Exact Parent Hamiltonian for Quantum Hall States on Lattice

JIE WANG

Temple University ⇒ Peking University

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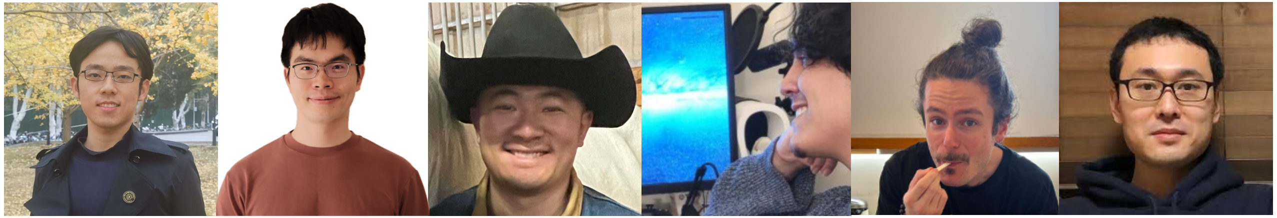


Xin Shen	Guangyue Ji	Jinjie Zhang	David Palomino	Bruno Mera	Tomoki Ozawa
Hua-Zhong Sci.Tech.	Temple U.	Lamma C.C.	Temple U.	U. Lisbon	Tohoku U.

Based on work: Exact Parent Hamiltonians for All Landau Levelss in a Half-flux Lattice (arXiv 2501.09742)

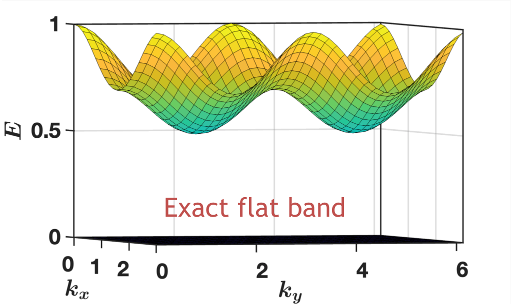
Xin Shen[*#], Gangue Ji[#], Jinjie Zhang, D. Palomino, Bruno Mera, T. Ozawa, Jie Wang*

Outline

- **Motivation**
  - Quantum geometry
  - FCI/FQH in continuum and lattice systems
- **Kapit-Mueller model (lowest lattice-LL state)**
  - Poisson summation (Perelomov)
- **Generalized Kapit-Mueller model (higher lattice-LLs)**
  - Summation rule for n_{th} LL
  - Model features
    - Gapped band: n even
    - Gapless (singular): n odd

Hofstadter model with:

- *Energy*: exact flat band
- *Wavefunction*: lattice LL state

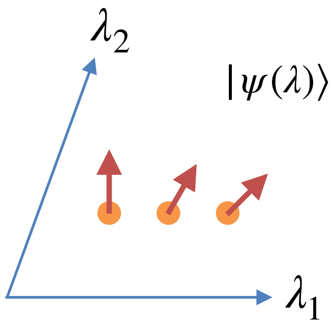


Lattice LL state: $|\varphi_{n\mathbf{k}}\rangle = \sum_{\mathbf{r}} \Phi_{n\mathbf{k}}(\mathbf{r})|\mathbf{r}\rangle$

$\Phi_{n\mathbf{k}}(\mathbf{r})$: standard n_{th} LL wavefunction

X. Shen, Ji, Zhang, Palomino, Mera, Ozawa, JW; arXiv (25)

Topological and geometric properties



<u>Topological properties:</u> how much quantum states change globally in parameter space

Chern number, Z2 invariant, classifying phase of matter …

<u>Geometric properties:</u> how much quantum states change locally in parameter space

Quantum metric (distance)

$$|\langle\psi_\lambda|\psi_{\lambda+\delta\lambda}\rangle|^2 = 1 - g^{ab}(\lambda)d\lambda_a d\lambda_b + \ldots$$

Berry curvature (phase)

