Group Invariant States as Quantum Many-Body Scars

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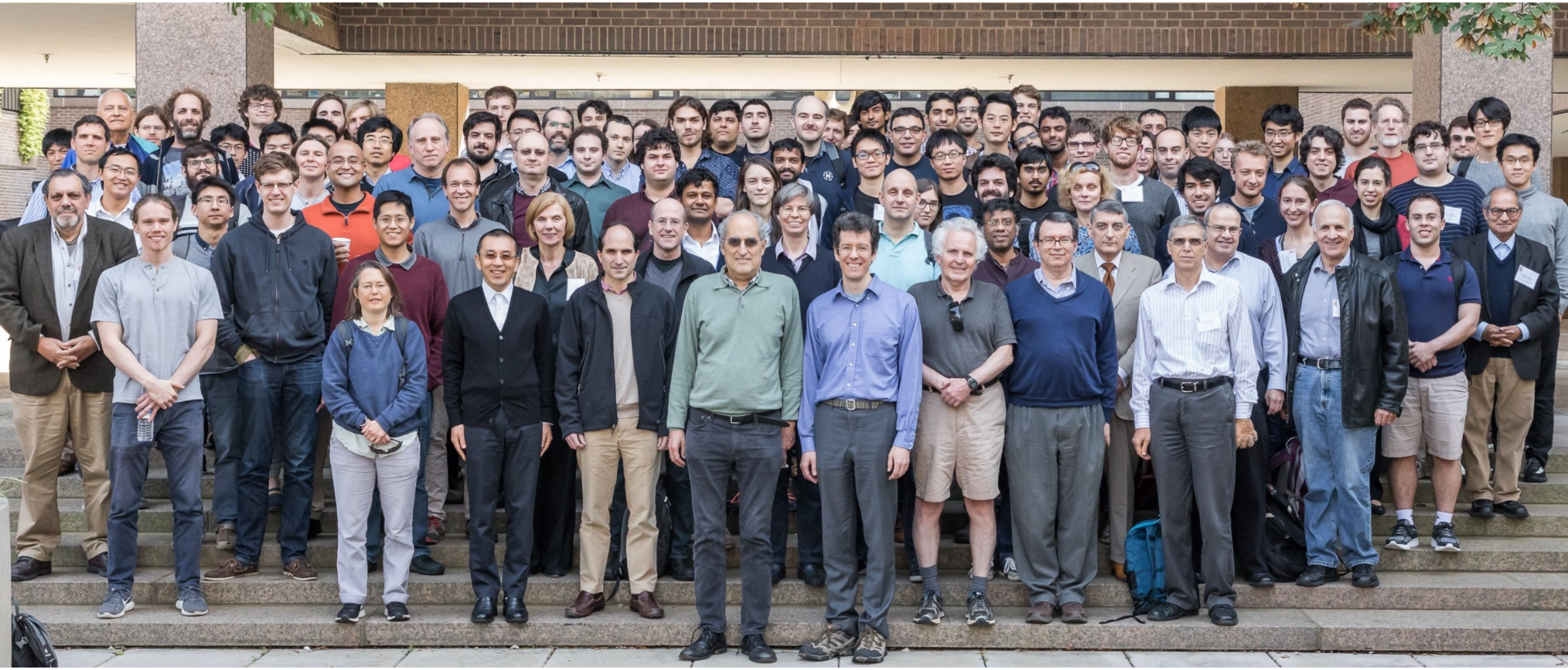
Hirosi and I

- I first met Hirosi in the late 1980s. I remember him well at the August 1990 Aspen workshop on Matrix Models and 2D Quantum Gravity.
- There I started collaborating with Bershadsky; this reduced my "Ooguri number" to 2.
- Since then Hirosi and I have intersected frequently and had similar research interests.
- Hirosi was a great host during my various visits to Japan, including Strings 2003, Tohru Eguchi's retirement symposium in 2017, Strings 2018, etc.

- This memorable photo was taken at the Tohru Eguchi 60th Birthday Symposium in Kyoto in March 2008

- Back in Kyoto in June 2016

- Group photo of the conference AdS/CFT@20 at Princeton University in November 2017



- Among Hirosi's many seminal contributions, let me highlight his work with N. Sasakura on generalizing successes of the matrix models in describing discretized random surfaces to 3D random geometries obtained by gluing tetrahedra

**Discrete and Continuum Approaches
to Three-Dimensional Quantum Gravity**

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Group Field Theory

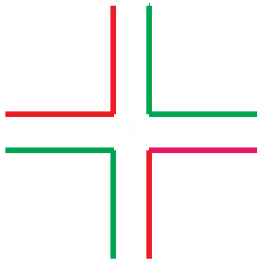
- Ooguri and Sasakura revisited the Ponzano-Regge model of 3D simplicial gravity. The 6j symbol of the SU(2) group labels a tetrahedron with the corresponding 6 edges.

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

- The wave functions on the SU(2) group manifold then serve as a "generalized index." This, along with work by Boulatov, and Ambjorn et al., was a major step towards Group Field Theory and Tensor Models.

