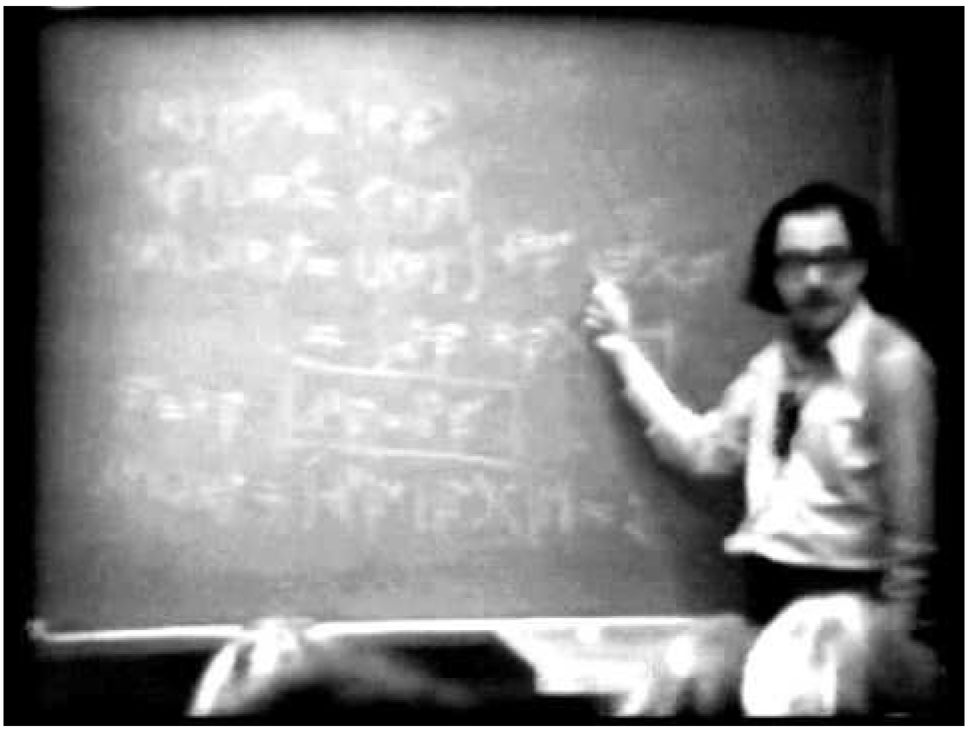

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Classical field theory

Master de Physique – M1 2021-2022

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Master de Physique M1
Classical Field Theory

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Contents

1. Introduction: the reason for this text and how to use it

These lecture notes are first of all intended to "my" students. I want to produce an introduction to Classical Field Theory, but of course, many excellent texts are available, some of them essential and famous[1], and others, more recent and still of lesser notoriety[2], but all with which mine cannot compete in terms of quality, completeness or depth of view. So what is the reason for that new text?

My experience in teaching at the University since almost thirty years already showed me, at least in France, that students have evolved, have changed. Their education is very different from the one we had, especially in mathematics. Their skills are different from those that we had. As a result, they sometimes make errors, or misunderstandings to which, as professors, we are not always prepared. Students can be astonishing! Writing a book in terms of a conversation between two students allows me to prevent the possible reader (expectedly a student) from several errors which I have seen in recent years, or it enables me to emphasize several points which a more traditional text would maybe fly over. I can even leave my two main characters make time to time an incomplete, or even wrong exposition of some notion, to correct it then. This is indicated in the text when the situation occurs and the corrected version is in that case given in the following paragraph. Exchanging ideas can time to time lead you to wrong or partially wrong conclusions that you have to correct then. I have experienced myself this situation very often in my life as a researcher, during discussions with colleagues. In the heat of the discussion, we can assert "truths" with authority, but which then appear inaccurate or even wrong!

In the first section, I introduce the characters, two students, one (she) who had a good education and is good in physics, and the other (he), who is maybe in advanced undergraduate studies, is working, but still may have difficulties here and there. They often meet and she answers his questions. In the beginning, I use a non standard format, a dialogue, and I imagine that both students write on a blackboard. For this I use "handwritten fonts". Progressively, I use less and less this stratagem, which may render the text chopped and unpleasant to read, and I slowly evolve towards a more academic exposition, but I keep the format of a conversation between the two main characters. The sections are divided in Days. Essentially, one Day could correspond to a 1h30 course and 1h30 of worked exercises, but this is not my intention to organize rigorously this "sectionning", so some *Days* will be shorter, others will be longer, depending on the intensity of the discussions among the two characters.

Considering that we are feeded by our readings, something that researchers and professors know, but that students might still overlook, I make up my mind to refer explicitly to literature, with quotations or excerpts of famous books.

[1] L. Landau et E. Lifchitz, Théorie des Champs, 3ème édition, Editions MIR, Moscou, 1970; B. Felsager, Geometry, Particles and Fields, Odense University Press, Odense 1981; N.A. Doughty, Lagrangian interactions, Addison Wesley, Redwood City, 1990; V. Rubakov, Classical Theory of Gauge Fields, Princeton University Press, Princeton, 2002; M. Burgess, Classical Covariant Fields, Cambridge University Press, Cambridge, 2003; D.E. Soper, Classical Field Theory, Dover, 2008.

[2] F. Scheck, Classical Field Theory, Springer, Berlin, 2012; J. Franklin, Classical Field Theory, Cambridge University Press, Cambridge, 2017; J.L. Lancaster, Introduction to Classical Field Theory, IOP ebooks, Morgan and Claypool Publishers, Bristol, 2018; H. Năstase, Classical Field Theory, Cambridge University Press, Cambridge, 2019.

☐ Parts of the text are written in helvetica font, starting with an open square and with a vertical bar in the left margin. These are supposed to be the notes given to the students by the professor. There are not always very detailed because the two students will then elaborate on the content. In the table of contents, these sections are also spotted by the open square.

When she writes on the blackboard, I use this handwritten font and a framed text...

...while when he write

Essentially all the dialogs between the two main characters are as many worked exercises, so the main text does not explicitly contain any proposed exercices, but when a part of dialog is convenient for an exercise, this is specified like this:

►**EXERCISE # – Spinor representation of rotations** – and the exercise is considered as finished with the symbol ◄.

A very fast first reading could be limited to only the "professor" sections in helvetica font.

In various occasions, I conceal my own opinions on Physics when *she* speaks. Physics is supposed to be an *exact science*. This doesn't mean that everything there is fixed forever. On the contrary, I believe that as practicing Physics as researchers, we develop personal views on various aspects, on what is more fundamental, what is secondary, on which theory – when several compete – has greater value. And all this depends a lot on our own experience[3]. Writing this text partially like a fiction also enables me to go a bit beyond physics, and to gently give my own opinion on various things, e.g. the society in which we live, but since this is a Physics text, I will not insist too much! Being a professor, I think that my role is not only to teach physics, but also to help for the emancipation of my students, maybe even to help them to work for a better world!

The text is written without having been corrected. I am not native from an English speaking country, but I read, speak, and write English basically everyday. When I teach in English, very often there are only non native English students. And we all communicate in that language, that we call English, which is only an approximation of English. When we communicate using this language, our colleagues, from England, Ireland, Scotland, etc are fair enough to understand what we say! And this is the language used in this text.

Last, but not least, what do I mean with Classical field theory? Rather than trying to explain myself my own point of view of this question let me quote someone who elaborated upon this very clearly. This is the beginning of the Introduction of Burgess's Classical Covariant Fields[4].

[3] This is why professors are not always interchangeable.

[4] M. Burgess, Classical Covariant Fields, Cambridge University Press, Cambridge, 2003.

In contemporary field theory, the word *classical* is reserved for an analytical framework in which the local equations of motion provide a complete description of the evolution of the fields. Classical field theory is a differential expression of change in functions of space and time, which summarizes the state of a physical system entirely in terms of smooth fields. The differential (holonomic) structure of field theory, derived from the action principle, implies that field theories are microscopically reversible by design: differential changes experience no significant obstacles in a system and may be trivially undone. (...)

When applied to quantum mechanics, the classical framework is sometimes called the *first quantization*. The first quantization may be considered the first stage of a more complete theory, which goes on to deal with the issues of many-particle symmetries and interacting fields. Quantum mechanics is classical field theory with additional assumptions about measurement. The term quantum mechanics is used as a name for the specific theory of the Schrödinger equation, which one learns about in undergraduate studies, but it is also sometimes used for any fundamental description of physics, which employs the measurement axioms of Schrödinger quantum mechanics, i.e. where change is expressed in terms of fields and groups. (...)

In the so-called *quantum field theory*, or *second quantization*, fields are promoted from c-number functions to operators, acting upon an additional set of states, called Fock space. (...) When one speaks about *quantum* field theory, one is therefore referring to this second quantization in which the fields are dynamical operators, spawning indistinguishable quanta.

2. Notations

▷ 2.1 Spacetimes and spacetime location

Spacetime is the arena where physical processes take place. Mathematically, spacetime is represented by a manifold M. In classical physics, space is a three-dimensional Euclidean space $\mathbb{E}^3$. In Special Relativity, spacetime is a four-dimensional *pseudo*-Euclidean, also called Minkowskian, manifold $\mathbb{M}^4$. In General relativity, this is a *pseudo*-Riemannian manifold referred to as $\mathbb{V}^4$. Other types of manifolds can be considered in physics, like Einstein-Weitzenböck $\mathbb{A}^4$, or Einstein-Cartan $\mathbb{U}^4$ manifolds.

Quoting Kevin Cahill[5] is instructive for the distinction between points and their coordinates:

> Points are physical, coordinates are metaphysical. When we change our system of coordinates, the points don't change, but their coordinates do. We'll often group the n coordinates x^i together and write them collectively as x without a superscript. Since the coordinates $x(p)$ label the point p, we sometimes will call them the point x. But p and x are different. The point p is unique with infinitely many coordinates x, x', x'', . . . in infinitely many coordinate systems.

Indices of Cartesian coordinates x^i in Euclidean space $\mathbb{E}^3$ will be denoted with lowercase Latin letters, $a, b, c \ldots$, or $i, j, k, \ldots$ and the associated normalized basis as $\{\mathbf{u}_a\}$ or $\{\mathbf{u}_i\}$. The corresponding metric tensor is denoted by δ_{ij} in $\mathbb{E}^3$ and represented, in matrix form, by

$$(\delta_{ij})_{i,j=1\ldots3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Most of the time, Euclidean indices i, j, k are used for "ordinary" space and vary from 1 to 3 (for physical reasons!) while a, b, c describe more abstract space components, e.g. they can describe group generators and they may have values from 1 to n. For example a, b, c will vary from 1 to 3 for the generators of $SO(3)$ or $SU(2)$, but from 1 to 8 for $SU(3)$. The Latin indices $a, b, c \ldots$ are most of the time Euclidean indices and the upper/lower position doesn't matter, i.e. $\sigma^a = \sigma_a$. A notable exception is the case of tetrad indices, like in $e_a{}^\mu$ (see below).

The matrix indices in the representation vector spaces of Lie generators are denoted by uppercase Latin letters, $A, B, C \ldots$ which vary from 1 to N. For example in the case of $SU(2)$ in the fundamental representation, A, B, C can vary from 1 to 2 for spin $\frac{1}{2}$, from 1 to 3 for spin 1, from 1 to 4 for spin $\frac{3}{2}$, etc. $A, B, C \ldots$ are also, most of the time, Euclidean indices, e.g. $\psi^A = \psi_A$.

In Minkowski spacetime $\mathbb{M}^4$, Cartesian coordinates x^α will carry lowercase Greek indices from the beginning of the alphabet, $\alpha, \beta, \gamma, \delta, \ldots$ and they vary from 0 to 3. The

[5] From K. Cahill, Physical Mathematics, Cambridge University Press, New-York, 2013. The excerpt here is borrowed from the online version at http://quantum.phys.unm.edu .

corresponding metric tensor is denoted as $\eta_{\alpha\beta}$. The signature of $\mathbb{M}^4$ is chosen $(+,-,-,-)$ and the metric tensor is represented, in matrix form, by

$$(\eta_{\alpha\beta})_{\alpha,\beta=0\ldots3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

An exception will be the Minkowskian tetrad Lorentz indices which, according to the most common use, are Latin indices a, b, c, like in $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$. If one specifies purely space Cartesian indices in Minkwoski spacetime, one uses Latin indices $i, j, k \ldots$, e.g. $x^\alpha = (x^0, x^i)^\mathrm{T}$, $i = 1, 2, 3$ and then $x^i = -x_i$. The context should be sufficient to avoid confusion with Euclidean indices in $\mathbb{E}^3$.

When arbitrary coordinates x^μ are used on $\mathbb{M}^4$, or when the manifold considered is more general than Minkowskian, the indices used are Greek letters from the middle and the end of the alphabet, $\kappa, \lambda, \mu, \ldots \rho, \sigma, \tau$ which also vary from 0 to 3. The metric tensor then depends on the coordinates and is either written $\eta_{\mu\nu}(x)$ in $\mathbb{M}^4$ or $g_{\mu\nu}(x)$ in more general manifolds.

▷ 2.2 Scalars, vectors and various other objects

Generically, we will denote various types of vectors with the same notation, in bold font. These may be

– ordinary $3d$ space vectors, like the velocity $\mathbf{v}$,

$$\mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}, \tag{3}$$

– "vectors" made of scalar fields, like Pauli complex spinors,

$$\boldsymbol{\psi} = \begin{pmatrix} \varphi_\uparrow \\ \varphi_\downarrow \end{pmatrix}, \tag{4}$$

or like vectors of the $2d$ representation space of $SU(2)$ or triplets of real scalar fields for the $3d$ representation space of $SO(3)$,

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}. \tag{5}$$

– The same notation can be used for the vectors which parametrize three-dimensional rotations, $\boldsymbol{\alpha}$, or boosts, $\boldsymbol{\phi}$,

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \end{pmatrix}, \qquad \boldsymbol{\phi} = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}. \tag{6}$$

Occasionally we will also write tensors in bold font, like $\mathbf{w} = w^{\mu\nu}\mathbf{e}_\mu \otimes \mathbf{e}_\nu$.

Scalar fields will generally be denoted as $\varphi(x)$ (real or complex scalar fields) or $\phi(x)$ (real scalar fields). Spinors will be denoted $\psi(x)$, possibly also $\boldsymbol{\psi}(x) = (\psi^A(x))_{A=1\ldots N}$.

The 4-potential is $A_\alpha(x) = (\phi/c, -\mathbf{A})$, the Faraday tensor $F_{\alpha\beta}(x) = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and their non Abelian generalizations written in components are given by $A^a{}_\alpha(x)$ or $F^a{}_{\alpha\beta}(x)$ while contracted with the Lie generators they are $\mathsf{A}_\alpha = A^a{}_\alpha \mathsf{t}_a$ and $\mathsf{F}_{\alpha\beta} = F^a{}_{\alpha\beta}\mathsf{t}_a$.

Operators and matrices are denoted in sanserif font (like above), e.g. matrix representations of Lie groups $\mathsf{D}_R(g(\alpha))$ or Lie generators in a given representation t^a_R. In the case of operators denoted by Greek letters, we use upright symbols, e.g. $\upsigma_a$, $\uptau_a$. Vectors of operators (Kronecker products or $\mathbf{\mathsf{P}} = -i\boldsymbol{\nabla}$ for example) are in bold sanserif, like $\mathbf{\mathsf{L}} = \mathbf{u}_i \otimes \mathsf{L}^i$ or $\boldsymbol{\upsigma} = \mathbf{u}_i \otimes \upsigma^i$.

Spacetime transformations will be denoted with sanserif font and square brackets to specify the type of transformation and the parameters of the transformation, e.g. $\mathsf{T}[\mathbf{a}]$, $\mathsf{R}[\boldsymbol{\alpha}]$ and $\mathsf{B}[\boldsymbol{\phi}]$ are respectively a space translation of vector $\mathbf{a}$, a space rotation of parameter $\boldsymbol{\alpha}$ and a boost of parameter $\boldsymbol{\phi}$. $\mathsf{U}_T[\mathbf{a}]$ is a unitary ray representation of a translation of parameter $\mathbf{a}$ and $\mathsf{S}[\Lambda]$ a Lorentz transformation in the spin representation (i.e. acting on a Dirac spinor).

▷ 2.3 Tensors

At each point $x \in M$ is defined the tangent space T_xM in which live vectors. The coordinate basis vectors define a holonomic basis, $\mathbf{e}_\mu \equiv \partial_\mu$ and an arbitrary vector $\mathbf{v} \in T_xM$ is written as $\mathbf{v} = v^\mu \mathbf{e}_\mu = v^\mu \partial_\mu$. Under an arbitrary change of coordinates $\{x^\mu\} \to \{x^{\mu'}\}$,

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu, \quad v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} v^\mu. \tag{7}$$

A more general tensor may be written e.g. as $\mathbf{w} = w^{\mu\nu}\mathbf{e}_\mu \otimes \mathbf{e}_\nu$ where $\otimes$ denotes the Cartesian product. Its components obey the transformation law

$$w^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu} w^{\mu\nu}. \tag{8}$$

Holonomic basis, means that the bracket of any two coordinate basis vectors vanishes, $[\mathbf{e}_\mu, \mathbf{e}_\nu] = [\partial_\mu, \partial_\nu] = 0$.

The dual basis in T^*_xM is denoted as $\{\mathbf{e}^\mu\}$. Dual vectors in the dual basis are written $\boldsymbol{\theta} = \theta_\mu \mathbf{e}^\mu$. Differentials of coordinate functions $dx^\mu = \mathbf{e}^\mu$ form basis covectors in the cotangent space corresponding to the coordinate basis in tangent space. These basis covectors transform like

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu, \quad \theta_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\theta_\mu. \tag{9}$$

Mixed tensors can also be constructed, e.g.

$$\mathbf{w} = w^{\mu\nu}{}_\kappa \mathbf{e}_\mu \otimes \mathbf{e}_\nu \otimes \mathbf{e}^\kappa = w^{\mu\nu}{}_\kappa \partial_\mu \otimes \partial_\nu \otimes dx^\kappa. \tag{10}$$

Under a change of coordinates, they obey mixed transformation laws

$$w^{\mu'\nu'}{}_{\kappa'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\frac{\partial x^\kappa}{\partial x^{\kappa'}} w^{\mu\nu}{}_\kappa. \tag{11}$$

The metric tensor $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$ maps two vectors onto a scalar, using the scalar product between a dual vector and a vector, $\mathbf{g}(\mathbf{u},\mathbf{v}) \equiv \mathbf{u}\cdot\mathbf{v}$ with $\mathbf{g}(\mathbf{u},\mathbf{v}) = g_{\mu\nu} u^\mu v^\nu = u_\mu v^\mu$.. The quantity $g_{\mu\nu}$ (component of the metric tensor) is usually referred to as metric tensor: $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$. In Minkowski spacetime, the metric tensor is denoted as $\eta_{\alpha\beta}$ as we said.

Levi-Civita symbol, $\varepsilon_{0123} = +1$. (Caution, the choice $\varepsilon^{0123} = +1$ is also very frequently used in the literature, and of course there are then differences, e.g. with the components of the dual electromagnetic tensor $\mathscr{F}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}$).

▷ 2.4 Differential geometry

Derivatives of vectors and tensors are not tensors, a property which is restored by the introduction of the connection coefficients and the notion of covariant derivative,

$$\nabla_\mu v^\sigma = \partial_\mu v^\sigma + \Gamma^\sigma{}_{\nu\mu} v^\nu, \tag{12}$$

$$\nabla_\mu v_\sigma = \partial_\mu v_\sigma - \Gamma^\nu{}_{\sigma\mu} v_\nu. \tag{13}$$

Tetrad coefficients $e_a{}^\mu$ and $e^a{}_\mu$ form a set of 4 vector fields (resp. covector fields) which relate (non holonomic) orthonormal basis $\{\mathbf{e}_a\}$ and cobasis $\{\mathbf{e}^b\}$ by linear combinations

$$\mathbf{e}_a = e_a{}^\mu \mathbf{e}_\mu, \ \mathbf{e}_\mu = e^a{}_\mu \mathbf{e}_a \tag{14}$$

to the coordinate basis $\{\mathbf{e}_\mu\} = \{\partial_\mu\}$ and cobasis $\{\mathbf{e}^\nu\} = \{dx^\nu\}$ in T_xM and T_x^*M respectively. Orthogonality relations are satisfied, $e_a{}^\mu e^b{}_\mu = \delta_a^b$ and $e_a{}^\mu e^a{}_\nu = \delta_\nu^\mu$ and vectors components can be written in the non holonomic basis, $v^a = e^a{}_\mu v^\mu$. Similar relations hold for covariant components and for higher order rank tensors, in particular

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu, \tag{15}$$

$$g^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu. \tag{16}$$

The $b-$component of the covariant derivative defines the connection coefficient $\omega^b{}_{a\mu}$,

$$\nabla_\mu v^b = \partial_\mu v^b + \omega^b{}_{a\mu} v^a, \tag{17}$$

$$\nabla_\mu v_a = \partial_\mu v_a - \omega^b{}_{a\mu} v_b, \tag{18}$$

with its relation to the natural connection

$$\omega^a{}_{b\mu} = e^a{}_\sigma \partial_\mu e_b{}^\sigma + \Gamma^\sigma{}_{\lambda\mu} e_b{}^\lambda e^a{}_\sigma. \tag{19}$$

Consistency requires a constraint called the tetrad compatibility postulate

$$\partial_\mu e_b{}^\nu + \Gamma^\nu{}_{\lambda\mu} e_b{}^\lambda - \omega^a{}_{b\mu} e_a{}^\nu \equiv \nabla_\mu e_b{}^\nu = 0. \tag{20}$$

The condition $\nabla_\mu \eta^{ab} = 0$ also demands the symmetry property $\omega^{ab}{}_\mu + \omega^{ba}{}_\mu = 0$.

The Fock-Ivanenko connection Ω_μ allows to write the covariant derivative of a spinor field ψ:

$$\nabla_\mu \psi = (\partial_\mu + \Omega_\mu)\psi. \tag{21}$$

It takes one of several forms:

$$\Omega_\lambda(x) = \frac{1}{4i}\omega_{bc\lambda}(x)\sigma^{bc} = \frac{1}{8}\omega_{bc\lambda}(x)[\gamma^b,\gamma^c] = \frac{1}{4}\omega^{bc}{}_\lambda(x)\gamma_b\gamma_c. \tag{22}$$

An important equality,

$$[\nabla_\mu,\nabla_\nu]v^\rho = R^\rho{}_{\sigma\mu\nu}v^\sigma - S^\lambda{}_{\mu\nu}\nabla_\lambda v^\rho, \tag{23}$$

defines the curvature,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\sigma\nu} + \Gamma^\rho{}_{\lambda\mu}\Gamma^\lambda{}_{\sigma\nu} - \partial_\nu\Gamma^\rho{}_{\sigma\mu} - \Gamma^\rho{}_{\lambda\nu}\Gamma^\lambda{}_{\sigma\mu} \tag{24}$$

and the torsion associated to the connection

$$S^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}. \tag{25}$$

Both curvature and torsion are tensors, since the expression in terms of commutator of covariant derivatives appears as a tensor identity. The Ricci tensor, obtained by contraction of the Riemann tensor,

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \tag{26}$$

plays an important role in General Relativity, and the Ricci scalar by a contraction of the Ricci tensor,

$$R = R^\rho{}_\rho. \tag{27}$$

A vector $\mathbf{v} = v^\mu\mathbf{e}_\mu$ can be transported along a curve $\gamma(s)$ parametrized by s with local tangent vector $\mathbf{t} = t^\mu\mathbf{e}_\mu$, $t^\mu = \frac{dx^\mu}{ds}$ if

$$\nabla_\mathbf{t}\mathbf{v} = t^\lambda\nabla_\lambda v^\mu\mathbf{e}_\mu = t^\lambda(\partial_\lambda v^\mu + \Gamma^\mu{}_{\nu\lambda}v^\nu)\mathbf{e}_\mu = 0. \tag{28}$$

A curve γ is said autoparallel if the tangent vector $\mathbf{t}$ is transported along γ by parallel transport,

$$t^\lambda(\partial_\lambda t^\mu + \Gamma^\mu{}_{\nu\lambda}t^\nu) = 0 \tag{29}$$

which, with the definition of t^μ gives

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0. \tag{30}$$

When a vector $\mathbf{v}$ is parallel transported along a curve with tangent $\mathbf{t}$ in a metric spacetime, its length is conserved

$$\frac{d}{ds}(g_{\mu\nu}v^\mu v^\nu) = 0 \tag{31}$$

and the scalar product $g_{\mu\nu}v^\mu w^\nu$ between two parallel transported vectors is also conserved,

$$\frac{d}{ds}(g_{\mu\nu}v^\mu w^\nu) = 0 \tag{32}$$

which implies that the angle between $\mathbf{v}$ and $\mathbf{w}$ is constant along γ.

In a metric spacetime, the shortest curve between two points defines the geodesic curve:

$$\delta \int_\gamma \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = 0. \tag{33}$$

This leads to the geodesic equation

$$\frac{d^2 x^\mu}{ds^2} + \frac{1}{2} g^{\mu\sigma} (g_{\lambda\sigma,\nu} + g_{\sigma\nu,\lambda} - g_{\lambda\nu,\sigma}) \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = 0 \tag{34}$$

where the notation $g_{\lambda\sigma,\nu}$ stands for the ordinary derivative, $g_{\lambda\sigma,\nu} = \partial_\nu g_{\lambda\sigma}$, while the frequent notation $g_{\lambda\sigma;\nu}$ is used for the covariant derivative, $g_{\lambda\sigma;\nu} = \nabla_\nu g_{\lambda\sigma}$.

If the connection is the natural connection (Christoffel symbols, denoted here with the `mathring` $\mathring{}$), also called Levi-Civita connection

$$\mathring{\Gamma}^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} (g_{\lambda\sigma,\nu} + g_{\sigma\nu,\lambda} - g_{\lambda\nu,\sigma}), \tag{35}$$

then the autoparallel curve is also the geodesic curve.

▷ 2.5 Differential forms

A p-form is the quantity denoted as $\overset{p}{\omega}$ which, upon integration over a p-dimensional domain of a d-dimensional manifold $\underset{p}{\Omega} \in \mathcal{M}$, delivers a scalar quantity s,

$$\int_{\underset{p}{\Omega}} \overset{p}{\omega} = s. \tag{36}$$

1-, 2 and 3-forms may be rewritten as

$$\overset{1}{\omega} = \omega_i dx^i, \quad \overset{2}{\omega} = \frac{1}{2!} \omega_{jk} dx^j \wedge dx^k, \quad \overset{3}{\omega} = \frac{1}{3!} \omega_{ijk} dx^i \wedge dx^j \wedge dx^k \tag{37}$$

with the wedge product satisfying $dx^j \wedge dx^k = -dx^k \wedge dx^j$. Their components transform like those of rank p covariant tensors.

The exterior derivative acting on a p-form

$$\overset{p}{\omega} = \frac{1}{p!} \omega_{1,\ldots,p} dx^1 \wedge \cdots \wedge dx^p, \tag{38}$$

delivers

$$d\overset{p}{\omega} = \frac{1}{p!} \frac{\partial \omega_{1,\ldots,p}}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge dx^p. \tag{39}$$

Acting on a product of forms it obeys a generalized Leibniz rule,

$$d(\overset{p}{\omega}\overset{q}{\theta}) = (d\overset{p}{\omega})\overset{q}{\theta} + (-1)^p \overset{p}{\omega} d\overset{q}{\theta}. \tag{40}$$

and $d\overset{0}{\omega}$ produces a gradient, $d\overset{1}{\omega}$ gives a curl, and $d\overset{2}{\omega}$ a divergence.

The Hodge product $\star$ acts on a p-form $\overset{p}{\omega}$ (with $p \leqslant d$). It delivers a $(d-p)$-form that we write $\overset{d-p}{\underset{\sim}{\omega}}$. There is a general expression for arbitrary manifold dimension,

$$\star \overset{d}{\omega} = \frac{1}{p!(d-p)!} \omega^{\mu_1 ... \mu_p} \sqrt{|\mathrm{det} g_{ab}|} \epsilon_{\mu_1 ... \mu_n} dx^{\mu_{p+1}} \wedge ... \wedge dx^{\mu_d}, \tag{41}$$

where

$$\overset{d}{\omega} = \frac{1}{p!} \omega_{\mu_1 ... \mu_p} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}. \tag{42}$$

and the totally anti-symmetric d-rank Levi-Civita symbol is defined as

$$\epsilon_{\mu_1 ... \mu_d} = \begin{cases} +1 & \text{if } \mu_1, ..\mu_d \text{ is an even permutation of } 1, ..d \\ -1 & \text{if } \mu_1, ..\mu_d \text{ is an odd permutation of } 1, ..d \\ 0 & \text{otherwise.} \end{cases}$$

The Poincaré lemma states that $d(d\overset{p}{\omega}) = 0$ for any form, or

$$d^2 = 0. \tag{43}$$

A closed form, which obeys $d\overset{p}{\theta} = 0$, can be written as $\overset{p}{\theta} = d(\overset{p-1}{\omega})$ and it is said exact (this is true locally). The form $\overset{p-1}{\omega}$ is a potentiel, since it is itself defined up to a $(p-2)$-form. An interesting case is that of a d- form, for which, automatically

$$d\overset{d}{\omega} = 0. \tag{44}$$

Stokes theorem takes the compact form

$$\int_{\underset{p+1}{\Omega}} d\overset{p}{\omega} = \int_{\partial \underset{p+1}{\Omega}} \overset{p}{\omega} \tag{45}$$

where $\partial \underset{p+1}{\Omega}$ is the boundary of the $(p+1)$-dimensional submanifold $\underset{p+1}{\Omega}$.

Algebra of the Dirac matrices

$$\tfrac{1}{2}\{\gamma^\alpha, \gamma^\beta\} = \eta^{\alpha\beta} 1_N \tag{46}$$

with the properties

$$\gamma_0 \gamma_\alpha \gamma_0 = \gamma_\alpha^\dagger, \quad \gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i. \tag{47}$$

There are various representations, the most common being the Weyl (or chiral) representation (here with $N = 4$)

$$(\gamma^0) = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad (\gamma^i) = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (\gamma^5) = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}, \tag{48}$$

and the Dirac representation,

$$(\gamma^0) = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad (\gamma^i) = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (\gamma^5) = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \tag{49}$$

given here in terms of Pauli matrices.

A less common representation is due to Majorana,

$$(\gamma^0) = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad (\gamma^1) = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad (\gamma^2) = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad (\gamma^3) = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}. \tag{50}$$

In the Majorana representation one has $\gamma_\alpha^* = -\gamma_\alpha$, i.e. all non vanishing elements of the gamma matrices are purely imaginary, a property which renders the Dirac equation real. This means that in this representation, we can find spinors with purely real components which play for spinors a role analogous to that of real scalar fields.

Here we choose

$$\varepsilon_{0123} = +1, \quad \gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma_5\gamma^d = -\gamma^d\gamma_5 \tag{51}$$

$$\sigma^{bc} = \frac{i}{2}[\gamma^b, \gamma^c], \tag{52}$$

then, the following relation holds

$$\gamma^a[\gamma^b, \gamma^c] = 2(\eta^{ab}\gamma^c - \eta^{ca}\gamma^b + i\varepsilon^{abcd}\gamma_5\gamma_d). \tag{53}$$

The action, a functional, is denoted as $S[q]$ (for a particle) or $S[\varphi]$ (for a real field) and $S[\varphi, \varphi^*]$ (for a complex field), e.g.

$$S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = \int_{t_1}^{t_2} (p\dot{q} - H(p, q, t))dt, \tag{54}$$

$$S[\varphi] = \int_{t_1}^{t_2} \int_{\Omega} \mathscr{L}(\varphi, \dot{\varphi}, \nabla\varphi, t)d^d r = \int_{t_1}^{t_2} \int_{\Omega} (\pi\dot{\varphi} - \mathscr{H}(\pi, \varphi, \nabla\varphi, t))d^d r \tag{55}$$

where $L(q, \dot{q}, t)$ is the Lagrangian of the particle (here with a single degree of freedom), $H(p, q, t)$ the Hamiltonian and $\mathscr{L}(\varphi, \dot{\varphi}, \nabla\varphi, t)$ the Lagrangian density ($\mathscr{H}(\pi, \varphi, \nabla\varphi, t)$ the Hamiltonian density) of the field (here a real scalar field in d space dimensions). Most of the time, the explicit dependence of the Lagrangian (density) w.r.t t will not be considered.

The functional derivative is denoted as

$$\frac{\delta S}{\delta \varphi}. \tag{56}$$

For a generic action

$$S[\varphi] = \int d^4x \mathscr{L}(\varphi, \partial\varphi, \ldots, \partial\partial\ldots\partial\varphi) \tag{57}$$

in terms of a Lagrangian density $\mathscr{L}(\varphi, \partial\varphi, \ldots, \partial\partial\ldots\partial\varphi)$, the Euler-Lagrange equations take the general form

$$\frac{\delta S}{\delta \varphi} = \frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \varphi)} + (-1)^m \partial_{\alpha_1}\partial_{\alpha_2}\ldots\partial_{\alpha_m} \frac{\partial \mathscr{L}}{\partial(\partial_{\alpha_1}\partial_{\alpha_2}\ldots\partial_{\alpha_m}\varphi)} = 0. \tag{58}$$

▷ 2.8 Continuous groups representations

There are various notations for the continuous group representations in the literature. Typically, we can see something like $\mathsf{D}_R(g) = \exp(\theta_a \mathsf{T}^a{}_R)$ or $\mathsf{D}_R(g) = \exp(i\theta_a \mathsf{t}^a{}_R)$ to denote the operator associated to a group element g in a given representation R in terms of the parameters of the group transformation and of the generators $\mathsf{T}^a{}_R$ or $\mathsf{t}^a{}_R$ in the specified representation. Of course in the two expressions above, the generators must obey $\mathsf{T}^a{}_R = i\mathsf{t}^a{}_R$. We can also find plus or minus signs in the definitions and at the end, this may be very obscure. Since this is a very heavy notation, the representation index R is usually forgotten. Generally, we will conform to standard use of the notations $\frac{1}{2}\sigma_a$ or $\frac{1}{2}\tau_a$ for the generators of $SU(2)$ and $\frac{1}{2}\lambda_a$ for those $SU(3)$.