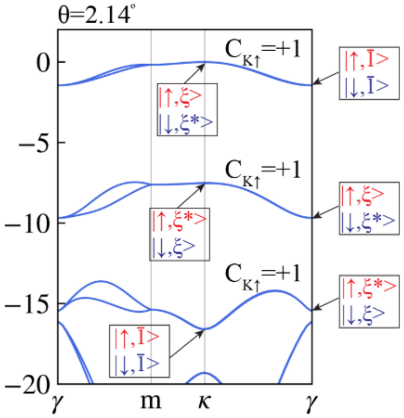
$$\Omega(\lambda)$$

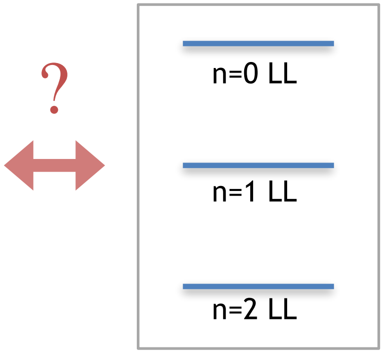
<u>Quantum geometric bound</u>: $\mathrm{Tr}g_{\mathbf{k}} \geq 2\sqrt{\det g_{\mathbf{k}}} \geq |\Omega_{\mathbf{k}}|$ for all $\mathbf{k}$ R. Roy PRB (14)

Geometry & Landau level mimicry

One approach/philosophy: what is the "LL" index of a Chern band?

- If LLL (n=0) —> Laughlin/Jain series mimicry
- If n=1 LL —> Moore-Read (non-Abelian)



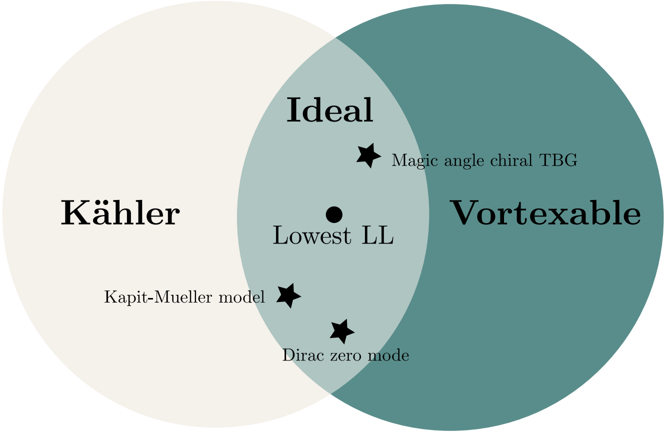


- <u>Useful</u>:
  - guided engineering materials for fractionalization

- <u>Not a necessary condition for FCI</u>:
  - topological order phase is insensitive to perturbation
  - Single particle property cannot decide many-body physics
  - FCI in C=0/ non-ideal bands (W. Yao group)

<u>Ideal geometry</u>: $\mathrm{Tr}g_{\mathbf{k}} = \Omega_{\mathbf{k}}$ for all $\mathbf{k}$

- A LLL condition: $\mathcal{N}_{\mathbf{k}}\ \mathcal{B}(\mathbf{r})\ \Phi_{0\mathbf{k}}(\mathbf{r})$
- Exact FCI for short-ranged interaction



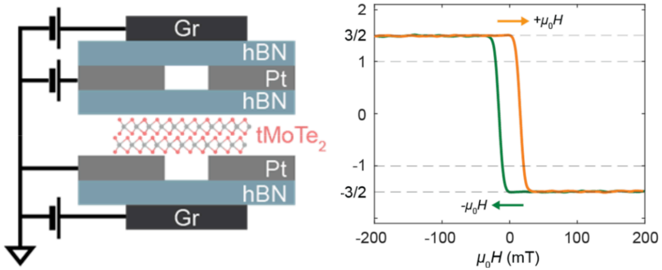
- <u>Ideal band</u>: **Jie Wang**, Cano, Millis, Zhao Liu, Bo Yang (PRL 21, 22)
- <u>Kahler band</u>: Bruno Mera & Tomoki Ozawa (PRB 21 * 3)
- <u>Vortexability</u>: P. Ledwith, A. Vishwanath, D. Parker (PRR 23)
- <u>Curved space Landau level</u>: Crepel, Estienne, Regnault (PRR 23)
- <u>Graphene based models</u>: Ledwith (PRR 20); Wan, Sarkar, Sun (PRL 23; NC 25)