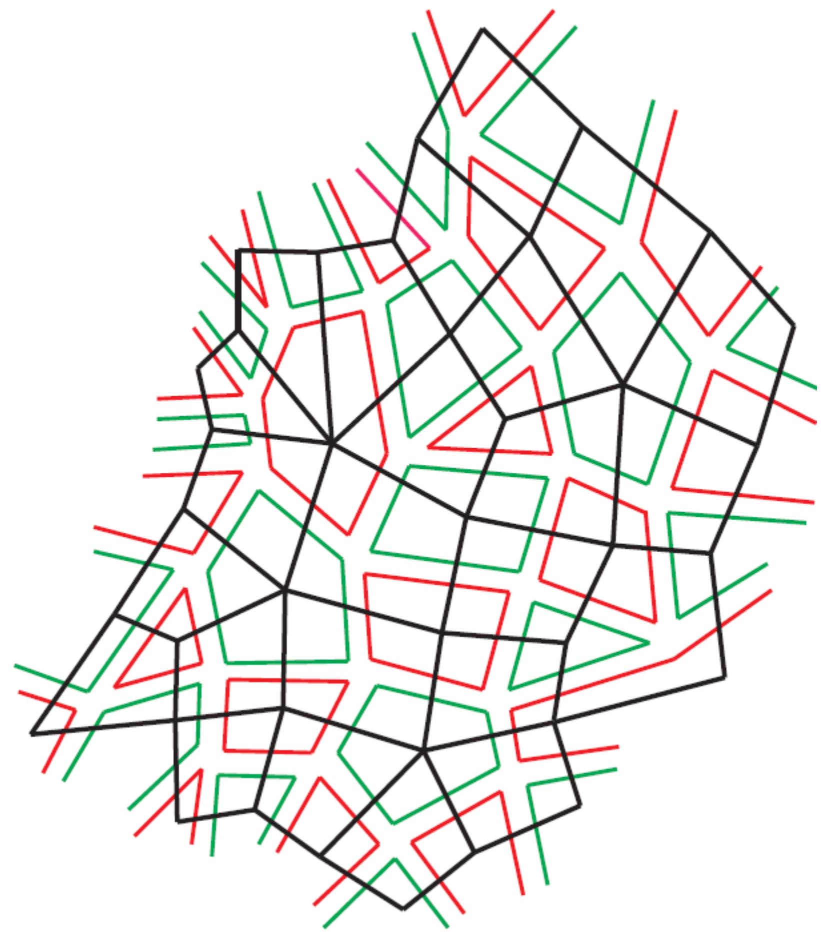
O(N) x O(N) Matrix Model

- Theory of real matrices ϕ^{ab} with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a \text{x} O(N)_b$ symmetry.
- The interaction is at least quartic: $g \operatorname{tr} \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is



- In d=0 or 1 special limits describe two-dimensional quantum gravity.

- In the large N limit where gN is held fixed we find planar Feynman graphs, and each index loop may be red or green.
- The dual graphs shown in black may be thought of as random surfaces tiled with squares whose vertices have alternating colors (red, green, red, green).



From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

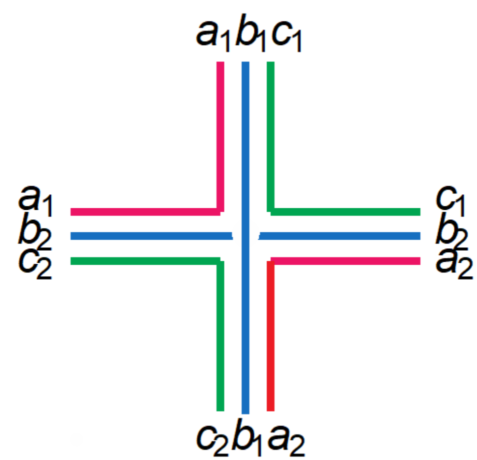
$$\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

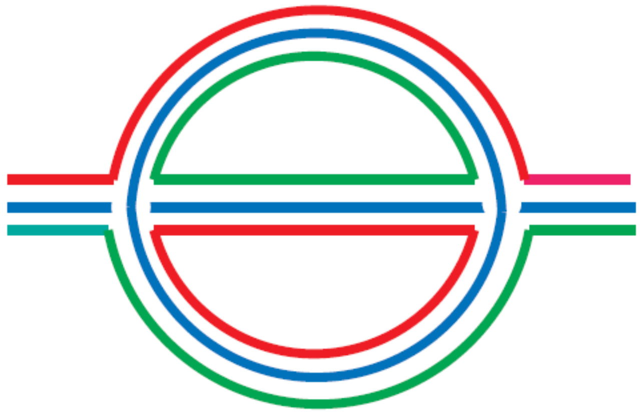
- Tetrahedral interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

Carrozza, Tanasa; IK, Tarnopolsky

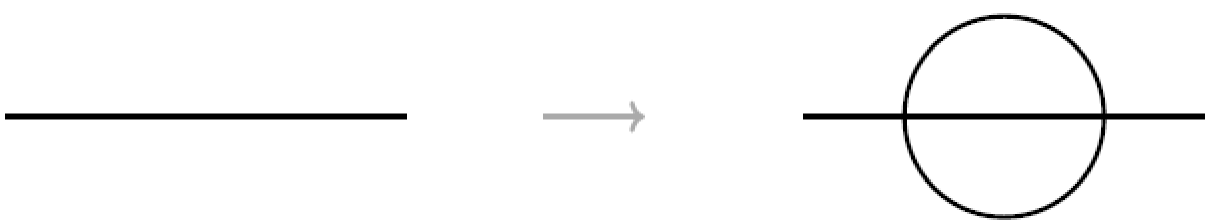
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$



- Leading correction to the propagator has 3 index loops



- Requiring that this “melon” insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.
- Melonic graphs obtained by iterating



Melonic O(N)^3 Tensor Model

- Quantum Mechanics of N^3 Majorana fermions
  IRK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

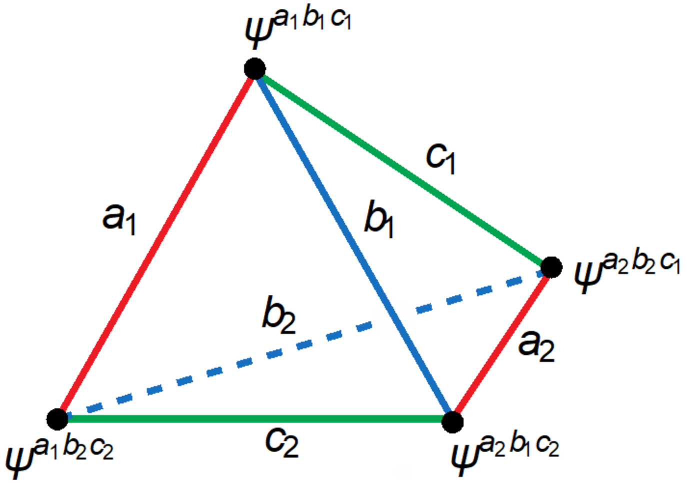
- Has $O(N)_a \times O(N)_b \times O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \qquad M_1, M_2, M_3 \in O(N)$$

