Callan-Symanzik method as a finite approach to QFT: non-renormalizable theory case

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1 Outline

1 Is it possible to proceed to all calculations in QFT without any divergences?

- In standard approaches in QFT $\rightarrow$ meet some **divergences** during calculations of loops...
- One of the goals of QFT is to compute **n-point Green's functions** $\rightarrow$ related to physical observables like particle lifetimes and cross-sections.
- Example: review the standard approach to the renormalization of these functions in the following theory:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The metric signature is $(- + ++)$.

1 Simple example in standard approach: ϕ^4 theory

- Consider the one-particle-irreducible (OPI) two- and four-point Green functions in dimensional regularization. In standard approach they are given by (up to one loop)

$$\Gamma^{(2)}(k) = i(k^2 + m^2) + \frac{\lambda\mu^{4-d}}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2},$$

$$\Gamma^{(4)}(\kappa_i) = -i\lambda\mu^{4-d} + \sum_{3 \text{ opt}} \frac{\lambda^2\mu^{8-2d}}{2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + m^2)} \frac{1}{(l + \kappa_i)^2 + m^2}.$$

- Since Green functions are directly related to observables, **they must be finite**.
- In the standard approach → regularise the infinities.
- Having regularised the UV divergent integrals → move to **renormalisation** → add counterterms to the Lagrangian and subtracts the divergences.

1 Simple example in standard approach: ϕ^4 theory

- After renormalisation we arrive to finite answers:

$$\bar{\Gamma}^{(2)} = i(k^2 + m_1^2) + O(\lambda^2),$$

$$\bar{\Gamma}^{(4)} = -i\lambda_1 + \sum_{3\ \text{opt}} \frac{i\lambda_1^2}{32\pi^2} \int_0^1 dx \cdot \ln\left(\frac{m_1^2}{x(1-x)\kappa_i^2 + m_1^2}\right) + O(\lambda^3),$$

where λ_1 and m_1 are physical and finite parameters now.

1 Is it possible to proceed to all calculations in QFT without any divergences? Yes!

- In standard approach, although it uses the UV divergent integrals, but at the end of the day, this is just a **mapping** between the well-defined set of **finite parameters**, that characterise the theory and the set of **experimental observables**.
- Thus, from this "mapping argument" point of view it is quite **natural to require** the existence of the formulation of **QFT without infinities** at all.
- We would like to explore such a procedure, which provides no divergent expressions at any stage of the computation in QFT.

1 Is it possible to proceed to all calculations in QFT without any divergences? Yes!

More fundamental motivation:

- **Hierarchy problem** (e.g. why the Higgs mass is so much smaller than the Planck scale?).
- Consider the bare mass m_0 for Higgs field. The quantum correction **shifts** m_0^2 by a **huge quadratically cutoff** dependent amount:

$$\delta m_0 \sim f\Lambda^2,$$

  where Λ is some characteristic mass scale (say, Planck mass) and where f denotes some dimensionless coupling.
- To have physical mass

$$m_P^2 = m_0^2 + \delta m_0,$$

  of order ~ 125 GeV $\rightarrow$ we require an extremely fine-tuned and highly unnatural cancellation between m_0^2 and δm_0.

2 Outline

2 Divergence-free methods

- Such divergence-free methods have been invented already in the past.
- We stick to the scheme which is based on **Callan-Symanzik** (differential) equations. C. G. Callan'1970; A. S. Blaer, K. Young'1974 $\rightarrow$ was designed to prove the validity of the standard multiplicative renormalization program.
- But the solution of these differential equations with boundary conditions $\rightarrow$ renormalized n-point OPI Green functions!

We have learned a great deal. First, we have shown that the multiplicative renormalization scheme actually produces renormalized Green functions which have a finite $\Lambda \rightarrow \infty$ limit. Second, we have shown that the renormalized Green functions satisfy a set of "renormalization group" equations,

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right] \bar{\Gamma}^{(n)}(p;\lambda,\mu) = -i\mu^2 \alpha(\lambda) \bar{\Gamma}^{(n)}_{\theta}(0;p;\lambda,\mu)\,,$$