Here, we will use the following notations for projective representations which are among the most important in QM:

– ray representation of translations,

$$(1 - i\delta a^a \mathsf{P}_a)\varphi(\mathbf{r}, t) = \varphi(\mathbf{r} - \delta\mathbf{a}, t), \tag{59}$$

– ray representation of rotations,

$$(1 - i\delta\alpha^a \mathsf{L}_a)\varphi(\mathbf{r}, t) = \varphi(\mathbf{r} - \delta\boldsymbol{\alpha} \times \mathbf{r}, t), \tag{60}$$

– ray representation of boosts,

$$(1 - i\delta v^a \mathsf{K}_a)\varphi(\mathbf{r}, t) = \varphi(\mathbf{r} - \delta\mathbf{v}t, t), \tag{61}$$

– ray representation of general Poincaré transformation

$$(1 + i\varepsilon^\alpha \mathsf{P}_\alpha - \tfrac{1}{2}i\omega^{\alpha\beta}\mathsf{L}_{\alpha\beta})\psi(x) = \psi(\Lambda^{-1}[\varepsilon, \omega]x) \tag{62}$$

(t ke care to the opposite signs) where the Poincaré transformation acts on space-time points according to $x \to x' = \Lambda[\varepsilon, \omega]x$ and the generators are $\mathsf{P}_\alpha = i\partial_\alpha$ and $\mathsf{L}_{\alpha\beta} = i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)$ with commutation relations

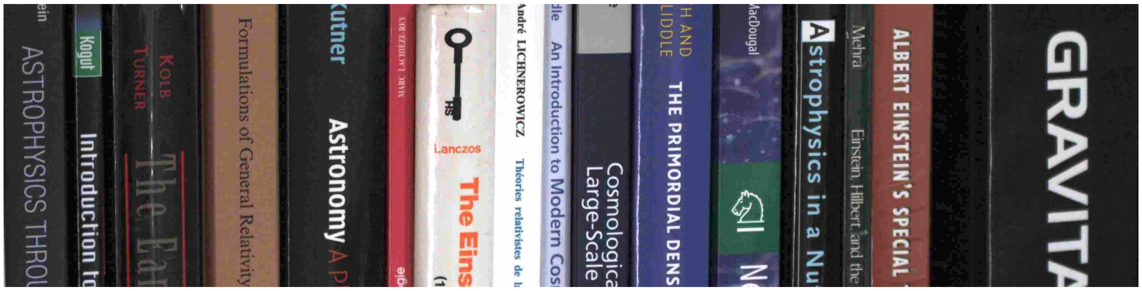
$$\begin{aligned}
\left[\mathsf{L}_{\alpha\beta}, \mathsf{L}_{\gamma\delta}\right] &= i(\eta_{\beta\gamma}\mathsf{L}_{\alpha\delta} - \eta_{\alpha\gamma}\mathsf{L}_{\beta\delta} - \eta_{\delta\alpha}\mathsf{L}_{\gamma\beta} + \eta_{\delta\beta}\mathsf{L}_{\gamma\alpha}), &(63)\\
\left[\mathsf{P}_\alpha, \mathsf{L}_{\gamma\delta}\right] &= i(\eta_{\alpha\gamma}\mathsf{P}_\delta - \eta_{\alpha\delta}\mathsf{P}_\gamma), &(64)\\
\left[\mathsf{P}_\alpha, \mathsf{P}_\beta\right] &= 0. &(65)
\end{aligned}$$

define the Lie algebra of the Poincaré group.

The font variations are subtle, but most of the time, the context is enough to know whether the object is a vector or a matrix or something else. We hope that the reader will be able to surf these notations variations!

The identity operator is generally denoted by 1. When this is the identity matrix, we may use 1_N in N dimensions.

Introduction to energy minimization



3. First evening

Aïssata is lying on the floor of her room. Books are everywhere, on the bed, on the small table near to the window, on the bookshelf, even in the small kitchen space. Titles are as strange as "Gravitation and Cosmology"[6], "Quantum Field Theory in a Nutshell"[7], "Théorie des Champs"[8], "La vie de Galilée"[9], "Physical Cosmology"[10], "Klassische Feldtheorie"[11] and many more. They have the seals of various University libraries. Some are open, some, from which one cannot read the titles, are piled up. Some of them are not in a too good shape, they have been read so many times!

Someone is knocking at the door.

– Aïssata: Yes?

Diego enters, bumping into a book on the floor.

– Aïssata: Hi Diego.

– Diego: What the hell is this, ... "Gravitation"[12], ... is there still something that you ignore on gravity Aïssata?

– Aïssata: I still ignore all important questions on gravitation! This is one of the most important problems in Physics...

– Diego: [not listening at all] Look, I have to talk to you. I got my first course in Classical Field Theory today. Great, the prof is great! She spoke about so many things in a single course, I am getting impatient to attend the next course. But I need your help if I don't want to be quickly overwhelmed by the multitude of concepts! You know a lot of Physics, and I have so many questions already after the first course.

– Aïssata: I was studying for my test the day after tomorrow you know, Aïssata complains,

– Diego: [ignoring again ...] The professor started saying that most of the formalism would be done using a variational approach and that a short reminder in Lagrange formalism would be needed. And we started with what she called a kind of pedestrian example in electrostatics. Look at the notes that she gave us.

[6] S. Weinberg, Gravitation and Cosmology, Wiley, 1972.
[7] A. Zee, Quantum Field Theory in a Nutshell, Princeton University Press, Princeton, 2003.
[8] L. Landau et E. Lifchitz, Théorie des Champs, 3ème édition, Editions MIR, Moscou, 1970.
[9] B. Brecht, La vie de Galilée, Théâtre complet, vol. 4, L'Arche, Paris, 1975.
[10] P.J.E. Peebles, Physical Cosmology, Princeton University Press, Princeton, 1993.
[11] W. Thirring, Klassische Feldtheorie, Springer, Wien, 1978.
[12] C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation, Princeton University Press, Princeton, 2017.

☐ 3.1 "Empirical" energy minimization

Consider a perfect cylindrical capacitor made of two infinite metallic cylinders with radii $r_1 = a$ and $r_2 = b$. The internal cylinder (r_1) is kept at an electrostatic potentiel V_0 and the external one (r_2) at 0. Imagine that you know *nothing* about the calculation of an electric field, except that it derives from a potential function, $\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r})$ and that the energy associated to this field is the volume integral of an *energy density* $\frac{1}{2}\varepsilon_0|\mathbf{E}(\mathbf{r})|^2$. What else can we reasonably say? The boundary conditions (BCs) impose that the potential function is such that $\phi(r_1) = V_0$ and $\phi(r_2) = 0$. Any smooth function satisfying these boundary conditions can be used as a *test function* to compute the energy stored in the capacitor. Of course, not *any function* will lead to the *actual energy* stored. For example the potentiel could decay linearly $\phi_1(r) = \alpha_1 r + \beta_1$ and the constants α_1 and β_1 chosen such that the boundary conditions are fulfilled, i.e. $\alpha_1 = \frac{V_0}{a-b}$ and $\beta_1 = -\frac{b}{a-b}V_0$. This allows to calculate easily the associated *putative* electric field $\mathbf{E}_1 = -\frac{d\phi_1}{dr}\mathbf{u}_r = \frac{V_0}{b-a}\mathbf{u}_r$ and the associated electric energy stored for a length h along the symmetry axis,

$$U_1 = \int_a^b dr \int_0^h dz \int_0^{2\pi} r d\varphi \left(\tfrac{1}{2}\varepsilon_0 E_1(r)^2\right) = 2\pi h \int_a^b \tfrac{1}{2}\varepsilon_0 E_1(r)^2 r dr = \tfrac{1}{2}\pi\varepsilon_0 h V_0^2 \frac{b+a}{b-a}. \quad (66)$$

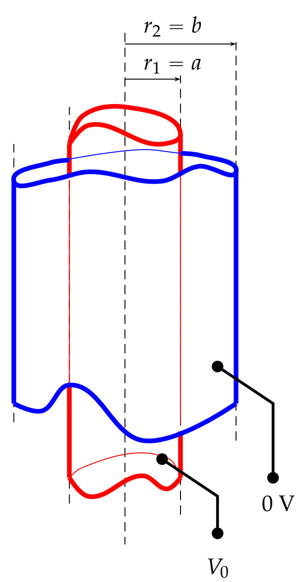


Figure 1. The cylindrical capacitor

Now, we said that *any* function with appropriate values at r_1 and r_2 would allow for similar calculations. Consider for example the set of functions parametrized by a single scalar κ, $\phi_\kappa(r) = \alpha_\kappa r^\kappa + \beta_\kappa$ with $\alpha_\kappa = \frac{V_0}{a^\kappa - b^\kappa}$ and $\beta_\kappa = -\frac{b^\kappa}{a^\kappa - b^\kappa}V_0$ to satisfy the BCs. It gives back the previous case when $\kappa = 1$. The energy stored, *if the potential function was $\phi_\kappa(r)$, would now be*

$$U_\kappa = \tfrac{1}{2}\kappa\pi\varepsilon_0 h V_0^2 \frac{b^\kappa + a^\kappa}{b^\kappa - a^\kappa}. \quad (67)$$

This is instructive to make a numerical estimate for various values of κ. In the next table, we give values of $U_\kappa/(\pi\varepsilon_0 h V_0^2)$ for various values of the ratio b/a and of κ.

b/a	actual value	$U_\kappa/(\pi\varepsilon_0 h V_0^2)$ $\kappa=1$	$\kappa=2$	$\kappa=0.5$	$\kappa=0.1$	$\kappa=-1$
1.1	10.4921	$\underline{10}.5000$	$\underline{10}.5238$	$\underline{10.49}40$	$\underline{10.4921}$	$\underline{10}.5000$
1.5	2.4663	$\underline{2}.5000$	$\underline{2}.6000$	$\underline{2.4}747$	$\underline{2.466}6$	$\underline{2}.5000$
2.	1.4427	$\underline{1}.5000$	$\underline{1}.6667$	$\underline{1.4}571$	$\underline{1.44}33$	$\underline{1}.5000$
4.	0.7213	$\underline{0}.8333$	1.1333	$\underline{0.7}500$	$\underline{0.72}25$	$\underline{0}.8333$

In the table we have underlined the digits which coincide with those given in the second column which corresponds to the actual value of the energy stored in the capacitor. Empirically, we have the intuition that *Nature* in this case is such that the energy stored is minimized.

– Diego: You see, there are all these words in italics in the notes of the professor, and there is this "actual value" in the table which comes from we don't know where. I feel that this example is instructive, but I am afraid maybe not to get the main message. Why did she introduce different potential functions to describe the capacitor? There is only one scalar potential in electrostatics as far as I know, and this is Coulomb potential, no?

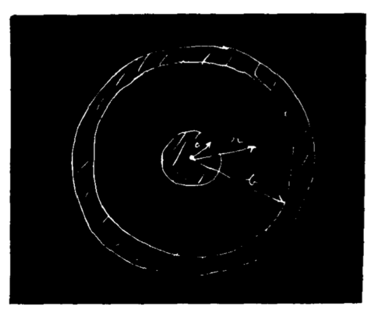
– Aïssata: Indeed, this is a very nice example. This is an argument of Feynman. Aïssata looks in a pile of books and opens Feynman's second volume[13]. This is the wonderful chapter 19, *The principle of least action*, that I urgently recommend she says.

"There is an interesting case when the only charges are on conductors. Then

$$U^* = \frac{\epsilon_0}{2}\int (\nabla\phi)^2\,dV.$$

Our minimum principle says that in the case where there are conductors set at certain given potentials, the potential between them adjusts itself so that integral U^* is least. What is this integral? The term $\nabla\phi$ is the electric field, so the integral is the electrostatic energy. The true field is the one, of all those coming from the gradient of a potential, with the minimum total energy.

"I would like to use this result to calculate something particular to show you that these things are really quite practical. Suppose I take two conductors in the form of a cylindrical condenser.



The inside conductor has the potential V, and the outside is at the potential zero. Let the radius of the inside conductor be a and that of the outside, b. Now we can suppose *any* distribution of potential between the two. If we use the *correct* $\underline{\phi}$, and calculate $\epsilon_0/2\int(\nabla\underline{\phi})^2\,dV$, it should be the energy of the system, $\frac{1}{2}CV^2$.

19–11

Figure 2. R.P. Feynman, Lectures on Physics, Vol 2, Addison Wesley, Reading, 1964, sec 19-11.

►EXERCISE 1 – **Energy stored in a cylindrical capacitor** –

So, let's start with the calculation of the actual value. Instead of assuming that you ignore all of electrodynamics, except the formula for the energy density,

$$u_{es} = \frac{1}{2}\varepsilon_0|\mathbf{E}(\mathbf{r})|^2, \tag{68}$$

[13] R.P. Feynman, Lectures on Physics, Vol 2, Addison Wesley, Reading, 1964.

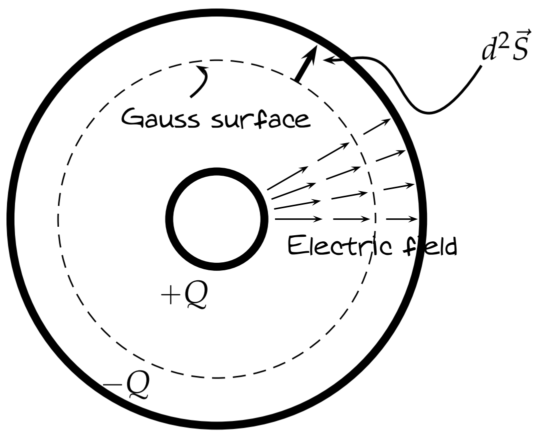
we now consider the problem of this cylindrical capacitor and solve it via the usual laws.

First the cylindrical symmetry tells you that the electric field if of the radial form

$$\mathbf{E}(\mathbf{r}) = E_r(r)\mathbf{u}_r \tag{69}$$

where $\mathbf{u}_r$ is the radial unit vector. The boundary conditions for the potential are such that the electric field points outwards, which means that a slice of height h of the internal conductor carries a charge $+Q$ ($Q = 2\pi r_1 h \sigma_1$ in terms of a surface charge density σ_1 on the internal cylinder), while the external conductor carries $-Q$ (hence carries a different charge density).

Let me draw a transverse section of the capacitor, like in the course of Feynman. She starts writing and drawing on her small blackboard which covers half the window of the small room.



Gauss law, applied to a cylindrical surface of radius $r_1 < r < r_2$,

$$\oint \vec{E}(\mathbf{r}) \cdot d^2\vec{S} = 2\pi r h E_r(r) = Q/\varepsilon_0$$

gives an energy density

$$u_{es} = \frac{1}{2}\varepsilon_0 (Q/2\pi\varepsilon_0 r h)^2$$

which, integrated over the volume between the conductors gives

$$U_{es} = \frac{Q^2}{4\pi\varepsilon_0 h}\ln(b/a).$$

– Aïssata: If we want to compare to previous expression (67), we have to form and simplify the quantity $U_{es}/(\pi\varepsilon_0 h V_0^2)$, i.e. to relate Q to V_0. This is done via the expression of the electrostatic potentiel, solution of $E_r(r) = -\frac{d\phi(r)}{dr}$ and satisfying $\phi(b) = 0$, hence

$$\phi(r) = -\frac{Q}{2\pi\varepsilon_0 h}\ln(r/b). \tag{70}$$

It follows that $Q = 2\pi\varepsilon_0 h V_0/(\ln(b/a))$ and eventually

$$U_{\text{actual}} = \frac{\pi\varepsilon_0 h V_0^2}{\ln(b/a)} \tag{71}$$

which leads, e.g. for $b/a = 2$, to the numerical value $U_{actual}/(\pi\varepsilon_0 h V_0^2) = 1.4427$ reported in the table.◂

– Diego: So what was the reason of these calculations with test functions and the appearance of the parameter κ?

– Aïssata: Your professor wanted to show you that if you *would ignore* the laws of electrodynamics, or, as discussed also by Feynman, if you know these laws, but the physical situation, the geometry of the conductors for example is so complicated that you cannot solve mathematically the known physical laws, you can use various trial functions for the electrostatic potential, and calculate the corresponding putative energy (this is the word used by your professor). This *variational approach* is very helpful for problems which are otherwise intractable. And it can always be implemented numerically if this is too hard to get an analytical solution.

To illustrate this, your professor has taken as an example a family of functions $\phi_\kappa(r)$ parametrized by the real κ. For each value of κ she has calculated the corresponding energy which *would be stored* in the capacitor and then she made a numerical comparison of a dimensionless quantity. By the way, don't you notice something with the figures given in the table?

– Diego: Let me check. I see, in each row, the values calculated with the test function are always larger than the actual one!

– Aïssata: Exact! The actual value is a lower bound (see the plot 3 that we can do e.g. for $b/a = 2$. On the left you have the functions ϕ_κ plotted against r/a for $\kappa = -5$, 1 and 4 and $V_0 = 2$ and on the right we can plot the dimensionless energy vs κ, the dashed horizontal line being the actual value). Electrodynamics in real life constrains the potential, or electric field, in such a way that it corresponds to the minimum for the energy actually stored. Of course, this is only an empirical observation here, since only a few test functions are used, but this "least energy behaviour" can be proven, you will see this later.

You may also notice that the best approximation in the table corresponds to $\kappa = 0.1$ which, in a sense, is the closest, among the test functions considered here, to the actual form in $-\ln r$ (a kind of limit for $\kappa = 0$).

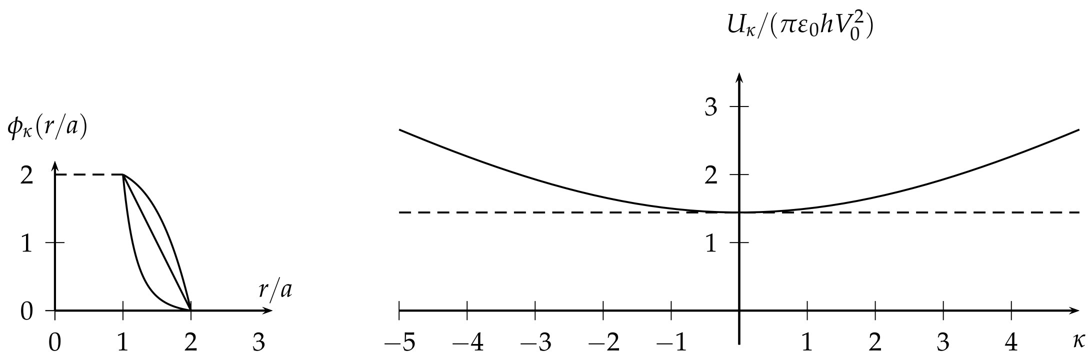


Figure 3. The potential profile (left) and the variational electrostatic energy stored in the capacitor (right).

– Diego: And why didn't she use directly test functions for the electric field, instead of the potential?

– Aïssata: This would be possible, of course, but you will also learn that as a field theory, this is simpler to treat electrodynamics in terms of potentials than in terms of electric (or magnetic) fields.

▷ 3.2 The *Psychological* problem with a global minimization

– Diego: OK, I got the message now. But there is still something hard to accept, I will try to explain my understanding. The professor then spoke about the application of a variational principle in particle dynamics. OK, that looks good to have such a principle in mechanics, but I must confess, … I didn't understand how it works. She said that a particle moves in such a way as to obey to a mysterious *least action principle*, due to Hamilton she said.

– Aïssata: This is the standard belief that the least action principle is due to Hamilton, but I haven't personally checked it.

– Diego: This is not what worries me! I understood that the action (this is even not the energy this time) is a global quantity, calculated over the whole trajectory between two points. So *how does the particle know* where to go in order to accumulate a minimal action?

A student in the class asked the question. The professor replied that she never said that the particle knows anything. She said, speaking the words slowly, that *Nature behaves in such a way that the action obeys a least action principle*. She added that she doesn't know if the particle *thinks*, but she believes it doesn't even have a brain! And everybody laughed.

– Aïssata: Well spotted! For all the questions that you ask, I strongly recommend the book of Lanczos[14] who had a deep view on the subject. In his introduction for example he poses the problem that you mentioned:

> Although it is tacitly agreed nowadays that scientific treatrises should avoid philosophical discussions, in the case of the variational principles of mechanics an exception to the rule may be tolerated, partly because these principles are rooted in a century which was philosophically oriented to a very high degree, and partly because the variational method has often been the focus of philosophical controversies and misinterpretations.
>
> Indeed, the idea of enlarging reality by including "tentative" possibilities and then selecting one of these by the condition that it minimizes a certain quantity, seems to bring a *purpose* to the flow of natural events. This is contradiction to the usual *causal* description of things.

She takes a book off the shelf and hands it to him.

– Diego: Aïssata, I feel exactly what is transcribed in this excerpt. I am lost with the new notions introduced. In fact I am not sure that I understood this action stuff, function, functional and all that... Newton dynamics is something that I understand: In absence of any force, a particle perseveres in its motion at constant velocity, both in direction and intensity (the uniform rectilinear motion). When a force is present, it modifies the velocity according to the fundamental principle of dynamics, or Newton's second law, $m\frac{d\mathbf{v}}{dt} = \mathbf{F}$. There is a local causality: a force acts locally, it bends the trajectory or accelerates the particle. This is clear to me.

– Aïssata: OK, let us discuss this point a few minutes. First, what do you really mean by "I understand"? The question deserves some attention. Let me show you a sentence that I like in this fiction[15] that I am reading,

> Même le plus génial des scientifiques ignore complètement pourquoi la pomme tombe de haut en bas et c'est à cette ignorance qu'il donne le nom de gravitation.[16]

Diego remains puzzled.

– Aïssata: Your professor gave you the correct answer I think. The theory gives you a *strategy* to get the actual trajectory followed by a particle under given circumstances. You can calculate a certain quantity, indeed, a kind of abstract quantity, the action. Then,

[14] C. Lanczos, The variational principles of mechanics, Dover, New-York, 1949.

[15] L'Ultime Question, Juli Zeh, Actes Sud, 2008, p. 228.

[16] Even the most brilliant of scientists has no idea why the apple falls down, and it is this ignorance that he calls gravitation.

requiring this quantity to be extremal gives you an equation for the trajectory. When you solve that equation and you compare to the trajectory measured in an experiment, you find that both coincide. This is what we mean when we say "I understand". What else would you like? Is there a difference with Newton's second law?

Yes, there is a difference: you are *used to* Newton's second law. This is the difference but you can also question this law. There is nothing we really *understand* in this relation, but we know that the conclusions that we can draw from this equation, when compared to *reality*, are in agreement with experiments. This is why we take this equation for *true*, at least as long as we do not make experiments in kind of extreme conditions where Newton's second law does no longer hold, e.g. high velocities or microscopic scales. We know now that at high velocities, special relativity is required to revisit the concepts of space and time and Newton's fundamental principle has to be modified. We also know that in the microscopic world, the concept of trajectory itself becomes meaningless and quantum mechanics tells us how to deal with the evolution of a physical system with an approach which has nothing to do with Newton's second law. These are examples of what Thomas Kühn calls scientific revolutions.

Another limit of a theory can be revealed by access to more accurate experimental devices, which are able to reach a better precision and to prove some discrepancies with the existing theory, calling for a revision of the theory. This can also happen when small discrepancies are not considered as damaging the main theory, since they are believed to probably require just tiny adjustments. But that might not be correct! This happened in the case of Mercury's perihelion advance. The famous 43 arc seconds per century. Barely 2% of a degree per century. Almost nothing! It was known in the nineteenth century already, . . . but was'nt understood before Einstein elaborated his General Theory of Relativity!

▷ 3.3 Notion of falsifiable theory

– Diego: This is why the professor started the course saying that Physics is an experimental science, and as such, is condemned to be called into question. She then said that what she will teach us will, some day, be revisited and maybe *falsified*. This is the word she used. Look, we say that Physics is an exact science and she says that she does'nt teach us the truth!

– Aïssata: Right, *falsified*, this is the word used when an experiment is in such a severe contradiction with a theory that the theory has to be abandoned. Any sensible physical theory must be falsifiable (it does not necessarily mean that it has been falsified yet!), i.e. has to make predictions that experiments can check or disprove. Of course a falsified theory, e.g. Newton's second law which fails at explaining Mercury's perihelion advance, can still be very useful. It still describes accurately Nature in a given domain, at a given scale and as such, although not the best theory available, its simplicity and its historical value, its degree of generality, make it of paramount importance and time must be dedicated to teach this theory, as well as other similar theories which are the pillars of Physics.

– Diego: So, Lagrange equations and the action formalism are a new theory after Newton's dynamics has been falsified I suppose.

– Aïssata: No, I wouldn't say that. This, and other alternative forms like Hamilton formalism, Poisson brackets, Hamilton-Jacobi approach, etc, together usually referred to as analytical mechanics, form a body of theories historically dedicated to dynamics, or mechanics, and rather constitute a *new formulation* of Newton's dynamics. They do not lead to different predictions concerning outcomes of experiments. They just define and manipulate different concepts, but at the end, they lead to the same equations for trajectories. The main gift of analytical mechanics lies in its possible extensions to fields outside of Newton's dynamics, namely relativistic mechanics, even relativistic gravitation, electrodynamics, quantum physics, etc. It is also very well designed to deal with *symmetries* and this is why most of recent theories (say from the second half of twentieth century, even earlier in which concerns unified field theories of gravitation and electrodynamics[17]) are formulated within a Lagrangian approach.

There is something that I would like to add concerning this question of falsifiabilty. *Not all theories are falsifiable.* This is an interesting aspect which is discussed in the famous book of Penrose[18]. There, he gives examples of theories that many physicists take for true, because of simplicity or aesthetic arguments, but which possibly cannot be falsified. Such an example that he discusses is the case of supersymmetry. This is a symmetry between fermions and bosons which has been incorporated in the standard model of fundamental interactions and which predicts that each particle (bosons and fermions), has a so-called superpatner (sfermions or bosinos) of equal mass. The problem is that such superpartners of equal masses have never been discovered in experiments! Theoreticians had to invent hypothetical symmetry breaking mechanisms to explain this absence: superpartners have masses beyond what is experimentally accessible. This strategy nevertheless poses severe problems. If experiments can be done at higher energies in the future and are still negative, the lower limit for the masses of superpartners will be pushed back, but the theory still not falsified. How far this process can survive?

▷ 3.4 An attempt to define scientific truth

– Diego: Is it still science?

– Aïssata: I think that the notion of scientific truth can be a delicate question. There is the idealization of the scientific approach that we learn at school: Science is made of propositions which are checked experimentally. Experiments can be reproduced. The scientific community has rigorous methods and can decide what is true and what is wrong. The scientific discourse is not a question of opinion. There could be opinions in interpretations of theories (we speak about the *orthodox* interpretation of quantum mechanics), but facts are not subject to interpretation (an object in the gravitation field falls down).

In my opinion, these statements have to be moderated and I think that science has to be considered as produced by the social community of scientists with its own codes and regulations, but also its own flaws. I would say that a scientific discourse is not an intangible truth, but a scientific truth, which means that it is valid in a certain domain, at a certain period and is the result of a large consensus within the scientific community. The discourse concerns science, because it is *in principle* falsifiable. Confronted to an experiment, it can be contradicted and abandoned. From this point of view, any scientific

[17] M.-A. Tonnelat, Les théories unitaires de l'électromagnétisme et de la gravitation, Gauthier-Villars, Paris, 1965; A. Lichnerowicz, Théories relativistes de la gravitation et de l'électromagnétisme, Ed. Jacques Gabay, Paris, 2008.

[18] R. Penrose, The Road to Reality: A Complete Guide to the Laws of the Universe, Vintage books, 2005, chap. 34.

discourse corresponds *most of the time* to something which is true in that sense. Of course, scientists do not individually check all the results that they take for true. There is a confidence in the codes of the community. New results appear in papers which are published after they have been accepted by peers. The process is not hundred percent safe, but results can still be discussed by the community after publication and if they appear to be wrong, the common belief of the community evolves, etc. But there are also flaws in the way the community auto-regulates itself. Maybe for prestige reasons, maybe because of conflicts of interest (this is particularly the case when there is a business associated to a science, e.g. drug industry and medicine), intentionally or not, some false results can be published. This should not happen in science, but unfortunately it does. As long as it is the behaviour of a small minority, the whole community still produces a valued science, but in any case this tendency must to be condemned. Above a certain fraction of fake publications, nothing could be trusted anymore! In a sense, this happened during the Covid-19 pandemia. Contradictory results were published on various effects of some molecules, maybe because of insufficient attention to the protocoles, maybe, worse, because of the ego of researchers, relayed by various media. The result was a disaster. It became difficult to find unbiased reliable experts and the door was open to all kinds of conspiracy theories[19].

There is another aspect of social community which may also be connected to esteem issues. This is a mode effect which pushes lots of scientists to work in the same fashionable fields. This is probably also a bias, less dramatic, but still questionable, because it contributes to narrow the fields of study and of knowledge.

Both keep silence for a short moment.

– Aïssata: On these words, I have to prepare my test, this is in two days. But we can meet again tomorrow afternoon if you want. I will be ready I think.

– Diego: Oh! Aïssata, you know everything for your test already. Can't we go on, I was in a very good mood to understand today, . . . and what you just told me is really calling for more thoughts.

– Aïssata: Go to the library, you will be in a nice place, especially made to work and think she says, smiling. You know, this is the building with plenty of books inside, she adds, ironically! She pushes him out and locks the door.

[19] https://www.franceculture.fr/emissions/lsd-la-serie-documentaire/la-grande-aventure-de-la-science-24-publish-or-perish

Introduction to Lagrangian and Hamiltonian formalisms

4. Day 1 – Lagrange formalism

The day after, Diego has his second class in the morning. The topic is on Lagrange equations. This is supposed to be an introductory reminder, since the main topic of the course will deal with fields and not with systems having discrete degrees of freedom. After lunch, Diego meets Aïssata again at her room.

– Diego: Hi Aïssata. We had the second class today. The professor mentioned in more details several concepts, Lagrangian, action, functional etc. Incredible how it looks good to rewrite the equations of Newton mechanics, ...

– Aïssata: ...not only Newtonian mechanics, Aïssata says

– Diego: ...yes, she said that. I think I understand most of the ideas, but if you agree, let us look at the mathematical aspects of variational formalism. I didn't follow the whole derivation of these Euler-Lagrange equations.

Diego shows his lecture notes to Aïssata. They both start reading aloud.

□ 4.1 Lagrangian formalism - a short reminder

Consider a point particle of mass m that we assume, for the sake of simplicity, described by a single degree of freedom denoted as $q(t)$ and called a *generalized coordinate*. The dynamical state of the particle is fully determined by the knowledge of $q(t)$ and its time derivative $\dot{q}(t)$ called the *generalized velocity* and is encoded in a quantity called the *Lagrangian*

$$L(q(t), \dot{q}(t), t). \tag{72}$$

This is a function of the generalized coordinate and velocity, and possibly explicitly of time. The generalized coordinate $q(t)$ may differ from a length (it can be an angle for example) and the dimensions of $\dot{q}(t)$ may therefore differ from those of a velocity. This is the case when one uses curvilinear coordinates.

We are seeking the trajectory followed by the particle between two known end points q_1 and q_2 at times t_1 and t_2. For that purpose, we build a new quantity, the *action* $S[q]$, which is a *functional* of $q(t)$, defined as

$$S[q] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)dt. \tag{73}$$

Hamilton's principle of *least action* states that the *actual trajectory* of the particle is that one which minimizes the action,

$$\frac{\delta S}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \tag{74}$$

This is *the equation of motion* provided that the Lagrangian is properly defined.

For a free particle, the Lagrangian takes the simple form $L = \frac{1}{2}m|\mathbf{v}|^2$. Indeed, the Lagrangian can only depend on $\mathbf{v}$ because of homogeneity of space. But space is also isotropic, so the actual dependence can only be on the modulus of the velocity. For further consistency, it appears that the correct dependence is with the square $|\mathbf{v}|^2$ and we *define* the mass of a particle such that the coefficient of $|\mathbf{v}|^2$ in the Lagrangian of the free particle is $\frac{1}{2}m$. With $L = \frac{1}{2}m|\mathbf{v}|^2$, equation (74) leads to $m\mathbf{v} = \text{const}$, which is consistent with the definition of an inertial frame as one in which a free particle moves at a constant velocity.

In the more general case of a particle subject to some external force $\mathbf{F}$ deriving from a scalar potential energy

$$\mathbf{F} = -\boldsymbol{\nabla} V, \tag{75}$$

one finds that the correct Lagrangian which describes the dynamics through (74) is the following:

$$L = \tfrac{1}{2}m|\mathbf{v}|^2 - V(\mathbf{r}). \tag{76}$$

This case is usually referred to as *conservative system*. There exist more complex situations that we will not describe here.

– Diego: This is just a "reminder" in the case of a single particle with only a one degree of freedom, but extensions to more general cases are obvious our professor said! The thing is that already at this level, I found mysterious the introduction of this quantity, the action, and the way she obtains the form of the Lagrangian is mysterious also.

– Aïssata: This is indeed a strange approach at first sight, but not that difficult you will see. If you're interested in more details, you can get satisfaction in the first volume of the Landau and Lifshitz series which probably inspired your professor.

She looks around her in the small room and eventually finds, under the bed, the book that she was looking for[20]. She opens it and goes through the first pages. Look, this is the first volume of a famous course in theoretical physics. It had an incredible influence all over the world and was translated in many languages. According to Wikipedia, more than a million volumes of the Course were sold by 2005. The Britannica says:

> a multivolume Course of Theoretical Physics, a major learning tool for several generations of research students worldwide.

