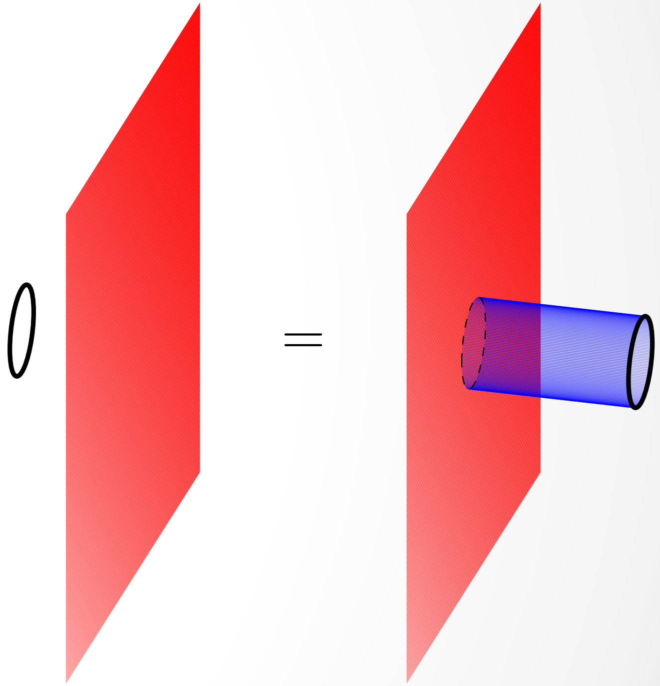
$$\eta \times D = D \times \eta = D, \qquad \eta^N = 1,$$

$$D(M) \times \overline{D}(M) = \sum_{Y \in H^2(M,\mathbb{Z}_N)} \eta(Y)$$

Properties of the Defects

Why are these duality defects non-invertible? Their action on operators in the theory does not preserve the dimension of the operator
[Kiadi-Ohmori-Zheng, Choi-Córdova-Lam-Hsin-Shao]

Sweeping D past a Wilson loop yields an η surface that ends on the duality defect.



Generalizes order/disorder map in 2d KW duality

The summation over surfaces appearing in the fusion of D with $\overline{D}$ is an example of a condensation defect.
[Gaiotto-Johnson-Freyd, Roumpedakis-Seifnashri-Shao]

[Roumpedakis-Seifnashri-Shao, Choi-Córdova-Lam-Hsin-Shao]

The fusion “coefficients” appearing in fusion rules are generally TQFTs

Concrete Applications

There is an expanding list of models in 4d with non-invertible symmetry:

- Lattice gauge theories (both discrete and continuous gauge groups) [Koide-Nagoya-Yamaguchi, Hiyashi-Tanizaki, Choi-Córdova-Lam-Hsin-Shao]
- Free Maxwell theory at special couplings [Choi-Córdova-Lam-Hsin-Shao]
- $\mathcal{N}=4$ SYM at special couplings (S-duality is not invertible at $\tau = i$) [Kiadi-Ohmori-Zheng, Choi-Córdova-Lam-Hsin-Shao]
- $PSU(N)$ YM at $\theta = \pi$, (protects multiple vacua) [Kiadi-Ohmori-Zheng]

Non-invertible symmetries also have anomalies, and these constructions are rapidly being generalized and explored! [Choi-Córdova-Lam-Hsin-Shao, Bhardwaj-Bottini-Schafer-Nameki-Tiwari]

Conclusions

Symmetry and anomalies are key concepts to understand universality in quantum field theory

From ordinary symmetries and their defects $U_g(\Sigma)$:

general defect dimension → higher-form symmetry

mixing of form degrees → higher-group symmetry

general fusion algebra → non-invertible symmetry

Great opportunity to apply general symmetry formalism to new and old problems of RG flow

Thanks for Listening!