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda) + \gamma_\theta(\lambda)\right] \bar{\Gamma}^{(n)}_{\theta}(q;p;\lambda,\mu) = -i\mu^2 \alpha(\lambda) \bar{\Gamma}^{(n)}_{\theta\theta}(0,q;p;\lambda,\mu)\,, \qquad (2.13)$$

which, together with the normalization conditions, allow one to systematically compute in a unique fashion (indeed, in a way which never encounters a divergent Feynman integral) the perturbation expansion of the renormalization parts. Thus, any two renormalization schemes which yield Green functions satisfying these equations will necessarily yield identical Green functions. The re-

C. G. Callan'1970

- In order to obtain CS equations, which only include renormalised (= finite) quantities, we firstly turn to the bare Lagrangian

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4.$$

- The CS approach is all based on the observation that differentiating the (bare) scalar field propagator with respect to m_0^2 yields (minus i times) two propagators:

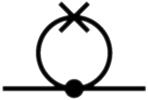
$$\frac{d}{dm_0^2}\left[\frac{-i}{k^2+m_0^2}\right] = -i\left(\frac{-i}{k^2+m_0^2}\right)^2.$$

- Obviously, **adding an extra propagator** to the diagram **reduces its degree** of divergence by two.

- Taking this derivative (and multiplying by $-i$) is now denoted as acting with a θ-**operation** on a propagator.
- **The algebraic** representation of the θ-operation is

$$\Gamma_{\theta}^{(n)}(k^2) \equiv -i \times \frac{d}{dm_0^2}\Gamma^{(n)}(k^2).$$

- Since this operation splits every propagator, one by one, in two parts $\rightarrow$ it equals to **inserting a new kind of "cross" vertex**, which comes with Feynman rule (-1):



2 Obtain CS equations

- In order to obtain equations $\to$ rewrite both sides of

$$\Gamma_\theta^{(n)}(k^2) \equiv -i \times \frac{d}{dm_0^2}\Gamma^{(n)}(k^2),$$

  in terms of renormalized quantities.
- Need to know the relation between bare and renormalized correlation functions. Recall that the renormalized field and bare one are connected as

$$\phi_{ph} = \frac{\phi_0}{\sqrt{Z}}, \quad \Gamma^{(n)}(\lambda_0, m_0) = Z^{n/2}\bar{\Gamma}^{(n)}(\lambda, m).$$

- Also introduce

$$\Gamma_\theta^{(n)}(\lambda_0, m_0) = Z^{n/2} Z_\theta \bar{\Gamma}_\theta^{(n)}(\lambda, m).$$

- Use the following decomposition of the "bare" total derivative in terms of "physical" partial derivatives

$$\frac{d}{dm_0^2} = \frac{\partial m^2}{\partial m_0^2}\frac{\partial}{\partial m^2} + \frac{\partial \lambda}{\partial m_0^2}\frac{\partial}{\partial \lambda}.$$

2 Obtain CS equations

- Then, the **first Callan-Symanzik equation** reads

$$2im^2(1+\gamma)\bar{\Gamma}_\theta^{(n)} = \left[n\gamma + \left(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda}\right)\right]\bar{\Gamma}^{(n)},$$

where

$$\beta \equiv 2m^2\left[\frac{\partial m^2}{\partial m_0^2}\right]^{-1}\frac{\partial\lambda}{\partial m_0^2},$$

$$\gamma \equiv m^2\left[\frac{\partial m^2}{\partial m_0^2}\right]^{-1}\frac{\partial \ln Z}{\partial m_0^2}.$$

- CS equation contains only finite quantities!

- Now, recall that for bare $\Gamma^{(2)}$ up to one loop in ϕ^4-theory we have

$$\Gamma^{(2)}(k) = i(k^2 + m^2) + \frac{\lambda\mu^{4-d}}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 + m^2},$$

  and here we need to apply two θ-operations in order to obtain finite value $\to$ we need one more CS equation!
- Later on, for non-renormalizable theory we will need **even more CS equations** $\to$ three CS equations and three θ-operations and etc...
- One can derive the most general form of CS equation:

$$2m^2 i(1+\gamma)\bar{\Gamma}^{(n)}_{\theta_1\ldots\theta_k} = $$
$$= \left[\left(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \sum_i \Omega_{L_i}\frac{\partial}{\partial L_i}\right) + n\gamma + (k-1)\gamma_\theta\right]\bar{\Gamma}^{(n)}_{\theta_1\ldots\theta_{k-1}},$$

  where L_i corresponds to new possible terms in Lagrangian and another definition was used

$$\Gamma^{(n)}_{\theta_1\ldots\theta_k} \equiv -i \times \frac{d}{dm_0^2}\Gamma^{(n)}_{\theta_1\ldots\theta_{k-1}}.$$