For those who are fans of citations counts, Landau, more than fiftty years after his death, is still cited almost 6000 times each year in research documents (according to Google Scholar). This is a kind of measure of his prominent influence, still nowadays. You should always refer to this course when you want to deepen some question, at least the volumes 1 (Mechanics), 2 (Classical Theory of Fields), 3 (Quantum Mechanics), 5 (Statistical Physics), and 7 (Theory of Elasticity) which, in my opinion, are still "essential classics" (the volumes 3, 5 and 7 are among the most cited). The others are maybe too specialized, or a bit out of date.

[20] L. Landau. and E. Lifshitz, Mechanics, Butterworth-Heinemann, Oxford, 1976.

This first volume is dedicated to mechanics and you see, instead of starting with the exposition of Newton laws as most of the books in classical mechanics do, Landau and Lifshitz introduce directly the least action principle. This is already at page 2! And at page 1, they define the notion of generalized coordinates, so, for the beginner this is just like a jump in the void! But this is incredibly stimulating as well. The construction of the Lagrangian for the free particle is given at page 5. In fact this is what allows Landau and Lifshitz to define the central notion of inertial frame.

§2. The principle of least action

The most general formulation of the law governing the motion of mechanical systems is the *principle of least action* or *Hamilton's principle*, according to which every mechanical system is characterised by a definite function $L(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s, t)$, or briefly $L(q, \dot{q}, t)$, and the motion of the system is such that a certain condition is satisfied.

Let the system occupy, at the instants t_1 and t_2, positions defined by two sets of values of the co-ordinates, $q^{(1)}$ and $q^{(2)}$. Then the condition is that the system moves between these positions in such a way that the integral

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, \mathrm{d}t \tag{2.1}$$

takes the least possible value.† The function L is called the *Lagrangian* of the system concerned, and the integral (2.1) is called the *action*.

The fact that the Lagrangian contains only q and $\dot{q}$, but not the higher derivatives $\ddot{q}$, $\dddot{q}$, etc., expresses the result already mentioned, that the mechanical state of the system is completely defined when the co-ordinates and velocities are given.

Figure 4. L. Landau. and E. Lifshitz, Mechanics, Butterworth-Heinemann, Oxford, 1976, p. 2

We can now draw some immediate inferences concerning the form of the Lagrangian of a particle, moving freely, in an inertial frame of reference. The homogeneity of space and time implies that the Lagrangian cannot contain explicitly either the radius vector $\mathbf{r}$ of the particle or the time t, i.e. L must be a function of the velocity $\mathbf{v}$ only. Since space is isotropic, the Lagrangian must also be independent of the direction of $\mathbf{v}$, and is therefore a function only of its magnitude, i.e. of $\mathbf{v}^2 = v^2$:

$$L = L(v^2). \tag{3.1}$$

Since the Lagrangian is independent of $\mathbf{r}$, we have $\partial L/\partial \mathbf{r} = 0$, and so Lagrange's equation is†

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) = 0,$$

whence $\partial L/\partial \mathbf{v} =$ constant. Since $\partial L/\partial \mathbf{v}$ is a function of the velocity only, it follows that

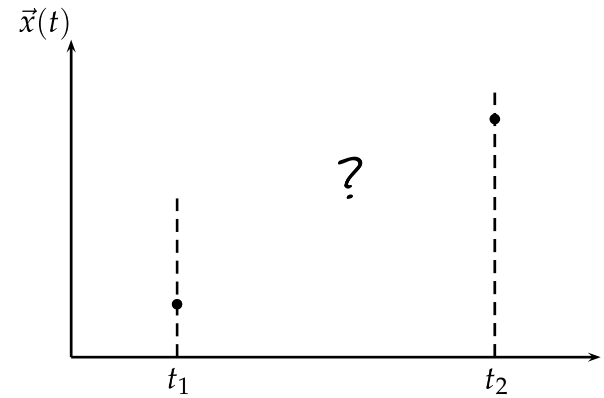
$$\mathbf{v} = \text{constant}. \tag{3.2}$$

Thus we conclude that, in an inertial frame, any free motion takes place with a velocity which is constant in both magnitude and direction. This is the *law of inertia*.

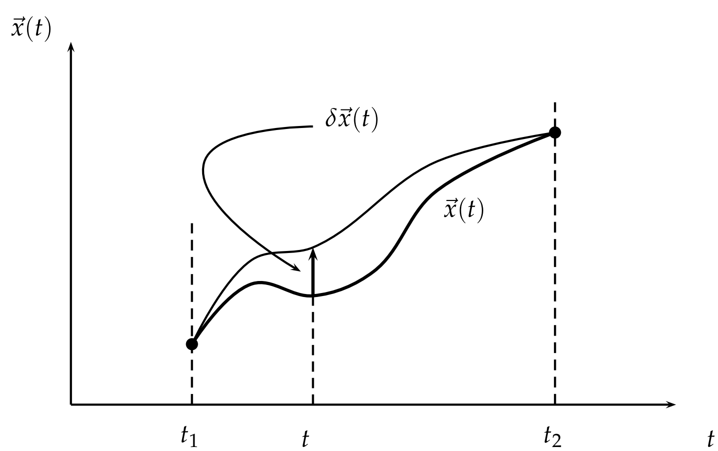
Figure 5. L. Landau. and E. Lifshitz, Mechanics, Butterworth-Heinemann, Oxford, 1976, p. 5

– Aïssata: Ok now, let's redefine things from the beginning, since you found it mysterious. Look! She draws on her blackboard a system of axes with a coordinate vs time and asks about the trajectory between two points that she picks arbitrarily at times denoted as t_1 and t_2.

Assume that we know the lagrangian (whatever it means), what is the trajectory of a particle between two fixed points $\vec{x}(t_1)$ and $\vec{x}(t_2)$?



She explains and completes the drawing while speaking, considering a hypothetical trajectory $\mathbf{x}(t)$, which, still unknown, is supposed to be the actual trajectory followed by the particle. Then she draws another curve between the same two points $\mathbf{x}_1$ and $\mathbf{x}_2$, the difference between the two curves being a function called $\delta\mathbf{x}(t)$, with the property that it vanishes in t_1 and t_2, when the two curves merge. This second trajectory is also hypothetical. We say it is a varied trajectory.



The key concept she says is that of *action*, a central piece in the calculus of variations. This is a quantity defined for the whole path instead of just locally, and she writes a formula in words:

$$\text{Action} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} dt \text{ (Lagrangian)} \tag{77}$$

– Aïssata: Let's postpone for later a discussion on the meaning of the quantity called Lagrangian. We only assume that it exists and is a function of position and velocity. This is a gentle function, we can calculate its derivatives, it behaves properly. So now, we write things in a more mathematical language, but this is the same as above:

Let $\vec{x}(t)$ be a curve in $\mathbb{R}^3$ which goes between points $\vec{x}_1$ and $\vec{x}_2$ at times t_1 and t_2. Let $L(\vec{x}, \dot{\vec{x}})$ be the Lagrangian. The action $S[\vec{x}]$ is defined as

$$\boxed{S[\vec{x}] = \int_{t_1}^{t_2} L(\vec{x}, \dot{\vec{x}}) dt.} \tag{78}$$

Hamilton principle states that the actual trajectory is the one which obeys $\delta S = 0$ to first order, when the trajectory $\vec{x}(t)$ is varied of an amount $\delta\vec{x}(t)$.

Feynman [21] has a nice illustration in the Lecture on Physics Aïssata says,

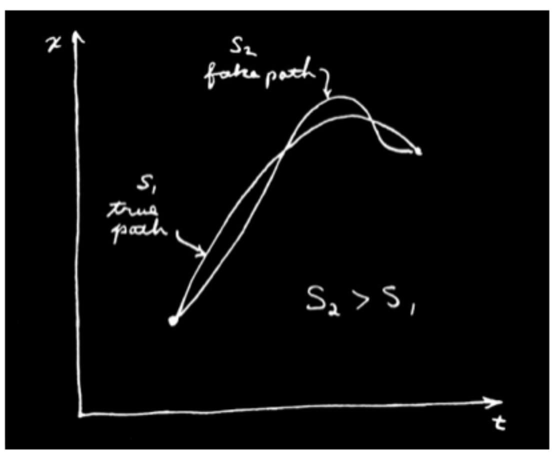


Fig. 19–7.

Figure 6. R.P. Feynman, Lectures on Physics, Vol 2, Addison Wesley, Reading, 1964, sec 19-1.

then she starts doing the variational calculation as it is done in standard texts, or as the professor did on the blackboard during the class.

[21] R.P. Feynman, Lectures on Physics, Vol 2, Addison Wesley, Reading, 1964, sec 19-1.

Expanding the Lagrangian under the integral, one has

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \vec{x}} \delta \vec{x} + \frac{\partial L}{\partial \dot{\vec{x}}} \delta \dot{\vec{x}} \right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \vec{x}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{x}}} \right) \delta \vec{x} dt + \left[\frac{\partial L}{\partial \dot{\vec{x}}} \delta \vec{x} \right]_{t_1}^{t_2} \tag{79}$$

The last term vanishes because the boundaries of the trajectory are kept fixed and, since $\delta S = 0$, the integral is zero. This is true for arbitrary synchronous variation $\delta \vec{x}(t)$ between the fixed boundaries, so the integrand vanishes. This is Euler-Lagrange equation:

$$\boxed{\frac{\delta S}{\delta \vec{x}} = \frac{\partial L}{\partial \vec{x}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{x}}} = 0.} \tag{80}$$

– Aïssata: This is beautiful isn't it?

– Diego: ...mmh, ...Again I have several questions. First, why do you write $S[\mathbf{x}]$ instead of $S(\mathbf{x})$ as we usually do? I am sure that there is some subtlety there. Then, I don't follow already the first line of calculation. To tell the truth, I don't know how you make a derivative w.r.t a vector. I see more or less, but I wouldn't be able to do the calculation myself. Also, there is this mysterious notation $\frac{\delta S}{\delta \mathbf{x}}$ instead of an ordinary derivative, ...

– Aïssata: OK, you are right, I was going fast. Starting from the easiest, your question on the derivative with respect to a vector. This is just a matter of notation. Assume that we use Cartesian coordinates, not to wonder too much with what the gradient is. $\frac{\partial L}{\partial \mathbf{x}}$ is just the usual gradient. She starts writing again.

L being a function of coordinates and velocity components,

$$\frac{\partial L}{\partial \vec{x}} = \vec{\nabla} L \tag{81}$$

which I could write $\vec{\nabla}_{\vec{x}} L$.

– Aïssata: Now, $\frac{\partial L}{\partial \dot{\mathbf{x}}}$ is the equivalent notation for the gradient w.r.t the components of the velocity,

$$\frac{\partial L}{\partial \dot{\vec{x}}} = \vec{\nabla}_{\dot{\vec{x}}} L = \frac{\partial L}{\partial v_i} \vec{u}_i \tag{82}$$

with $\vec{u}_i$ unit vector associated to coordinate x_i and summation over i understood.

– Diego: and I suppose that $v_i = \dot{x}_i$.

– Aïssata: Yes, of course. Now your next point. L is a function of several variables (the functions $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$), but in a sense, S is a function *of functions*. In mathematical language, a function f is a map from a set of numbers onto another set of numbers (these might be two different sets). For example a *wave function* is a map between, say, points in $\mathbb{R}^3$ and complex numbers in $\mathbb{C}$. Aïssata erases the blackboard and writes:

An example of function φ:

$$\begin{aligned} \varphi: \quad \mathbb{R}^3 &\to \mathbb{C} \\ \vec{x} &\mapsto \varphi(\vec{x}). \end{aligned} \tag{83}$$

Of course, the vector $\vec{x}$ stands for x, y, z.

– Aïssata: Note that physicists say "a function $\varphi(\mathbf{x})$" where mathematicians say "a function φ". Indeed $\varphi(\mathbf{x})$ is just the value taken by the function at $\mathbf{x}$, but we know what we mean!

L is a function of six variables, say x, y and z and $\dot{x}$, $\dot{y}$ and $\dot{z}$, or $\mathbf{x}$ and $\dot{\mathbf{x}}$ for short. And possibly of time also but this is not essential here. We usually write $L(q_i, \dot{q}_i)$ where i runs from 1 to 3 for obvious reasons, and the case of N particles can also be treated the same way, with i now running from 1 to $3N$.

▷ 4.3 The notion of functional

– Aïssata: A functional is an extension of that object. Essentially, this is a map between a space of functions with appropriate properties, call E that space, and a set of numbers, e.g. $\mathbb{R}$.

She adds on the blackboard:

A functional F:

$$\begin{aligned} F: \quad E &\to \mathbb{R} \\ g &\mapsto F[g]. \end{aligned} \tag{84}$$

where $g \in E$ is a function.

– Aïssata: We usually use brackets for functionals, like $S[q]$, defined by the integral of some density, not to be confused with the density itself, then denoted with parenthesis. We could also denote the value of the functional as $\langle g, F\rangle$ for example or whatever you find convenient. Other possible examples of functionals are $F[g] = g(0)$, or $F[g] = \int_{-\infty}^{\infty} g(x)dx$. The function g is supposed to have properties such that there is no ill-defined $g(0)$ or $\int_{-\infty}^{\infty} g(x)dx$.

If you want to know more about functionals and functional derivatives in only two pages, you can learn in that book for example. This is not a book dedicated at mathematical questions, but it makes abundant use of functionals and introduces the basic definitions.

She takes a book off the shelf and hands it to him. I have to read my notes once more for my test tomorrow. We will meet as usual. In the afternoon if you want. And she brings him to the door.

Diego had to obey and to leave Aïssata. He went to the library where he was starting to feel comfortable. He had a look at the book that she lent him[22]. "Statistical Field Theory", strange title he thought, there is *classical* field theory, *quantum* field theory, and now *statistical* field theory! He went through the table of contents and the index and didn't

[22] Excerpt from G. Parisi, Statistical Field Theory, Addison-Wesley, Redwood City, 1988, p. 18.

find any entry with "functional" or "functional derivative". No other choice than flipping through the book that Aïssata gave him, but rapidly he found what he was looking for in the appendix of chapter 2, at pages 18 and 19. Diego spent an hour to learn how to manipulate this new concept of functional and went back home.

5. Interlude: Functional derivative

▷ 5.1 Functional derivative

The day after, Diego, being impatient, decides to look for Aïssata directly at the end of her exam, at noon. As usual, she's going out of the lecture hall after most of the students, probably chatting with the professor or other students about the correct answers to the test. She's just incredible, Diego thinks. And I am sure that she will get highest grades, he says, whispering.

– Diego: Hi Aïssata. How are you doing? How was it?

– Aïssata: I think that I answered most of the questions correctly she says, ...

– Diego: What was it?

– Aïssata: Differential geometry she says. Exterior calculus, $p-$forms formulation of electrodynamics and static solutions for **E** and **B** fields in specific examples of curved spacetimes.

– Diego: ... mmh ... sounds interesting.

Diego doesn't know what differential geometry is about but he feels impatient to address the other topic of conversation.

– Diego: You know, I would like to go on the discussion of yesterday if you have some free time. I have studied by myself yesterday and I have made some progress.

– Aïssata: I was expecting to have time for a quick lunch Aïssata says, looking at Diego's disapointing face. OK, let's go to my room, we will nibble something, I think I still have an egg or two, bread, and some fruit maybe.

They leave the campus under the gaze of the other students of the master of physics' class. Just arrived at Aïssata's room, Diego goes to the blackboard, without even giving time to Aïssata to take off her jacket:

– Diego: I have questions on the functional derivative, he says, writing on the blackboard and proud to show what he had read in Parisi's book. He continues : Like the variation df of a function between two neighbouring points x and $x + \delta x$ defines the derivative of the function, say

$$df = f(x + \delta x) - f(x) + O(\delta x^2) = \frac{df}{dx}\delta x, \tag{85}$$

where the derivative is denoted as

$$f'(x) = \frac{df}{dx}, \tag{86}$$

the functional derivative is defined from the variation of a functional δF when its argument (a function now) is varied infinitesimally,

Appendix to Chapter 2

Functionals were introduced in the last century by the great Italian mathematician, Vito Volterra.[15] The basic idea behind them is very simple: in the same way that a function $f(\ )$ associates a number x with another number, i.e., $f(x)$, a functional $F[\ ]$ associates a function $g(\ )$ with a number, i.e., $F[g]$.

Very elementary examples of functionals are the definite integral (e.g., $I[g] \equiv \int_{-\infty}^{+\infty} g(x)\, dx$), the value of the function at a given point (e.g., $F_0[g] = g(0)$), and the maximum of a function in a given interval (e.g., $M[g] = \max_{0 \le x \le 1} [g(x)]$). The functional $S[g] \equiv -\int dx\, g(x) \ln |g(x)|$ is the entropy of $g(\ )$ if the function g is positive definite and normalized to one (i.e., $\int dx\, g(x) = 1$).

Generally speaking we must be careful when dealing with functionals; very often they can take an infinite value even when g seems to be "well behaved" (e.g., $I[g] = \infty$ if $g = 1/(1+x^2)^{1/2}$). If we do not work within well-defined mathematical framework, paradoxical results may be obtained. However, in most cases it is sufficient to consider the functionals as acting only on a restricted space of functions (e.g., $I[g]$ will be well defined when g is a bounded measurable function with fast decrease at infinity, $F_0[g]$ and $M[g]$ when g is a continuous function).

If for simplicity we skip all the mathematical details, we can define the functional derivative of a functional by starting from the Taylor expansion. We suppose that

$$F[g + \varepsilon h] = F[g] + \varepsilon \int dx\, \frac{\delta F}{\delta g(x)}\, h(x) + O(\varepsilon^2)$$

for any reasonable $h(x)$, where $\delta F/\delta g(x)$ is by definition the functional derivative; we stress that the functional derivative is a function of x and a functional of g. In this book all the functional derivatives we need can be computed by inspection. For example, we easily obtain

$$\frac{\delta I}{\delta g(x)} = 1\,, \qquad \frac{\delta}{\delta g(x)} \int_{-\infty}^{\infty} f(g(x))\, dx = f'(g(x))$$

$$\frac{\delta F_0}{\delta g(x)} = \delta(x)\,, \qquad \frac{\delta}{\delta g(x)} [f(g(0))] = f'(g(0))\delta(x)$$

$$\frac{\delta}{\delta g(x)} \int_{-\infty}^{+\infty} \left(\frac{dg}{dy}\right)^2 dy = -2\, \frac{d^2 g(x)}{dx^2}\,.$$

Figure 7. G. Parisi, Statistical Field Theory, Addison-Wesley, Redwood City, 1988, p. 18.

Functional derivative:

$$\delta F = F[g + \delta g] - F[g] \text{ to 1st order} \tag{87}$$

where g is a function, $\delta g = \epsilon h$ is a perturbation with h a function al ϵ a small parameter.

– Diego: Parisi gives a few examples in his book. I was able to do the calculation in the case which seems to be similar to our dynamics of point particle. Let me show you. He writes

As $F[g] = \int_{x_1}^{x_2} f(g(x))dx$ where g is a function of x and f, the density of F, a function of g. Ex

$$F[g+\epsilon h] = \int_{x_1}^{x_2} f(g(x)+\epsilon h(x))dx \simeq \int_{x_1}^{x_2} f(g(x))dx + \epsilon \int_{x_1}^{x_2} h(x) \left.\frac{df}{dg}\right|_x dx \tag{88}$$

where $\left.\frac{df}{dg}\right|_x$ means the function $\frac{df}{dg}$ evaluated at x. U

$$\boxed{\delta F = \epsilon \int_{x_1}^{x_2} \frac{\delta F}{\delta g} h(x)dx} \tag{89}$$

It follow

$$\frac{\delta F}{\delta g} = \left.\frac{df}{dg}\right|_x . \tag{90}$$

– Diego: But setting this to zero is *not* Euler-Lagrange equation (80)! Diego says, making a funny, but disapointed face.

– Aïssata: Of course, it depends on your initial assumption

$$F[g] = \int_{x_1}^{x_2} f(g(x))dx \tag{91}$$

which does not apply in the case of particle dynamics. Imagine another functional, with a different f function which now depends on g and some of its derivatives, say, and she starts writing

►EXERCISE 2 – **Euler-Lagrange equation for** $F[g] = \int_{x_1}^{x_2} f(g(x), g'(x), g''(x))dx$ –

Now $f(g(x), g'(x), g''(x))$ with primes for standard derivatives w.r.t. x (again, this is not particle dynamics). Expand

$$\delta F = \epsilon \int_{x_1}^{x_2} \left(h(x)\frac{\partial f}{\partial g} + h'(x)\frac{\partial f}{\partial g'} + h''(x)\frac{\partial f}{\partial g''} \right) dx. \tag{92}$$

– Aïssata: At this point you cannot identify this equation with (89) where the integral is in terms of $h(x)$, not in terms of its derivatives. So you have to integrate by parts to get an $h(x)$ from the second term, and you have to do it twice for the third term:

$$\begin{aligned}
\delta F &= \epsilon \left(\int_{x_1}^{x_2} h(x)\frac{\partial f}{\partial g}dx + \left[h(x)\frac{\partial f}{\partial g'} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} h(x)\frac{d}{dx}\frac{\partial f}{\partial g'}dx \right. \\
&\quad \left. + \left[h'(x)\frac{\partial f}{\partial g''} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} h'(x)\frac{d}{dx}\frac{\partial f}{\partial g''}dx \right) \\
&= \epsilon \left(\int_{x_1}^{x_2} h(x) \left(\frac{\partial f}{\partial g} - \frac{d}{dx}\frac{\partial f}{\partial g'} \right) dx - \left[h(x)\frac{d}{dx}\frac{\partial f}{\partial g''} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} h(x)\frac{d^2}{dx^2}\frac{\partial f}{\partial g''}dx \right).
\end{aligned}$$

– Aïssata: All integrated terms vanish because of the variation assumption that the end points are fixed, i.e. the function $h(x)$ vanishes at the boundaries, $h(x_1) = h(x_2) = 0$ (we may have to add similar constraints on the derivatives, but I omit this question, not essential here). Then collecting all the integrals, you get

$$\delta F = \epsilon \int_{x_1}^{x_2} h(x) \left(\frac{\partial f}{\partial g} - \frac{d}{dx}\frac{\partial f}{\partial g'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial g''} \right) dx = \epsilon \int_{x_1}^{x_2} \frac{\delta F}{\delta g} h(x) dx \tag{93}$$

thus

$$\frac{\delta F}{\delta g} = \frac{\partial f}{\partial g} - \frac{d}{dx}\frac{\partial f}{\partial g'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial g''} \tag{94}$$

– Aïssata: and you see that the functional derivative is crucially depending on the form of the functional. ◄

– Aïssata: Wikipedia gives you plenty of examples. Functional derivatives obey ordinary rules of differential calculus,

$$\frac{\delta(F + \alpha G)}{\delta g} = \frac{\delta F}{\delta g} + \alpha \frac{\delta G}{\delta g}, \tag{95}$$

$$\frac{\delta(FG)}{\delta g} = \frac{\delta F}{\delta g} G + F \frac{\delta G}{\delta g} \tag{96}$$

where the product is defined in the sense $FG[g] = F[g]G[g]$.

You may also find in the physics literature (again, look e.g. in Wikipedia) a definition in terms of "delta function" like physicists usually do, considering that the variation of the test function $g(x)$ is just a hit somewhere, say at $x = y$. Applying the same calculation as in equation (92) and for the same example, I mean, expanding to first order, you get that

With $\delta F = F[g(x) + \epsilon\delta(x - y)] - F[g(x)]$, one has

$$\delta F = \epsilon \int_{x_1}^{x_2} \delta(x - y) \left(\frac{\partial L}{\partial g} - \frac{d}{dx}\frac{\partial L}{\partial g'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial g''} \right) dx = \epsilon \left(\frac{\partial L}{\partial g} - \frac{d}{dx}\frac{\partial L}{\partial g'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial g''} \right)_y \tag{97}$$

With the definition

$$F[g + \epsilon\delta] = F[g] + \epsilon \int \delta(x - y) \frac{\delta F}{\delta g} dx + O(\epsilon^2) = F + \epsilon \left. \frac{\delta F}{\delta g} \right|_y + O(\epsilon^2) \tag{98}$$

This gives the correct result like in equation (94) and has the advantage of being very similar to first order expansion of a function, compare the last line on the blackboard

$$F[g + \epsilon\delta] = F + \epsilon \left. \frac{\delta F}{\delta g} \right|_y + O(\epsilon^2) \tag{99}$$

with the well known 1st-order expansion of a function in standard notations:

$$f(x + \epsilon) = f + \epsilon \frac{df}{dx} + O(\epsilon^2). \tag{100}$$

– Diego: ..., and in the case of particle dynamics, the relevant functional is the action S, defined by the time integral of the Lagrangian which depends on the coordinates, say $q(t)$ instead of $g(x)$, and on the velocity $\dot{q}(t)$ instead of $g'(x)$. So we are in a simpler scenario than in equation (92), since we have

Sub $F[g] \to S[q]$ and $g(x) \to q(t)$ and $f(g(x), g'(x)) \to L(q(t), \dot{q}(t))$. From direct translation of. equation (94), with only the coordinate it first time derivative $\dot{q}(t)$ as an argument of L, the functional derivative is now

$$\frac{\delta S}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}. \tag{101}$$

Hamilton principle demands that the particle follow the least action principle holds, thus

$$\boxed{\frac{\delta S}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0} \tag{102}$$

which is the Euler-Lagrange equation!

Diego is excited by his success. Yes! he says with a winning gesture. And I understood that in the case of the dynamics of a particle of kinetic energy K and potential energy V, the Lagrangian is given by $L = K - V$.[23]

Euler-Lagrange equation give $\frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q}$ and $\frac{\partial L}{\partial \dot{q}} = \frac{\partial K}{\partial \dot{q}}$ because kinetic energy de $\dot{q}$, $K = \frac{1}{2}m\dot{q}^2$, while potential energy de q, so one get Newton second law $-\frac{\partial V}{\partial q} = m\ddot{q}$.

– Diego: and Euler-Lagrange equations are then the standard equations of motion, because $-\frac{\partial V}{\partial q}$ is the $q-$ component of the force.

[23] Caution, the statement which follows is not always true.

– Aïssata: Take care Diego. This is correct in most cases in Cartesian coordinates, but imagine polar coordinates for example, $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\varphi}\mathbf{u}_\varphi$, so that the kinetic energy has a more complicated expression, and assume a potential energy $V(r, \varphi)$, leading to a Lagrangian

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r, \varphi). \tag{103}$$

Then the Euler-Lagrange equation for $q = r$ and $q = \varphi$ respectively (we call these variables *generalized coordinates*) lead to

$$-\frac{\partial V}{\partial r} = m(\ddot{r} - r\dot{\varphi}^2) \tag{104}$$

$$-\frac{\partial V}{\partial \varphi} = m(r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi}). \tag{105}$$

The first of these is indeed the radial projection $F_r = ma_r$ of Newton's second law $\mathbf{F} = m\mathbf{a}$, but the l.h.s. of the second equation is not the gradient operator, and the r.h.s. is not the angular projection of the acceleration. Instead, this equation, which of course is still correct, is $rF_\varphi = mra_\varphi$ with F_φ the angular component of the force.

We will say more on tensor formalism later, but for now, could you accept the following notation: Instead of $\mathbf{v}$, I temporarily denote $(v^i)_{i=1,2,3}$ an ordinary vector with just three space components in Cartesian coordinates with *upperscripts*, as a column vector:

$$(v^i)_{i=1,2,3} = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \tag{106}$$

With a column vector available, we can act on the right with a matrix, as in standard matrix calculus. In some situations, it may be helpful to act with a matrix, but on the left. For that we need a line vector instead. So I can also introduce the notion of a transposed vector $(v^i)^{\mathrm{T}}_{i=1,2,3}$ when its components are written in line:

$$(v^i)^{\mathrm{T}}_{i=1,2,3} = (v^1\ v^2\ v^3) \tag{107}$$

and that of a *dual* vector or *covector*, with *subscripts* instead of upperscripts, but also as a line vector

$$(v_i)_{i=1,2,3} = (v_1\ v_2\ v_3) = (v^1\ v^2\ v^3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{108}$$

At that point, there is no difference between the transposed and the dual vector, because we use Cartesian coordinates, so $v^i = v_i$. The position of the index tells you whether it is a contravariant vector (upperscript) or a covariant vector (subscript). The link between both types of components is given by the rules of matrix multiplication, $v_i = \delta_{ij}v^j$. In this latter formula, I use the so-called Einstein summation convention on *repeated* indices, here j, which means that the exact meaning is

$$v_i = \delta_{ij}v^j = \sum_{j=1}^{3} \delta_{ij}v^j. \tag{109}$$

According to the convention, the summed indices are dummy indices, one being an upperscript and the other a subscript. We will surely talk about that again later, but I have to emphasize that the appearance of the Kronecker δ_{ij} here is connected to the fact that we are using Cartesian coordinates in an Euclidean space.

With this in mind, this is clear that we can write the velocity squared in the kinetic energy as $|\mathbf{v}|^2 = v_i v^i$. The important thing comes now. The generalization to arbitrary components requires the introduction of a *metric tensor* g_{ij} instead of the identity matrix or Kronecker symbol in equations (108) and (109). I will show you with an example which is very familiar and will help you to understand: the case of polar coordinates.

Define the vector of infinitesimal generalized coordinates in cylindrical coordinates,

$$(dq^i)_{i=1,2,3} = \begin{pmatrix} dq^1 \\ dq^2 \\ dq^3 \end{pmatrix} = \begin{pmatrix} dr \\ d\varphi \\ dz \end{pmatrix}. \tag{110}$$

As you know, the distance squared between two neighbouring points writes in cylindrical coordinates as

$$dl^2 = dr^2 + r^2 d\varphi^2 + dz^2 \tag{111}$$

as given by Pythgoras theorem, and it corresponds to a scalar product (like what we have written for the velocity squared),

$$dl^2 = dq_i dq^i \tag{112}$$

which demands that the associated covector is

$$(dq_i)_{i=1,2,3} = (dq_1\ dq_2\ dq_3) = (dr\ d\varphi\ dz) \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{113}$$

or

$$dq_i = g_{ij} dq^j \tag{114}$$

with the g_{ij} being the components of the metric tensor matrix $(g_{ij})_{i,j=1,2,3}$ in (113). We can still write the line element squared as

$$dl^2 = g_{ij} dq^i dq^j, \tag{115}$$

or, in explicit matrix form,

$$dl^2 = (dr\ d\varphi\ dz) \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dr \\ d\varphi \\ dz \end{pmatrix}. \tag{116}$$

Although we are still in an Euclidean space, a non trivial metric tensor appears due to our use of local coordinates. Here the conventional notation for the object $(g_{ij})_{i,j=1,2,3}$ with two subscript indices is convenient w.r.t. Einstein summation convention. This is called a second rank covariant tensor. I know that I haven't told you yet what a tensor is! Most of the time, physicists denote vectors or tensors only by their components. For example you will see very often "Let x^i be a vector" or "Let x_i be a covector" instead of "Let $(x^i)_{i=1,2,3}$ be a vector" or "Let $(x_i)_{i=1,2,3}$ be a covector". Similarly you will see often "Let w^{ij} be a second rank contravariant tensor" instead of "Let $(w^{ij})_{i,j=1,2,3}$ be a second rank contravariant tensor". This is like the way we, physicists, denote functions or functionals, we do not confuse between a vector and its components, we know what we mean and we go straight to the target!

So now you see that in the example in (103), when you use non Cartesian coordinates, the kinetic energy contains *metric factors* which are not visible in Cartesian coordinates because $g_{ij} = \delta_{ij}$ there. More precisely in polar coordinates,

$$K = \tfrac{1}{2}m\dot{q}_i\dot{q}^i = \tfrac{1}{2}mg_{ij}\dot{q}^i\dot{q}^j = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) \tag{117}$$

instead of your claim that $K = \frac{1}{2}m\dot{q}^2$! These metric factors are responsible for the fact that the radial Euler-Lagrange equation is indeed the radial component of Newton's second law while the angular one has an extra r factor. This is a very simple case, but there are situations where the metric tensor has many more non zero components (even possibly in some non standard manifolds, non vanishing non diagonal terms).

– Diego: I understand. This is why the generalized coordinates don't necessarily have the dimensions of a length! The dimension of the components of the metric tensor play their role.

– Aïssata: Correct. But the most important thing is that, contrary to Newton equations, the form of the equations of motion in Lagrange formalism is the same in arbitrary generalized coordinates, say $q'_i = f(q_1, q_2, \dots)$ (we call them holonomic coordinates). This means that if we write Lagrange equation with the q^i's coordinates,

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0, \tag{118}$$

then, written in terms of the ${q'}^i$'s, they are just the same:

$$\frac{\partial L}{\partial {q'}^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}'^i} = 0. \tag{119}$$

Let us take the example of polar coordinates again for an illustration. Newton second law in two dimensions

$$m\frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla}V \tag{120}$$

written in Cartesian coordinates with $\mathbf{v} = \dot{x}\mathbf{u}_x + \dot{y}\mathbf{u}_y$ and $\boldsymbol{\nabla}V = \frac{\partial V}{\partial x}\mathbf{u}_x + \frac{\partial V}{\partial y}\mathbf{u}_y$ leads to

$$m\frac{d\dot{x}}{dt} = -\frac{\partial V}{\partial x}, \tag{121}$$

$$m\frac{d\dot{y}}{dt} = -\frac{\partial V}{\partial y}. \tag{122}$$

Both equations are of the form $m\frac{d\dot{q}^i}{dt} = -\frac{\partial V}{\partial q^i}$ with $q^i = x, y$. But in polar coordinates with $x = r\cos\varphi$ and $y = r\sin\varphi$, we have now $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\varphi}\mathbf{u}_\varphi$ and $\boldsymbol{\nabla}V = \frac{\partial V}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial V}{\partial \varphi}\mathbf{u}_\varphi$ leading to the equations of motion in components form

$$m\left(\frac{d\dot{r}}{dt} - 2r\dot{\varphi}^2\right) = -\frac{\partial V}{\partial r}, \tag{123}$$

$$m\left(2\dot{r}\dot{\varphi} + r\frac{d(\dot{r}\varphi)}{dt}\right) = -\frac{1}{r}\frac{\partial V}{\partial \varphi}. \tag{124}$$

These equations *do not* have the form $m\frac{d\dot{q}^i}{dt} = -\frac{\partial V}{\partial q^i}$ with $q^i = r, \varphi$.