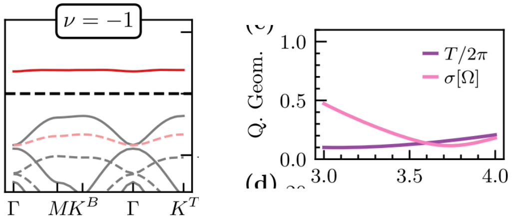
<u>Higher LL condition</u>:
Theory of Generalized Landau levels and its Implications to non-Abelian states
Zhao Liu#, Bruno Mera#, Fujimoto, Tomoki Ozawa, Jie Wang*; PRX (25)

Ideal bands in moire and Hofstadter

Continuum system

- Twisted MoTe2





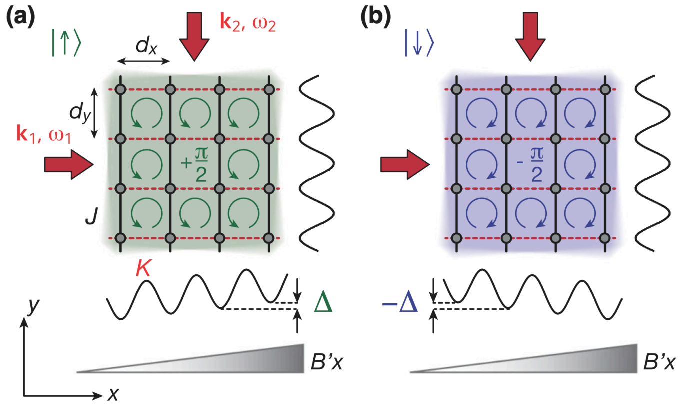
$$T \equiv \int d^2\mathbf{k}\, |\mathrm{tr}g_{\mathbf{k}} - \Omega_{\mathbf{k}}|$$

Xiao-dong Xu group (23); Kinfai Mak & Jie Shan group (23); Ting-xin Li group (23).

- Chiral magic angle TBG (exact ideal band; Ledwith et al; PRR 21)
- Higher C ideal band in twisted multiple layer graphene (JW, Liu; PRL 21)
- 2D material under strain (K. Sun 23)

Lattice/ Hofstadter system

- Kapit-Mueller model (ideal band)
- Generalized Kapit-Mueller model (higher LL)