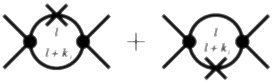
- Recall the bare 4-point function

$$\Gamma^{(4)} = -i\lambda + \sum_{3\text{ opt}} \frac{\lambda^2}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2+m^2)} \frac{1}{(l+\kappa_i)^2+m^2}.$$

- Here it is enough to only consider the one CS equation

$$2im^2(1+\gamma)\bar{\Gamma}^{(4)}_\theta = \left[4\gamma + \left(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda}\right)\right]\bar{\Gamma}^{(4)},$$

so firstly one needs to find $\bar{\Gamma}^{(4)}_\theta$. It is shown in figure below:



2 Use CS equations: derive 4-point correlation function in ϕ^4 theory

- These diagrams are given by

$$[\bar{\Gamma}^{(4)}_{\theta}]_{\lambda^2} = -\sum_{3\text{ opt}} \lambda^2 \int_0^1 dx \frac{1}{32\pi^2\left(m^2 + \kappa_i^2 x(1-x)\right)}.$$

- Next, use $\bar{\Gamma}^{(4)}(0) = -i\lambda$ at $\kappa_i^2 = 0$, one can find

$$-\frac{3i\lambda^2}{16\pi^2} = -i\left([\beta]_{\lambda^2} + 4\lambda[\gamma]_{\lambda}\right).$$

- And turning back to CS equation:

$$\frac{\partial}{\partial m^2}\left[\bar{\Gamma}^{(4)}\right]_{\lambda^2} = -\frac{i\lambda^2}{32\pi^2}\sum_{3\text{ opt}}\int_0^1 dx \frac{1}{x(1-x)\kappa_i^2 + m^2} + \frac{3i\lambda^2}{32\pi^2}\cdot\frac{1}{m^2},$$

one finds

$$\bar{\Gamma}^{(4)} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\sum_{3\text{ opt}}\int_0^1 dx \ln\frac{m^2}{x(1-x)\kappa_i^2 + m^2} + \mathcal{O}\left(\lambda^3\right).$$

The result:

This answer coincides with the result from standard approach, but now it was obtained in a **manifestly finite way**!

- CS for ϕ^4 (for theory with two fields, and etc) $\to$ see **S. Mooij, M. Shaposhnikov 2110.05175, 2110.15925**
- Next orders, more loops? $\to$ CS method is recursive, so order by order, one can recover the usual results (up to all orders) for n-point functions! $\to$ see **S. Mooij, M. Shaposhnikov 2110.05175, 2110.15925**

3 Outline

3 Non-renormalizable theory

- Consider now

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$
$$+ \frac{\xi}{M^2}\phi(\Box^2\phi) + \frac{g}{6!M^2}\phi^6 + \frac{f}{3!M^2}\phi^3\Box\phi.$$

- We take into account all possible operators of M^6 dimension $\to$ thus expect, that in each separate order the model is renormalizable!
- The λ and λ^2 orders for 2- and 4-point correlation functions were already obtained in **S. Mooij, M. Shaposhnikov 2110.05175, 2110.15925** $\to$ we consider $1/M^2$ and λ/M^2 orders to illustrate the CS method in the non-renormalizable case!
- At each step (=order) we compare our answers from CS method with the answers from standard approach $\to$ spoiler: they coincide!

- The boundary conditions are

$$\bar{\Gamma}^{(2)}|_{k^2=0} = im^2,$$

$$\left[\frac{d}{dk^2}\bar{\Gamma}^{(2)}\right]_{k^2=0} = i,$$

$$\left[\frac{d^2}{d(k^2)^2}\bar{\Gamma}^{(2)}\right]_{k^2=0} = -\frac{4i\xi}{M^2},$$

$$\bar{\Gamma}^{(4)}|_{\kappa_i^2=0} = -i\lambda,$$

$$\left[\frac{d}{d\kappa_i^2}\bar{\Gamma}^{(4)}\right]_{\kappa_i^2=0} = -\frac{if}{M^2},$$