On the other hand, with the Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y)$, the equations of motion (118) automatically lead to (121) and (122) when $q^i = x, y$ and *the same equation of motion* (118) applied to (103) lead, as we have shown earlier to (104) and (105), which are indeed the same as (123) and (124). This is true more generally than in this example of course, and this is one of the main advantages of Lagrange formalism.

▷ 5.4 Charged particle in an electromagnetic field

– Aïssata: There is another important physical situation where your claim on the form of the Lagrangian is not correct. There are indeed cases where the "potential energy" does not depend only on the coordinates. The case of a charged particle (of electric charge e) in an electromagnetic field is a notable exception. Indeed in this case you can convince yourself that in order to recover the equation of motion in the presence of a Lorentz force

$$e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = m\mathbf{a}, \tag{125}$$

you need to start from a Lagrangian

$$L = \frac{1}{2}m|\mathbf{v}|^2 - e(\phi(\mathbf{r}, t) - \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{v}). \tag{126}$$

– Diego: You say I can convince myself, . . . this is not obvious to me. Could you elaborate a bit more?

►EXERCISE 3 – **Equation of motion for a charged particle in a EM field** –

– Aïssata: Okay. The question is to apply Euler-Lagrange equations to the above Lagrangian (126). The cross product in the Lorentz force is the difficult part. This is probably easier to write things in terms of components.

With this in mind, this is clear that we can write the velocity squared in the kinetic energy as $|\mathbf{v}|^2 = v_j v^j$ and that the scalar product in the last term in the Lagrangian is $\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{v} = A_j(x^i, t)v^j$. We can then write

$$L = \frac{1}{2}mv_j v^j - e\phi(x^i, t) + eA_j(x^i, t)v^j. \tag{127}$$

The derivatives are then

$$\frac{\partial L}{\partial x^i} = -e\frac{\partial \phi}{\partial x^i} + e\frac{\partial A_j}{\partial x^i}v^j, \tag{128}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = m\frac{dv_i}{dt} + e\frac{dA_i}{dt}. \tag{129}$$

The total derivative is also $\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x^j}v^j$ and the equation of motion is thus

$$m\frac{dv_i}{dt} = -e\left(\frac{\partial \phi}{\partial x^i} + \frac{\partial A_i}{\partial t}\right) + e\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)v^j. \tag{130}$$

This is exactly the equation of motion (125).

– Diego: Is it clear that the last term is the cross product of velocity by the magnetic field? There is something which looks like the curl of $\mathbf{A}$, indeed, but the cross product with $\mathbf{v}$ is not so obvious.

– Aïssata: You can prove it, again in component form. The expected $i-$component is $e\mathbf{v} \times \mathbf{B}|_i = e\epsilon_{ijk}v^j B^k$ with the Levi-Civita symbol ϵ_{ijk} which equals $+1$ if ijk are 123 up to an even number of permutations, it equals -1 for an odd number of permutations, and it equals 0 if two or three indices coincide. Again, with Cartesian coordinates in Euclidean space, the position of indices as upperscripts or subscripts is not essential (but this is not true in general if we work with the Levi-Civita *tensor*!), but we still use the Einstein summation convention, so up or down position is important for us. With the Levi-Civita symbol, a curl is given by $B^k = \epsilon^{klm}\frac{\partial A_m}{\partial x^l} = \epsilon^{klm}\partial_l A_m$. So now you just have to write things, maths do the rest:

$$
\begin{aligned}
e\mathbf{v}\times\mathbf{B}|_i &= e\epsilon_{ijk}\epsilon^{klm}\frac{\partial A_m}{\partial x^l}v^j && (131)\\
&= e\left(\frac{\partial A_j}{\partial x^i}-\frac{\partial A_i}{\partial x^j}\right)v^j && (132)
\end{aligned}
$$

where I have used the property

$$\epsilon_{ijk}\epsilon^{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{133}$$

and you recognize our formula (130)

$$m\left.\frac{d\mathbf{v}}{dt}\right|_i = e(-\boldsymbol{\nabla}\phi - \partial_t\mathbf{A})_i + e\mathbf{v}\times\mathbf{B}|_i. \tag{134}$$

as the $i-$component of (125) ◄

▷ 5.5 Dynamic momentum and canonical momentum

– Diego: Yes, I recognize the form of the electric field component in the first term at the r.h.s. I have the feeling that what you just explained has something to do with the difference between canonical momentum and dynamic momentum. Am I right?

– Aïssata: This is also a good question, and this is the right place to ask about it indeed. First, what do you mean by *dynamic momentum*?

– Diego: Our professor defined it as the quantity that we usually denote as $\mathbf{p} = m\mathbf{v}$, which enables to write Newton's second law as

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{135}$$

or the kinetic energy as

$$K = \frac{|\mathbf{p}|^2}{2m}. \tag{136}$$

She said that it can also be called *kinematic momentum*, *mechanical momentum* or simply the *linear momentum*.

– Aïssata: And how do you define the *canonical momentum*?

– Diego: *She* defined the canonical momentum from the Lagrangian,

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}. \tag{137}$$

We still ignore why this definition and will learn more about this with the Hamiltonian formalism she said. What worries me is that she said that the two definitions may not coincide.

– Aïssata: OK, so let us forget the origin of this latter definition for a while, but accept it as it is. Assuming a single particle and $L = \frac{1}{2}m|\mathbf{v}|^2 - V(\mathbf{r})$, you get that

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} \tag{138}$$

so this is indeed the same as the dynamical momentum. If now you look at components in non Cartesian coordinates, the situation becomes more tricky. Consider the Langrangian (103) in polar coordinates. We have that

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \tag{139}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}. \tag{140}$$

Indeed, the radial component p_r coincides with the radial component of the linear momentum, but the angular component p_φ on the other hand doesn't have the dimensions of a linear momentum. Instead, it corresponds to the magnitude of the *angular momentum* $\mathbf{L} = m\mathbf{r} \times \mathbf{v} = mr\mathbf{u}_r \times (\dot{r}\mathbf{u}_r + r\dot{\varphi}\mathbf{u}_\varphi) = mr^2\dot{\varphi}\mathbf{u}_z$.

Now, consider a charged particle in an EM field with Lagrangian (126), the canonical momentum is now

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + e\mathbf{A}(\mathbf{r}, t), \tag{141}$$

which *is not* the dynamic momentum. To make it clear, most authors usually introduce the notation $\boldsymbol{\pi} = m\mathbf{v}$ for the dynamic momentum and the canonical momentum is thus

$$\mathbf{p} = \boldsymbol{\pi} + e\mathbf{A}(\mathbf{r}, t). \tag{142}$$

$\mathbf{p}$ is sometimes called *conjugate momentum*, or *generalized momentum*. It is linked to conservation laws as your professor will tell you.

– Diego: You mean that the dynamic momentum is not conserved? I thought it was conserved, as in scattering experiments!

– Aïssata: You are right, it is *generally* conserved, but not always! In particular in the presence of a magnetic field, there are some subtleties. Feynman discusses them, but this is for later.

▷ 5.6 Damped harmonic oscillator

►EXERCISE 4 – **Euler-Lagrange equation for the damped harmonic oscillator** –

– Aïssata: You can also worry about problems in which there is a source of dissipation. This is a hard problem and there is no general answer. But for example the case of the damped oscillator can be solved if you make the correct guess for the Lagrangian. Consider a one-dimensional oscillator with friction. The equation of motion is, with obvious notations,

$$m\ddot{x}(t) = -kx(t) - \eta\dot{x}(t). \tag{143}$$

The solution $x(t)$ is damped by a factor of $e^{-\eta t/2m}$ and if you just form the Lagrangian with $K - V$ (there is no potential energy associated to friction), you get something which vanishes exponentially fast at long times. Instead of that, you can define a modified Lagrangian function, amplified by the inverse of the square of the damping exponential, and for which I keep, maybe improperly, the notation L,

$$L = \left(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2\right)e^{\eta t/m} \tag{144}$$

The Euler-Lagrange equation then leads to the correct equation of motion,

$$(m\ddot{x} + \eta\dot{x} + kx)e^{\eta/t} = 0 \tag{145}$$

up to an exponential multiplicative factor which doesn't play any role there. ◄

– Diego: As I understand, you are seeking for the correct Lagrangian in terms of which you recover the known equations of motion. But what if you ignore these equations of motion and you want to build a new theory?

– Aïssata: This is an excellent question Diego. It appears that very often, symmetry considerations (Lorentz symmetry, gauge symmetry) give you enough informations to constrain the form of the Lagrangian, or at least to constrain families of Lagrangians, but you will see this when you will be more advanced in the course.

Time has passed, they have forgotten the eggs and bread, concentrated as they were on their discussion. Diego had enough to think about for a first course and, after warmly thanking Aïssata, he decides to go back to his place.

6. Day 2 – Hamilton formalism

After Lagrange formalism, Diego attented his third course, on Hamilton formalism. Before visiting his friend Aïssata, he decides to study the notes given by his professor. After all, Aïssata and his professor advised many times to work *before* he comes with questions, because when one finds an answer by oneself, it is properly recorded in the memory and it is a kind of training for intellectual skills.

When she introduced this new section, the professor said that although most of field theory is formulated on the basis of Lagrangian theory, there are branches in Physics where one is more used to the manipulation of Hamiltonians. A noticeable example is Quantum Mechanics. Most of the time students encounter QM first formulated in the Hamiltonian formalism. This is also amazingly elegant, she said, and it brings on the front a few subtleties with functional derivatives when you apply it to field theory.

□ 6.1 Hamiltonian formalism

Let $L(q^i, \dot{q}^i)$ be the Lagrange function for a mechanical system. The Euler-Lagrange equations are

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0. \tag{146}$$

Instead of the variables q^i and $\dot{q}^i$, we are now interested in writing the dynamics in terms of new variables, the old q^i's and the associated *canonical momenta*

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{147}$$

Note the position of indices, the quantity L is a scalar, hence the derivative w.r.t a contravariant coordinate leads to a covariant quantity.

In order to change the variables as indicated, we introduce a *Legendre transformation*

$$X \equiv L(q^i, \dot{q}^i) - \sum_i p_i \dot{q}^i \tag{148}$$

where we emphasize the summation, writing it explicitly. We require the independence of the new function X in terms of the $\dot{q}^i$'s,

$$\frac{\partial X}{\partial \dot{q}^i} = 0 = \frac{\partial L}{\partial \dot{q}^i} - p_i. \tag{149}$$

Therefore (147) is recovered and the new function, X, has the right dependence, $X = X(q^i, p_i)$.

In the case of a conservative system where $L = \frac{1}{2} m g_{ij} \dot{q}^i \dot{q}^i - V(q^i)$, the canonical momentum is

$$p_i = m g_{ij} \dot{q}^j. \tag{150}$$

In Cartesian coordinates it equals the usual *mechanical momentum*. Then, the function X takes the simple form $X = -K - V$ and one usually defines, instead of X, the *Hamiltonian* $H(p_i, q^i) = -X$ which takes in this case the meaning of the total energy of the system:

$$H(p_i, q^i) = \sum_i p_i \dot{q}^i - L(q^i, \dot{q}^i). \tag{151}$$

The Hamilton equations follow from Hamilton principle,

$$\begin{aligned}
S[p_i, q^i] &= \int dt (p_i \dot{q}^i - H(p_i, q^i)), && (152)\\
\frac{\delta S}{\delta p_j} &= \frac{\partial}{\partial p_j}(p_i \dot{q}^i - H(p_i, q^i)) &&\\
&= \dot{q}^j - \frac{\partial H}{\partial p_j} = 0, && (153)\\
\frac{\delta S}{\delta q^j} &= \frac{\partial}{\partial q^j}(p_i \dot{q}^i - H(p_i, q^i)) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}^j}(p_i \dot{q}^i - H(p_i, q^i)) &&\\
&= -\frac{\partial H}{\partial q^j} - \frac{d}{dt} p_j = 0. && (154)
\end{aligned}$$

It yields

$$\begin{aligned}
\dot{q}^i &= \frac{\partial H}{\partial p_i}, && (155)\\
\dot{p}_i &= -\frac{\partial H}{\partial q^i}. && (156)
\end{aligned}$$

This is a set of coupled first order differential equations while Euler-Lagrange equations are second-order differential equations. They are usually called canonical equations of motion. They obviously have the same content as the Euler-Lagrange equations.

A third equation also follows for the time variation,

$$\frac{dH}{dt} = \sum_i \left(p_i \ddot{q}^i + \dot{p}_i \dot{q}^i - \frac{\partial L}{\partial q^i} \dot{q}^i - \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i \right) - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} \tag{157}$$

and it has to do with the conservation of the energy as we will see later.

After having studied his course, Diego had an appointment to have lunch with Aïssata. They meet at the university restaurant and start discussing about the political tensions in the country and the police violence against street protests which is increasing worryingly.

Aïssata is a concerned person, maybe partly because of her life trajectory, maybe just because she dreams about a world where people would have equal rights! The Police she says is not the institution it pretends to be[24]. It is supposed to protect the population and for law enforcement. Surely, it does this latter point, but instead of a protection of the population, it protects the power, whatever it is, and helps people who have the power to keep it. If these people act against the interest of the majority, so does the police. Diego mentions a book [25] from an American writer, Edward Abbey, who opens his novel with a harsch description of the role of the Police in the American society.

Prologue: The Aftermath

When a new bridge between two sovereign states of the United States has been completed, it is time for speech. For flags, bands and electronically amplified techno-industrial rhetoric. For the public address.

The people are waiting. The bridge, bedecked with bunting, streamers and Day-Glo banners, is ready. All wait for the official opening, the final oration, the slash of ribbon, the advancing limousines. No matter that in actual fact the bridge has already known heavy commercial use for six months.

Long files of automobiles stand at the approaches, strung out for a mile to the north and south and monitored by state police on motorcycles, sullen, heavy men creaking with leather, stiff in riot helmet, badge, gun, Mace, club, radio. The proud tough sensitive flunkies of the rich and powerful. Armed and dangerous.

Figure 8. E. Abbey, The Monkey Wrench Gang, Dream Garden Press, 1985, p. 11.

After a while they move again to physics.

– Diego: We have elaborated Hamilton formalism in the last course. This looks nice, but I don't really see why this is different from Lagrange formalism. It looks like a reformulation essentially.

[24] https://lundi.am/Mobilisations-contre-les-violences-policieres.

[25] E. Abbey, The Monkey Wrench Gang, Dream Garden Press, 1985.

– Aïssata: This is correct, but there can be some advantages. While Lagrange formalism is nicely adapted to the analysis of conserved quantities, the Hamiltonian itself is a physical quantity which we usually manipulate already in undergraduate courses (at the least as a total energy), so even if we are not used to it, we know the concept of total energy since the first course on classical mechanics. And remember, *understanding* is a bit *being accustumed to*.

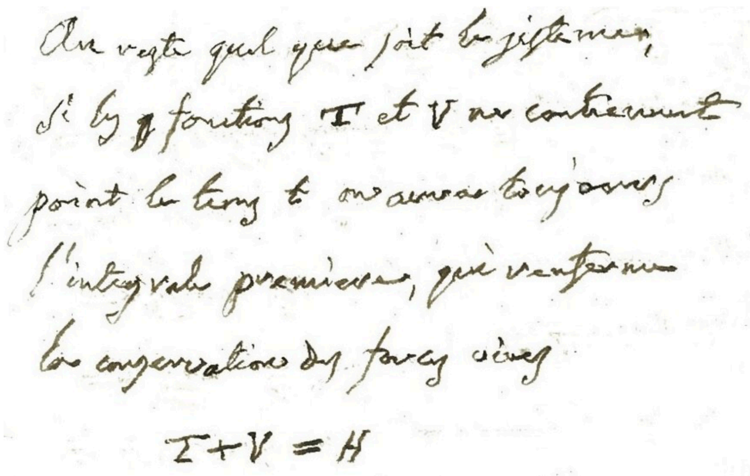

Au reste quel que soit le systême,
si les fonctions T et V ne contiennent
point le tems t on aura toujours
l'integrale premiere, qui renferme
la conservation des forces vives
$T+V=H$

Figure 9. A 1813 handwritten manuscript by J.-L. Lagrange, found in the 1815 edition of "La mécanique analytique". From J. Perez, La clé du mystère de la lettre H, https://images.math.cnrs.fr/La-cle-du-mystere-de-la-lettre-H.html

As a side comment, a pleasant story is that the notation H is due to, ... Lagrange who probably referred to Huygens![26] Hamilton coined many words, quaternions, scalars, tensors, but not Hamiltonian! The notation $H = T + V$ appears for the first time in a handwritten manuscript in 1813 by Lagrange, when Hamilton was eight years old. But Hamilton later kept the notations of Lagrange, H, T (for the kinetic energy), V for the potential energy.

Coming back to our main concern, there are also more obvious links to quantum mechanics when one starts from Hamilton formalism. You should have a look at Goldstein[27]. As far as I remember, he gives various reasons in favor of the Hamilton formalism.

After lunch, they decide to go to the library in the small working room that Aïssata reserves whenever possible. This is a room equipped with a computer connection and a blackboard. But before that, they borrow the Goldstein and Aïssata goes through the table of contents to find the convenient paragraph. She shows to Diego the relevant part of Goldstein's book.

– Diego: Could we now spend a bit of time to revisit the examples that you had used to illustrate the Lagrangian formalism?

[26]P. Iglesias, Histoire dH. In Symmétries et Moments, (Annexe B), Hermann Editeur, Paris, 2000; G.M. Tuynman, The Hamiltonian?. In: Kielanowski P., Bieliavsky P., Odesskii A., Odzijewicz A., Schlichenmaier M., Voronov T. (eds) Geometric Methods in Physics. Trends in Mathematics. Birkhäuser, Cham, 2014.

[27]H. Goldstein, Classical Mechanics (second editiopn), Addison Wesley, 1980, p. 239.

CHAPTER 8

The Hamilton Equations of Motion

The Lagrangian formulation of mechanics was developed largely in the first two chapters, and most of the subsequent discussion has been in the nature of application, but still within the framework of the Lagrangian procedure. In this chapter we resume the formal development of mechanics, turning our attention to an alternative statement of the structure of the theory known as the Hamiltonian formulation. Nothing new is added to the physics involved; we simply gain another (and more powerful) method of working with the physical principles already established. The Hamiltonian methods are not particularly superior to Lagrangian techniques for the direct solution of mechanical problems. Rather, the usefulness of the Hamiltonian viewpoint lies in providing a framework for theoretical extensions in many areas of physics. Within classical mechanics it forms the basis for further developments, such as Hamilton–Jacobi Theory and perturbation approaches. Outside classical mechanics, the Hamiltonian formulation provides much of the language with which present day statistical mechanics and quantum mechanics is constructed. We shall assume in the following chapters that the mechanical systems are holonomic and that the forces are monogenic, that is, derived either from a potential dependent on position only, or from velocity-dependent generalized potentials of the type discussed in Section 1–5.

Figure 10. H. Goldstein, Classical Mechanics (second edition), Addison Wesley, 1980, p. 239

▷ 6.2 Motion in polar coordinates

– Aïssata: Ok, Aïssata says. Ok, let's start with the simple 1−particle in polar coordinates. We had written the Lagrangian (103)

$$L(r,\varphi,\dot r,\dot\varphi) = \tfrac{1}{2}m(\dot r^2 + r^2\dot\varphi^2) - V(r,\varphi). \tag{158}$$

First calculate the canonical momenta: $p_r = \frac{\partial L}{\partial \dot r} = m\dot r$ and $p_\varphi = \frac{\partial L}{\partial \dot\varphi} = mr^2\dot\varphi$ which we had already in (139) and (140).

Then, build the Hamiltonian (eliminate the $\dot q^i$'s for the p_i's)

$$\begin{aligned} H(r,\varphi,p_r,p_\varphi) &= p_r\left(\frac{p_r}{m}\right) + p_\varphi\left(\frac{p_\varphi}{mr^2}\right) - \tfrac{1}{2}m\left[\left(\frac{p_r}{m}\right)^2 + \left(\frac{p_\varphi}{mr^2}\right)^2\right] + V(r,\varphi) \\ &= \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r,\varphi). \end{aligned} \tag{159}$$

Now the Hamilton equations of motion follow from (155) and (156). First we use (155) and get $\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot r$ and $\frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2} = \dot\varphi$ which, again, are (139) and (140). Then with (156), we form

$$\frac{\partial H}{\partial r} = -\frac{p_\varphi^2}{mr^3} + \frac{\partial V}{\partial r} = -\dot{p}_r, \tag{160}$$

$$\frac{\partial H}{\partial \varphi} = \frac{\partial V}{\partial \varphi} = -\dot{p}_\varphi \tag{161}$$

If you want to show the equivalence with the standard equation of motion, you can eliminate p_r for $\dot{r}$ and p_φ for $\dot{\varphi}$ to get

$$m(\ddot{r} - r\dot{\varphi}^2) = -\frac{\partial V}{\partial r}, \tag{162}$$

$$m(2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi}) = -\frac{\partial V}{\partial \varphi}, \tag{163}$$

which are the equations of motion already found in (104) and (105).

– Diego: So nothing really new!

– Aïssata: Yes, nothing really new, this is another formalism to treat the same problem, as we have discussed earlier. But the spirit of Hamilton equations is not to solve the second order equation of motion, but rather to solve the first order one in p_i's, then substitute p_i's in terms of $\dot{q}_i$'s if you need.

▷ **6.3 Charged particle in an electromagnetic field**

►EXERCISE 5 – **Hamilton equations for a charged particle in an electromagnetic field** –

– Aïssata: Now the case of the particle in a magnetic field. Let's write again the Lagrangian (127),

$$L = \tfrac{1}{2}m|\mathbf{v}|^2 - e(\phi - \mathbf{A}\cdot\mathbf{v}) \tag{164}$$

in component form in Cartesian coordinates

$$L = \tfrac{1}{2}mv_iv^i - e\phi + eA_iv^i \tag{165}$$

where we omit the explicite dependence of the potentials on x^i and t. We define the canonical momenta

$$p_i = \frac{\partial L}{\partial v^i} = mv_i + eA_i, \tag{166}$$

$$\text{or} \quad v_i = \frac{p_i - eA_i}{m}. \tag{167}$$

This is the component form of (141).

Then, we build the Hamiltonian along the same lines as before,

$$\begin{aligned} H &= p_iv^i - L \\ &= p_i\left(\frac{p^i - eA^i}{m}\right) - \tfrac{1}{2}m\left(\frac{p_i - eA_i}{m}\right)\left(\frac{p^i - eA^i}{m}\right) \\ &\quad +e\phi - eA_i\left(\frac{p^i - eA^i}{m}\right) \\ &= \frac{1}{2m}(p_i - eA_i)(p^i - eA^i) + e\phi \end{aligned} \tag{168}$$

where there are some simplifications when you expand the intermediate lines, but you easily get the last line.

The first set of Hamilton equations again gives back the definition of the canonical momentum,

$$\frac{\partial H}{\partial p_i} = \frac{1}{m}(p^i - eA^i) = v^i \tag{169}$$

and the second set corresponds to the equation of motion,

$$\frac{\partial H}{\partial x^i} = \frac{1}{m}\left(-e\frac{\partial A_j}{\partial x^i}\right)(p^j - eA^j) + e\frac{\partial \phi}{\partial x^i} = -\dot{p}_i \tag{170}$$

with

$$\dot{p}_i = m\dot{v}_i + e\frac{dA_i}{dt} = m\dot{v}_i + e\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x^j}v^j\right). \tag{171}$$

Collecting all terms together, we recover (130):

$$m\dot{v}_i = -e\frac{\partial \phi}{\partial x^i} - e\frac{\partial A_i}{\partial t} + e\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)v^j. \tag{172}$$

– Diego: In each case, the first set of equations, $\frac{\partial H}{\partial p} = \dot{q}$, is a kind of definition of the canonical momentum, like $\frac{\partial L}{\partial \dot{q}} = p$. ◄

– Aïssata: Yes, this is correct. In some cases, you build the Hamiltonian directly, without going through the Lagrangian first. Remember the meaning of total energy which is often something already known. So you need to be able to have the canonical momentum without L.

▷ **6.4 Damped harmonic oscillator**

►EXERCISE 6 – **Hamilton equations for the damped harmonic oscillator** –

– Aïssata: The third example that we had considered was that of a damped harmonic oscillator with Lagrangian (144)

$$L = \left(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2\right)e^{\eta t/m} \tag{173}$$

from which one deduces the canonical momentum

$$p = m\dot{x}e^{\eta t/m}, \quad \text{or} \quad \dot{x} = \frac{p}{m}e^{-\eta t/m}. \tag{174}$$

Note that the solution shows that $\dot{x}$ decays exponentially with time while p remains essentially constant. This is another clear distinction between canonical momentum p and dynamical momentum $m\dot{x}$. The Hamiltonian follows

$$H = \frac{p^2}{2m}e^{-\eta t/m} + \tfrac{1}{2}kx^2e^{\eta t/m}. \tag{175}$$

Although the signs in the exponentials may appear surprising at first glance, both terms are equally exponentially decaying. The first Hamilton equation, as usually, gives back an already known relation,

$$\frac{\partial H}{\partial p} = \frac{p}{m}e^{-\eta t/m} = \dot{x} \tag{176}$$

while the second leads to

$$\frac{\partial H}{\partial x} = kxe^{\eta t/m} = -\frac{d}{dt}p = -(m\ddot{x} + \eta\dot{x})e^{\eta t/m} \tag{177}$$

$$\text{or} \quad (m\ddot{x} + \eta\dot{x} + kx)e^{\eta/t} = 0. \tag{178}$$

which is the equation of motion (145). ◄

Happy with these new things that he learnt from his friend, Diego left the library and Aïssata. He had his Judo training in the evening and wanted to have a small snack and time to drink enough water to prevent from cramps.

7. Day 3 – Lagrangian approach to scalar fields

The day after, Aïssata and Diego meet at the library, their meeting place "by default". This is where they find each other when they haven't agreed on an appointment before.

They both have a day free of classes and Diego brings the content of the next course that he attended. It is now question of fields and of the action principle revisited in the case of continuous fields rather that discrete degrees of freedom.

☐ 7.1 From discrete to continuum Lagrangian formulation

Consider a 1d chain of N identical masses m bounded by identical springs of stiffness k. At rest, the distance between masses is denoted as a and the deviation from rest position for the n^{th} mass is $\phi_n(t)$ (see Figure 11).

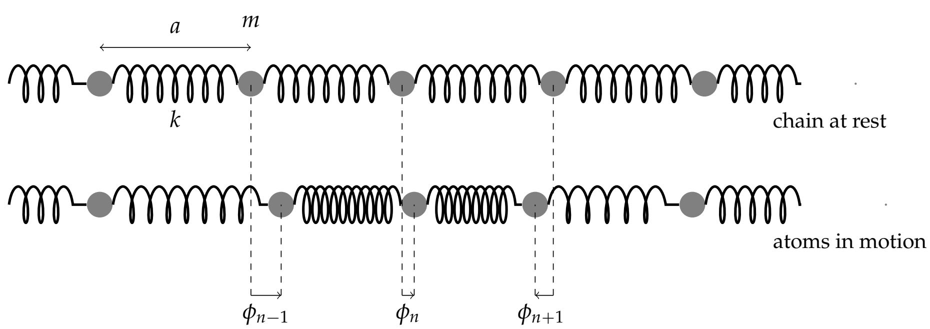


Figure 11. The one-dimensional chain of atoms.

The total kinetic energy is

$$K = \tfrac{1}{2} m \sum_n \dot{\phi}_n^2(t) \tag{179}$$

while the potential energy, neglecting boundary terms (we could either choose periodic boundary conditions of fixed BC's if needed) reads as

$$V = \tfrac{1}{2} k \sum_n (\phi_{n+1}(t) - \phi_n(t))^2 \tag{180}$$

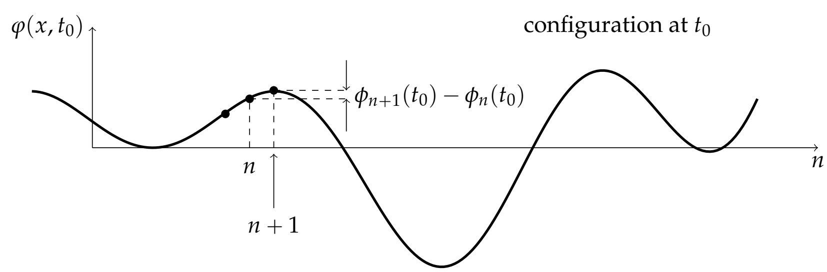


Figure 12. A typical configuration of the field $\varphi(x, t)$ at fixed t_0.

so that the Lagrangian of the whole chain is given by

$$L = \tfrac{1}{2} a \sum_n \left[\frac{m}{a} \dot{\phi}_n^2(t) - ka \left(\frac{\phi_{n+1}(t) - \phi_n(t)}{a} \right)^2 \right]. \tag{181}$$

Note that n here is a particle index, not a coordinate component index, therefore the summation is explicit.

We denote $\mu = m/a$ the mass per unit length and $Y = ka$ the Young modulus. Both quantities are mechanical properties of the material. Clearly, we see appearing a derivative in the continuum limit in the expression above,

$$\lim_{a \to 0} \frac{\phi_{n+1}(t) - \phi_n(t)}{a} = \frac{d\varphi(x,t)}{dx} \tag{182}$$

where the $\phi_n(t)$'s discrete variables approach, when $a \to 0$, a continuous function $\varphi(x,t)$ with $x = na$,

$$\phi_n(t) \underset{a \to 0}{\to} \varphi(x,t). \tag{183}$$

In the same limit, the discrete sum becomes an integral $\sum_n a \to \int dx$ and the Lagrangian takes the form of a functional

$$L[\varphi] = \tfrac{1}{2} \int dx \left[\mu \left(\frac{\partial \varphi}{\partial t} \right)^2 - Y \left(\frac{\partial \varphi}{\partial x} \right)^2 \right], \tag{184}$$

where, again, we have not specified the BC's. This is important that the Lagrangian L is no longer a function, but a functional defined as an integral over *space*. The quantity under the integral may naturally be called a *Lagrangian density* and denoted as $\mathscr{L}$. This density is here a function of the variables $\partial_t \varphi$ and $\partial_x \varphi$.

$$\mathscr{L}(\partial_t \varphi, \partial_x \varphi) = \tfrac{1}{2} \left[\mu \left(\frac{\partial \varphi}{\partial t} \right)^2 - Y \left(\frac{\partial \varphi}{\partial x} \right)^2 \right]. \tag{185}$$

Generically, it could also depend on the function φ itself. In terms of it, the action is still a functional, but it is now defined as an integral over *space and time*,

$$S[\varphi] = \int dt \int dx \, \mathscr{L}(\varphi, \partial_x \varphi, \partial_t \varphi). \tag{186}$$

As a consequence, the least action principle leads this time to an additional term in the Euler-Lagrange equation

$$\frac{\delta S}{\delta \varphi} = \frac{\partial \mathscr{L}}{\partial \varphi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial (\partial_x \varphi)} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial (\partial_t \varphi)} \right) = 0. \tag{187}$$

In the present case of the 1d chain of atoms, it leads to the wave equation

$$\mu \frac{\partial^2 \varphi}{\partial t^2} - Y \frac{\partial^2 \varphi}{\partial x^2} = 0. \tag{188}$$

It is clear that a natural extension to arbitrary number of space dimensions would deliver $d-$dimensional integrals,

$$L[\varphi] = \int d^d r \, \mathscr{L}(\varphi, \boldsymbol{\nabla} \varphi, \partial_t \varphi), \tag{189}$$

$$S[\varphi] = \int dt \int d^d r \, \mathscr{L}(\varphi, \boldsymbol{\nabla} \varphi, \partial_t \varphi), \tag{190}$$

with the corresponding Euler-Lagrange equations

$$\frac{\delta S}{\delta \varphi} = \frac{\partial \mathscr{L}}{\partial \varphi} - \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathscr{L}}{\partial (\boldsymbol{\nabla} \varphi)} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial (\partial_t \varphi)} \right) = 0. \tag{191}$$

Such fields can be called "Newtonien fields" (see e.g. N.A. Doughty, Lagrangian interactions, Addison Wesley, Redwood City, 1990) or "Galilean fields" as well as they obey Galilean invariance. Relativistic extensions obeying Lorentz invariance will treat space and time on the same footing as we will see later. We will call these "relativistic fields". The usual denomination of "classical fields" does not refer to Galilean vs Lorentz invariance, but to *non quantized* fields (in the spirit of second quantization), since we will have no problem to deal with Schrödinger equation as deriving from a *classical* field theory approach.

– Diego: My first question concerns the example of the chain of atoms given in the notes of my professor. Once at home, I have studied it again. When I use the discrete variables $\phi_n(t)$, I can use the Euler-Lagrange equation (74) for each variable to get an equation of motion, but if I use the continuum limit version, I mean the field $\varphi(x,t)$, apparently, I have to use equation (187). Is it clear that it gives the same result?

– Aïssata: I see your point. Ok, when you use the discrete variables and the Lagrangian (181), you get

Discrete variables Lagrangian:

$$L = \tfrac{1}{2} a \sum_n \left[\frac{m}{a} \dot{\phi}_n^2(t) - ka \left(\frac{\phi_{n+1}(t) - \phi_n(t)}{a} \right)^2 \right]. \tag{95}$$

Resulting E-L equation of motion via (74) :

$$m\ddot{\phi}_n(t) = k(\phi_{n+1}(t) - 2\phi_n(t) + \phi_{n-1}(t)). \tag{192}$$

Field Lagrangian:

$$L[\varphi] = \tfrac{1}{2} \int dx \left[\mu \left(\frac{\partial \varphi}{\partial t} \right)^2 - Y \left(\frac{\partial \varphi}{\partial x} \right)^2 \right] \tag{98}$$

Resulting E-L equation of motion via (187):

$$\mu \frac{\partial^2 \varphi}{\partial t^2} - Y \frac{\partial^2 \varphi}{\partial x^2} = 0. \tag{102}$$

Now, notice that

$$\phi_{n+1}(t) - 2\phi_n(t) + \phi_{n-1}(t) \underset{a \to 0}{\to} a^2 \partial_x^2 \varphi(x,t) + O(a^3) \tag{193}$$

With $\mu = m/a$ and $Y = ka$, the two equations of motion coincide.