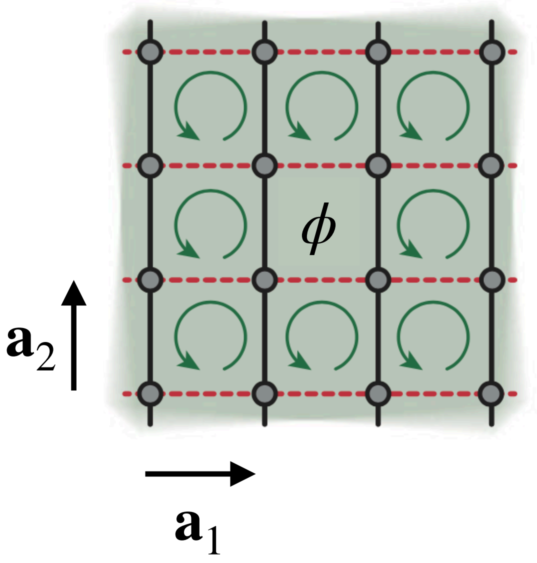


Ultra-cold atom realization of artificial gauge field

D. Jaksch and P. Zoller; NJP 03.

M. Aidelsburger, Atala, Lohse, Barreiro, Paredes, I Bloch; PRL 13.

Kapit-Mueller model: lattice realization of ideal band



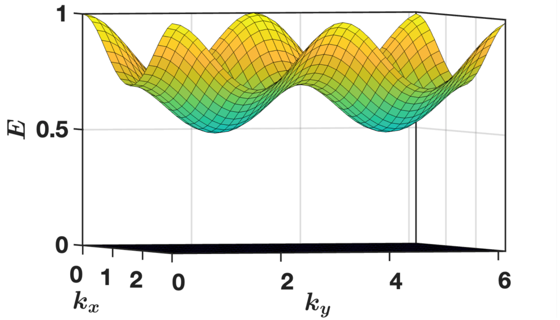
ϕ unit of flux per unit cell: $\mathbf{a}_1 \times \mathbf{a}_2 = \phi$
A generic lattice point is denoted by $\mathbf{d} = m\mathbf{a}_1 + n\mathbf{a}_2$

<u>Hofstadter Hamiltonian</u>: $H = \sum_{m,n} J(\mathbf{d})\hat{t}(\mathbf{d})$

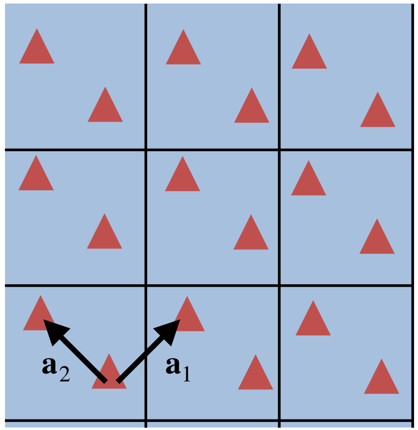
- $\hat{t}(\mathbf{d})$: hopping operator, $\hat{t}(\mathbf{d})\hat{t}(\mathbf{d}') = e^{i\mathbf{d}\times\mathbf{d}'}\hat{t}(\mathbf{d}')\hat{t}(\mathbf{d})$
- $J(\mathbf{d})$: hopping parameter, complex number, $J(\mathbf{d}) = J^*(-\mathbf{d})$

Q: What set of $\{J(\mathbf{d})\}$ gives ideal flat band?

A: (Kapit-Mueller) $J(\mathbf{d}) = (-1)^{m+n+mn}\exp\left[-(\phi^{-1} - 1)\mathbf{d}^2/4\right]$



Kapit-Mueller model: Poisson summation (Perelomov)



One magnetic unit cell
(Unit area)

Lattice site

Lattice unit cell area $\mathbf{a}_1 \times \mathbf{a}_2 = \phi < 1$

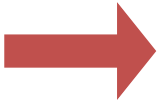
$$\mathbf{d} = m\mathbf{a}_1 + n\mathbf{a}_2$$

Perelomov (1971): On the Completeness of Coherent States.

<u>Poisson summation (Perelomov):</u> $\sum_{m,n} \eta_{m,n} W_0(\mathbf{d}) \cdot \Phi_0(\mathbf{d}) = 0$

- $\eta_{m,n} = (-1)^{m+n+mn}$
- $W_0(\mathbf{d}) = \exp\left[-(\phi^{-1} - 1)\mathbf{d}^2/4\right]$, Gaussian decay (very fast)
- $\Phi_0(\mathbf{d}) = f(d)\exp(-\mathbf{d}^2/4)$, the standard LLL wavefunction function
- Properly sum a LLL wf on lattice can exactly annihilate it (over-complete).

$$\boxed{\sum_{m,n} \eta_{m,n} W_0(\mathbf{d})} \cdot \Phi_0(\mathbf{d}) = 0$$



$$H = \sum_{m,n} \eta_{m,n} W_0(\mathbf{d}) \hat{t}(\mathbf{d})$$

$$H\,|\,\varphi_0\rangle = 0$$

Kapit-Mueller; PRL (15)

$$|\,\varphi_0\rangle = \sum_{\mathbf{d}} \Phi_0(\mathbf{d})\,|\,\mathbf{d}\rangle$$

Generalized Kapit-Mueller Model

New lattice sum rule on half-flux $\phi = 1/2$ lattice:

Xin Shen
Hua-Zhong Sci.Tech.

