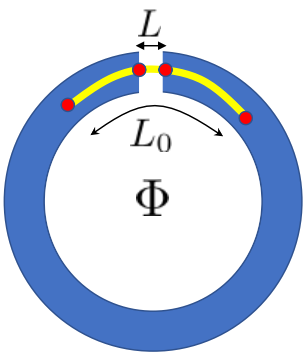
Solid state theory **Homework 1** **Due 03/30/2026**

1. Detecting Majorana Zero Modes



Consider the configuration shown in the figure. A semiconductor wire is placed on top of a superconducting ring (blue) interrupted by a Josephson junction (white). When the wire lies on top of the superconducting regions, it enters a topological superconducting phase with Majorana zero modes localized at its ends (red dots). The presence of these Majorana zero modes can be detected through the doubled periodicity of the critical current across the Josephson junction as a function of the superconducting phase difference between the two sides. The phase difference ϕ is controlled by threading a magnetic flux Φ through the ring, and is given by $\phi = \frac{2\pi\Phi}{\Phi_0}$.

We model the wire as two Kitaev chains, a left segment and a right segment, coupled through a weak link across the Josephson junction. We take the pairing amplitude Δ on the left segment to be real and positive, while the pairing on the right segment differs by a superconducting phase ϕ. The full Hamiltonian is

$$H = H_L + H_R + H_c. \tag{1}$$

The Hamiltonian for the left segment is

$$H_L = -\mu \sum_{j=1}^{N} c_j^\dagger c_j - \sum_{j=1}^{N-1} \left(t\, c_{j+1}^\dagger c_j + t\, c_j^\dagger c_{j+1} \right) + \sum_{j=1}^{N-1} \left(\Delta\, c_j^\dagger c_{j+1}^\dagger + \Delta\, c_{j+1} c_j \right), \tag{2}$$

while the Hamiltonian for the right segment is

$$H_R = -\mu \sum_{j=N+1}^{2N} c_j^\dagger c_j - \sum_{j=N+1}^{2N-1} \left(t\, c_{j+1}^\dagger c_j + t\, c_j^\dagger c_{j+1} \right) + \sum_{j=N+1}^{2N-1} \left(\Delta e^{i\phi} c_j^\dagger c_{j+1}^\dagger + \Delta e^{-i\phi} c_{j+1} c_j \right). \tag{3}$$

The two segments are coupled by a weak hopping term

$$H_c = -t' c_N^\dagger c_{N+1} - t' c_{N+1}^\dagger c_N. \tag{4}$$

(a) Show that when $\mu = 0$ and $t = \Delta$, the Hamiltonian H_L is in the topologically nontrivial phase and supports Majorana zero modes at its two ends. Identify the Majorana operators corresponding to these zero modes.

(b) Show that H_R can be mapped to the same form as H_L by defining

$$\tilde{c}_j = e^{-i\phi/2} c_{j+N}, \qquad \tilde{c}_j^\dagger = e^{i\phi/2} c_{j+N}^\dagger.$$

Using this mapping, identify the Majorana operators corresponding to the zero modes of H_R.

(c) Rewrite H_c in terms of the Majorana operators for the left and right segments. In the weak-link limit $t' \ll t, \Delta$, project onto the low-energy subspace spanned by the two Majorana zero modes at the junction, and obtain the corresponding effective Hamiltonian. Diagonalize this reduced Hamiltonian, plot its eigenenergies as a function of ϕ, and determine its periodicity.

(d) In an ordinary Josephson junction, the energy is a 2π-periodic function of ϕ. In the present Majorana-wire Josephson junction, the period is doubled. Briefly explain the physical origin of this doubled periodicity.

2. Majorana Braiding and Non-Abelian Statistics [arXiv:1006.4395]

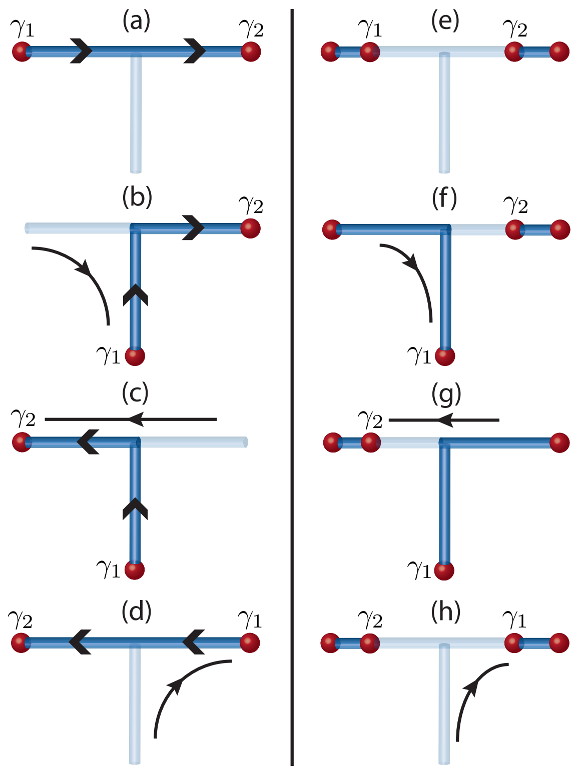


Figure 1: A T-junction allows for adiabatic exchange of two Majorana fermions bridged by either a topological region (dark blue lines) as in (a)-(d), or a non-topological region (light blue lines) as in (e)-(h). The arrows along the topological regions in (a)-(d) are useful for understanding the non-Abelian statistics as outlined in the main text.

In this problem, we study how Majorana zero modes in semiconducting wires realize non-Abelian braiding. You are encouraged to read the discussion around T-junction braiding in the reference above.