– Diego: So things are correct, but this is important not to forget the additional term in the field version of the Euler-Lagrange equations.

– Aïssata: Yes. In Physics, and particularly in field theory, you will see very often that the function f, the density of the functional F, may depend on another function, say g – which itself depends on several variables, say x and y – and on its derivatives with respect possibly to all these variables. This was the case of the action (186) defined by your professor in your notes. The relevant functional, may then be defined as a multi-dimensional integral, here two-dimensional,

$$F[g] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(g(x,y), \partial_x g(x,y), \partial_y g(x,y)) dx dy. \tag{194}$$

Then, expanding in terms of $\delta g = g + \epsilon h$ as we have done before in the case of discrete systems, and performing again integration by parts, this time for both variables, with the assumption that the variation function h vanishes at all boundaries $h(x_1, y) = h(x_2, y) = h(x, y_1) = h(x, y_2) = 0$, you get

$$\begin{aligned}
\delta F &= \epsilon \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left(h\frac{\partial f}{\partial g} + \partial_x h \frac{\partial f}{\partial(\partial_x g)} + \partial_y h \frac{\partial f}{\partial(\partial_y g)} \right) dxdy \\
&= \epsilon \int_{x_1}^{x_2} \int_{y_1}^{y_2} h \left(\frac{\partial f}{\partial g} - \frac{\partial}{\partial x}\frac{\partial f}{\partial(\partial_x g)} - \frac{\partial}{\partial y}\frac{\partial f}{\partial(\partial_y g)} \right) dxdy
\end{aligned} \tag{195}$$

thus

$$\frac{\delta F}{\delta g} = \frac{\partial f}{\partial g} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial(\partial_x g)}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial(\partial_y g)}\right) \tag{196}$$

and again you see that the precise form of the functional is important since it determines the form of the functional derivative.

Think in terms of an explicite example, imagine that x is a space coordinate and y is time. Instead of a Lagrangian, one usually prefers to speak about a *Lagrangian density*, and to denote it as calligraphic, $\mathscr{L}(g, \partial_x g, \partial_t g)$, like your professor did, with a function of two variables $g(x, t)$. The Lagrangian L is the space (only) integral of the density, $L = \int dx \mathscr{L}$. The action is then defined as

$$S[g] = \int_{x_1}^{x_2} \int_{t_1}^{t_2} \mathscr{L}(g(x,t), \partial_x g(x,t), \partial_t g(x,t)) dxdt \tag{197}$$

instead of equation (194) and the Euler-Lagrange equation, instead of (196), will be

$$\frac{\delta S}{\delta g} = \frac{\partial \mathscr{L}}{\partial g} - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial(\partial_x g)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial(\partial_t g)}\right) = 0. \tag{198}$$

You probably recognize the equations given in the notes of your professor.

– Diego: We should discuss examples of field theoies, do you agree?

– Aïssata: Yes, you are right, this is the right method to really understand how things work.

▷ 7.2 An example of real scalar field: Korteveg - de Vries equation

►EXERCISE 7 – **Korteveg - de Vries equation** –

– Aïssata: We consider an infinite linear chain (along the x-axis) made of identical pendulums of lengths ℓ, oscillating in planes perpendicular to Ox and mutually coupled by perfect torsion wires of constant C which produce a potential energy quadratic in the difference $\theta_n - \theta_{n-1}$ for pendulums n and $n-1$.

Being a discrete system, the Lagrangian L of the chain is written as a sum of terms,

$$L = \sum_n \left(\tfrac{1}{2}\ell^2 \dot{\theta}_n^2 - mg\ell(1-\cos\theta_n) - \tfrac{1}{2}C(\theta_{n+1}-\theta_n)^2 \right). \tag{199}$$

The distance between pendulums is denoted as a. In the continuum limit, $a \to 0$, the lagrangian $L = \sum_n aL_n$ becomes an integral over a Lagrangian density,

$$\begin{aligned}
L &= \frac{m\ell^2}{a} \sum_n a \left(\tfrac{1}{2}\theta_n^2 - \frac{g}{\ell}(1-\cos\theta_n) - \tfrac{1}{2}\frac{Ca^2}{m\ell^2} \right) \\
&= \frac{m\ell^2}{a} \int dx \left(\tfrac{1}{2}\left(\frac{\partial\theta}{\partial t}\right)^2 - \tfrac{1}{2}c_0^2\left(\frac{\partial\theta}{\partial x}\right)^2 - \omega_0^2(1-\cos\theta) \right)
\end{aligned} \tag{200}$$

with $c_0^2 = Ca^2/m\ell^2$ and $\omega_0^2 = g/\ell$. The Euler-Lagrange equation leads to

$$\frac{\partial^2\theta}{\partial t^2} - c_0^2\frac{\partial^2\theta}{\partial x^2} + \omega_0^2 \sin\theta = 0. \tag{201}$$

The solutions at small oscillations $\sin\theta \simeq \theta$ are sinusoidal plane waves $\theta(x,t) = \theta_0 e^{i(kx-\omega t)}$ with a dispersion relation $\omega = \sqrt{c_0^2k^2 + \omega_0^2}$. There are also exact solutions, called solitons, at arbitrary amplitudes. We use the variable $z = x - vt$ and build the derivatives $\partial_t\theta = \partial_z\theta\partial_t z = -v\partial_z\theta$ and $\partial_x\theta = \partial_z\theta\partial_x z = \partial_z\theta$, leading to the equation of motion

$$\frac{d^2\theta}{dz^2} = \frac{\omega_0^2}{c_0^2 - v^2}\sin\theta. \tag{202}$$

This equation can be integrated after multiplication by $d\theta/dz$, leading to

$$\tfrac{1}{2}\left(\frac{d\theta}{dz}\right)^2 - \frac{\omega_0^2}{c_0^2 - v^2}(1-\cos\theta) = 0 \tag{203}$$

where the constant of integration is fixed by the condition of a fixed θ at infinity. We can interpret the second term as a potential energy $V(\theta)$ of a unit mass of coordinate θ moving in time z. When $c_0^2 - v^2 > 0$, the potential is always negative while it is always positive in the other case. In the first situation, propagation is allowed. Assuming that this condition is realized, one has

$$\frac{\sqrt{2}\omega_0}{\sqrt{c_0^2 - v^2}}dz = \pm\frac{d\theta}{\sqrt{1-\cos\theta}}. \tag{204}$$

Using the result $\int \frac{d\theta}{\sqrt{1-\cos\theta}} = \sqrt{2}\ln(\tan\frac{\theta}{4})$, we find

$$\frac{\sqrt{2}\omega_0}{\sqrt{c_0^2 - v^2}}(z - z_0) = \pm\sqrt{2}\ln\tan\frac{\theta}{4} \tag{205}$$

or, in terms of the original variables

$$\theta(x,t) = 4\,\mathrm{Arctan}\,\exp\left(\pm\frac{\omega_0}{c_0}\frac{x - vt - z_0}{\sqrt{1 - v^2/c_0^2}}\right). \tag{206}$$

The sign $+$ in the solution corresponds to a soliton, the sign $-$ to an antisoliton. These localized solutions which propagate without deformation have been observed on the water surface in channels in Edinburgh in 1834 by an hydrodynamician, Russel. In 1895, Korteveg and de Vries proposed an equation to explain this phenomenon[28] which then became famous after 1953 and the numerical simulations of Fermi, Pasta, Ulam and Tsingou in Los Alamos[29]. ◄

[28] M. Peyrard and T. Dauxois, Physique des solitons, EDP Sciences, Paris, 2004.

[29] For an account of the underestimated contribution of Mary Tsingou, see T. Dauxois, Fermi, Pasta, Ulam, and a mysterious lady, Physics Today, January 2008, p55.

– Diego: As another example, can we come back to the cylindrical capacitor of Feynman that we studied the first day?

– Aïssata: Sure. This is the case of a real scalar field theory in $3d$.

– Diego: We had the energy density $u_{es} = \frac{1}{2}\varepsilon_0|\nabla\phi(\mathbf{r})|^2$ which plays the role of the density, ...

►**EXERCISE 8 – Lagrangian formulation of electrostatics –**

– Aïssata: even better, you can also assume that there are free charges with a density $\rho(\mathbf{r})$. The Lagrangian density now is

$$\mathscr{L}_{es}(\phi, \nabla\phi) = \frac{1}{2}\varepsilon_0|\nabla\phi(\mathbf{r})|^2 - \rho(\mathbf{r})\phi(\mathbf{r}), \tag{207}$$

– Diego: ... and assuming that the functional built from it is an extremum, applying equation (191), the Euler-Lagrange equation is simple because there are only space variables. It reads as

$$\frac{\partial\mathscr{L}_{es}}{\partial\phi} - \nabla\cdot\frac{\partial\mathscr{L}_{es}}{\partial(\nabla\phi)} = 0 \tag{208}$$

hence, one gets

$$\frac{\partial\mathscr{L}_{es}}{\partial\phi} = -\rho, \tag{209}$$

$$\nabla\cdot\frac{\partial\mathscr{L}_{es}}{\partial(\nabla\phi)} = \varepsilon_0\nabla^2\phi. \tag{210}$$

Great, this is Poisson equation!

$$\varepsilon_0\nabla^2\phi = -\rho. \tag{211}$$

This justifies the empirical approach treated in the Feynman, Diego says.

– Aïssata: Correct! I would like to complete with one remark. You asked earlier why we were not building the relevant functional in terms of electric field instead of potential. You can do that, but the price to pay is to introduce a Lagrange multiplier to enforce the correct relation between electric field and the scalar potential: $\mathbf{E} = -\nabla\phi$. So, instead of (207) you define now

$$\mathscr{L}_{es}(\mathbf{E}, \phi, \nabla\phi, \lambda) = \frac{1}{2}\varepsilon_0|\mathbf{E}(\mathbf{r})|^2 - \rho(\mathbf{r})\phi(\mathbf{r}) - \lambda(\mathbf{E} + \nabla\phi) \tag{212}$$

and treat all three quantities ϕ, $\mathbf{E}$ and λ as varying fields. Euler-Lagrange equations are now

$$\begin{aligned}
\frac{\delta}{\delta\mathbf{E}}\int d^3r\mathscr{L}_{es}(\mathbf{E}, \phi, \nabla\phi, \lambda) &= \varepsilon_0\mathbf{E} - \lambda = 0 \\
\frac{\delta}{\delta\phi}\int d^3r\mathscr{L}_{es}(\mathbf{E}, \phi, \nabla\phi, \lambda) &= -\rho + \nabla\lambda = 0 \\
\frac{\delta}{\delta\lambda}\int d^3r\mathscr{L}_{es}(\mathbf{E}, \phi, \nabla\phi, \lambda) &= \mathbf{E} + \nabla\phi = 0. \qquad (213)
\end{aligned}$$

$$\tag{214}$$

The first two equations combine into an equation that you know

$$\varepsilon_0\nabla\cdot\mathbf{E} = \rho \tag{215}$$

and the last one is the relation between $\mathbf{E}$ and ϕ built in on purpose. The two equations of course imply Poisson equation. ◄

Aïssata then opens her laptop and looks at Wikipedia. She starts making a table with various examples of functional derivatives in Physics and even completes with her own examples.

Functional	Functional derivative
$I[g] = \int_{-\infty}^{+\infty} g(x)dx$	$\frac{\delta I}{\delta g(y)} = 1$
$F_0[g] = g(0)$	$\frac{\delta F_0}{\delta g(y)} = \delta(y)$
$F[\varphi] = \int d^3r(\frac{1}{2}a\|\varphi(\mathbf{r})\|^2 + \frac{1}{4}b\|\varphi(\mathbf{r})\|^4 + \frac{1}{2}c\|\vec{\nabla}\varphi(\mathbf{r})\|^2)$	$\frac{\delta F}{\delta\varphi(\mathbf{r})} = a\varphi(\mathbf{r}) + b\|\varphi(\mathbf{r})\|^2\varphi(\mathbf{r}) - c\vec{\nabla}^2\varphi(\mathbf{r})$
$J[\rho] = \int \left(\frac{1}{2}\int \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{\|\mathbf{r}-\mathbf{r}'\|}d^3r'\right) d^3r$	$\frac{\delta J}{\delta\rho(\mathbf{r}')} = \int \frac{\rho(\mathbf{r})}{\|\mathbf{r}-\mathbf{r}'\|}d^3r$
$S[\phi] = \int dt \int \mathscr{L}(\phi, \dot{\phi}, \vec{\nabla}\phi)d^3r$	$\frac{\delta S}{\delta\phi} = \frac{\partial\mathscr{L}}{\partial\phi} - \vec{\nabla}\left(\frac{\partial\mathscr{L}}{\partial(\vec{\nabla}\phi)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\phi}}\right)$
$H[\phi, \pi] = \int \mathcal{H}(\phi, \vec{\nabla}\phi, \pi)d^3r$	$\frac{\delta H}{\delta\phi} = \frac{\partial\mathcal{H}}{\partial\phi} - \vec{\nabla}\left(\frac{\partial\mathcal{H}}{\partial(\vec{\nabla}\phi)}\right)$ $\frac{\delta H}{\delta\pi} = \frac{\partial\mathcal{H}}{\partial\pi}$

▷ **7.4 Born-Infeld approach to electrostatics**

– Diego: I read somewhere that in the early XXth century, physicists were worried about the fact that the electric field of a point charge was a diverging quantity, and they were looking for alternatives to Maxwell electrodynamics to possibly repair this.

►EXERCISE 9 – **Born-Infeld approach to electrostatics** –

– Aïssata: This is true. One of these attempts was made in the thirties by Born and Infeld[30], after suggestions around 1910 by Mie to prevent from infinities. What Born and Infeld do is essentially to propose a non linear variant of the lagrangian (207):

$$\mathscr{L}_{\mathrm{BI}}(\phi, \nabla\phi) = -\varepsilon_0 E_0^2 \Big(1 - \frac{|\nabla\phi(\mathbf{r})|^2}{E_0^2}\Big)^{1/2} - \rho\phi(\mathbf{r}). \tag{216}$$

Of course, we simplify considerably the original theory here to consider only the case of electrostatics. The equation of motion follows as

$$\nabla \cdot \Big(\nabla\phi(\mathbf{r})\Big(1 - \frac{|\nabla\phi(\mathbf{r})|^2}{E_0^2}\Big)^{-1/2}\Big) = -\rho(\mathbf{r})/\varepsilon_0 \tag{217}$$

where you immediately see the deviation from ordinary Poisson equation which would be recovered in the limit $E_0 \to \infty$. The case of the point charge e is easy to study. You solve the above equation for $\mathbf{r} \neq 0$,

$$\frac{1}{r^2}\frac{d}{dr}\Big[r^2\frac{d\phi}{dr}\Big(1 - \frac{1}{E_0^2}\Big(\frac{d\phi}{dr}\Big)^2\Big)^{-1/2}\Big] = 0. \tag{218}$$

Upon integration, we get

$$\frac{d\phi}{dr} = \frac{\kappa}{r^2}\sqrt{1 - \frac{1}{E_0^2}\left(\frac{d\phi}{dr}\right)^2}. \tag{219}$$

Taking the square of this equation and solving for $E(r) = -d\phi/dr$ we get

$$E(r) = \frac{e}{4\pi\varepsilon_0 r^2}\left(1 + \frac{e^2}{16\pi^2\varepsilon_0^2 E_0^2 r^2}\right)^{-1/2} \tag{220}$$

[30] H.F.M. Goenner, On the history of unified field theories, Part II (ca. 1930 - ca. 1965), Living Rev. Relativity **17**, 5, 2014.

where the constant κ was fixed to recover the Coulombian form at large distances. At short distances we get the following electric field

$$E(r) \sim E_0(1 - (8\pi^2\varepsilon_0^2 E_0^2/e^2)r^4), \quad r \to 0 \tag{221}$$

which is bounded to the constant E_0. You can find arguments for the value of this maximum electric field, e.g. in Wikipedia. As far as I know, this theory is no longer considered anymore as a viable alternative to Coulomb law. ◄

– Diego: What is the reason for the strange form of the Lagrangian Aïssata?

– Aïssata: This doesn't come out of the blue. You will learn later that the Lagrangian of a free particle in Special Relativity takes the form (668)

$$L = -mc^2(1 - v^2/c^2)^{1/2} \tag{222}$$

where c is the maximum value for the velocity v. The simplified version (216) of the Lagrangian proposed by Born and Infeld is built on the same form.

– Diego: OK, thank you Aïssata for this nice illustration of real scalar field theories. After the course on the chain of atoms, we had a few notes on another example which is a complex scalar field now. Let me show you:

□ 7.5 Scalar Quantum Mechanics

The case of non relativistic quantum mechanics can be treated that way. The wave function $\varphi(\mathbf{r}, t)$ is a complex number. It is a complex scalar field, that means that it is defined over space and time and takes complex values.

The action is thus a functional of φ and φ^* (two real fields are needed to describe a complex field and one could use $\Re\varphi$ and $\Im\varphi$ as independent fields, but usually one uses φ and its complex conjugate instead):

$$S[\varphi, \varphi^*] = \int dt d^3r \left(i\hbar\varphi^*\partial_t\varphi - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\varphi^* \cdot \boldsymbol{\nabla}\varphi - V(\mathbf{r})\varphi^*\varphi \right) \tag{223}$$

The functional derivative w.r.t. φ^* gives

$$\frac{\delta S}{\delta\varphi^*} = i\hbar\partial_t\varphi - V(\mathbf{r})\varphi + \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\varphi. \tag{224}$$

Assuming the least action principle $\delta S = 0$ with the action (223) thus delivers Schrödinger equation. This is a variational formulation of non relativistic quantum mechanics.

– Aïssata: This is a nice application. Actually, I had this application in mind when I spoke about complex fields! The Lagrangian density in parentheses at the r.h.s. of (223) was written down by Jordan and Wigner,

$$\mathscr{L}_{\mathrm{JW}} = i\hbar\varphi^*\dot{\varphi} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\varphi^* \cdot \boldsymbol{\nabla}\varphi - V(\mathbf{r})\varphi^*\varphi, \tag{225}$$

and indeed, there is a variational formulation, and not only of non relativistic equation. For Klein-Gordon equation also, and even with more sophisticated fields, there is one for Dirac equation. The field there, called Dirac spinor, is a more elaborate mathematical object that you will surely encounter in your curriculum. Maybe in this course even[31].

[31] See the Part on Relativistic Field Theory in this course.

– Diego: But look, before going to more difficult things, there is a problem with your Jordan-Wigner Lagrangian. It is not Hermitian Diego says!

– Aïssata: You are right, but this is not necessarily a problem. The Lagrangian or Lagrangian density is not a measurable quantity! And it is not uniquely defined. For example I could have defined the Jordan-Wigner Lagrangian as

$$\mathscr{L}_{\mathrm{JW}} = \tfrac{1}{2} i\hbar(\varphi^* \dot{\varphi} - \dot{\varphi}^* \varphi) - \frac{\hbar^2}{2m} \boldsymbol{\nabla}\varphi^* \cdot \boldsymbol{\nabla}\varphi - V(\mathbf{r})\varphi^*\varphi. \tag{226}$$

You can check that the Euler-Lagrange equation again leads to the correct Schrödinger equation.

– Diego: Ok. I believe you, Diego says, smiling. So could you show me the Lagrangian of Klein-Gordon equation Diego asks, forgetting about the previous point.

▷ **7.6 Klein-Gordon equation**

– Aïssata: OK. This is also called Klein-Gordon-Fock equation, and other authors also (including Schrödinger himself) discussed this equation in 1926. We have not yet used symmetries to constrain the Lagrangians, so I will show you a pedestrian approach. I must also warn you that with KG equation we are no longer in the situation of Galilean invariance, but Lorentz invariance instead. I will nevertheless keep a non-symmetric notation for time and space coordinates for a while. You may remember that Klein-Gordon equation is intended to be the "wave equation associated to the energy-momentum relation" of Special Relativity,

$$p_\mu p^\mu = (E/c, -\mathbf{p}) \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix} = \frac{E^2}{c^2} - |\mathbf{p}|^2 = m^2c^2. \tag{227}$$

You have enough knowledge in basic Quantum Mechanics, so I can proceed. Aïssata stands up and starts writing on the blackboard available in the study room of the library,

Substitute E by $i\hbar\partial_t$ and $\vec{p}$ by $-i\hbar\vec{\nabla}$ acting on some complex wave amplitude $\varphi(\vec{r}, t)$ in Eq.(227).

You get

$$-\frac{1}{c^2}\partial_t^2\varphi + \vec{\nabla}^2\varphi = \left(\frac{mc}{\hbar}\right)^2 \varphi. \tag{228}$$

Demand that this equation of motion is the EL equation for some still unknown Lagrangian $\mathscr{L}_{KG}$.

Simple guess:

i) $\left(\frac{mc}{\hbar}\right)^2 \varphi$ is $\frac{\partial \mathscr{L}_{KG}}{\partial \varphi^*}$. This is OK if the Lagrangian is $\mathscr{L}_{KG} = \left(\frac{mc}{\hbar}\right)^2 \varphi^*\varphi +$ some function f of $\dot{\varphi}, \vec{\nabla}\varphi$, and cc.

ii) $\frac{1}{c^2}\partial_t^2\varphi$ is $-\partial_t \frac{\partial \mathscr{L}_{KG}}{\partial \dot{\varphi}^*}$. OK if $f(\dot{\varphi}, \vec{\nabla}\varphi, \text{and cc}) = -\frac{1}{c^2}\dot{\varphi}^*\dot{\varphi} + g(\vec{\nabla}\varphi, \text{and cc})$.

iii) $\vec{\nabla}^2\varphi$ is $\vec{\nabla} \cdot \frac{\partial \mathscr{L}_{KG}}{\partial(\vec{\nabla}\varphi^*)}$. Ok for $g(\vec{\nabla}\varphi, \text{and cc}) = (\vec{\nabla}\varphi^*) \cdot (\vec{\nabla}\varphi)$.

So a possible Lagrangian is

$$\boxed{\mathscr{L}_{KG} = (\vec{\nabla}\varphi^*) \cdot (\vec{\nabla}\varphi) - \frac{1}{c^2}\dot{\varphi}^*\dot{\varphi} + \left(\frac{mc}{\hbar}\right)^2 \varphi^*\varphi.} \tag{229}$$

– Diego: Amazing! Diego says, enthusiastic.

▷ 7.7 Coupling of scalar matter fields to the EM field

►**Exercise 10** – **Coupling of scalar QM to the EM field** –

– Aïssata: The standard minimal coupling scheme, very well known, although not fully understood at this level, can be extended to Schrödinger and KG equations. Remember that in non relativistic classical dynamics, we have discussed the Hamiltonian of a charged particle in an EM field (168) which just says that in the Hamiltonian $H = |\mathbf{p}|^2/(2m) + V$, we have to make the substitutions

$$H \rightsquigarrow H - e\phi \quad \text{and} \quad \mathbf{p} \rightsquigarrow \mathbf{p} - e\mathbf{A} \tag{230}$$

to get the corresponding Hamiltonian in the presence of an EM field,

$$H = (\mathbf{p} - e\mathbf{A})^2/(2m) + V + e\phi. \tag{231}$$

Proceeding along the same lines for a charged particle in an EM field leads to Schrödinger equation

$$\mathsf{H} - V - e\phi(\mathbf{r}, t) = \frac{1}{2m}(\mathsf{P} - e\mathbf{A}(\mathbf{r}, t))^2, \tag{232}$$

$$i\hbar\partial_t \varphi_{\text{Sch.}}(\mathbf{r}, t) = \left[\frac{1}{2m}(-i\hbar\boldsymbol{\nabla} - e\mathbf{A}(\mathbf{r}, t))^2 + V + e\phi(\mathbf{r}, t)\right] \varphi_{\text{Sch.}}(\mathbf{r}, t), \tag{233}$$

and for the Klein-Gordon case, the same substitution (230) in the relativistic equation $(H - V)^2 = c^2|\mathbf{p}|^2 + m^2c^4$ gives

$$(\mathsf{H} - V - e\phi(\mathbf{r}, t))^2 = c^2(\mathsf{P} - e\mathbf{A}(\mathbf{r}, t))^2 + m^2c^4, \tag{234}$$

$$(i\hbar\partial_t - V - e\phi(\mathbf{r}, t))^2 \varphi_{\text{KG}}(\mathbf{r}, t) = c^2(-i\hbar\boldsymbol{\nabla} - e\mathbf{A}(\mathbf{r}, t))^2 \varphi_{\text{KG}}(\mathbf{r}, t) + m^2c^4\varphi_{\text{KG}}(\mathbf{r}, t). \tag{235}$$

The same minimal coupling prescription would apply for spinor equations, like Dirac equation which you will study later. ◄

A bit tired, but happy, Aïssata proposes to go for a drink. Diego is delighted and suggests to go downtown in a pub where he has his habits.

Half an hour later, they enter the pub, go directly to the bar. Aïssata orders a pint of Punk IPA and Diego an *agua de coco*. He also orders a few snacks. They choose a table in the back of the pub.

– Diego: Aïssata, you mentioned beauty about the variational formalism. Our professor also mentioned that. Then she stopped and looked at us like if she had a kind of a secret to tell us. "I have always been one of these physicists who claim that some theories are beautiful she said. And I was ready to accept that a good theory is one which is beautiful. In a certain sense, I was considering beauty as a criterion which an acceptable theory *had to* satisfy. Among such theories, gauge theory is an example of a beautiful theory. I have nothing very original in this, most physicists would tell you the same". For example, according to Dirac[32],

> What makes the theory of relativity so acceptable to physicists in spite of its going against the principle of simplicity is its great mathematical beauty. This is a quality which cannot be defined, any more than beauty in art can be defined, but which people who study mathematics usually have no difficulty in appreciating.

or, in the same vein, Weyl said[33] that

> My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful.

Dirac formulated this very definitively at the end of a talk at Moscow University in 1955, summarizing his philosophy of physics, "Physical laws should have mathematical beauty"[34] , and according to Schweber, beauty and simplicity were so important for Dirac that his evaluation of a theory could be very sarcastic

> Recent work by Lamb, Schwinger, Feynman and others has been very successful in setting up rules for handling the infinities and subtracting them away, so as to leave finite residues which can be compared with experiments, but the resulting theory is an ugly and incomplete one, and cannot be considered as a satisfactory solution to the problem of the electron.

– Aïssata: I think that most of the theoretical physicists consider as an important one the question about beauty of physical laws. As another example, in the famous second volume of Landau and Lifshitz series for example, general relativity is described as the most beautiful physical theory,[35]

The theory of gravitational fields, constructed on the basis of the theory of relativity, is called the *general theory of relativity*. It was established by Einstein (and finally formulated by him in 1915), and represents probably the most beautiful of all existing physical theories. It is remarkable that it was developed by Einstein in a purely deductive manner and only later was substantiated by astronomical observations.

Figure 13. L. Landau. and E. Lifshitz, The Classical Theory of Fields, Butterworth-Heinemann, Oxford, 1976, p. 245

[32] P.A.M. Dirac, Sammlung, Cambridge University Press, 1995, p. 908.

[33] Hermann Weyl: Legacy, https://www.ias.edu/hermann-weyl-legacy

[34] quoted in S.S. Schweber, QED and the men who made it, Princeton University Press, Princeton, 1994, p. 70; or in S. Hossenfelder, Lost in Math, Basic books, New-York, 1997.

[35] L. Landau. and E. Lifshitz, The Classical Theory of Fields, Butterworth-Heinemann, Oxford, 1976, p. 245.

and Chandrasekhar wrote an article entitled "The General Theory of Relativity: Why is it the most beautiful of all existing theories"[36] where he tries to elaborate on this comment from Landau and Lifshitz.

– Diego: Maybe these "beauty" qualifiers are widespread among physicists, but I want to comment on another point of view. Our professor added, "recently I read a book by Sabine Hossenfelder, *Lost in Math*[37], which has completely modified my views on this question of aesthetics. Regarding appreciation of beauty, she has this statement:"

> Scientists are human. Humans are influenced by the communities they are part of. Therefore scientists are influenced by the communities they are part of (...) it leads me to conjecture that the laws of nature are beautiful because physicists constantly tell each other those laws are beautiful.

"But more fundamental, Sabine Hossenfelder is questioning the problems of beauty, or of naturalness according to which dimensionless numbers should all be typically of order one. She again formulates this as "

> In summary, numbers that are very large, very small, or very close together are not natural. In the standard model, the Higgs mass is not natural, which makes it ugly.

And then, our prof continued "Naturalness is a well accepted criterion for beauty. A famous counterexample is the case of gravitation, the structure constant of which is by orders of magnitude smaller than the structure constants of the other interactions. This is known as the hierarchy problem and physicists don't like it. This is supposed to call for an explanation. Sabine Hossenfelder is revisiting this question. Nature the professor says is what it is. Physical laws describe the behaviour of nature and allow to make predictions, but for which reason should a successful theory be beautiful? The logical demand is that a theory makes predictions, and that experiments are made to check these predictions. The more accurate the predictions, the better the theory. That's all. At least, this is how I understand Sabine Hossenfelder's argument and I must confess that it has completely modified my personal appreciation of Physics. I still like aesthetic in a theory she added, but I now believe that this is only an unnecessary additional property, not a demand."

Aïssata seems concentrated for a moment.

– Aïssata: I must read that book she says! This might also change my own view on Physics... I had not thought along these lines of reasoning, but your professor touches here a very important point, for sure. This is really disturbing, ... this goes against many claims in Physics, like what your professor told you. I have to find a mentor for an internship next semester, I should ask her.

After a few minutes of silence, moving to another subject Diego asks: I have never asked you. Where do you come from Aïssata?

– Aïssata: This is a long story you know. To make it short, I was born in the Land of upright men, Burkina Faso. I grew up in the countryside, then I went to Ouagadougou for my undergrad studies and, since I had some skills in mathematics, a professor advised me to go to the African Institute for Mathematical Sciences in Ghana. So I moved to AIMS to prepare a master in Sciences. There I met ... someone, ..., and we both decided to apply for a PhD in Europe.

[36] S. Chandrasekhar, J. Astrophys. Astr. **5**, 3, 1984.

[37] S. Hossenfelder, Lost in Math, Basic books, New-York, 1997.

At AIMS, we had professors coming from all over the world and that helped to get opportunities and PhD grants to support our stay here. Once arrived here, I decided to attend some of the courses of the master programme to consolidate my knowledge. This is easier for me than for many of other students coming from abroad, specially from Africa, most of the time from poor countries, and who have to make small jobs, not paid correctly, to support their own studies.

– Diego: I know that, there are many foreign students in our class. Beside the university, most of them have a job, sometimes a night job, not very well paid, . We try to help them for their studies. We have organized a kind of system to make copies of our notes when someone cannot attend a course. The professors say that we are not there to compete, but to help each other! University is not competition, this is knowledge and knowledge can be shared. And I adhere to this point of view.

After a short silence, Diego asks

– Diego: Look, you said “the Land of upright men". Is it an official name?

– Aïssata: We were a former French colony, named Haute-Volta. Our famous hero, Thomas Sankara , who was anti-imperialist, feminist, pan-africanist, socialist, ecologist, etc, became president in the eighties and decided to change the name of the country to stop referring to ourselves as being under European domination. The name Burkina-Faso comes from two of the main languages spoken in the county, Burkina means “integrity” in moré and Faso means “Country, Fatherland” in dioula. Thomas Sankara was killed in 1987. He is still a hero for many Africans, a kind of African Che Guevara. For women and for progressists.

Aïssata then keeps silence, locked in her thoughts. The evening goes on, smoothly. They leave the pub and go back to their own places. The atmosphere has become sad suddenly.