The following valid for any LL index n,

$$\sum_{m,n} \eta_{m,n} W_n(\mathbf{d}) \Phi_n(\mathbf{r}) = 0$$

where $W_n(\mathbf{r}) = \bar{z}^n \exp\left(-\mathbf{d}^2/4l_B^2\right)$.

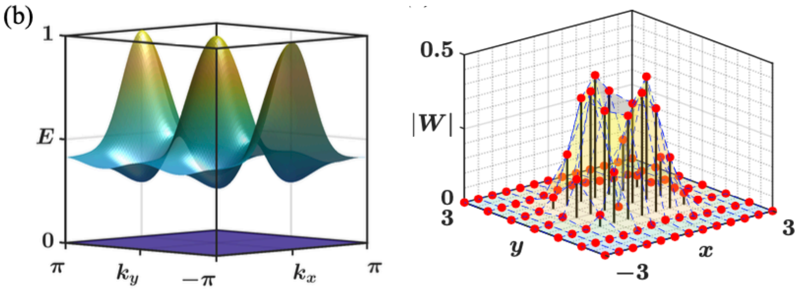
Zero mode operator & exact parent Hamiltonian:

$$\hat{H}_n = \hat{D}_n^\dagger \hat{D}_n$$

$$\hat{D}_n \equiv \sum_{\mathbf{d}} W_n(\mathbf{d}) \hat{t}(\mathbf{d}) \text{ satisfying } \hat{D}_n |\Phi_n\rangle = 0.$$

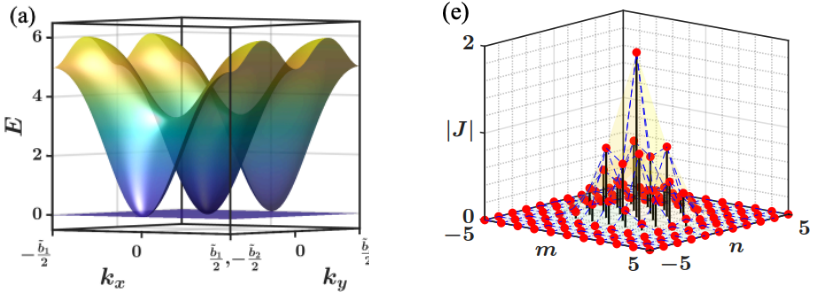
Still a Hofstadter model with 'local' hopping
(all to all, Gaussian decay amplitude).

Even n series: gapped



n=2

Odd n series: gapless (next slide)

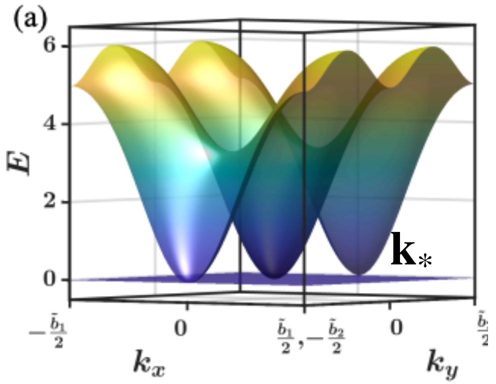


n=1

Xin Shen, Gangyue Ji, J. Zhang, David, Mera, Ozawa, JW (2025)

Singular flatband for odd series

For example: band structure of n=1



Features for the gapless points (analytically derived):

- Energetically: quadratic band touching
- Non-analyticity of Bloch wavefunction (singular)
- Symmetry (inversion) protected

To see why it is a singular band

- Two band system, so two-component w.f.:

$$|\varphi_{n\mathbf{k}}\rangle = \sum_{m,n} \Phi_{n\mathbf{k}}(\mathbf{d})\,|\,\mathbf{d}\rangle = \begin{pmatrix} \Phi_{n\mathbf{k}}(\mathbf{0}) \\ \Phi_{n\mathbf{k}}(\mathbf{a}_1) \end{pmatrix}$$