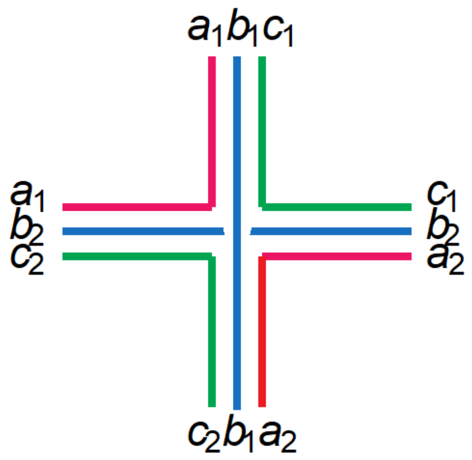
- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}] , \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}] , \qquad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.



- This is equivalent to



- The triple-line Feynman graphs are produced using the propagator

$O(N)^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

- The number of distinct terms is

$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\text{SYK}} = N^3$

$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

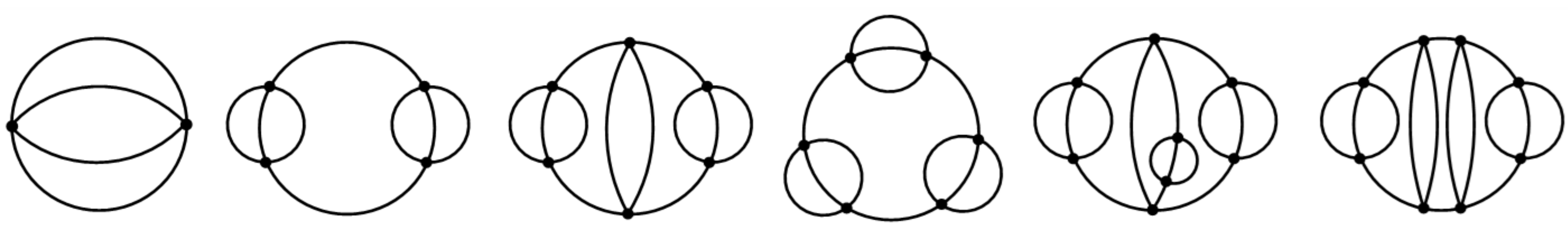
- No $SO(N)^3$ invariant states for odd N.
- Their number grows very rapidly for even N IRK, Milekhin, Popov, Tarnopolsky

N	# singlet states
2	2
4	36
6	595354780

Table 1: Number of singlet states in the $O(N)^3$ model

$$\#\text{singlet states} \sim \exp\left(\frac{N^3}{2}\log 2 - \frac{3N^2}{2}\log N + O(N^2)\right)$$

- Large N dynamics in the singlet sector is similar to SYK. Same melonic Schwinger-Dyson eqns.



Qubit Hamiltonian

- Convenient to introduce operator basis which breaks the third O(N) to U(N/2)

$$\bar{c}_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} + i\psi^{ab(2k+1)}\right), \quad c_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} - i\psi^{ab(2k+1)}\right),$$

$$\{c_{abk}, c_{a'b'k'}\} = \{\bar{c}_{abk}, \bar{c}_{a'b'k'}\} = 0, \quad \{\bar{c}_{abk}, c_{a'b'k'}\} = \delta_{aa'}\delta_{bb'}\delta_{kk'},$$

$$a, b = 0, 1, \ldots, N-1, \text{ and } k = 0, \ldots, \tfrac{1}{2}N - 1$$

- Operators $c_{abk}, \bar{c}_{abk}$ correspond to qubit number $N^2k + Nb + a$
- The Hamiltonian couples N/2 sets of N^2 qubits

$$H = 2\left(\bar{c}_{abk}\bar{c}_{ab'k'}c_{a'bk'}c_{a'b'k} - \bar{c}_{abk}\bar{c}_{a'bk'}c_{ab'k'}c_{a'b'k}\right)$$

- The Cartan generators of U(N/2) are

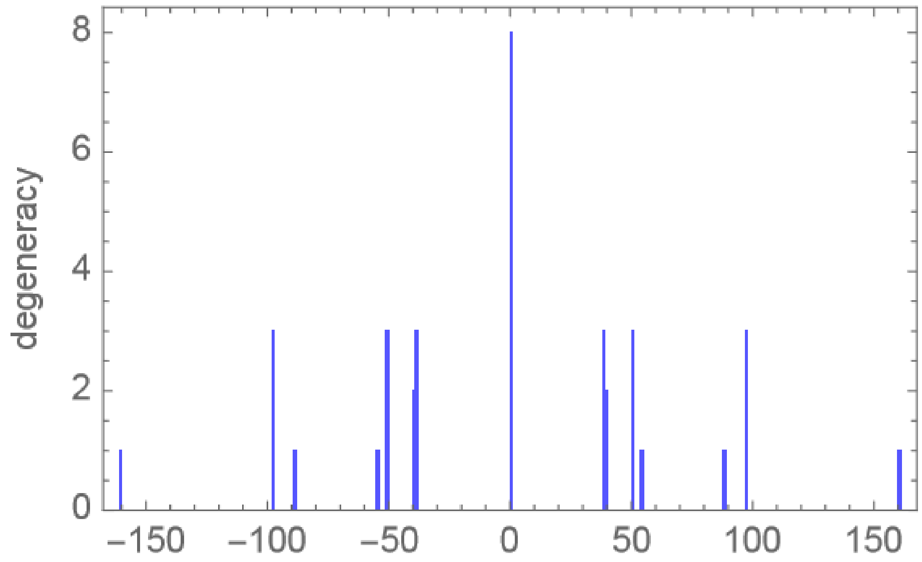
$$Q_k = \sum_{a,b} \frac{1}{2} [\bar{c}_{abk}, c_{abk}] \ , \qquad k = 0, \ldots, \frac{1}{2}N - 1$$