We begin with Kitaev's toy model for a spinless p-wave superconducting chain:

$$H = -\mu \sum_{x=1}^{N} c_x^\dagger c_x - \sum_{x=1}^{N-1} (t c_x^\dagger c_{x+1} + |\Delta| e^{i\phi} c_x c_{x+1} + h.c.), \tag{5}$$

where c_x is a spinless fermion operator. In the special limit $\mu = 0$ and $t = |\Delta|$, it is convenient to write

$$c_x = \frac{1}{2} e^{-i\frac{\phi}{2}} (\gamma_{B,x} + i\gamma_{A,x}), \tag{6}$$

where $\gamma_{\alpha,x} = \gamma^\dagger_{\alpha,x}$ are Majorana operators satisfying

$$\{\gamma_{\alpha,x}, \gamma_{\alpha',x'}\} = 2\delta_{\alpha\alpha'}\delta_{xx'}.$$

In this limit, the Hamiltonian becomes

$$H = -it \sum_{x=1}^{N-1} \gamma_{B,x}\gamma_{A,x+1}. \tag{7}$$

Thus $\gamma_{B,x}$ and $\gamma_{A,x+1}$ pair into finite-energy bulk fermions, while $\gamma_{A,1}$ and $\gamma_{B,N}$ remain unpaired. These end Majoranas form a nonlocal zero-energy fermion

$$d_{\text{end}} = \frac{1}{2}(\gamma_{A,1} + i\gamma_{B,N}),$$

leading to two degenerate ground states $|0\rangle$ (with $d_{\text{end}}|0\rangle = 0$) and $|1\rangle = d^\dagger_{\text{end}}|0\rangle$.

A T-junction in Fig. 1 allows one to adiabatically exchange two Majorana zero modes γ_1 and γ_2. The detailed wire manipulations are discussed in the reference above. The important point for us is that, after exchange, the Majorana operators transform in the same way as Majoranas bound to vortices in a 2D $p + ip$ superconductor:

$$\gamma_1 \to \gamma_2, \qquad \gamma_2 \to -\gamma_1.$$

This is the basic signature of non-Abelian braiding.

(a) Show that the transformation

$$\gamma_1 \to \gamma_2, \qquad \gamma_2 \to -\gamma_1$$

is generated by the unitary operator

$$U_{12} = \exp\left(\frac{\pi}{4}\gamma_2\gamma_1\right).$$

Then derive the matrix representation of U_{12} in the basis $\{|0\rangle, |1\rangle\}$, up to an overall phase.

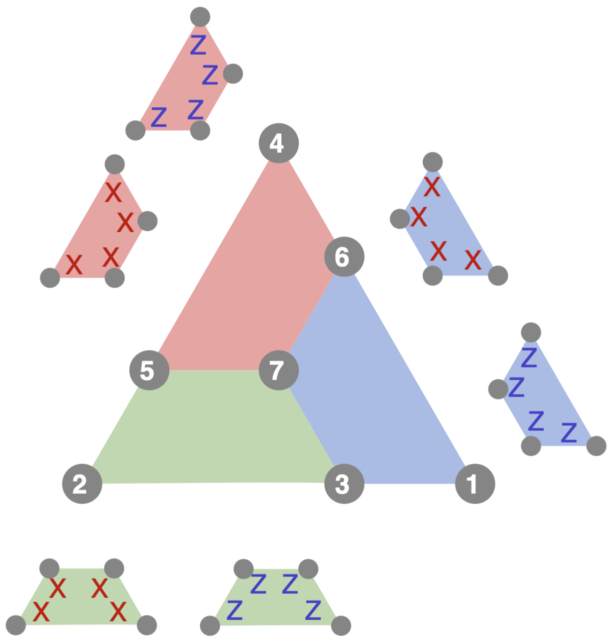
(b) For a system with more than two Majorana zero modes, different exchanges are represented by different unitary operators U_{ij}. Explain why these operators generally do not commute, and why this implies non-Abelian statistics.

(c) In the semiconducting-wire realization, one may deform the exchange process so that the Hamiltonian remains purely real throughout the adiabatic evolution. In this description, the exchange ends with the sign of the pairing reversed relative to the initial Hamiltonian. Determine a gauge transformation acting on c and $c^\dagger$ that restores the Hamiltonian to its original form, and verify that it induces

$$\gamma_1 \to \gamma_2, \qquad \gamma_2 \to -\gamma_1.$$

Hint: Consider multiplying c and $c^\dagger$ by suitable $U(1)$ phase factors.

3. Color code on an open surface [arXiv:quant-ph/0605138]



The color code can be implemented on any 3-colorable open surface. We will examine the color code on the simplest such surface—a triangle—divided into three trapezoids. Place the qubits at the vertices, resulting in a total of 7 qubits. The stabilizers (Hamiltonian terms) are products of either all X or all Z operators on these trapezoids, as depicted in the provided figure. There are 6 independent stabilizers. The stabilizers commute with each other, ensuring that the ground states are the simultaneous +1 eigenstates of these stabilizers.

(a) Calculate the ground state degeneracy of the system. **Hint**: Determine this by counting the number of physical qubits and independent stabilizers.

(b) Find a basis for the ground-state subspace.

(c) Identify all independent non-trivial logical operators. These operators commute with all stabilizers but are not products of stabilizers. Logical operators are considered "not independent" if one can be derived from another by multiplying by stabilizers. Focus your analysis on finding the logical X and Z operators for the encoded logical qubit(s).

(d) Define the code distance as the minimum number of Pauli matrices required to form any non-trivial logical operator. Determine the code distance for this color code. **Hint**: the code distance is not 7.