- Inversion property of LL w.f.:

$$\Phi_{n,\mathbf{k}}(\mathbf{r}) = (-1)^{n+1}\Phi_{n,-\mathbf{k}}(-\mathbf{r})$$

- Both two components vanish identically when:
  - n is odd
  - $\mathbf{k} \to \mathbf{k}_*$

Opening gap while preserving exact flat band

Breaks inversion opens the gap while remains flat bands

Linearly mixing even/odd parity zero mode operator

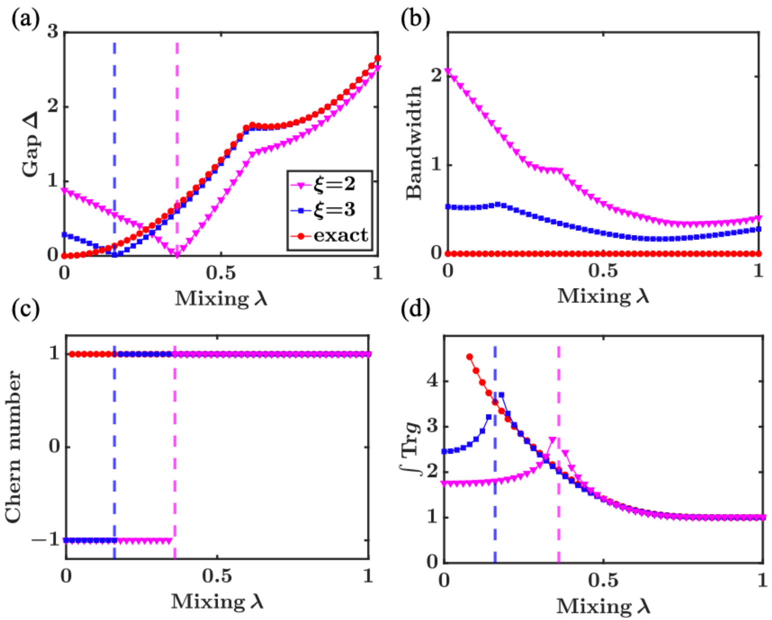
$$\hat{D}_\lambda = (1-\lambda)\hat{D}_1 + \lambda\hat{D}_0$$

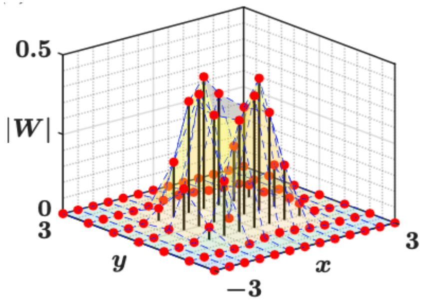
Gapped Hamiltonian with exact flat band:

$$H_\lambda = \hat{D}_\lambda^\dagger \hat{D}_\lambda$$

Proof based on SVD form of zero mode operator (see paper).

- Exact limit ($\zeta = \infty$): still Gaussian decay;
- Truncation effect ($\zeta < \infty$): see right figures.