- For the oscillator vaccuum

$$c_{abk} \, |\text{vac}\rangle = 0 \ , \qquad Q_k \, |\text{vac}\rangle = -\frac{N^2}{2} \, |\text{vac}\rangle$$

- The $SO(N)^3$ invariant states appear in the sector where all these charges vanish: each set of N^2 qubits is at half filling.
- This reduces the number of states but it still grows rapidly. For N=4 there are 165636900, while for N=6 over 7.47 * 10^29

Singlet Energies for N=4



- For N=6, over 595 million states packed into energy interval <1932. The singlet gaps should be tiny. Pakrouski, IRK, Popov, Tarnopolsky
- Finding the spectrum seems like a good problem for a 108 qubit quantum computer.

Singlet Sector Simplification

- Appears in many group invariant quantum many-body systems.
- For example, the SU(N) invariant sector of Hermitian matrix quantum mechanics is described by wave functions of N eigenvalues which act as the free fermions. Brezin et al.

$$L = \mathrm{Tr}\{\tfrac{1}{2}\dot{\Phi}^2 - U(\Phi)\}$$

- The Vandermonde determinant $\prod_{i<j}(\lambda_i - \lambda_j)$ appears in $\Psi(\lambda) = \Delta(\lambda)\chi_{sym}(\lambda)$

$$\left(\sum_{i=1}^{N} h_i\right)\Psi(\lambda) = E\Psi(\lambda) \qquad h_i = -\frac{1}{2\beta^2}\frac{d^2}{d\lambda_i{}^2} + U(\lambda_i)$$

Compact c=1

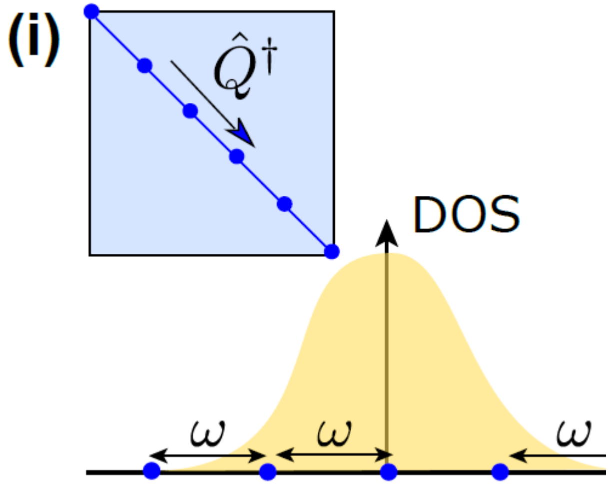
- The contribution of the SU(N) singlet sector to the finite temperature free energy exhibits the R->1/R duality in the double scaling limit. Gross, IRK
- Match with the continuum calculation. Bershadsky, IRK
- The singlet sector describes the modified XY model on random surfaces, where the BKT vortices are excluded. Gross, IRK
- Such modified Villain models are being used to construct a variety of lattice models that have additional symmetries. Sulejmanpasic, Gattringer; Gorantla, Lam, Shao, Seiberg; …

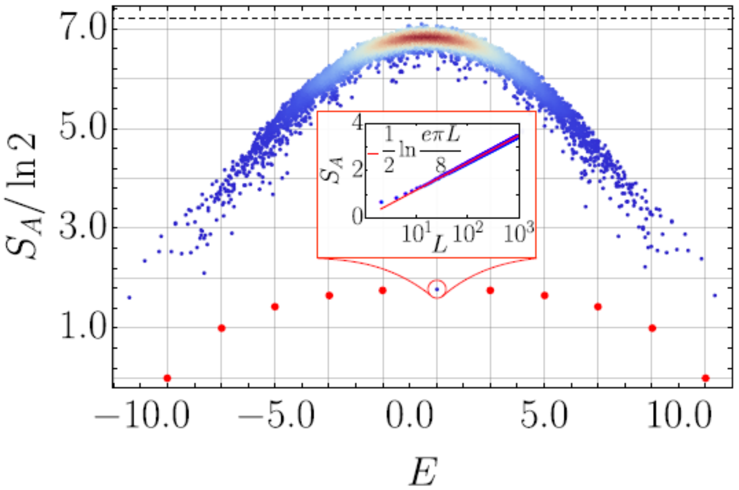
Quantum Many-Body Scars

- Over the past few years have been an active area in Condensed Matter Physics. Several reviews Serbyn, Abanin, Papic; Moudgalya, Bernevig, Regnault; Chandran, Iadecola, Khemani, Moessner
- Scars do not thermalize with the rest of the states and constitute a violation of the Eigenstate Thermalization Hypothesis.
- The Hilbert space “fractures”

$$\mathcal{H} = \mathcal{H}_{\rm therm} \oplus \mathcal{H}_{\rm scar}$$

- Schematic equidistant scar spectrum for a special scarred Hamiltonian: Serbyn et al.; Schecter and Iadecola





- The scars are characterized by lower entanglement entropy than the typical states.
- In some cases, the scar sector is invariant under a "large" group whose rank is proportional to the number of lattice sites.