$$\bar{\Gamma}^{(6)}|_{\kappa_j^2=0} = \frac{ig}{M^2}.$$

3 Non-renormalizable theory: two-point correlation function

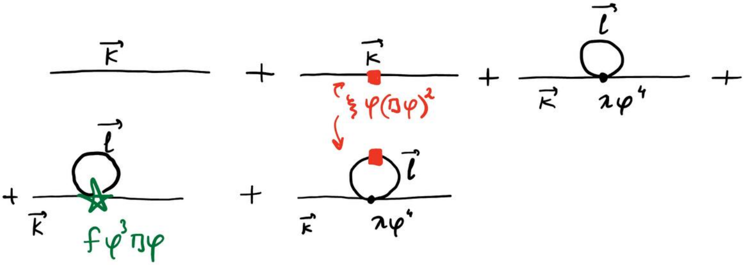


Figure: Two-point function (up to λ, $1/M^2$, λ/M^2 orders).

- Consider, for example

$$[\Gamma^{(2)}]_{1/M^2} \sim \int \frac{d^4 l}{(2\pi)^4} \frac{(-i)}{l^2 + m^2} [(ik_\mu)^2 + (il_\mu)^2].$$

- Applying three θ-operations to mentioned diagram (and corresponding expression), we obtain (after integration):

$$[\bar{\Gamma}^{(2)}_{\theta\theta\theta}]_{f/M^2} = \left(\frac{k^2 + 2m^2}{16m^4\pi^2}\right).$$

- Now, write all CS equations for two-point correlation function...

$$1)\ \ 2m^2 i(1+\gamma)\bar{\Gamma}^{(2)}_{\theta\theta\theta} = \Big[\Big(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_\xi\frac{\partial}{\partial(\xi/M^2)} + \Omega_g\frac{\partial}{\partial(g/M^2)} + \Omega_f\frac{\partial}{\partial(f/M^2)}\Big) + 2\gamma + 2\gamma_\theta\Big]\bar{\Gamma}^{(2)}_{\theta\theta},$$

$$2)\ \ 2m^2 i(1+\gamma)\bar{\Gamma}^{(2)}_{\theta\theta} = \Big[\Big(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_\xi\frac{\partial}{\partial(\xi/M^2)} + \Omega_g\frac{\partial}{\partial(g/M^2)} + \Omega_f\frac{\partial}{\partial(f/M^2)}\Big) + 2\gamma + \gamma_\theta\Big]\bar{\Gamma}^{(2)}_{\theta},$$

$$3)\ \ 2m^2 i(1+\gamma)\bar{\Gamma}^{(2)}_{\theta} = \Big[\Big(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_\xi\frac{\partial}{\partial(\xi/M^2)} + \Omega_g\frac{\partial}{\partial(g/M^2)} + \Omega_f\frac{\partial}{\partial(f/M^2)}\Big) + 2\gamma\Big]\bar{\Gamma}^{(2)}.$$

$$3)\ \ 2m^2 i(1+\gamma)[\bar{\Gamma}^{(2)}_{\theta}]_{k^2=0} = \Big[\Big(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_\xi\frac{\partial}{\partial(\xi/M^2)} + \Omega_g\frac{\partial}{\partial(g/M^2)} + \Omega_f\frac{\partial}{\partial(f/M^2)}\Big) + 2\gamma\Big][\bar{\Gamma}^{(2)}]_{k^2=0}.$$

$$2)\ \ 2m^2 i(1+\gamma)[\bar{\Gamma}^{(2)}_{\theta\theta}]_{k^2=0} = \Big[\Big(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_\xi\frac{\partial}{\partial(\xi/M^2)} + \Omega_g\frac{\partial}{\partial(g/M^2)} + \Omega_f\frac{\partial}{\partial(f/M^2)}\Big) + 2\gamma + \gamma_\theta\Big][\bar{\Gamma}^{(2)}_{\theta}]_{k^2=0},$$

$$1)\ \ 2m^2 i(1+\gamma)[\bar{\Gamma}^{(2)}_{\theta\theta\theta}]_{k^2=0} = \Big[\Big(2m^2\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda} + \Omega_\xi\frac{\partial}{\partial(\xi/M^2)} + \Omega_g\frac{\partial}{\partial(g/M^2)} + \Omega_f\frac{\partial}{\partial(f/M^2)}\Big) + 2\gamma + 2\gamma_\theta\Big][\bar{\Gamma}^{(2)}_{\theta\theta}]_{k^2=0}.$$