Red: exact model

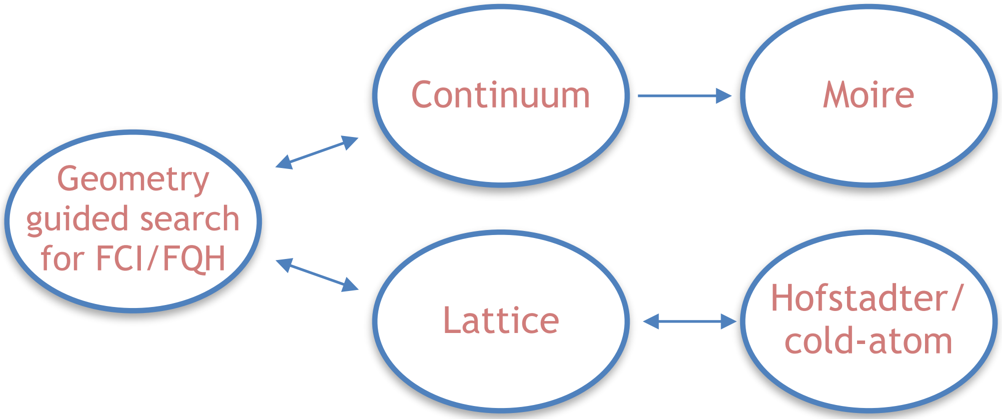
Blue and pink: truncation effect

Pink: truncate more

Xin Shen, Gangyue Ji, J. Zhang, David, Mera, Ozawa, JW (2025)

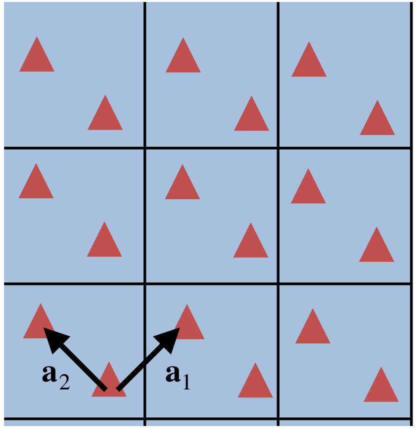
Conclusion and acknowledgements

- Exact Parent Hamiltonian for lattice LL states from generalized Poisson summation rule
- Inversion protected singular quadratic band touching for odd series



- ***Ideal bands:***
  JW*, J. Cano, A. Millis, Z. Liu*, B. Yang*; PRL (21).
- ***Generalized Landau levels:***
  Z. Liu†, B. Mera†, M. Fujimoto, T. Ozawa, JW*; PRX (25).
- ***Generalized Kapit-Mueller models:***
  X. Shen*†, G. ji†, Zhang, Palomino, Mera, Ozawa, JW*; arXiv 2501.09742.

Appendix: Proof of higher LL sum rule



Step I: $0 = \sum_{\mathbf{r}} \eta_{\mathbf{r}} \Phi_0^l(\mathbf{r}) \Rightarrow \; 0 = \sum_{\mathbf{r}} \eta_{\mathbf{r}} \Phi_{2n}^l(\mathbf{r}) = \sum_{\mathbf{r}} \eta_{\mathbf{r}} \left[(\bar{z} - l^2 \partial_z)^{2n} f(z) \right] e^{-\frac{1}{2l^2}|z|^2} = 0$

- A sum rule of any even LL of $l_B = l$
- This can be proved by "squeezing" the LLL (amounts to tuning the mass tensor/ metric of LL)

Is zero, following the LLL sum rule

Step II: derive a sum rule for all LLs including odd LL (e.g. n=1)

To illustrate, we take $n = 1$ above, which gives $\sum_{\mathbf{r}} \eta_{\mathbf{r}} \left[\bar{z}(\bar{z} - 2l^2 \partial_z) f(z) \right] e^{-\frac{1}{2l^2}|z|^2}$ plus $l^2 \sum_{\mathbf{r}} \eta_{\mathbf{r}} (\partial^2 f)(z) e^{-\frac{1}{2l^2}|z|^2}$

$$\sum_{\mathbf{r}} \eta_{\mathbf{r}} \bar{z} e^{-\frac{1}{2}|z|^2 (l^{-2} - l_B^{-2})} \left[(\bar{z} - 2l^2 \partial_z) f(z) \right] e^{-\frac{1}{2l_B^2}|z|^2} \xrightarrow[\bar{z} - 2l^2 \partial_z = \hat{a}^\dagger]{\text{If } 2l^2 = l_B^2} \sum_{\mathbf{r}} \eta_{\mathbf{r}} \bar{z} e^{-\frac{1}{2}|z|^2 (l^{-2} - l_B^{-2})} \cdot \Phi_1^{l_B = \sqrt{2}l}(\mathbf{r}) = 0$$

Sum rule for 1LL (new)!

Straightforward to generalize this procedure to higher ones.