From Tensor Models to Scars

- Generalize the Majorana tensor model to have $O(N_1) \times O(N_2) \times O(N_3)$ symmetry
- The traceless Hamiltonian is

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N_1N_2N_3(N_1 - N_2 + N_3)$$

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$a = 1, \ldots, N_1;\ b = 1, \ldots, N_2;\ c = 1, \ldots, N_3$$

- The Hilbert space has dimension $2^{[N_1N_2N_3/2]}$
- The eigenstates of H form irreducible representations of the symmetry.

A Fermionic Matrix Model

- For $N_3=2$ this is a fermionic matrix model with symmetry $O(N_1) \times O(N_2) \times U(1)$

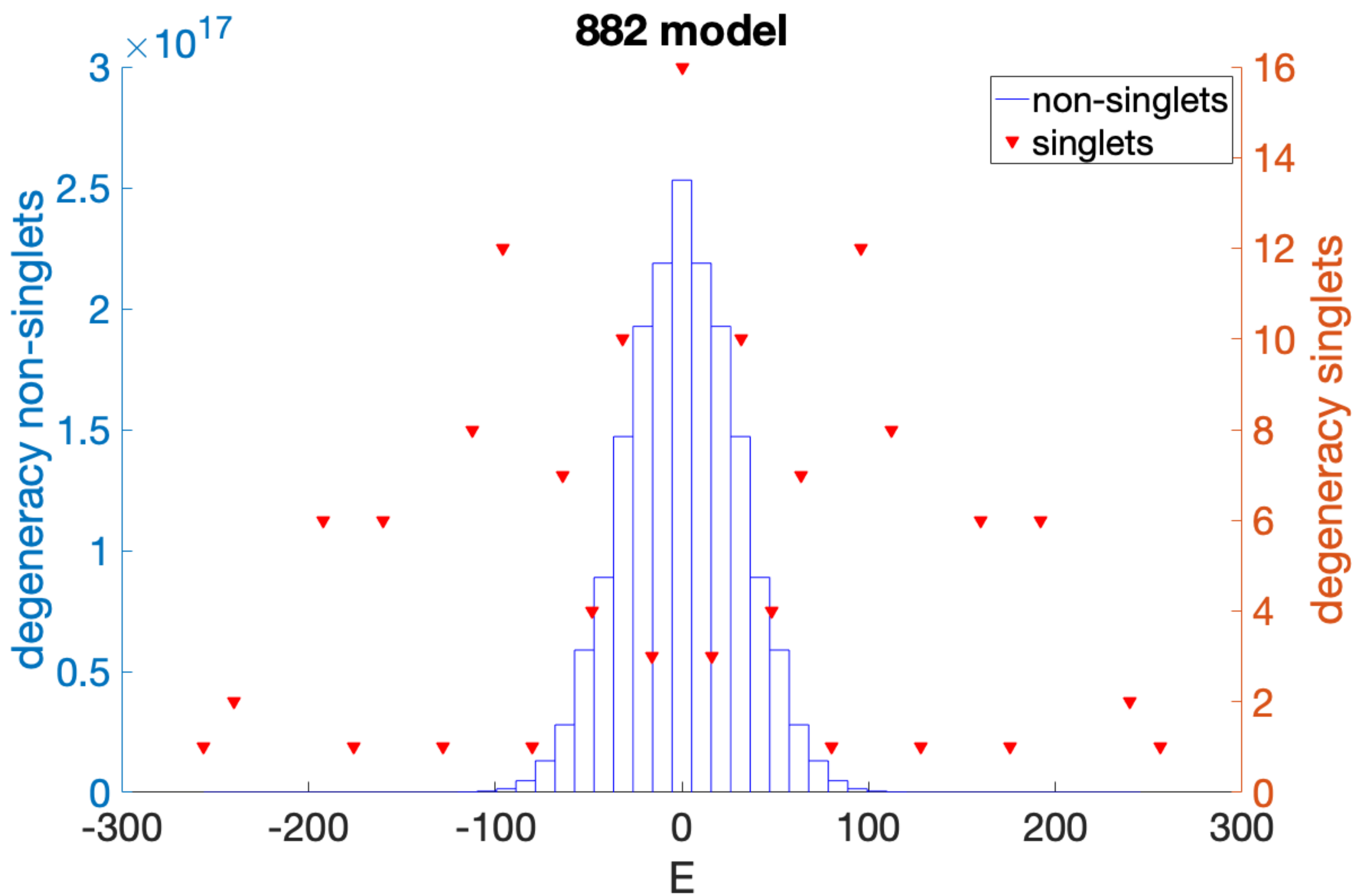
$$\bar{\psi}_{ab} = \frac{1}{\sqrt{2}}\left(\psi^{ab1} + i\psi^{ab2}\right), \qquad \psi_{ab} = \frac{1}{\sqrt{2}}\left(\psi^{ab1} - i\psi^{ab2}\right)$$

$$\{\bar{\psi}_{ab}, \bar{\psi}_{a'b'}\} = \{\psi_{ab}, \psi_{a'b'}\} = 0, \qquad \{\bar{\psi}_{ab}, \psi_{a'b'}\} = \delta_{aa'}\delta_{bb'}$$

- Describes qubits on a N_1 x N_2 lattice with non-local couplings. IRK, Milekhin, Popov, Tarnopolsky
- A useful example for studying bounds on eigenvalues of fermionic Hamiltonians. Hastings, O'Donnell

Complete Spectrum

- The $SO(N)^2$ singlets "scar" the histogram.