8. Day 4 – Hamitonian approach to scalar fields

□ 8.1 Hamiltonian formulation of field theory

For a scalar field theory, we saw that the Lagrangian density is a function of the field $\varphi(\mathbf{r}, t)$ and field derivatives,

$$\mathscr{L}(\varphi, \partial_t\varphi, \boldsymbol{\nabla}\varphi) \tag{236}$$

and Euler-Lagrange equations are

$$\frac{\partial\mathscr{L}}{\partial\varphi} - \frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)} - \boldsymbol{\nabla}\cdot\frac{\partial\mathscr{L}}{\partial(\boldsymbol{\nabla}\varphi)} = 0. \tag{237}$$

Defining the Hamiltonian density

$$\mathcal{H}(\pi, \varphi, \nabla\varphi) = \pi\partial_t\varphi - \mathscr{L} \tag{238}$$

and demanding its independence in terms of $\partial_t\varphi$ specifies the density of canonical momentum

$$\pi = \frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)}. \tag{239}$$

The action (190) reads now as a functional of *two* fields, namely $\pi(\mathbf{r},t)$ and $\varphi(\mathbf{r},t)$,

$$S[\pi,\varphi] = \int dt \int d^d r (\pi \partial_t \varphi - \mathcal{H}). \tag{240}$$

That least action principle leads to

$$\frac{\delta S}{\delta \pi} = \partial_t \varphi - \frac{\partial \mathcal{H}}{\partial \pi} = 0, \tag{241}$$

$$\frac{\delta S}{\delta \varphi} = -\frac{\partial \mathcal{H}}{\partial \varphi} - \partial_t \pi - \boldsymbol{\nabla} \cdot \left(\frac{\partial(-\mathcal{H}}{\partial(\boldsymbol{\nabla}\varphi)} \right) = 0 \tag{242}$$

and the equations of motion follow, like in the case of the discrete system

$$\partial_t \varphi = \frac{\partial \mathcal{H}}{\partial \pi}, \tag{243}$$

$$\partial_t \pi = -\frac{\partial \mathcal{H}}{\partial \varphi} + \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{H}}{\partial(\boldsymbol{\nabla}\varphi)}. \tag{244}$$

They don't look similar to the equations of motion of discrete systems, like Euler-Lagrange (237) differs from its discrete counterpart (146). Again, there are these gradients at the r.h.s. of the second equation which differ from simple extension of the corresponding discrete degrees of freedom case. On the other hand, if one uses functional derivatives instead of ordinary derivatives, we note that the total Hamiltonian is a functional

$$H[\pi,\varphi] = \int d^3 r\, \mathcal{H}(\pi,\varphi,\boldsymbol{\nabla}\varphi), \tag{245}$$

hence its functional derivatives w.r.t the fields π and φ are given by

$$\frac{\delta H}{\delta \pi} = \frac{\partial \mathcal{H}}{\partial \pi}, \tag{246}$$

$$\frac{\delta H}{\delta \varphi} = \frac{\partial \mathcal{H}}{\partial \varphi} - \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{H}}{\partial(\boldsymbol{\nabla}\varphi)} \tag{247}$$

and eventually Hamilton equations can be rewritten in a more friendly form

$$\frac{\delta H}{\delta \pi} = \dot{\varphi}, \tag{248}$$

$$\frac{\delta H}{\delta \varphi} = -\dot{\pi} \tag{249}$$

with $\dot{\varphi} \equiv \partial_t \varphi$ and $\dot{\pi} = \partial_t \pi$ (note that we usually use the dot to denote time partial derivatives and the prime for the space partial derivatives). These equations are now pretty close to (155) and (156). The last equation (157) has an equivalent as

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}. \tag{250}$$

▷ 8.2 Discrete versus continuous Hamiltonian formalism: worked examples

Diego had class on Hamiltonian formalism applied to field theory in the morning and spent the afternoon to treat the example of the string of atoms via this new approach, to check whether it was indeed consistent with the Lagrange approach (although he had no doubt)! He went to the library and gave Aïssata's name to have access to the working room. He started to write at the blackboard,

String of atoms: $L = \sum_n aL_n = \sum_n \frac{1}{2}a(\mu\dot{\phi}_n^2 - Y((\phi_{n+1}-\phi_n)/a)^2)$

Canonical momentum $p_n = \frac{\partial L}{\partial \dot{\phi}_n} = a\frac{\partial L_n}{\partial \dot{\phi}_n} = \mu a \dot{\phi}_n$

Take care: $p_n/a \rightarrow_{a\to 0} \pi$ in the continuum limit (a momentum density).

Hamiltonian:

$$H = \sum_n a\left(\frac{1}{2\mu}\left(\frac{p_n}{a}\right)^2 + \frac{1}{2}Y\left(\frac{\phi_{n+1}-\phi_n}{a}\right)^2\right) \tag{251}$$

and Hamilton equations of motion

$$\dot{\phi}_n = \frac{\partial H}{\partial p_n} = \frac{1}{\mu a}p_n, \tag{252}$$

$$\dot{p}_n = -\frac{\partial H}{\partial \phi_n} = Ya\frac{\phi_{n+1}-2\phi_n+\phi_{n-1}}{a^2}. \tag{253}$$

Derivative of (252) incorporated in (253) delivers the correct equation

$$\mu\ddot{\phi}_n = Y\frac{\phi_{n+1}-2\phi_n+\phi_{n-1}}{a^2} \tag{254}$$

which, in the continuum limit, is nothing but

$$\mu\partial_t^2\varphi = Y\partial_x^2\varphi. \tag{255}$$

Then, he worked out the same problem through the field theory approach.

Momentum density $\pi = \partial\mathscr{L}/\partial(\partial_t\varphi) = \mu\partial_t\varphi$.

Hamiltonien density from (185)

$$\mathcal{H} = \pi\partial_t\varphi - \mathscr{L} = \frac{1}{2\mu}\pi^2 + \frac{1}{2}Y(\partial_x\varphi)^2 \tag{256}$$

Hamilton equations

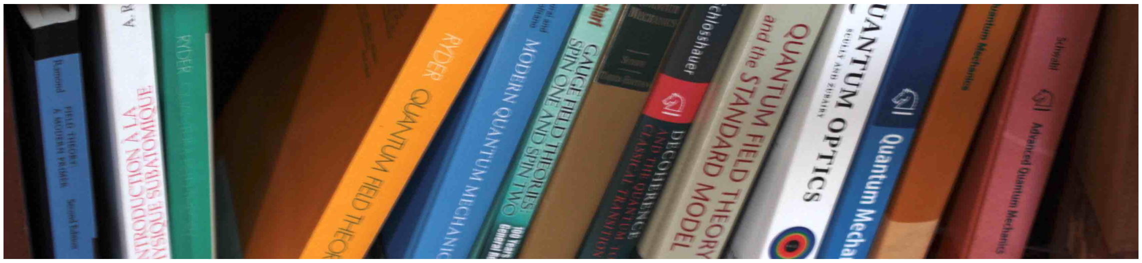
$$\partial_t\varphi = \frac{\partial\mathcal{H}}{\partial\pi} = \frac{\pi}{\mu}, \tag{257}$$

$$\partial_t\pi = -\frac{\partial\mathcal{H}}{\partial\varphi} + \partial_x\frac{\partial\mathcal{H}}{\partial(\partial_x\varphi)} = -0 + Y\partial_x^2\varphi. \tag{258}$$

Inserting (257) in (258) leads to (255).

Happy to be successful in this exercise that he solved by himseelf, Diego has understood the importance of the additional term in the r.h.s. of the second Hamilton equation, since this is the one which plays a role in the present case.

Space-time symmetries, Lie groups, Lie algebras and their representations



9. Day 5 – Symmetry and conservation laws in classical mechanics

Thee next day, Diego is ready for a new course. The professor said that she would speak about conservation properties in particle dynamics, before going to the corresponding problem in field theory. She also mentioned that she would have to introduce abstract notions of mathematics to go deeper with the notion of symmetry in Physics.

□ 9.1 Conservation laws in particle dynamics and space-time symmetries

In classical physics, when a quantity is conserved, its numerical value does not change as times evolves. This is the case for example for the total mechanical energy of a dissipationless system. The conservation of mechanical energy is linked, via Noether theorem, to the invariance of physical laws under time translation. In analytical mechanics, you can prove this using Euler-Lagrange equations.

Let $L(\mathbf{r}_n, \mathbf{v}_n, t)$ be the Lagrangian of a system of N particles labelled by the index n. The variation of L for a variation of its arguments is

$$\delta L = \frac{\partial L}{\partial t} dt + \sum_n \left(\frac{\partial L}{\partial \mathbf{r}_n} d\mathbf{r}_n + \frac{\partial L}{\partial \mathbf{v}_n} d\mathbf{v}_n \right) \equiv \frac{dL}{dt} dt. \tag{259}$$

Substituting Euler-Lagrange equations $\frac{\partial L}{\partial \mathbf{r}_n} = \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}_n}$ and the definition $\mathbf{p}_n = \frac{\partial L}{\partial \mathbf{v}_n}$ leads to

$$-\frac{\partial L}{\partial t} = \frac{d}{dt}\left(\sum_n \mathbf{p}_n \cdot \mathbf{v}_n - L \right). \tag{260}$$

If the Lagrangian does not depend on time explicitly, there is invariance under time translation, since the Lagrangian does contain all the information of the dynamics of the system. It follows that

$$\sum_n \mathbf{p}_n \cdot \mathbf{v}_n - L \equiv E_{\text{tot}} = \text{const.} \tag{261}$$

This is the expression of the conservation of total energy in particle dynamics. This property appears, as we announced, to be directly connected to the *symmetry under time translation*.

We will now use space translation symmetry. In this case, the equations governing the system's behaviour do not change under the transformation

$$\forall n, \quad \delta \mathbf{r}_n = \boldsymbol{\varepsilon}, \quad \delta \mathbf{v}_n = 0. \tag{262}$$

This requires that the Lagrangian, hence the potential energy, depends only on the relative positions of the bodies, $V(\mathbf{r}_1, \mathbf{r}_2, \dots) = V(\{\mathbf{r}_l - \mathbf{r}_m\})$ and not on their absolute positions $\mathbf{r}_n$. Therefore, one has

$$\delta L = \sum_n \frac{\partial L}{\partial \mathbf{r}_n} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \frac{d}{dt} \sum_n \frac{\partial L}{\partial \mathbf{v}_n}. \tag{263}$$

Space translation being a symmetry, $\delta L = 0$ leads to the conservation law

$$\frac{d}{dt} \sum_n \mathbf{p}_n = 0, \quad \sum_n \mathbf{p}_n \equiv \mathbf{P}_{\text{tot}} = \text{const}. \tag{264}$$

Finally, let us now consider space rotation symmetry, i.e. invariance under a rotation of all coordinates by the same amount $\delta\theta$ around an axis $\mathbf{u}$, with $\delta\boldsymbol{\theta} = \delta\theta\mathbf{u}$,

$$\forall n, \quad \delta \mathbf{r}_n = \delta\boldsymbol{\theta} \times \mathbf{r}_n, \quad \delta \mathbf{v}_n = \delta\boldsymbol{\theta} \times \mathbf{v}_n. \tag{265}$$

The Lagrangian variation is

$$\delta L = \sum_n \left(\frac{\partial L}{\partial \mathbf{r}_n} \cdot \delta\boldsymbol{\theta} \times \mathbf{r}_n + \frac{\partial L}{\partial \mathbf{v}_n} \cdot \delta\boldsymbol{\theta} \times \mathbf{v}_n \right) = \frac{d}{dt} \sum_n \mathbf{p}_n \cdot \delta\boldsymbol{\theta} \times \mathbf{r}_n, \tag{266}$$

which requires the relation

$$\delta\boldsymbol{\theta} \cdot \frac{d}{dt} \sum_n \mathbf{r}_n \times \mathbf{p}_n = 0, \quad \sum_n \mathbf{r}_n \times \mathbf{p}_n \equiv \mathbf{L}_{\text{tot}} = \text{const}. \tag{267}$$

An important caveat is in order now: the momentum $\mathbf{p}_n$ which appears in these equations is the *canonical* momentum, since it is defined by the derivative of the Lagrangian w.r.t. the coordinate. This is the one which is associated to conservations laws, and not the kinematic momentum. Similarly, this is the momentum of the canonical momentum or the *canonical angular momentum*, not the ordinary kinematic angular momentum which is associated to rotational symmetry. And for time translation symmetry, this is the *Hamiltonian*, not just the energy which plays the key role.

▷ 9.2 From symmetry to Group theory

After this new course, Diego goes to the library where he is sure to meet Aïssata in her working room.

– Diego: Hi Aïssata. How are you doing? Ready for your daily tutoring? Today we started the course on conserved quantities and symmetry properties.

– Aïssata: Let's go on, it will change my mind. You arrive on purpose, I was studying a part of Mathematics applied to Physics for which I am not an expert and working with you will help me: this is Group theory for which I have hesitation to be enthusiastic. Although many physicists consider it as fundamental, for example in Schwartz and Schwartz book[38] they say

> As it happens, there is a beautiful branch of mathematics, called group theory, which is ideally suited to the description of symmetry. (...) By now it is part of the standard toolkit of an educated physicist.

I know I should love it, but this is stronger than me, it frightens me. The discussion with you will probably help me to clarify my thoughts on this. So, please, go ahead!

– Diego: Thank you. My first question is a bit of semantic nature. There seems to be a connection between various concepts when one deals with symmetry in Physics: *Symmetry*, *Conservation* and *Invariance*. These words all appear in my professor's notes. Also, I am used to speak about symmetry of something, an object. For example a bottle has a cylindrical symmetry, or a rotational symmetry about a vertical axis, but the professor mentions symmetry of physical laws rather than symmetry of objects.

– Aïssata: Well spotted Diego. This is a question you *must* ask. As usual, I recommend to go to the fundamental sources. Feynman has a last chapter called "Symmetry in physical laws" in the first volume of the his lectures on Physics[39]. She goes to the Physics' section of the library, moves between the bookshelves as if she knew precisely what she's looking for. Indeed, this is the case, she goes straight to the textbooks section, chooses a book and opens it at section 52. She reads:

But our main concern here is not with the fact that the *objects* of nature are often symmetrical. Rather, we wish to examine some of the even more remarkable symmetries of the universe—the symmetries that exist in the *basic laws themselves* which govern the operation of the physical world.

First, what *is* symmetry? How can a physical *law* be "symmetrical"? The problem of defining symmetry is an interesting one and we have already noted that Weyl gave a good definition, the substance of which is that a thing is symmetrical if there is something we can do to it so that after we have done it, it looks the same as it did before. For example, a symmetrical vase is of such a kind that if we reflect or turn it, it will look the same as it did before. The question we wish to consider here is what we can do to physical phenomena, or to a physical situation in an experiment, and yet leave the result the same. A list of the known operations under which various physical phenomena remain invariant is shown in Table 52–1.

Figure 14. R.P. Feynman, Lectures on Physics, Vol 1, Addison Wesley, Reading, 1964, Sec. 52. Here from the online version available at the web site https://www.feynmanlectures.caltech.edu . The table 52-1 to which Feynman refers to is presented in Fig. 9.

– Aïssata: To Feynman's statement: "The question we wish to consider here is what we can do to physical phenomena, or to a physical situation in an experiment, and yet leave the result the same", I would add that the physical law must reflect this *invariance* of the result. Here, you see the second of the words you were speaking about. You may

[38] P.M. Schwartz and J.H. Schwartz, Special Relativity, from Einstein to strings, Cambridge University Press, Cambridge, 2004.

[39] R.P. Feynman, Lectures on Physics, Vol 1, Addison Wesley, Reading, 1964.

also have a look at the book of Brading and Castellani[40]:

> The real turning point in the use of symmetry in science came, however, with the introduction of the *group* concept and with the ensuing developments in the theory of transformation groups. This is because the group-theoretic definition of symmetry as 'invariance under a specified group of transformations' allowed the concept to be applied much more widely, not only to spatial figures but also to abstract objects such as mathematical expressions – in particular, expressions of physical relevance such as dynamical equations. Moreover, the technical apparatus of group theory could then be transferred and used to great advantage within physical theories.
>
> The first explicit study of the invariance properties of equations in physics is connected with the introduction, in the first half of the nineteenth century, of the transformational approach to the problem of motion in the framework of analytical mechanics. Using the formulation of the dynamical equations of mechanics due to Hamilton (known as the Hamiltonian or canonical formulation), Jacobi developed a procedure for arriving at the solution of the equations of motion based on the strategy of applying transformations of the variables that leave the Hamiltonian equations invariant, thereby transforming step by step the original problem into new ones that are simpler but perfectly equivalent (for further details see Lanczos, 1949).[4] Jacobi's canonical transformation theory, although introduced for the 'merely instrumental' purpose of solving dynamical problems, led to a very important line of research: the general study of physical theories in terms of their transformation properties.

Figure 15. C. Brading and E. Castellani, Symmetries in Physics, Philosophical Reflections, Cambridge University Press, Cambridge, 2003

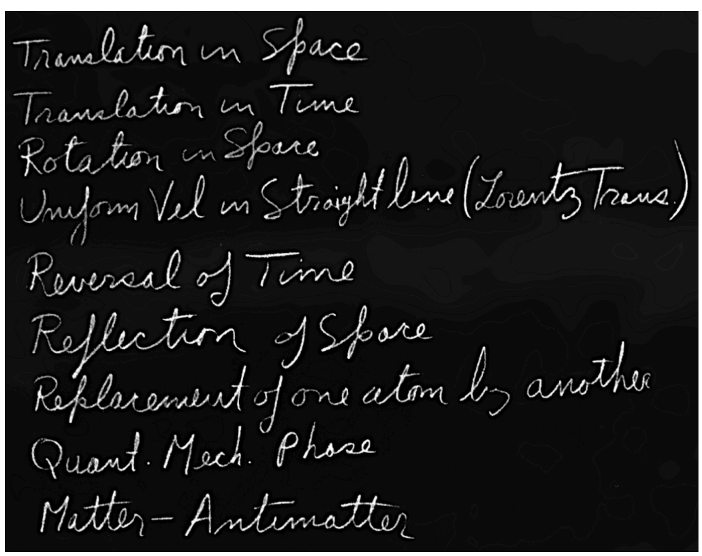


Figure 16. R.P. Feynman, Lectures on Physics, Vol 1, Addison Wesley, Reading, 1964, Sec. 52

[40]C. Brading and E. Castellani, Symmetries in Physics, Philosophical Reflections, Cambridge University Press, Cambridge, 2003.

– Aïssata: It is maybe time for me to learn more on Group theory, because your question refers me to where Brading and Castellani discuss the fact that historically, *symmetries* of physical laws means *invariance* w.r.t transformations under a symmetry group.

Coming back to Feynman, he proposes a list of Symmetry operations, the first three of which only are mentioned in your notes.

▷ 9.3 Gauge choice and conserved quantities

– Diego: Now Aïssata, could you elaborate more on the subtlety between canonical and kinematic momenta for the conservation laws please.

– Aïssata: OK, I will propose an illustration using gauge freedom in the case of a very basic problem. This is the uniform electric field $\mathbf{E}$ in which you release a point charge e from rest somewhere and you want to know the velocity v acquired, say at some distance d apart.

This problem is often investigated by considering a perfect (infinite) capacitor with parallel plates perpendicular to the axis x which create the electric field $\mathbf{E} = E\mathbf{u}_x$. Let us assume that $e > 0$ and that the particle is initially at rest at $x = 0$. Newton's law tells us

$$m\frac{dv(t)}{dt} = eE, \quad \text{hence} \quad v(t) = \frac{eE}{m}t, \quad x(t) = \frac{1}{2}\frac{eE}{m}t^2, \tag{268}$$

where the initial conditions $x(t=0) = 0$, $v(t=0) = 0$ were used. The time delay needed to reach d is given by $t_d = \sqrt{\frac{2md}{eE}}$, and the velocity at this time is

$$v(t_d) = \sqrt{\frac{2eEd}{m}}. \tag{269}$$

One can prefer a faster solution to the same problem using conservation of the energy. The variation of kinetic energy between $x = 0$ and $x = d$ is given by $K(d) - K(0) = \frac{1}{2}mv^2(d)$ and it must be compensated by the variation of potential energy $e(\phi(0) - \phi(d)) = eEd$, hence

$$v(d) = \sqrt{\frac{2eEd}{m}}. \tag{270}$$

This second solution, mathematically elegant, implies a presumption that we will now analyse within Lagrange formalism. As we know now, the Lagrangian of the point charge e in an electromagnetic field is

$$L(\mathbf{x}, \mathbf{v}, t) = \frac{1}{2}m|\mathbf{v}(t)|^2 - q(\phi(\mathbf{x}) - \mathbf{A}(\mathbf{x}) \cdot \mathbf{v}(t)) \tag{271}$$

(the EM field is static, hence there is, *a priori*, no time dependence in the potentials) and the canonical momentum is given by

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}. \tag{272}$$

From the point of view of conservation laws, this Lagrangian formalism is very useful, because if a coordinate q^i doesn't appear in L, the conjugate momentum is conserved:

$$\frac{\partial L}{\partial q^i} = 0 \quad \text{implies} \quad \frac{d}{dt}\frac{\partial L}{\partial v^i} = 0 \quad \text{then} \quad \frac{dp_i}{dt} = 0, \quad \text{or} \quad p_i = \text{const} \tag{273}$$

and if time t doesn't appear in L, the Hamiltonian corresponds to the total energy and is conserved,

$$\frac{\partial L}{\partial t} = 0 \quad \text{or} \quad \frac{d}{dt}(\mathbf{p}\cdot\mathbf{v} - L) = 0 \quad \text{then} \quad \frac{dH}{dt} = 0, \quad \text{or} \quad H = \text{const.} \tag{274}$$

This being clearly stated, let us revisit the uniform electric field, first in the standard gauge (the perfect capacitor problem), then using a somehow less common treatment, but which corresponds to the same physical situation of an uniform and static electric field.

In the standard gauge, $\mathbf{E}$ is associated to a scalar potential $\phi(x) = -Ex$ (fixing $\phi(0) = 0$). The Lagrangian is

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + eEx. \tag{275}$$

It doesn't depend on t, so one has $H = \text{const}$ and $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ leading to

$$\begin{aligned} H &= p\dot{x} - L \\ &= \frac{1}{2}m\dot{x}^2 - eEx = \text{const} = 0 \end{aligned} \tag{276}$$

with the value zero corresponding to the initial conditions.

It yields

$$\dot{x}^2(d) = \frac{2eEd}{m} \tag{277}$$

and this is just the method of energy conservation. *Note that the canonical momentum* $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ *coincides with the kinematic momentum and is not conserved (the particle gets accelerated).*

In an alternative gauge, we do not define the source of the field but it is supposed to be associated to a vector potential $A(t)$ such that $E = -\frac{dA}{dt}$, or $\mathbf{A}(t) = -Et\mathbf{u}_x$. We assume that time variations are slow enough to allow to neglect all induction effets, otherwise $\mathbf{E}$ wouldn't be uniform. The Lagrangian is

$$L(\dot{x}, t) = \frac{1}{2}m\dot{x}^2 - eEt\dot{x}. \tag{278}$$

It doesn't depend on x, hence

$$p_x = m\dot{x} + eA = m\dot{x} - eEt = \text{const} = 0, \tag{279}$$

which *differs from the kinematic momentum*. It follows the same expression for the velocity at distance d:

$$\dot{x}(d) = \frac{eEt_d}{m}. \tag{280}$$

Note that $H = p_x\dot{x} - L$ with $\dot{x} = (p_x + eEt)/m$ and $p_x = 0$ leads after simplifications to $H = \frac{1}{2}m\dot{x}^2$ and is not a conserved quantity. This is just the kinetic energy because there is no potential energy in this gauge.

– Diego: This is illuminating! The choice of gauge in standard electrodynamics has consequences on the conserved quantities, this is incredible!

– Aïssata: You have to take care that the physical quantities are the same with the two approaches, of course. For example in both cases, the kinematic momentum is the same, the kinetic energy is the same. But not the canonical momentum nor the Hamiltonian. These latter quantities which appear in the conservation equations are not automatically physical quantities (in the sense of measurable). These are gauge dependent quantities!

This discussion also emphasizes the role of potentials which *can have a measurable physical meaning when they are expressed in a specific gauge*[41].

– Diego: Thank you Aïssata, you have clarified the things. The rest of the course was more technical, this was devoted to stress-energy in field theory and I am sure that we will have to rediscuss that later.

☐ 9.4 Stress-energy for Newtonian scalar fields

Fo a scalar Newtonian field $\varphi(\mathbf{r},t)$, as we know, the Euler-Lagrange equation reads as

$$\frac{\partial\mathscr{L}}{\partial\varphi}-\partial_t\left(\frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)}\right)-\boldsymbol{\nabla}\cdot\left(\frac{\partial\mathscr{L}}{\partial(\boldsymbol{\nabla}\varphi)}\right)=0 \tag{281}$$

and the Hamiltonian density

$$\mathcal{H}=\pi\partial_t\varphi-\mathscr{L}=\frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)}\partial_t\varphi-\mathscr{L}. \tag{282}$$

Its space integral leads to the total energy (or the Hamiltonian)

$$E=\int d^3r\mathcal{H}. \tag{283}$$

This is a conserved quantity, which means that there exists a continuity equation for $\mathcal{H}$,

$$\frac{\partial\mathcal{H}}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{s}=0 \tag{284}$$

where $\mathbf{s}$ is an energy density current that we are looking for. In order to do so, we develop the combination

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial t}&=\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)}\partial_t\varphi-\mathscr{L}\right),\\&=\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)}\right)\partial_t\varphi+\frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)}\partial_t^2\varphi-\frac{\partial\mathscr{L}}{\partial t}.\end{aligned} \tag{285}$$

We can expand the last term of the r.h.s.

$$\frac{\partial\mathscr{L}}{\partial t}=\frac{\partial\mathscr{L}}{\partial\varphi}\partial_t\varphi+\frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)}\partial_t^2\varphi+\frac{\partial\mathscr{L}}{\partial(\boldsymbol{\nabla}\varphi)}\partial_t(\boldsymbol{\nabla}\varphi) \tag{286}$$

which simplifies the previous expression into

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial t}&=-\boldsymbol{\nabla}\cdot\left(\frac{\partial\mathscr{L}}{\partial(\boldsymbol{\nabla}\varphi)}\right)\partial_t\varphi-\frac{\partial\mathscr{L}}{\partial(\boldsymbol{\nabla}\varphi)}\cdot\boldsymbol{\nabla}(\partial_t\varphi)\\&=-\boldsymbol{\nabla}\cdot\left(\frac{\partial\mathscr{L}}{\partial(\boldsymbol{\nabla}\varphi)}\partial_t\varphi\right)\end{aligned} \tag{287}$$

[41] B. Berche, D. Malterre, E. Medina, Am. J. Phys. **84**, 616, 2016.

and allows to identity the energy density current

$$\mathbf{s} = \frac{\partial \mathscr{L}}{\partial(\boldsymbol{\nabla}\varphi)}\partial_t\varphi. \tag{288}$$

Written in a matrix form, $\mathcal{H}$ is the "time-time" component, while $\mathbf{s}$ is the time-space part of something. There remains a space-time part which is the momentum density $\mathbf{g}$ and a space-space part, the stress tensor, T^{ij} with our standard notation that i and $j = 1, 2, 3$ for the three space components. One writes

$$\begin{pmatrix} \mathcal{H} & \mathbf{s} \\ \mathbf{g} & T^{ij} \end{pmatrix} \tag{289}$$

and by inspection of $\mathbf{s}$, one gets $\mathbf{g}$ by interchange $\partial_t \leftrightarrow \boldsymbol{\nabla}$,

$$\mathbf{g} = \frac{\partial \mathscr{L}}{\partial(\partial_t\varphi)}\boldsymbol{\nabla}\varphi \tag{290}$$

and T^{ij} by inspection of $\mathcal{H}$,

$$T^{ij} = \frac{\partial \mathscr{L}}{\partial(\partial_j\varphi)}\partial^i\varphi - \delta^{ij}\mathscr{L}. \tag{291}$$

The latter expression is the one which will be easily generalized to relativistic notations later.

– Aïssata: You are right Diego. This will be revisited when you will deal with Noether theorems in the relativistic case, since the tensor formalism will enable to write things in a more compact form.

Then, switching to something else, Diego says

– Diego: Aïssata, I wanted to ask you something. You said that you came from Africa with your friend. I never saw him, is he also studying in Physics?

– Aïssata: She, . . . She is in maths. Preparing her PhD in topology.

– Diego: Ah! I didn't understand, I am sorry.

– Diego: You don't have to apologize, there is no problem. You will meet her some day, I am sure. But it seems that I spend more time with you than with her now, she adds, smiling.

10. Day 6 – Conserved currents in field theories

After a few days without meeting again, Diego looks for Aïssata at the university restaurant and finds her, sitting alone at a table, a book in one hand, an apple in the other. She has finished her lunch, but Diego asks to join her.

– Diego: Hello Aïssata. How was your trip to Berlin?

– Aïssata: Hi Diego. This was really great. That was my first trip there and I was excited to be in such a historical place. I was looking everywhere to find remains of the wall, but except in a few places where they kept pieces of the wall, you couldn't imagine how life was in the sixties to eighties. Now Berlin is a very "trendy" place.

– Diego: You know, we have a professor who was young in 1989, and when he heard about the events in Berlin in this night of november, he just took his car and made the thousand kilometers to participate to that historical moment when the wall felt down.

– Aïssata: Wouaw! Impressive. You know, even for us, Africans, far away from Europe, this was a historical moment. People were expecting that something good could come out from these events in Leipzig and Berlin. Some kind of progressive alternative to communism, based on freedom and solidarity. All what USSR didn't achieve! I cannot imagine how they were deceived when a few years later this historical moment appeared to be just the opposite, the starting point of the triumph of the hardest capitalism and individualism. But I think that we are again at the dawn of something which could be the collapse of a system. The climate change doesn't affect too much rich countries, but there are already massive disasters all around the world. The crazy race to economics growth will soon seal its fate. I hope that this time, something positive for human kind will emerge. We need solidarity, more than ever, and this is not compatible with capitalism and profit for a minority.

– Diego: I admire you Aïssata. You know, you are always observing and analyzing the world from far above. You have experienced hard times in your own life but you still have hope in the future. This is a nice way of seeing things.

– Aïssata: Thank you Diego. But I am sure that you wanted to talk about physics. Let's go to work together Aïssata says.

☐ 10.1 Conserved currents from the Maxwell equations of motion

Conserved currents in field theories can be obtained from the equations of motion themselves. A most powerful approach will be presented later with Noether theorems. Here we assume the validity of the equations of motion for a given theory, say Maxwell equations, or Schrödinger equation, then we build a continuity equation. This is not an essential property to have a relativistic or a newtonian theory there.

Consider the paradigmatic example of electrodynamics. Assuming the inhomogeneous Maxwell equations

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{292}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\partial_t \mathbf{E}, \tag{293}$$

we form the quantity

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \mathbf{j} = \varepsilon_0 \partial_t (\boldsymbol{\nabla} \cdot \mathbf{E}) + \frac{1}{\mu_0} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \frac{1}{\mu_0 c^2} \boldsymbol{\nabla} \cdot (\partial_t \mathbf{E}) \tag{294}$$

The term in the middle of the r.h.s. is identically zero and the last two terms cancel each other. It follows the standard

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{295}$$

which, once integrated over a volume Ω, leads to

$$\frac{dQ_\Omega}{dt} = \int_\Omega d^3 r \partial_t \rho = -\oint_{\partial\Omega} \mathbf{j} \cdot d^2\mathbf{Surf} \tag{296}$$

where Q_Ω is the total charge in Ω and where Stokes theorem is used at the r.h.s. If the volume Ω goes to infinity, the current density vanishes (there are no sources at infinity, by hypothesis) and the conservation of the total charge follows.

– Diego: This was an easy lecture, but our professor said that the same strategy could be applied to Schrödinger equation. She even said that we have "necessarily" seen this during our undergraduate course in QM. So none of the students in the class was crazy enough to ask the question, but we don't remember how to proceed in that case. There is no source term like ρ and $\mathbf{j}$ in Schrödinger equation, so there is no starting point...

– Aïssata: Happily, you didn't ask the question. You should know how to proceed, this is indeed part of basic courses, and I am sure also that you have seen this earlier in your curriculum. Do you remember what the probability density is in QM?

– Diego: Yes, this is Born assumption, the probability density is given by $|\varphi(\mathbf{x})|^2$.

– Aïssata: Right. So you can call $\rho(\mathbf{x},t) = |\varphi(\mathbf{x},t)|^2$ and you try to write $\partial_t \rho(\mathbf{x},t)$ as the divergence of something which you will identify as the associated probability current density, up to a minus sign. For that purpose you use the Schrödinger equation multiplied by φ^* and the complex conjugate expression. After a bit of algebra, it follows that

$$i\hbar\partial_t(\varphi^*\varphi) = -\boldsymbol{\nabla}\cdot\left(\frac{\hbar^2}{2m}(\varphi^*\boldsymbol{\nabla}\varphi - (\boldsymbol{\nabla}\varphi^*)\varphi)\right) \tag{297}$$

which identifies

$$\mathbf{j} = \frac{-i\hbar}{2m}(\varphi^*\boldsymbol{\nabla}\varphi - (\boldsymbol{\nabla}\varphi^*)\varphi). \tag{298}$$

Again, use has been made of the equation of motion. This means that if the equation of motion is modified, for example because of the presence of an interaction with an electromagnetic field, the expression for the current density may have to be modified accordingly.

– Diego: Can we consider the case of the Zeeman interaction for example?

– Aïssata: This is not a simple problem, but let's discuss this case in a simplified version. Electrons are particles which carry not only an electric charge as you know, but also a spin $\frac{1}{2}$. In non-relativistic Quantum Mechanics, they obey the Pauli equation which is a generalization of the Schrödinger equation with an essential innovation: the existence of the spin, i.e. a new degree of freedom, coupled to space-time degrees of freedom in the Hamiltonian through interactions such as the spin-orbit interaction and the Zeeman interaction. In the presence of such interactions, spin is not conserved, but spin carries angular momentum and the total angular momentum is conserved. This is at the origin e.g. of the Einstein - de Haas experiment.

Here the starting point is given by the known interaction terms present in the Pauli equation. We will focus attention here only on the consequences of the Zeeman interaction between the magnetic moment associated to the spin of the electron $\frac{e}{m}\mathbf{s}$ (with Landé factor $g_e = 2$ and $\mathbf{s} = \frac{1}{2}\hbar\sigma$) and an external magnetic field $\mathbf{B}$. The state of the particle is described by a Pauli spinor $\psi = \begin{pmatrix} \varphi_\uparrow \\ \varphi_\downarrow \end{pmatrix}$, and, according to (168) the Hamiltonian reads as

$$\mathsf{H} = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2\mathbb{1} + V\mathbb{1} - \frac{e}{m}\mathbf{s}\cdot\mathbf{B}. \tag{299}$$

Following standard textbooks [42] one can build a continuity equation where the charge

[42] L.D. Landau and E.M. Lifshitz, Quantum Mechanics, Butterworth Heinemann, Third English edition, Oxford, 1977; A.S. Davydov, Quantum Mechanics, Pergamon Press, Oxford, 1965.

density and the charge current density are defined according to

$$\rho(\mathbf{r},t) = e\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t), \tag{300}$$

$$\mathbf{J}(\mathbf{r},t) = \frac{-ie\hbar}{2m}(\psi^\dagger(\mathbf{r},t)[\boldsymbol{\nabla}\psi(\mathbf{r},t)] - [\boldsymbol{\nabla}\psi^\dagger(\mathbf{r},t)]\psi(\mathbf{r},t)) - \frac{e^2}{m}\mathbf{A}\psi^\dagger\psi + \frac{e}{m}\boldsymbol{\nabla}\times(\psi^\dagger\mathbf{s}\psi). \tag{301}$$

As we had already anticipated, when a new interaction (here the Zeeman interaction, but also the terms involving the potential vector) appears in the problem, the conserved current gets modified. Here, there are two types of additional contributions, those in **A** and also the rotor of the magnetization associated to the electron density produces an additional charge current density. Note that this last term may be forgotten during the standard derivation using the wave equation and its complex conjugate, since the divergence of a curl vanishes, and some care must be taken to establish the full current density. Nevertheless, this addititonal term, sometimes called "spin term", is compulsory in order to obtain a conservation equation[43].