- Firstly, we find all γ, β and Ω_L from these equations, using boundary conditions $\rightarrow$ find out, that there are **several sets of suitable** γ, β and Ω_L!
- Next, having γ, β and Ω_L $\rightarrow$ find renormalized two-point function from CS equations.
- Note, that we have used all found sets of γ, β and Ω_L and, surely, obtain the same answer for $\bar{\Gamma}^{(2)}$:

$$[\bar{\Gamma}^{(2)}]_{1/M^2} = -2ik^4\xi.$$

3 Non-renormalizable theory: four-point correlation function

- We solve corresponding equations and use corresponding boundary conditions to find **4-point** function in $1/M^2$ order:

$$[\bar{\Gamma}^{(4)}]_{1/M^2} = -i(s+t+u)f.$$

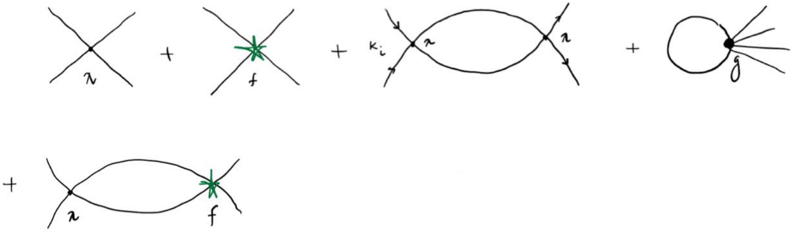


Figure: Four-point function (up to λ, λ^2, $1/M^2$, λ/M^2).

- Six-point function obtains one-loop correction only at λ/M^2 order!
  This order $\rightarrow$ in progress now...

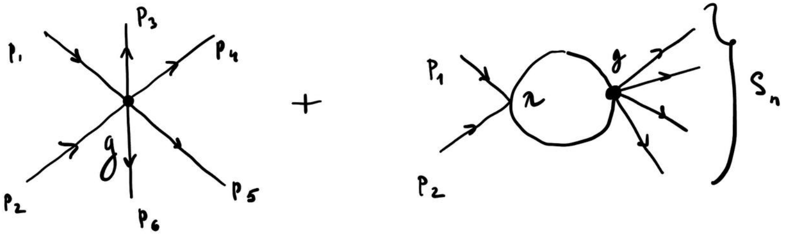


Figure: Six-point function (up to $1/M^2$, λ/M^2).

4 Results and outlook

- Currently we consider **non-renormalizable theory** and would like to find corresponding 2-, 4- and 6-point correlation functions there in fully finite way!
- One should add all M^6 dimension terms (in one loop) $\to$ can find all β, γ, etc, as well as all Green functions in this order (in one loop).
- $1/M^2$ in one loop is already done, λ/M^2 in one loop is in progress.
- The CS method as it stands cannot work for massless particles $\to$?
- It would be also interesting to see what happens with naturalness in other formulations of finite QFT.

THIS IS THE END OF PRESENTATION

Questions: are welcomed

THANK YOU FOR YOUR ATTENTION

- Consider the model with concrete realisations of "UV physics":

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}\left(\partial_\mu \Phi\right)\left(\partial^\mu \Phi\right)$$
$$- \frac{m^2}{2}\phi^2 - \frac{M^2}{2}\Phi^2 - \frac{\lambda_\phi}{4!}\phi^4 - \frac{\lambda_{\phi\Phi}}{4}\phi^2\Phi^2 - \frac{\lambda_\Phi}{4!}\Phi^4.$$

- Assume $m \ll M$.
- Physics involving the field $\Phi \to$ a toy representation of "new physics" living at large energy scales.
- We see that even after subtracting the formal UV divergences, $\bar{\Gamma}^{(2\phi)}$ still receives large contributions of order M^2:

$$\bar{\Gamma}^{(2\phi)} = i\left(k^2 + m^2\right) - \frac{i\lambda_\phi m^2}{32\pi^2}\left(1 + \ln\frac{\mu^2}{m^2}\right) - \frac{i\lambda_{\phi\Phi} M^2}{32\pi^2}\left(1 + \ln\frac{\mu^2}{M^2}\right).$$

- Therefore, it seems that heavy scale physics of order M^2 has **a dramatic influence** on the physics of order m^2.