Towards Hubbard Model

- Can also think of the first index as labeling the lattice site, and the second as labeling spin. When N_2=2, there are two spin states, up and down. The model is beginning to resemble a non-local Hubbard model, but need to add quadratic hopping terms. Pakrouski, Pallegar, Popov, IRK
- Imaginary hopping terms are SO(N) generators

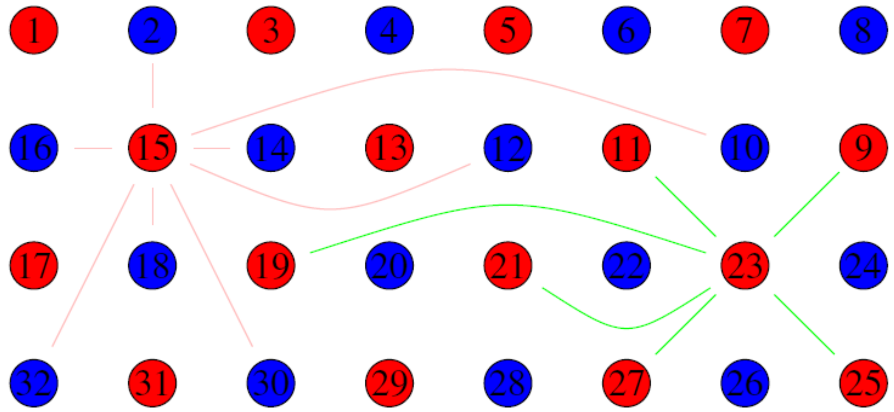
$$T^O_{kl} = i\sum_\sigma (c^\dagger_{k\sigma} c_{l\sigma} - c^\dagger_{l\sigma} c_{k\sigma}) \qquad \sigma = \uparrow, \downarrow$$

- Adding them to H keeps SO(N) singlets as eigenstates but mixes up the non-singlets.

- A simple transformation leads to a model with a real nearest neighbor hopping parameter:

$$H_{nn} = t \sum_{\langle ij \rangle \sigma} (c_{i\sigma}^{\dagger} c_{j\sigma} + h.c.)$$

- This transformation is possible on a bi-partite lattice



Scars without Pain

- There are Hamiltonians that are not symmetric under a large group G, yet some of their eigenstates are invariant. These are the scars!
- Examples include (deformations of) the Hubbard model

$$T = it \sum_{j=1}^{N-1} \sum_{\sigma \in \{\uparrow,\downarrow\}} \left(c_{j\sigma}^{\dagger} c_{j+1,\sigma} - c_{j+1,\sigma}^{\dagger} c_{j\sigma} \right) - \sum_{j=1}^{N} \left(\mu_{\downarrow} c_{j\downarrow}^{\dagger} c_{j\downarrow} + \mu_{\uparrow} c_{j\uparrow}^{\dagger} c_{j\uparrow} \right)$$

$$V = U \sum_{j=1}^{N} n_{j\uparrow} n_{j\downarrow} = U \sum_{j=1}^{N} c_{j\uparrow}^{\dagger} c_{j\uparrow} c_{j\downarrow}^{\dagger} c_{j\downarrow} \ .$$

- The SO(4) symmetry of the Hubbard model is made manifest by introducing 4 Majorana fermions on each lattice site

$$c_{j\uparrow} = \frac{\psi_j^1 - i\psi_j^2}{\sqrt{2}} \,, \quad c_{j\downarrow} = \frac{\psi_j^3 - i\psi_j^4}{\sqrt{2}}$$

- For special values $\mu_\uparrow = \mu_\downarrow = \frac{U}{2}$

$$H_{Hub} = it \sum_j \sum_{A=1}^{4} \psi_j^A \psi_{j+1}^A - U \sum_j \psi_j^1 \, \psi_j^2 \, \psi_j^3 \, \psi_j^4$$

- Add symmetry breaking terms which annihilate the SO(N) singlets, e.g.

$$\widetilde{H}_{\text{int}} = \sum_{\langle j,k \rangle} T_{jk} \left(i \sum_{A<B} r_{AB} \, \psi_j^A \psi_j^B \right) T_{jk}$$

Pseudospin

- The scars are states of maximum pseudospin or spin.
- After transforming to imaginary hopping, the pseudospin $\widetilde{\mathrm{SU}}(2)$ is generated by C.N. Yang, S.C. Zhang

$$\eta^{+} = \sum_{j} c^{\dagger}_{j\uparrow} c^{\dagger}_{j\downarrow} = \frac{1}{2} \sum_{j,\sigma,\sigma'} c^{\dagger}_{j\sigma} c^{\dagger}_{j\sigma'} \epsilon_{\sigma\sigma'}$$

$$\eta^{-} = (\eta^{+})^{\dagger}, \quad \eta^{3} = \frac{1}{2}(Q - N) \qquad Q = \sum_{i=1}^{N} n_i$$

$$n_{i\uparrow} = c^{\dagger}_{i\uparrow} c_{i\uparrow}\,, \quad n_{i\downarrow} = c^{\dagger}_{i\downarrow} c_{i\downarrow}\,, \quad n_i = n_{i\uparrow} + n_{i\downarrow}$$

- It commutes with the rotation group $\mathrm{SU}(2)$ and with the SO(N) that acts on the lattice index.