The spin is a contribution to the total angular momentum and this latter quantity is of course conserved. Our discussion thus suggests that there should be a way to write a conservation equation where spin components would appear explicitly, together with other sources of angular momentum. The "spin density" components $s^a(\mathbf{r},t) = \psi^\dagger s^a\psi$ are, in a sense equivalent to the charge density, except that they carry the spin index a. Deriving such a conservation equation could be the purpose of some interesting homework[44].

11. Day 7 – Finite dimensional representations of Lie groups

□ 11.1 Lie groups and Lie algebras

• Group: A group is a collection of objects $g \in G$ and a multiplication or composition operation $\times$ under which G is closed, i.e. for g_l and g_m in G, $g_m \times g_l$ is also in G. For the composition operation you can also find the notation $g_m \circ g_l$, or just $g_m g_l$, with the meaning that g_l acts first, then only g_m acts (say on the right). The multiplication is associative, $g_n(g_m g_l) = (g_n g_m)g_l$, there is a neutral element e such that $g_l e = e g_l = g_l$ and all elements have an inverse in G such that $g_l g_l{}^{-1} = g_l{}^{-1} g_l = e$. We discuss further properties below.

• Representation: A linear representation R assigns to each element $g \in G$ a linear operator $\mathsf{D}_R(g)$ (which depends on the representation) which maps the group multiplication law to the multiplication of operators, i.e. with the properties:

(1) $\mathsf{D}_R(e) = 1$ with e the identity element of G, and 1 the identity operator,

(2) $\mathsf{D}_R(g_m)\mathsf{D}_R(g_l) = \mathsf{D}_R(g_m g_l)$ to preserve the group structure.

If a representation is irreducible, the vectors on which the operators act are completely mixed up. In a matrix representation, if the matrix has a block diagonal structure, the representation is on the contrary reducible.

[43] B. Berche and E. Medina, Eur. J. Phys. **34**, 161, 2013.

[44] We will discuss this when the effect of spin-orbit interactions will be considered in condensed matter systems.

• Lie group: A Lie group is a group of elements which depend in a continuous and differentiable manner on a set of real parameters θ^a, $a = 1, \dots n$. We denote the elements $g(\theta^a)$ (instead of discrete elements g_l) and choose the parameters such that $g(0) = e$, the identity element.

• Lie algebra: In the neighborhood of the identity, we write

$$\mathrm{D}_R(g(\theta)) \simeq 1 + i\theta_a \mathrm{t}^a{}_R, \quad \mathrm{t}^a{}_R = -i \left.\frac{\partial \mathrm{D}_R}{\partial \theta_a}\right|_{\theta=0}. \tag{302}$$

The $\mathrm{t}^a{}_R$'s are the generators of the group G in the representation R. Far from identity we have

$$\mathrm{D}_R(g(\theta)) = e^{i\theta_a \mathrm{t}^a{}_R}. \tag{303}$$

Note that the sign in the exponential (even the i) is a matter of convention. For Lie groups, the representation is unitary, therefore, with our convention, $\mathrm{t}^a{}_R$ are hermitian operators.

Given two such group elements with parameters α_a and β_a, $\mathrm{D}_R(g_l) = e^{i\alpha_a \mathrm{t}^a{}_R}$ and $\mathrm{D}_R(g_m) = e^{i\beta_b \mathrm{t}^b{}_R}$, the demand $\mathrm{D}_R(g_l)\mathrm{D}_R(g_m) = \mathrm{D}_R(g_l g_m)$ requires that there are γ_c's such that $\mathrm{D}_R(g_l)\mathrm{D}_R(g_m) = e^{i\gamma_c \mathrm{t}^c{}_R}$, with $\gamma_c = \alpha_c + \beta_c - \frac{1}{2}\alpha_a \beta_b f^{ab}{}_c$ which implies the structure of Lie algebra

$$[\mathrm{t}^a, \mathrm{t}^b] = i f^{ab}{}_c \mathrm{t}^c \tag{304}$$

where the form of the generators depends on the representation, but not the Lie algebra structure, i.e. the values of the structure constants $f^{ab}{}_c$ (note the use of the summation over repeated indices). Equation (304) is the central piece of Lie algebra.

– Diego: Aïssata, the last course was devoted to a pretty short introduction to Lie groups, representations, Lie algebras. Could you give me more details and explanations of what we are doing there and why?

– Aïssata: Maybe I should first remind you a few basic ingredients on Lie groups and Lie algebras and then come back to their use in Physics to show you for example how the generators of space-time/spacetime symmetries appear in the theory.

– Diego: Yes, please proceed, because I am not used to this language which is new for me.

– Aïssata: It is useful to consider infinitesimal transformations to address these questions. In fact, maybe you ignore it, but you heard about Lie groups and Lie algebras, and about their representations already in your courses on angular momentum in Quantum Mechanics. Even the term of Lie algebra, for sure, was used in these courses. Indeed in quantum mechanics, symmetries are represented by unitary operators, since they preserve the norm of quantum states. When infinitesimal transformations are considered, they can be written in the vicinity of the identity transformation, this is an essential property. The same type of strategy holds for any kind of continuous group in a certain sense.

First you have to understand that a representation is a correspondence between operators (we will start, to be concrete and because this is the most useful way to speak about representations, by considering that these operators can be represented by matrices) and group elements. Each group element has an associated matrix. Using the property of continuity of Lie groups with a set of parameters, and the fact that the identity matrix is associated to the neutral element of the group[45], as I said earlier, we can expand matrices representing group elements in the neighborhood of the identity matrix. The deviation from identity defines the *generators* of the group and these generators obey specific commutation relations which define an algebra. You have to understand the status of the various quantities in (304). The t^a's are operators, e.g. matrices, and the $f^{ab}{}_c$'s are just numbers. To avoid confusion, one could write operators with hats, like $\hat{\mathsf{t}}_a$ or $\hat{\mathsf{D}}_R(g)$ but this is usually not done in the literature, since the context is enough to understand what we are speaking about.

– Diego: Aïssata, could you give me examples of what is a Lie group, and of what is not a Lie group?

▷ 11.2 The group of permutations

– Aïssata: Maybe an example of what is not a Lie group first. Imagine a collection of three ordered objects ♀, ⚥, ♂ and consider the group of permutations among these elements, i.e. all the possible manners to write these three elements. A permutation sends ♀, ⚥, ♂ onto any of these different manners. There are 3! permutations, including the neutral element which sends ♀, ⚥, ♂ onto itself. This is clear that the permutations form a group, as you can check writing explicitly the table of the group. This is a discrete group. It is easy to find a 3×3 representation, associating a column vector to each configuration. For example if we denote by indexing with the final state, $P_{⚥♀♂}$, the permutation which sends the configuration ♀ ⚥ ♂ onto ⚥ ♀ ♂, a representation is such that

$$\mathsf{D}_R(P_{⚥♀♂}) \begin{pmatrix} ♀ \\ ⚥ \\ ♂ \end{pmatrix} = \begin{pmatrix} ⚥ \\ ♀ \\ ♂ \end{pmatrix}, \quad \mathsf{D}_R(P_{⚥♀♂}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{305}$$

and you can build the five remaining matrices.

– Diego: And now an example of a Lie group?

▷ 11.3 An example of Lie group: $SU(2)$

– Aïssata: An example of a very useful Lie group in Physics is $SU(2)$, the Special Unitary group of 2×2 matrices U, i.e. 2×2 matrices with complex entries and determinant 1. This is an example of a group defined directly by matrices and by their multiplication in the first place, which provides directly a representation (we call it a fundamental representation).

Any matrix in $SU(2)$ has the property $\mathsf{U}^\dagger \mathsf{U} = \mathsf{U}\mathsf{U}^\dagger = 1$ and can be written as

$$\mathsf{U} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C} \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1. \tag{306}$$

If we write $\alpha = a + ib$ and $\beta = c + id$ with a, b, c, d reals, we have automatically

$$\begin{aligned} \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} &= a1 + ib \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + ic \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + id \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= a1 + ib\sigma_3 + ic\sigma_2 + id\sigma_1 \end{aligned} \tag{307}$$

[45]We ignore here the situations in which the group is not simply connected.

where the σ_a's are the Pauli matrices and provide a representation of $SU(2)$. They obey the following algebra

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma^c \tag{308}$$

with structure constants $f_{abc} = \epsilon_{abc}$, the Levi-Civita symbol. You have probably noticed that I keep upper indices and lower indices to conform to the summation rule, but, owing to the Euclidean structure of the underlying manifold of the algebra, the indices position doesn't matter here. There is thus no distinction between co- and contravariant components and the Einstein summation convention is over repeated indices, whatever their positions, although we still take care about them. The *abc* indices are like Cartesian indices[46] and they are raised and lowered with the Kronecker δ, e.g. $\sigma_a = \delta_{ab}\sigma^b$ and the Levi-Civita symbol is such that $\epsilon^{abc} = \epsilon_{abc}$. The same holds in your professor's notes, e.g. in equations (303) and (304) where e.g. $\theta_a = \delta_{ab}\theta^b$ and $f^{ab}{}_c = \delta_{cd} f^{abd}$ which is also f^{abc} [47].

▷ 11.4 Terminology of matrix Lie groups

– Aïssata: Matrix Lie groups are classified and they have quite obscure names. I can make a bit of terminology:

- ☐ General Linear group:

$$GL(N,\mathbb{R}) = \{\text{real } N\times N \text{ invertible matrices } \mathsf{A}, \text{ i.e. } \det\mathsf{A} \neq 0\}$$

- ☐ Special linear:

$$SL(N,\mathbb{R}) = \{\mathsf{A}\in GL(N,\mathbb{R}), \quad \det\mathsf{A} = 1\}$$

- ☐ Orthogonal:

$$O(N) = \{\mathsf{A}\in GL(N,\mathbb{R}), \quad \mathsf{A}^{\mathrm{T}} = \mathsf{A}^{-1}\}, \quad \text{then} \quad \det\mathsf{A} = \pm 1$$

- ☐ Special Orthogonal:

$$SO(N) = \{\mathsf{A}\in O(N), \quad \det\mathsf{A} = 1\}$$

- ☐ Unitary:

$$U(N) = \{\mathsf{A}\in GL(N,\mathbb{C}) \quad \mathsf{A}^{\mathrm{T}} = \mathsf{A}^{-1}\}, \quad \text{then} \quad |\det\mathsf{A}| = 1$$

- ☐ Special Unitary:

$$SU(N) = \{\mathsf{A}\in U(N), \quad \det\mathsf{A} = 1\}$$

The space on which $\mathsf{D}_R(g)$ acts (e.g. a vector space $\{\phi^A\}_{A=1\ldots N}$ of finite dimension) is the *basis of the representation*. The typical examples that we are studying are those of matrix representations where $\mathsf{D}_R(g)$ are $N\times N$ matrices. A change of representation changes the form of the matrix. Some authors speak about the vector space on which the operators act as *the representation*. For example we can read that vectors or tensors are representations of the Lorentz group[48].

[46] They could have been denoted xyz and this done often in the case of the Pauli matrices.
[47] This is why most of the authors would simply write $[\mathrm{t}_a, \mathrm{t}_b] = i f_{abc}\mathrm{t}_c$ instead of (304).
[48] e.g. M.D. Schwartz, Quantum Field Theory and the Standard Model, Cambridge University Press, Cambridge, 2014, p. 158.

– Diego: I understand that equation (304) seems to follow from the calculus, but it appears a bit mysterious. Is there a reason behind it?

– Aïssata: You have a good feeling, yes, there is a simple intuition for that. Remember that because of the group structure, the multiplication of group elements $g_l g_m$ is another element of the group. Thus, the difference $g_l g_m - g_m g_l$ can be written as a linear combination of other group elements and this property is also inherited by the operators or matrices $\mathsf{D}_R(g_l)\mathsf{D}_R(g_m)$ and by the corresponding generators.

The Lie algebra is a "robust structure" (i.e. independent of the representation) characteristic of the Lie group. We talk about an algebra instead of a group, because while group elements can only be multiplied, the generators of the algebra can be multiplied, but also added. Note that in the case of an Abelian group (when group elements commute, i.e. we also say commutative group) the structure constants are all vanishing and the commutators of the generators all equal zero.

▷ **11.5 Representations of** $SO(3)$

– Aïssata: Now, let us proceed with another physically important example. A case, well-known to all students, is the Lie algebra of angular momentum, called $\mathfrak{so}(3)$, which is associated to the rotation group in 3 dimensions.[49] A finite rotation $R_{\alpha\mathbf{u}_x}$ of angle α around the x axis is *represented* by a matrix in Cartesian coordinates

$$\mathsf{D}_{\alpha\mathbf{u}_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \tag{309}$$

and the corresponding infinitesimal rotation $R_{\delta\alpha\mathbf{u}_x}$ of angle $\delta\alpha$ around the same axis by the matrix

$$\mathsf{D}_{\delta\alpha\mathbf{u}_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\delta\alpha \\ 0 & \delta\alpha & 1 \end{pmatrix}. \tag{310}$$

Take care that in the following, I will consider *active transformations* which act on vectors and are maybe more frequently used in undergrad studies, i.e. with the opposite signs for the angles of rotations than in the case of passive transformations where the axes are rotated, instead of the vectors. We will come back to this question for the discussion about the Lorentz group.

This is an exemple of a group element written in the 3−dimensional representation of $SO(3)$. Due to the 2π periodicity the rotation group is a *compact* group. In the vicinity of the identity, a group element is written as [50]

$$\mathsf{D}_{\delta\alpha\mathbf{u}_x} = \mathbb{1}_3 + \delta\alpha\mathsf{M}_x \tag{311}$$

[49] The group is called $SO(3)$ and the algebra $\mathfrak{so}(3)$, but we are flexible on this notation and terminology.

[50] A caveat is in order here with respect to equation (302). Some authors choose a different sign in (302), have or do not have an i, therefore, the explicite expressions for the generators can change (their hermiticity properties also) and the commutation relations are obviously modified accordingly. For the matrix representation of the generators, we conform here to the choice of e.g. F. Laloë, Cours de DEA sur les symetries, cel-00092953, https://cel.archives-ouvertes.fr/cel-00092953/document

with M_x which therefore has the matrix representation

$$\mathsf{M}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{312}$$

Infinitesimal rotations around y and z axes respectively lead in a similar manner to the matrices M_y and M_z,

$$\mathsf{D}_{\delta\alpha\mathbf{u}_y} = \begin{pmatrix} 1 & 0 & \delta\alpha \\ 0 & 1 & 0 \\ -\delta\alpha & 0 & 1 \end{pmatrix}, \quad \mathsf{M}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{313}$$

$$\mathsf{D}_{\delta\alpha\mathbf{u}_z} = \begin{pmatrix} 1 & -\delta\alpha & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{M}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{314}$$

This is now easy to obtain by simple matrix calculation the algebra among the M_i's, e.g.

$$[\mathsf{M}_x, \mathsf{M}_y] = \mathsf{M}_z \tag{315}$$

and cyclic permutations of x, y, z.

It is useful to introduce the generators L_a's, $\mathsf{M}_a = -i\mathsf{L}_a$, in terms of which the algebra takes the well known form, that you should recognize,

$$[\mathsf{L}_a, \mathsf{L}_b] = i\epsilon_{ab}{}^c\mathsf{L}_c \tag{316}$$

with $\mathsf{L}_a = \mathsf{L}_x, \mathsf{L}_y, \mathsf{L}_z$ for $a = 1, 2, 3$ and as usually ϵ_{abc} the antisymmetric symbol.

– Diego: This is the algebra of the angular momentum, or of the Pauli matrices!

– Aïssata: Correct, up to factors $\frac{1}{2}$ in the case of Pauli matrices. You will see that there is a link a bit later. I will tell you more, Aïssata says, smiling.

– Diego: Aïssata, why did you change the ijk indices for the abc indices?

– Aïssata: Very often, the generators of a Lie algebra act upon objects which live in abstract spaces. For example, this was the case with $SU(2)$ that we discussed earlier. The rotation group in $3D$ acts on ordinary vectors and there are exactly three generators, for rotations around each of the three Cartesian axes, this is why the usual notation xyz or ijk taking values from 1 to 3 is often used there.

But what would you say about $SO(4)$, rotations in $4D$ space? There are now six generators, because a rotation is properly defined by the plane in which the rotation takes place rather than by the invariant remaining subspace and there are six of these planes in $4D$. I prefer in this case the use of abc indices, which vary from 1 to 6 for $SO(4)$, or even from 1 to 8 for $SU(3)$ which possesses eight generators! So, in order to anticipate on further notations, I try to conform to abc indices for the generators of Lie algebras, although you will see that the generators can be related to physical objects which may be ordinary vectors, and in this case I will conform to the standard ijk choice.

– Diego: Notations, notations, ... Diego says, sighing.

– Aïssata: But let us go on with rotations in $3D$. In terms of the L_a's generators, the infinitesimal rotation around an arbitrary vector $\mathbf{u}$ takes the form

$$\mathsf{D}_{\delta\alpha\mathbf{u}} = \mathbb{1}_3 - i\delta\alpha\mathbf{u}\cdot\mathbf{L} = \mathbb{1}_3 - i\delta\alpha^a\mathsf{L}_a \tag{317}$$

with L the *vector of matrices* $\mathsf{L} = \mathbf{u}_x \otimes \mathsf{L}_x + \mathbf{u}_y \otimes \mathsf{L}_y + \mathbf{u}_z \otimes \mathsf{L}_z$ and

$$\mathsf{L}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathsf{L}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \mathsf{L}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{318}$$

Take care to an expression like $\mathbf{u}_x \otimes \mathsf{L}_x$, the matrix L_x doesn't act on the vector $\mathbf{u}_x$ and this is the reason of the Kronecker matrix $\otimes$ product. The matrix and the vector are not written in the same basis! Expanding the Kronecker product, we have

$$\mathbf{u}_x \otimes \mathsf{L}_x = \begin{pmatrix} \mathsf{L}_x \\ 0 \\ 0 \end{pmatrix} \tag{319}$$

which has nothing to do with the operation $\mathsf{L}_x \mathbf{u}_x$ which would lead to an ordinary vector

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{320}$$

if the matrix was acting on the vector!

These newly introduced operators are hermitian, $\mathsf{L}_a^\dagger = \mathsf{L}_a$ and the representation (317) is unitary. Take care to the sign in (317), as compared with (302). The commutation relations would be modified if the sign (which is conventional) would be changed and I conform here to our choice in Physics as I said before[51].

– Diego: I don't recognize the generators of rotation algebra that we have studied in Quantum Mechanics. There I remember that we have specifically built a basis in which L_z takes a diagonal form, together with the square of the angular momentum, and in the above formulas, none of the components L_a has a diagonal structure.

– Aïssata: Well spotted Diego. Equation (316) is an example of the more generic commutation relations known in angular momenta algebra in quantum mechanics, you are right, and the form which I have shown is called the *adjoint* representation. We built it as the representation which acts on vectors of ordinary space:

$$\begin{aligned} \exp(-i\alpha \mathbf{u}_x \cdot \mathsf{L}) = e^{-i\alpha \mathsf{L}_x} \quad &= \quad \exp\left[-i\alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \right] \\ &= \quad 1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix}^2 + \dots \\ &= \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = \mathsf{D}_{\alpha \mathbf{u}_x} \end{aligned} \tag{321}$$

and we have the property

$$\mathsf{D}_{\alpha \mathbf{u}_x} \mathbf{x} = \mathbf{x}' \tag{322}$$

[51] Mathematicians like (311).

with $\mathbf{x}' = \mathbf{x}\cos\alpha + x(1-\cos\alpha)\mathbf{u}_x + \sin\alpha(\mathbf{u}_x \times \mathbf{x})$ as given by the Rodrigues rotation formula.

It is equivalent, but different from the one that you are used to, which acts on quantum states. This latter representation that you know in QM is based on Casimir operators:

A Casimir operator of a Lie group is an operator which commutes with all the generators. There exists an irreducible representation in which the Casimir is proportional to the identity operator and the coefficients of proportionality can be used to label the irreducible representations.

Let us now denote the $SO(3)$ generators as J_a, $a = 1,2,3$, and the commutation relations

$$[\mathsf{J}_a, \mathsf{J}_b] = i\epsilon_{ab}{}^{c}\mathsf{J}_c. \tag{323}$$

With this new notations, J_1, J_2, J_3 correspond to the previous L_x, L_y and L_z. The *adjoint* representation given above is simply

$$(\mathsf{J}_a)_{bc} = -i\epsilon_{abc} \tag{324}$$

with a the label of the generator and b and c the line and column indices. You can easily convince yourself that this reproduces correctly the matrices (318). For an arbitrary Lie algebra with structure constants f_{abc}, one would have in the adjoint representation $(\mathsf{J}_a)_{bc} = -if_{abc}$ which has the dimension of the number of generators (3 here, but for example 8 for $SU(3)$ as indicated above).

With the algebra (323), the Casimir is the square of the angular momentum

$$\mathsf{J}^2 = (\mathsf{J}_1)^2 + (\mathsf{J}_2)^2 + (\mathsf{J}_3)^2, \tag{325}$$

since $[\mathsf{J}^2, \mathsf{J}_a] = 0$. It has the form $j(j+1)\mathbb{1}$ in the irreducible representations of dimensions $(2j+1)$, with $\mathbb{1}$ the $(2j+1)\times(2j+1)$ identity matrix. The irreducible representations are conveniently labelled by the values of j and are called the *fundamental* representations.

– Diego: And the representations of the group are given by the matrices representing the generators J_a's for $j=0$, $j=1$, $j=2$, etc, with respective dimensions 1×1, 3×3, 5×5, etc.

For example for $j=0$, the irreducible representation is one-dimensional and the commutations relations $[\mathsf{J}_a, \mathsf{J}_b] = i\epsilon_{ab}{}^{c}\mathsf{J}_c$ are satisfied with all $\mathsf{J}_a = 0$. For $j=1$ we have the standard representation in terms of 3×3 matrices. I know how we build them. First you notice that since $[\mathsf{J}^2, \mathsf{J}_a] = 0$, it is possible to find a basis which diagonalizes simultaneously J^2 and one of the J_a's. One only of these three components, because they do not commute with each other. The convention is to choose J^2 and J_3 and we denote $|j,m\rangle$ the corresponding basis states. At fixed j, J^2 is a multiple of the identity matrix and the corresponding factor is denoted as $j(j+1)$, even though we don't know yet what j is[52]. Then, we build *ladder operators* $\mathsf{J}_\pm = \mathsf{J}_1 \pm i\mathsf{J}_2$ which have the property that $[\mathsf{J}^2, \mathsf{J}_\pm] = 0$, $[\mathsf{J}_3, \mathsf{J}_\pm] = \pm\mathsf{J}_\pm$. Manipulating these commutation relations and matrix elements $\langle j,m|\mathsf{J}_+\mathsf{J}_-|j,m\rangle$ or their complex conjugate, it is easy to prove that[53]

$$\mathsf{J}_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}|j,m\pm 1\rangle \tag{326}$$

[52]It is real, because J^2 is hermitian.

[53]There could be a phase multiplying the factor $\sqrt{j(j+1)-m(m\pm 1)}$, but the convention is to take it equal to unity.

and that $-j \leqslant m \leqslant j$, $2j \in \mathbb{N}$. The expressions for the $(2j+1) \times (2j+1)$ matrices representing the $\mathsf{J}_a's$ follow, e.g. for $j = 1$:

$$\mathsf{J}_1^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{J}_2^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathsf{J}_3^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{327}$$

These matrices also obey (323)!

– Aïssata: Correct, both (318) and (327) are 3−dimensional representations of $SO(3)$. There is just a change of basis between the two forms which are equivalent, i.e. $\exists \mathsf{P}$ such that $\mathsf{P}\mathsf{J}_a^{(1)}\mathsf{P}^\dagger = \mathsf{L}_a$. You can work out the correct expression which makes it:

$$\mathsf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{328}$$

The latter representation (327) acts on states $|j, m\rangle$, i.e. quantum states. It is also called the spherical basis representation, since the states $|j, m\rangle$ are in fact the spherical harmonics,

$$Y_j^m(\theta, \varphi) = \langle \theta, \varphi | j, m \rangle. \tag{329}$$

An important difference is that the adjoint representation has automatically a dimension given by the number of generators while the spherical representations are of dimensions $(2j+1)$. For example there is a representation for $j = 2$ given by the matrices

$$\mathsf{J}_1^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \tag{330}$$

$$\mathsf{J}_2^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & -i\sqrt{6} & 0 & 0 \\ 0 & i\sqrt{6} & 0 & -i\sqrt{6} & 0 \\ 0 & 0 & i\sqrt{6} & 0 & -2i \\ 0 & 0 & 0 & 2i & 0 \end{pmatrix}, \tag{331}$$

$$\mathsf{J}_3^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{332}$$

– Diego: Clearly, this representation cannot act on ordinary $3D$ vectors!

– Aïssata: Yeah! You got the point!

– Aïssata: There is another important point which you may miss, although you have given the answer already. You should remember that half-integer values of j are also allowed (by the way, by half-integer, I mean here half-*odd* integer, since an integer is also a half-integer! So $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2\ldots$ are all half-integers). For $j = \frac{1}{2}$, you obtain for example the spin $\frac{1}{2}$ matrices in the fundamental representation which are half the Pauli matrices,

$$\mathsf{J}_1^{(\frac{1}{2})} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{J}_2^{(\frac{1}{2})} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathsf{J}_3^{(\frac{1}{2})} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{333}$$

and for $j = \frac{3}{2}$ you get

$$\mathsf{J}_1^{(\frac{3}{2})} = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \tag{334}$$

$$\mathsf{J}_2^{(\frac{3}{2})} = \frac{1}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}, \tag{335}$$

$$\mathsf{J}_3^{(\frac{3}{2})} = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \tag{336}$$

You should remember about the group $SU(2)$, the group of unitary matrices of determinant 1 that I introduced a bit earlier:

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{337}$$

The group generators of $SU(2)$ obey the same algebra than those of $SO(3)$. This is something that you have noticed yourself. So, $SU(2)$ is an alternative way to speak about rotations in $3D$.

Furthermore, like $SO(2)$ (rotations in $2D$) has a deep connection with phase transformations of unit complex numbers, $U(1)$, there is a close connection between $SO(3)$ and quaternions. There is an isomorphism between $SU(2)$ and the 3 dimensional unit sphere $\mathbb{S}^3$,

$$\varphi : \quad \mathbb{S}^3 \subset \mathbb{R}^4 \quad \to SU(2) \tag{338}$$

$$(a, b, c, d) \quad \mapsto \begin{pmatrix} a + ib & -c + id \\ c + id & a - ib \end{pmatrix} \tag{339}$$

with $a^2 + b^2 + c^2 + d^2 = 1$. $\mathbb{S}^3$ is said to be the manifold of $SU(2)$. A basis of $SU(2)$ is given by the Pauli matrices, we have learned that a few minutes ago,

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3), \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{340}$$

The link with rotations in $\mathbb{E}^3$ is made transparent using Rodrigues rotation formula, which gives, as we have seen with $SO(3)$, the vector $\mathbf{x}'$ obtained after applying a rotation $\mathsf{R}_{\mathbf{n}}[\theta]$ of angle θ around an axis of unit vector $\mathbf{n}$ on an initial vector $\mathbf{x}$ in terms of components along $\mathbf{x}$, along $\mathbf{n}$ and perpendicular to both $\mathbf{x}$ and $\mathbf{n}$:

$$\mathbf{x}' = \mathsf{R}_{\mathbf{n}}[\theta]\mathbf{x} = \mathbf{x}\cos\theta + \mathbf{n}\,(\mathbf{n}\cdot\mathbf{x})(1-\cos\theta) + (\mathbf{n}\times\mathbf{x})\sin\theta. \tag{341}$$

So, one first considers a unit vector with 4 Cartesian components, $u \in \mathbb{S}^3$, and denote its components as $u = (u_0, \mathbf{u})^{\mathrm{T}}$ with $u_0 = \cos\frac{\theta}{2}$. Therefore, $u^2 = 1 = u_0^2 + |\mathbf{u}|^2$ leads to write $\mathbf{u} = \mathbf{n}\sin\frac{\theta}{2}$ with $|\mathbf{n}| = 1$. We contemplate rotations $\mathsf{R}_{\mathbf{n}}[\theta]$ in $\mathbb{E}^3$. There is a one-to-one correspondence between $\mathsf{R}_{\mathbf{n}}[\theta]$ and $(u, -u)$ (two opposite directions of $\mathbb{S}^3$ correspond to the same rotation, because $\theta \to \theta + 2\pi$ leads to $\cos\frac{\theta}{2} \to -\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2} \to -\sin\frac{\theta}{2}$ and thus to $u \to -u$). Technically there is a isomorphism between $SO(3)$ and $\mathbb{S}^2/\mathbb{Z}^2$ (the 3-sphere with identification of opposite points). Now, we want to find $SU(2)$ matrices there. For all u, we consider the 2×2 matrix

$$\mathsf{U}_{\mathbf{n}}[\theta] = u_0 1 - i\mathbf{u}\cdot\boldsymbol{\sigma} \tag{342}$$

with $\boldsymbol{\sigma}$ the vector of Pauli matrices. It is clear that $U_{\mathbf{n}}[\theta] \in SU(2)$ (calculate its determinant to check). Using the properties of Pauli matrices, it is easy to compute

$$\mathsf{U}_{\mathbf{n}}[\theta] = \cos\frac{\theta}{2}1 - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2} = e^{-i\frac{\theta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{343}$$

This is performed via the identification of the Taylor expansion of the two expressions and use of $(\boldsymbol{\sigma})^2 = 1$.

The link with $\mathbb{E}^3$ rotations is made clear if, for any vector $\mathbf{x}$ with components (x_1, x_2, x_3) we build the matrix

$$\mathsf{X} = \mathbf{x}\cdot\boldsymbol{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \tag{344}$$

and we consider the map

$$\mathsf{X} \mapsto \mathsf{X}' = \mathsf{U}_{\mathbf{n}}[\theta]\mathsf{X}\mathsf{U}^{\dagger}_{\mathbf{n}}[\theta] \tag{345}$$

which preserves the determinant, $\det\mathsf{X}' = \det\mathsf{X}$. After some algebra, you get that

$$\mathsf{X}' = (\mathbf{x}\cos\theta + \mathbf{n}\,(\mathbf{n}\cdot\mathbf{x})(1-\cos\theta)1 + (\mathbf{n}\times\mathbf{x})\sin\theta)\cdot\boldsymbol{\sigma} \tag{346}$$

which is nothing but the Rodrigues rotation formula in terms of $SU(2)$ matrices. It follows that the transformation $\mathsf{X} \to \mathsf{X}'$ in $SU(2)$ describes a rotation which maps $\mathbf{x}$ to $\mathbf{x}'$, i.e. by a rotation of angle θ around $\mathbf{n}$. To the product of matrices $\mathsf{U}_{\mathbf{n}}[\theta]\mathsf{U}_{\mathbf{n}'}(\theta')$ in $SU(2)$ there corresponds the product of rotations in $SO(3)$ and there is an homomorphism which maps U and $-\mathsf{U}$ on the same rotation. Technically, $SU(2)$ is said to be the *double cover* of $SO(3)$.

– Diego: Impressive! So essentially, $SO(3)$ and $SU(2)$ both describe rotations in $3D$. And they have the same Lie algebra (323).

– Aïssata: Yes, both groups describe rotations in $3D$, but you see that they describe *rotations of different mathematical objects*. A Lie algebra on the other hand can be common to several groups. This is an example here. You will see other examples with the Lorentz group and its connection with $SO(3)\times SO(3)$.

– Diego: You also mentioned quaternions Aïssata. I heard about this as a generalization of complex numbers.

– Aïssata: This is true. Quaternions were invented by Hamilton and you can think about them in terms of 4−dimensional complex numbers. You may look e.g. at Schwichtenberg[54]. You can define the set of quaternions as follows

$$\mathbb{Q} = \{q = a1 + b\text{i} + c\text{j} + d\text{k}, \quad \text{i}^2 = \text{j}^2 = \text{k}^2 = -1, \quad \text{ijk} = -1\} \tag{347}$$

The complex conjugate is extended to a quaternion conjugate $q^\dagger = a1 - b\text{i} - c\text{j} - d\text{k}$ and there is a mapping between unit quaternions $q^\dagger q = 1 = a^2 + b^2 + c^2 + d^2$ and $SU(2)$ matrices via the identification $\text{i} = \sigma_1$, $\text{j} = \sigma_2$ and $\text{k} = \sigma_3$:

$$q = \begin{pmatrix} a + ib & c - id \\ c + id & a - ib \end{pmatrix} \tag{348}$$

with $\det q = q^\dagger q$.

I don't really know much more on the use of quaternions in Physics, but if you are interested, you can surely learn a lot more in specialized books[55], but I leave you work it by yourself. It is time to leave now!

▷ 11.7 $SO(4)$ and the energies and degeneracies of the hydrogen atom bound states

– Aïssata: There is a famous application of group theory in Quantum Mechanics which was proposed by W. Pauli as early as 1926 to build the spectrum of the Hydrogen atom. I think that it is instructive to learn about it. A nice treatment is proposed in Schiff[56], but a nice mathematician's perspective can be found also in the recent book *Lectures on Quantum Mechanics* by P.L. Bowers[57].

In classical mechanics, one knows that Kepler problem

$$H = \frac{\mathbf{p}^2}{2m} - \frac{\kappa}{r} \tag{349}$$

has bound solutions which are closed ellipses. The conservation of the energy H and of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ give enough constraints to show that the trajectories lie in a plane (the plane of the ecliptic), but not to impose the closure of the ellipses. For that, there should exist another constant of the motion which can be used to determine the orientation of the major axis of the ellipse in the ecliptic plane. Such a conserved quantity is known, this is the Lenz vector

$$\mathbf{M} = \frac{\mathbf{p} \times L}{m} - \frac{\kappa}{r}\mathbf{r}. \tag{350}$$

This vector has a magnitude $\kappa\epsilon$ in terms of the eccentricity ϵ of the trajectory, and it lies along the major axis.