Eta-pairing states

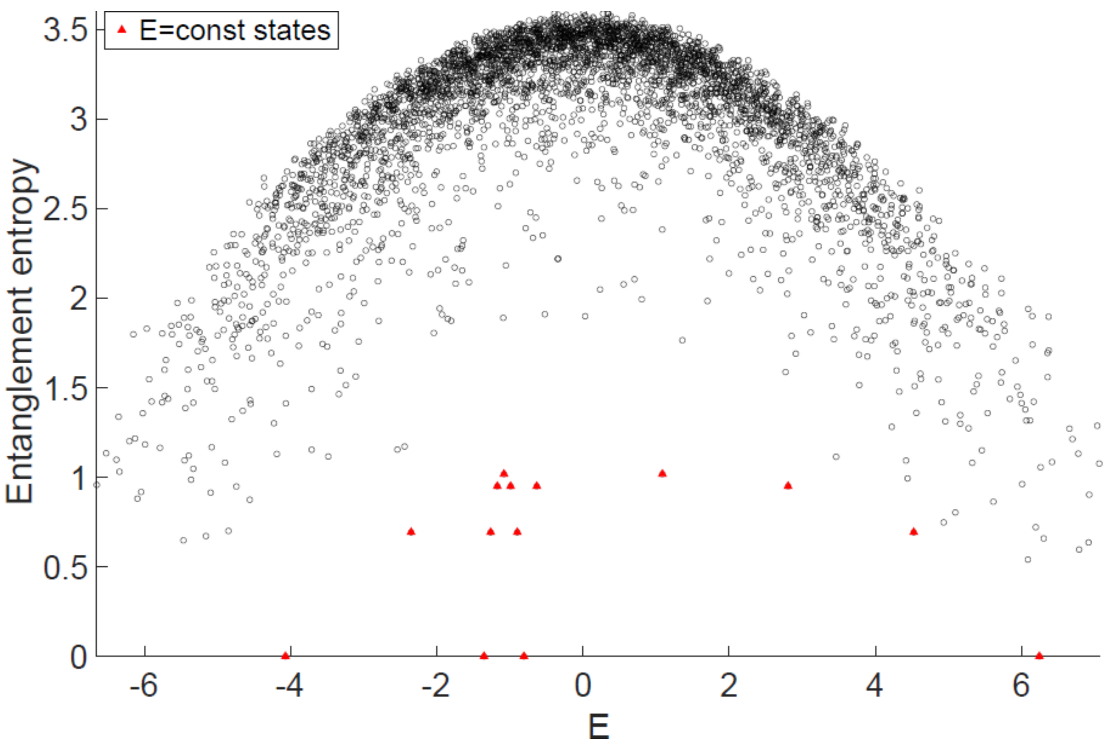
- There are N+1 states that are SU(2) invariant and form a multiplet of pseudospin N/2 Yang, Zhang

$$|n^\eta\rangle = \frac{(\eta)^n}{\sqrt{\frac{N!n!}{(N-n)!}}}|0\rangle \ , \qquad n = 0, \ldots, N$$

- The fact that they are also O(N) invariant was pointed out recently. Pakrouski et al.
- In fact, they are invariant under a bigger group $\widetilde{\mathrm{Sp}}(N)$
- They are highly excited, equally spaced states that play the role of scars in the (deformed) Hubbard model. Mark, Motrunich; Moudgalya, Regnault, Bernevig

Low Entanglement

- The scar states are distinguished by their low entanglement entropy when the system is divided into two parts. For the 6 site chain:



Majorana Scars

- Consider a lattice system with an even number M of Majorana fermions on each lattice site IRK, K. Pakrouski, F. Popov, Z. Sun (paper in preparation)

$$\psi_j^A, A = 1, 2, \cdots, M \qquad \{\psi_i^A, \psi_j^B\} = \delta^{AB}\delta_{ij}$$

- The generators of SO(N) and SO(M) are

$$T_{ij} = \frac{1}{2}\sum_{A=1}^{M}[\psi_i^A, \psi_j^A], \quad J^{AB} = \frac{1}{2}\sum_{j=1}^{N}[\psi_j^A, \psi_j^B]$$

- Complex fermions

$$c_{j\alpha} = \frac{\psi_j^{2\alpha-1} - i\psi_j^{2\alpha}}{\sqrt{2}} \qquad \text{Cartan}: \quad h_\alpha = \sum_j c_{j\alpha}^\dagger c_{j\alpha} - \frac{N}{2}$$

Scars as SO(N) singlets

- Constructed by acting with

$$\text{Positive roots}: \quad \zeta^\dagger_{\beta\gamma} = \sum_j c^\dagger_{j\beta} c_{j\gamma}, \quad \eta^\dagger_{\beta\gamma} = \sum_j c^\dagger_{j\beta} c^\dagger_{j\gamma}$$

- For M=6 the generalizations of eta-pairing states are explicitly written as Nakagawa, Katsura, Ueda

$$|k_{12}, k_{13}, k_{23}\rangle = C_{\boldsymbol{k}}(N)(\eta^\dagger_{12})^{k_{12}}(\eta^\dagger_{13})^{k_{13}}(\eta^\dagger_{23})^{k_{23}}|0\rangle$$

$$k_T \equiv k_{12}+k_{13}+k_{23} \leq N$$

- There are $\binom{N+3}{3}$ such eta-states.

- There are also $\binom{N+3}{3}$ zeta-states:

$$|k_{12}, k_{13}, k_{23}\rangle^{\zeta} = C_{\boldsymbol{k}}(N)(\eta_{12}^{\dagger})^{k_{12}}(\zeta_{13}^{\dagger})^{k_{13}}(\zeta_{23}^{\dagger})^{k_{23}}|0^{\zeta}\rangle$$

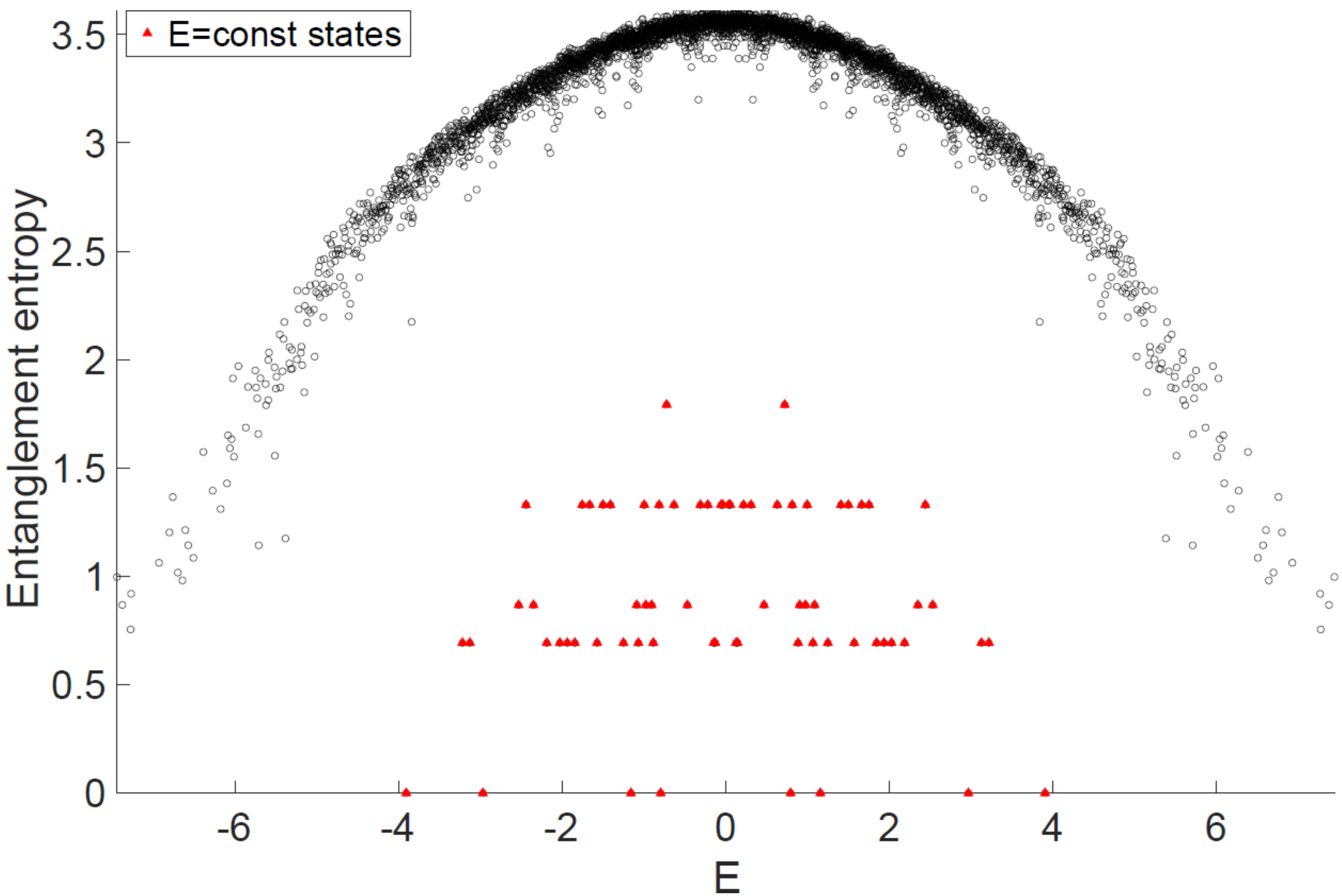
$$|0^{\zeta}\rangle \equiv c_{1,M/2}^{\dagger}c_{2,M/2}^{\dagger}\cdots c_{N,M/2}^{\dagger}|0\rangle$$

$$k_T \equiv k_{12}+k_{13}+k_{23} \leq N$$

- They are generalizations of the spin N/2 states for M=4 (the usual Hubbard model).
- It is not hard to do the counting of SO(N) invariants for M>6, but the wave functions cannot be written as explicitly.

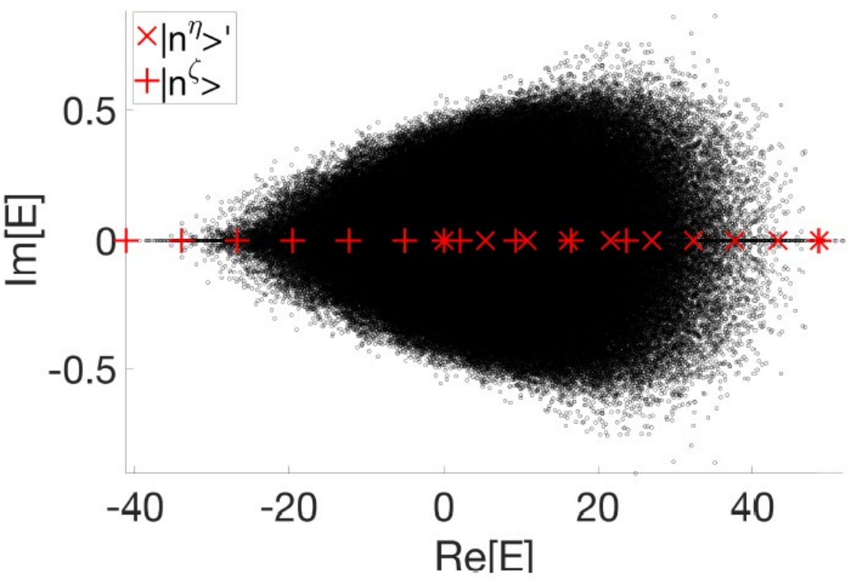
Spectrum of M=6 with 4 sites

$$H = it\sum_{\langle j,k\rangle} T_{jk} + \sum_{\alpha}\mu_\alpha h_\alpha + U(2i)^{M/2}\sum_j \psi_j^1\psi_j^2\cdots\psi_j^M + \sum_{\langle j,k\rangle} T_{jk}\left(i\sum_{A<B} r_{AB}\,\psi_j^A\psi_j^B\right)T_{jk}$$



Non-Hermitian Hamiltonians

- The group theoretic approach to scars continues to work when non-Hermitian terms are added to the Hamiltonians, e.g. the tJU model.
- The energies of scars continue to be real



Comments

- The scar states, which are invariant under the large Lie group acting on the lattice sites, are decoupled from all the non-singlet states. Only the latter thermalize.
- This decoupling is preserved by the TOT perturbations and may approximately survive some other perturbations.
- The Group theoretic approach to scars applies to non-Hermitian Hamiltonians.
- Recently the idea “group singlets are scars” was generalized by Moudgalya and Motrunich who connected it with the Shiraishi-Mori embedding formalism.
- Scar states in QFT? In AdS/CFT?

Happy Birthday, Hirosi!