[54] J. Schwichtenberg, Physics from Symmetry, Springer, Cham, 2015.

[55] C. Doran and A. Lasenby, Geometric algebra for Physicists, Cambridge University Press, Cambridge, 2003.

[56] L.I. Schiff, Quantum Mechanics, MacGraw-Hill, New-York, 1968, p. 234.

[57] P.L. Bowers, Lectures on Quantum Mechanics, Cambridge University Press, Cambridge, 2020.

The hydrogen atom corresponds to the quantum mechanical version of Kepler problem, with $\kappa = e^2 = |q_e|^2/(4\pi\varepsilon_0)$. The QM version of the Lenz vector requires symmetrization, since $\mathbf{p}$ and $\mathbf{L}$ do not commute,

$$\mathbf{M} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{\kappa}{r}\mathbf{r}. \tag{351}$$

After a considerable amount of computation, Schiff says, the following properties can be found

$$[\mathbf{M}, \mathsf{H}] = 0, \quad \mathbf{L} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{L} = 0, \tag{352}$$

$$\mathbf{M}^2 = \frac{2\mathsf{H}}{m}(\mathbf{L}^2 + \hbar^2 1) + \kappa^2 1. \tag{353}$$

The *difficult to calculate* (Schiff) commutation relations also follow,

$$[\mathsf{L}_i, \mathsf{L}_j] = i\hbar\epsilon_{ij}{}^k\mathsf{L}_k, \quad [\mathsf{M}_i, \mathsf{L}_j] = i\hbar\epsilon_{ij}{}^k\mathsf{M}_k, \quad [\mathsf{M}_i, \mathsf{M}_j] = -\frac{2i\hbar}{m}\epsilon_{ij}{}^k\mathsf{H}\mathsf{L}_k. \tag{354}$$

Since H commutes with the L_i's and M_i's, we can work in a subspace of the Hilbert space corresponding to a fixed energy E. Then, it is convenient to rescale $\mathbf{M}$

$$\mathbf{M}' = \left(\frac{m}{-2E}\right)^{1/2}\mathbf{M} \tag{355}$$

to write

$$[\mathsf{M}'_i, \mathsf{M}'_j] = i\hbar\epsilon_{ij}{}^k\mathsf{L}_k. \tag{356}$$

The six generators L_i's and M'_i's now form a closed algebra, the identification of which is not obvious in the present notations. But if we relabel the angular momentum components $\mathsf{L}_k = \varepsilon^{ij}{}_k\mathsf{x}_i\mathsf{p}_j$, with $i = 1, 2, 3$ such that $\mathsf{L}'{}_{ij} = \epsilon_{ij}{}^k\mathsf{L}_k = \mathsf{x}_i\mathsf{p}_j - \mathsf{x}_j\mathsf{p}_i$, (or $\mathsf{L}_k = \frac{1}{2}\epsilon^{ij}{}_k\mathsf{L}'_{ij}$) and $\mathsf{M}'_i = \mathsf{L}'_{i4}$, and we further invent fourth components x_4 and p_4, such that now

$$\begin{aligned} \mathsf{L}'_{ij} = \mathsf{x}_i\mathsf{p}_j - \mathsf{x}_j\mathsf{p}_i, \qquad & \text{with} \quad [\mathsf{x}_i, \mathsf{p}_j] = i\hbar\delta_{ij}, \quad \text{with} \quad i = 1, 2, 3, 4 \\ & \text{and} \quad [\mathsf{L}'_{ij}, \mathsf{L}'_{kl}] = i\hbar(\delta_{ik}\mathsf{L}'_{jl} - \delta_{il}\mathsf{L}'_{jk} - \delta_{jk}\mathsf{L}'_{il} + \delta_{jl}\mathsf{L}'_{ik}) \end{aligned} \tag{357}$$

is equivalent to the original commutation relations (354). The six generators $\mathsf{L}'{}_{ij}$ constitute a generalization of the algebra provided by the components of $\mathbf{L}$ from three to *four* dimensions. The underlying Lie group is thus $SO(4)$, the Special Orthogonal group of 4×4 real matrices with determinant $+1$ which describe rotations in four dimensions. There are six generators, because there are six independent planes of rotations there. This is a dynamical symmetry, because there is no physical meaning for the fictitious fourth components x_4 and p_4.

– Diego: I don't really see why you changed the single index in the angular momentum L_k for two indices instead in L'_{ij}?

– Aïssata: This is just what I said. A rotation is better described by the plane ij in which it operates than by the perpendicular axis k. Obviously in 3D there are three main axes and three main planes, but in 4D that we consider now you see that there are four main axes, but six principal planes defined by pairs of these axes. The reason why we still use a single index for the angular momentum is due to the pre-eminence of the 3D vision of our ordinary space, but the two indices version is more general.

– Diego: I understand. But how do you get the commutation relations among these objects?

– Aïssata: We know the commutation relations $[\mathsf{x}_i, \mathsf{p}_j] = i\hbar\delta_{ij}$ and we proceed to the calculation.

►EXERCISE 11 – **Calculation of** $[\mathsf{L}'_{ij}, \mathsf{L}'_{kl}]$ –

– Diego: Okay, let me try to do it?

$$\begin{aligned}[\mathsf{L}'_{ij}, \mathsf{L}'_{kl}] &= [\mathsf{x}_i\mathsf{p}_j - \mathsf{x}_j\mathsf{p}_i, \mathsf{x}_k\mathsf{p}_l - \mathsf{x}_l\mathsf{p}_k] \\ &= [\mathsf{x}_i\mathsf{p}_j, \mathsf{x}_k\mathsf{p}_l] - [\mathsf{x}_i\mathsf{p}_j, \mathsf{x}_l\mathsf{p}_k] - [\mathsf{x}_j\mathsf{p}_i, \mathsf{x}_k\mathsf{p}_l] + [\mathsf{x}_j\mathsf{p}_i, \mathsf{x}_l\mathsf{p}_k].\end{aligned} \tag{358}$$

Then I use relations like $[A, BC] = B[A, C] + [A, B]C$ to form e.g. $[\mathsf{x}_i\mathsf{p}_j, \mathsf{x}_k\mathsf{p}_l] = \mathsf{x}_i[\mathsf{p}_j, \mathsf{x}_k\mathsf{p}_l] + [\mathsf{x}_i, \mathsf{x}_k\mathsf{p}_l]\mathsf{p}_j = \mathsf{x}_i(\mathsf{x}_k[\mathsf{p}_j, \mathsf{p}_l] + [\mathsf{p}_j, \mathsf{x}_k]\mathsf{p}_l) + (\mathsf{x}_k[\mathsf{x}_i, \mathsf{p}_l] + [\mathsf{x}_i, \mathsf{x}_k]\mathsf{p}_l)\mathsf{p}_j = \mathsf{x}_i(-i\hbar)\delta_{jk}\mathsf{p}_l + \mathsf{x}_k(i\hbar)\delta_{il}\mathsf{p}_j$. I can repeat the same calculation for the three remaining terms. If I made no mistake, it leads to

$$\begin{aligned}[\mathsf{L}'_{ij}, \mathsf{L}'_{kl}] &= i\hbar(-\mathsf{x}_i\delta_{jk}\mathsf{p}_l + \mathsf{x}_k\delta_{il}\mathsf{p}_j + \mathsf{x}_i\delta_{jl}\mathsf{p}_k - \mathsf{x}_l\delta_{ik}\mathsf{p}_j + \mathsf{x}_j\delta_{ik}\mathsf{p}_l - \mathsf{x}_k\delta_{jl}\mathsf{p}_i - \mathsf{x}_j\delta_{il}\mathsf{p}_k + \mathsf{x}_l\delta_{jk}\mathsf{p}_i) \\ &= i\hbar(\delta_{ik}\mathsf{L}'_{jl} - \delta_{il}\mathsf{L}'_{jk} - \delta_{jk}\mathsf{L}'_{il} + \delta_{jl}\mathsf{L}'_{ik}).\end{aligned} \tag{359}$$

as announced. ◄

– Aïssata: Now the energy levels of the hydrogen atom follow. From the original vector $\mathbf{L}$ and $\mathbf{M}'$, we build the operators

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{L} + \mathbf{M}'), \quad \mathbf{B} = \tfrac{1}{2}(\mathbf{L} - \mathbf{M}') \tag{360}$$

which obey

$$[\mathsf{A}_i, \mathsf{A}_i] = i\hbar\epsilon_{ij}{}^k\mathsf{A}_k, \quad [\mathsf{B}_i, \mathsf{B}_i] = i\hbar\epsilon_{ij}{}^k\mathsf{B}_k, \quad i, j, k = 1, 2, 3, \tag{361}$$

$$[\mathbf{A}, \mathbf{B}] = 0, \quad [\mathbf{A}, \mathsf{H}] = [\mathbf{B}, \mathsf{H}] = 0. \tag{362}$$

$\mathbf{A}$ and $\mathbf{B}$ define two commuting $SU(2)$ algebras, and the possible eigenvalues for $\mathbf{A}^2$ and $\mathbf{B}^2$ are thus respectively $\hbar^2 a(a+1)$ and $\hbar^2 b(b+1)$ with $a, b = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. There are two Casimir operators which may be chosen to be

$$\mathsf{C} = \mathbf{A}^2 + \mathbf{B}^2 = \tfrac{1}{2}(\mathbf{L}^2 + \mathbf{M}'^2), \quad \mathsf{C}' = \mathbf{A}^2 - \mathbf{B}^2 = \tfrac{1}{2}(\mathbf{L}^2 - \mathbf{M}'^2), \tag{363}$$

but (353) shows that $\mathsf{C}' = 0$, so only the part of $SO(4)$ with $\mathbf{A}^2 = \mathbf{B}^2$ is relevant for the hydrogen atom, therefore

$$\mathsf{C} = 2a(a+1)\hbar^2, \quad a = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \tag{364}$$

Collecting now (353), (355) and (363), we obtain

$$E = -\frac{m\kappa^2}{2\hbar^2(2a+1)^2}, \tag{365}$$

where one may wish to identify $2a + 1 = n \in \mathbb{N}^*$.

The orbital degeneracies are given by $\deg(a) \times \deg(b)$ with $a = b$, hence

$$\begin{aligned} n = 1 &\quad \deg(0) \times \deg(0) = 1, \\ n = 2 &\quad \deg(1/2) \times \deg(1/2) = 2^2, \\ n = 3 &\quad \deg(1) \times \deg(1) = 3^2, \\ n = 4 &\quad \deg(3/2) \times \deg(3/2) = 4^2, \\ &\ldots \end{aligned} \tag{366}$$

and the values of l in the eigenvalues $\hbar^2 l(l+1)$ of $\mathbf{L}^2$ follow from $\mathbf{L} = \mathbf{A} + \mathbf{B}$ with $a = b$, hence

n	$\mathbf{L} = \mathbf{A} + \mathbf{B}$	l	deg.	orbitals
1	$0 + 0$	0	1	1s
2	$\frac{1}{2} + \frac{1}{2}$	0, 1	4	2s, 2p
3	$1 + 1$	0, 1, 2	9	3s, 3p, 3d
...				

and you mainly recover the known results from purely algebraic techniques.

12. Day 8 – Galilean group

The day before, Diego and Aïssata went home very late, and they are again early in the morning at the library. Both are excited by their discussions and they want to go on as soon as possible.

– Diego: So far so good! We have discussed rotations and $SO(3)$, but what for translations as they play an important role in the Galileo or Poincaré groups? Like the angular momentum is associated to rotations, I know that the linear momentum will be the generator of translations. We discussed the link between linear momentum and translation in the context of conservation equations.

– Aïssata: You are right, and this is true that conserved quantities are good candidates for being Lie group generators, but you will see slight differences with what we discussed yesterday. The translation group is not compact (typically, you can make a translation of an arbitrary amount while rotation angles are limited to finite ranges, with periodicity. This is the meaning of "compact".). As a consequence, there is no unitary finite dimensional representation of translations.

– Diego: And the same is probably true with the so-called *boosts*, i.e. the change of frame at a constant velocity which leaves the laws of dynamics unchanged. See our notes, we had a short course on Galilean symmetry, I will show you, Diego says, looking for his notes in his school bag.

□ 12.1 Galilean transformations

We call Galilean invariance the fact that one cannot determine whether an inertial reference frame is in motion w.r.t another inertial frame of reference by the means of an experiment of mechanics. This is an experimental fact, discussed by Galileo Galilei in the Discourses. This is also a consequence of Newton's laws of motion (otherwise, they wouldn't have been considered as physical laws, since they wouldn't have correctly described reality!) and of his conceptions of space and time. The terminology "Galilean invariance" or "Principle of Galilean Relativity" came much later. These were coined by Poincaré. This states that

> *the laws of dynamics take the same mathematical formulation in all inertial frames, i.e. those frames in which the principle of inertia is valid.*

The Principle of Inertia of course stipulates that in absence of any force acting on a body, its center of mass perseveres in its motion at constant velocity in direction and intensity, the so-called uniform rectilinear motion. The main pieces are the following:

(i) According to Newton, time is absolute and homogeneous, which means that in two reference frames which are synchronized at $t = t' = 0$, one has $t' = t$ always.

(ii) We call Galilean transformation the coordinate transformation between two reference frames $\mathscr{R}$ and $\mathscr{R}'$ in which $\mathscr{R}'$ moves at constant velocity $\mathbf{u}$ w.r.t $\mathscr{R}$ (passive transformation),

$$\mathbf{r}' = \mathbf{r} - \mathbf{u}t \tag{367}$$
$$t' = t \tag{368}$$

This specific transformation is also called a boost.

(iii) Assume that $\mathscr{R}$ is inertial. The Galilean transformation extends the principle of inertia to $\mathscr{R}'$, because if a point mass has a position $\mathbf{r}(t) = \mathbf{v}t$ in $\mathscr{R}$ with $\mathbf{v}$ constant (i.e. a uniform motion which means that there is no force acting on the mass), then $\mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{u}t$ leads to $\mathbf{v}'(t) = \mathbf{v} - \mathbf{u}$ which is also constant. Therefore the mass m also has a uniform motion in $\mathscr{R}'$.

(iv) The second law of Newton in $\mathscr{R}$ states that

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{369}$$

for an arbitrary external force $\mathbf{F}$ under which a point mass m acquires a velocity $\mathbf{v}(t)$. In $\mathscr{R}'$, the Galilean transformation leads to

$$\frac{d\mathbf{p}'}{dt'} = m\frac{d}{dt}(\mathbf{v}(t) - \mathbf{u}) = \frac{d\mathbf{p}}{dt} = \mathbf{F}. \tag{370}$$

The second law takes the same mathematical form in $\mathscr{R}'$ under the Galilean invariant force $\mathbf{F}' = \mathbf{F}$.

(v) As a consequence, if the transformation between the two frames is *not* Galilean, but with an arbitrary relative velocity $\mathbf{u}(t)$, there appear new forces in $\mathscr{R}'$, called inertial forces, because then

$$\frac{d\mathbf{p}'}{dt'} = \mathbf{F}' - m\frac{d\mathbf{u}(t)}{dt}. \tag{371}$$

This is the source of centrifugal forces for example.

– Aïssata: This is a nice summary of classical dynamics in less than two pages! Very concise! Boosts will be an important ingredient of the Galileo group, but we can start already with isometries. Let's see how it does work. I go on again following the excellent lecture notes of Franck Laloë which, as far as I know, are unfortunately unpublished[58] and if you really want to know more, I can recommend you the famous texts by Levy-Leblond[59], or a very complete master's thesis by D. Hansen that you can find on internet[60]

But let me first give a bit of vocabulary. Mathematicians call Euclidean group or Isom-

[58]F. Laloë, Cours de DEA sur les symetries, cel-00092953, https://cel.archives-ouvertes.fr/cel-00092953/document

[59]e.g. J.-M. Levy-Leblond, Galilei Group and Galilean Invariance, in Group Theory and its Applications Volume II, ed. by E.M. Loebl, Academic Press, 1971.

[60]D. Hansen, On non-relativistic field theory and geometry, Master's thesis, The Niels Bohr Intsitute and University of Copenhagen, 2016.

etry group (and they denote $E(N)$ or ISO(N) in N dimensions) the group of isometries of Euclidean space (translations, rotations and reflections). When you exclude reflections, you get the rigid motions, i.e. those in which we are mostly interested in as Physicists (at least in classical Physics). The corresponding group is caller either Special Euclidean or proper isometries and denoted e.g. as ISO$^+(N)$. This is the one that we will discuss now, if you agree.

▷ 12.2 Finite dimensional representations of the translation group

– Aïssata: Finite translations are maps from $M \to M$ (here the manifold is the 3−dimensional Euclidean space $\mathbb{E}^3$) which send x to $x' = \mathsf{T}[a]x = x + a$, or, in vector form, $\mathbf{x} \mapsto \mathsf{T}[\mathbf{a}]\mathbf{x} = \mathbf{x} + \mathbf{a}$ with a constant vector $\mathbf{a}$. They form a group in themsleves. Consider now an infinitesimal translation along the x−axis of an amount δa_x which maps $(x, y, z)^{\mathrm{T}}$ to $(x + \delta a_x, y, z)^{\mathrm{T}}$. Close to unity (when $\delta a_x = 0$), the representation obeys

$$\mathsf{D}_{\delta a_x} = 1 + \delta a_x \mathsf{Q}_x. \tag{372}$$

In order to get a matrix representation, we introduce *Galilean* 4−*component vectors*

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \tag{373}$$

which are such that the translation is performed via

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta a_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \tag{374}$$

and enable to identify the generator

$$\mathsf{Q}_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{375}$$

The two other generators are obtained similarly via translations along y− and z−axes,

$$\mathsf{Q}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{Q}_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{376}$$

These generators obey an Abelian algebra.

$$[\mathsf{Q}_i, \mathsf{Q}_j] = 0 \tag{377}$$

and they are *not* antihermitian matrices, $(i\mathsf{Q}_i)^\dagger \neq i\mathsf{Q}_i$, thus $\mathsf{D}_{\delta a_i}$ are *not* unitary representations.

Another side remark is that there is no constraint to fix the sign of the generators. The algebra (377) would equally be satisfied by the generators $-\mathsf{Q}_i$'s[61] . This was not the case with rotations. Changing the sign of the M_i's would change the sign at the r.h.s of the algebra (315), which means that the sign of the angular momentum is fixed by the Lie algebra (323).

If we consider general displacements comprising a rotation of angle α around $\mathbf{u}$ *followed by* a translation of $\mathbf{a}$, the transformation matrix in the same basis, instead of matrix in equation (374), will take the general form

$$\begin{pmatrix} & & & a_x \\ & (\mathsf{R}[\alpha\mathbf{u}]) & & a_y \\ & & & a_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\mathsf{R}[\alpha\mathbf{u}]$ the usual rotation matrix.

Translations and rotations lead to the Euclidean algebra (here in $3D$)

$$E(3) = \mathbb{R}^3 \ltimes SO(3) \tag{378}$$

where $\ltimes$ means semi-direct product of the groups $\mathbb{R}^3$ and $SO(3)$.

▷ 12.3 Generators of boosts and finite dimensional representations of the Galilean group

– Diego: We have now rotations *and* translations. We are not far from the Galileo group which describes classical mechanics. For that, we still need changes of frames at a constant relative velocity, the boosts.

– Aïssata: Like translations, Galileo transformations are non compact transformations. They admit, like space translations, non unitary finite dimensional representations (and unitary infinite dimensional representations, but we will discuss this later). We call Galilean group the group of space-time transformations of classical mechanics. It comprises as we know space and time translations, space rotations and boosts. Before studying the case of relativistic boosts, which is more interesting for its physical applications, we can illustrate the non relativistic case in one space dimension. There, we will ignore rotations of course and a general infinitesimal Galileo transformation is the following, built from a "Galilean 3-vector" $(t, x, 1)^{\mathrm{T}}$ (space-time $+1$):

$$\begin{pmatrix} t' \\ x' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta t \\ \delta v & 1 & \delta a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} t + \delta t \\ x + t\delta v + \delta a \\ 1 \end{pmatrix}. \tag{379}$$

Writing the 3×3 matrix as

$$1 + \delta a \mathsf{Q}_x + \delta t \mathcal{L}_t + \delta v \mathsf{N}_x \tag{380}$$

we can identify the three generators in a real space representation

$$\mathsf{Q}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}_t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{N}_x = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{381}$$

[61] There is a footnote on this question in Weinberg's The quantum theory of fields, Vol. I, Cambridge University Press, Cambridge, 1995, and since Steven Weinberg does never leave anything like an approximate statement, we may assume that this is an important issue!

They obey the commutation relations

$$[\mathsf{Q}_x, \mathcal{L}_t] = 0, \quad [\mathsf{N}_x, \mathcal{L}_t] = \mathsf{Q}_x, \quad [\mathsf{Q}_x, \mathsf{N}_x] = 0. \tag{382}$$

Note that these generators are not anti-hermitian, hence this representation is not unitary. A few words on the notations used is welcome. The Q_i's are the generators associated to space translation, they will later be related to the linear momentum components. The symbol $\mathcal{L}_t$ (for Liouvillian) refers to time translation, hence to the dynamics. This will later be in correspondence to the Hamiltonian. The N_i's are for the "boosts" for which we will later use the symbols K_i's and eventually, we have already introduced the M_i's which are related to the angular momenta L_i's or J_i's.

– Diego: And precisely, what happens if you also consider rotations. The algebra becomes richer I suppose.

– Aïssata: You are right. But if we are interested in incorporating the angular momentum in the algebra, we have at least to consider two space dimensions, and if we want to have non Abelian rotations, we have to go directly to the $3D$ case, and she starts writing:

►EXERCISE 12 – **Finite dimensional representations of the Galileo group in $3D$** –
A generic infinitesimal transformations reads as

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \delta t \\ \delta v_x & 1 & -\delta\theta_z & \delta\theta_y & \delta a_x \\ \delta v_y & \delta\theta_z & 1 & -\delta\theta_x & \delta a_y \\ \delta v_z & -\delta\theta_y & \delta\theta_x & 1 & \delta a_z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix}$$

and the 10 generators become 5×5 matrices:

$$\mathsf{Q}_x = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \ \mathsf{Q}_y = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \ \mathsf{Q}_z = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix},$$
$$\mathcal{L}_t = \begin{pmatrix} 0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix},$$
$$\mathsf{M}_x = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&-1&0\\0&0&1&0&0\\0&0&0&0&0 \end{pmatrix}, \ \mathsf{M}_y = \begin{pmatrix} 0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0 \end{pmatrix}, \ \mathsf{M}_z = \begin{pmatrix} 0&0&0&0&0\\0&0&-1&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix},$$
$$\mathsf{N}_x = \begin{pmatrix} 0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \ \mathsf{N}_y = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}, \ \mathsf{N}_z = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0 \end{pmatrix}. \tag{383}$$

New commutation relations involving the M_i's appear which shows that the boosts alone don't have a group structure. You can check that the following commutation relations, corresponding to the *Galilean algebra*

$$\mathrm{Gal}(3,1) = \mathbb{R}^4 \ltimes (SO(3) \ltimes \mathbb{R}^3) \tag{384}$$

follow:

$$[\mathsf{Q}_i, \mathsf{Q}_j] = 0, \tag{385}$$
$$[\mathsf{Q}_i, \mathcal{L}_t] = 0, \tag{386}$$
$$[\mathsf{Q}_i, \mathsf{M}_j] = \epsilon_{ij}{}^k \mathsf{Q}_k, \tag{387}$$
$$[\mathsf{Q}_i, \mathsf{N}_j] = 0, \tag{388}$$
$$[\mathcal{L}_t, \mathsf{M}_i] = 0, \tag{389}$$
$$[\mathcal{L}_t, \mathsf{N}_i] = -\mathsf{Q}_i, \tag{390}$$
$$[\mathsf{M}_i, \mathsf{M}_j] = \epsilon_{ij}{}^k \mathsf{M}_k, \tag{391}$$
$$[\mathsf{M}_i, \mathsf{N}_j] = \epsilon_{ij}{}^k \mathsf{N}_k, \tag{392}$$
$$[\mathsf{N}_i, \mathsf{N}_j] = 0. \tag{393}$$

Note that here we prefer the notation i, j, k rather than a, b, c to label the generators, since Galileo transformations are really space-time transformations and the labels indeed refer to space coordinates. This is just a matter of choice and the same could have been done in the case of rotations in 3 dimensions and the group $SO(3)$. Of course, the label names are only conventions. ◄

– Diego: I have seen in the literature that the rotation generators can be described by antisymmetric two-indices objects and the commutation relations look a bit different. Isn't it what we have done earlier during the discussion on $SO(4)$?

– Aïssata: This is correct. Instead of the notation that I used here, you can find L_{jk} such that $\mathsf{M}_i = \frac{1}{2}\epsilon_i{}^{jk}\mathsf{L}_{jk}$ or $\mathsf{L}_{jk} = \epsilon_{jk}{}^i \mathsf{M}_i$. Then, you can work out with products of ϵ's to get e.g. $[\mathsf{Q}_j, \mathsf{L}_{kl}] = \delta_{lj}\mathsf{Q}_k - \delta_{kj}\mathsf{Q}_l$.

– Diego: Thank you Aïssata. I think that now I understand better this notion of representation. But there is still one piece with which I am not fully satisfied. In our lectures on Quantum Mechanics, we have learnt that the generator of translations is the momentum. We had the occasion to elaborate on this together. We saw also in QM that the translation operator, acting on wave functions, can be written in terms of the operator $\mathbf{P}$ as

$$e^{-i\mathbf{a}\cdot\mathbf{P}} \tag{394}$$

assuming here $\hbar = 1$. I have two problems to make the connection with what you are telling me. First this doesn't seem to be a matrix, but rather a differential operator and I have the idea that this is a different representation. Second, $\mathbf{P}$ is of course hermitian, therefore $e^{-i\mathbf{a}\cdot\mathbf{P}}$ is unitary, Diego says, with pride in the voice!

To be more specific, Diego says, resuming his ideas, I had a course on time translation and space translations and rotations in quantum mechanics, with the appearance of important operators like the Hamiltonian, the canonical momentum and the canonical angular momentum respectively to describe these symmetries.

– Aïssata: You make good points Diego. As we have seen, non compact groups don't admit finite dimensional unitary representations and this is the case of the translation group. But you are right, the expression $e^{-i\mathbf{a}\cdot\mathbf{P}}$ is unitary and it is a representation of the translation group which acts on real scalar fields or on wave functions. But this is an infinite-dimensional representation (a differential operator as you noticed). So that there is no contradiction eventually. Unitarity at the price of infinite dimension is a property which is due to the non compact character of the group of translations, a property shared also by the Galilean or the Poincaré group of course, but also by the Lorentz group in which concerns the boosts.

– Diego: This is exactly the content of the next lecture that we had and which was dedicated to these questions, but what appears strange to me is that it was not necessarily applied to quantum mechanics. Look:

□ 12.4 Infinite dimensional unitary representations and the Galilean group

Non compact groups don't admit finite dimensional unitary representations and this is the case of the translation group. We would like to find a unitary representation (thus it will be an infinite dimensional one) of the Galilean group, which acts on real scalar fields, like the temperature field or the density in a fluid.

In the case of translations, close to the identity, we write the translation $\mathsf{U}_T[\delta \mathbf{a}] = 1 + i[\dots]$ where $[\dots]$ is something hermitian, still to be found, which we would be able to exponentiate to get a finite transformation. The notation U_T anticipates the unitary character of the representation and the subscript T is for "translation".

In ordinary space, the action of the (matrix) group element $\mathsf{T}[\mathbf{a}]$ which sends $\mathbf{x}$ onto $\mathbf{x}+\mathbf{a}$, is

$$\mathsf{T}[\mathbf{a}]\mathbf{x} = \mathbf{x} + \mathbf{a}. \tag{395}$$

Acting on a real field denoted as $F(\mathbf{x})$, the effect of the translation is given by

$$F'(\mathbf{x}) = \mathsf{U}_T[\mathbf{a}]F(\mathbf{x}) = F(\mathsf{T}^{-1}[\mathbf{a}]\mathbf{x}) = F(\mathbf{x}-\mathbf{a}) \tag{396}$$

where $\mathsf{T}^{-1}[\mathbf{a}] = \mathbf{x} - \mathbf{a}$ was used.

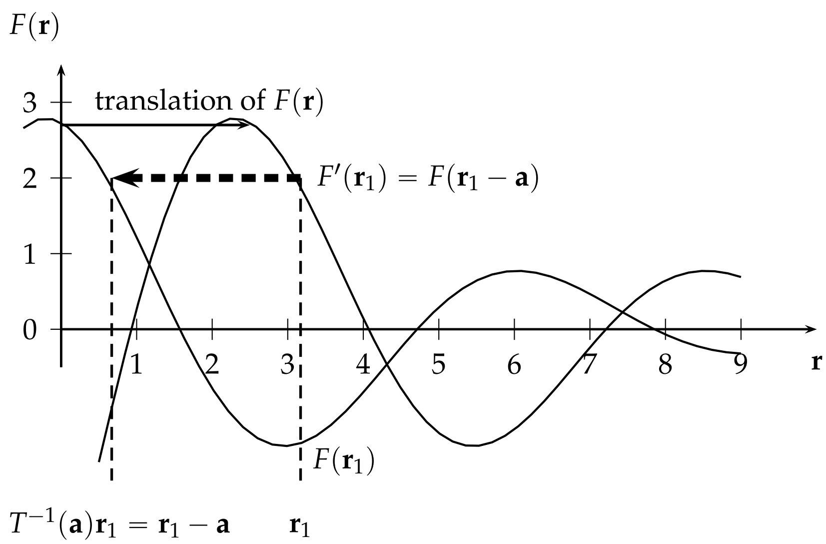


Figure 17. Translation of the field F under $\mathsf{U}_T[\mathbf{a}]$.

Expanding the r.h.s. in Taylor series

$$F(\mathbf{x}-\mathbf{a}) = \sum_{n=0}^{\infty} \frac{(-i\mathbf{a})^n}{n!}(-i\nabla)^n F(\mathbf{x}) \equiv e^{-i\mathbf{a}\cdot\mathsf{P}}F(\mathbf{x}) \tag{397}$$

with $\mathsf{P} = -i\boldsymbol{\nabla}$ identifies

$$\begin{aligned} \mathsf{U}_T[\delta\mathbf{a}] &= 1 - i\delta\mathbf{a}\cdot\mathsf{P}, & (398)\\ \mathsf{U}_T[\mathbf{a}] &= e^{-i\mathbf{a}\cdot\mathsf{P}}, \quad \mathsf{U}_T[\mathbf{a}]F(\mathbf{x}) = F(\mathbf{x}-\mathbf{a}). & (399)\end{aligned}$$

This is a unitary infinite dimensional representation (not a matrix representation) of the translation group given in terms of a differential operator and acting on scalar fields.

We can proceed along the same lines of reasoning to find an infinite dimensional representation of the rotation group. Therefore, we contemplate the map

$$\mathbf{x} \mapsto \mathbf{x}' = \mathsf{R}[\boldsymbol{\alpha}]\mathbf{x} \tag{400}$$

where $\boldsymbol{\alpha} = \alpha\mathbf{n}$ with α an angle of rotation around an axis defined by the unit vector $\mathbf{n}$ and where $\mathbf{x}' = \mathbf{x}\cos\alpha + \mathbf{n}\,(\mathbf{n}\cdot\mathbf{x})(1-\cos\alpha) + (\mathbf{n}\times\mathbf{x})\sin\alpha$ is the rotated vector as given by Rodrigues formula. Like before, we first consider the infinitesimal rotation in ordinary space (the infinitesimal rotation angle is denoted $\delta\alpha$ and $\mathsf{R}[\delta\boldsymbol{\alpha}]\mathbf{x} = \mathbf{x} + \delta\boldsymbol{\alpha}\times\mathbf{x}$ as given by the first order application of Rodigues formula) and we write the action of the rotation on a field as

$$F'(\mathbf{x}) = \mathsf{U}_R[\delta\boldsymbol{\alpha}]F(\mathbf{x}) = F(\mathsf{R}^{-1}[\delta\boldsymbol{\alpha}]\mathbf{x}), \tag{401}$$

such that

$$\begin{aligned} \mathsf{U}_R[\delta\boldsymbol{\alpha}]F(\mathbf{x}) &= F(\mathbf{x} - \delta\boldsymbol{\alpha}\times\mathbf{x})\\ &= F(\mathbf{x}) - (\delta\boldsymbol{\alpha}\times\mathbf{x})\cdot\boldsymbol{\nabla}F(\mathbf{x})\\ &= (1 - i\delta\boldsymbol{\alpha}\cdot(\mathbf{x}\times(-i\boldsymbol{\nabla})))F(\mathbf{x}) & (402)\end{aligned}$$

where we recognize $\mathsf{L} = \mathbf{x}\times(-i\boldsymbol{\nabla})$. We now have the infinite dimensional representations of the rotation group via

$$\begin{aligned} \mathsf{U}_R[\delta\boldsymbol{\alpha}] &= 1 - i\delta\boldsymbol{\alpha}\cdot\mathsf{L}, & (403)\\ \mathsf{U}_R[\boldsymbol{\alpha}] &= e^{-i\boldsymbol{\alpha}\cdot\mathsf{L}}, & (404)\\ \mathsf{U}_R[\boldsymbol{\alpha}]F(\mathbf{x}) &= F(\mathbf{x}\cos\alpha + \mathbf{n}\,(\mathbf{n}\cdot\mathbf{x})(1-\cos\alpha) - (\mathbf{n}\times\mathbf{x})\sin\alpha) & (405)\end{aligned}$$

where the sign in $-(\mathbf{n}\times\mathbf{x})\sin\alpha$ is for the opposite rotation R^{-1}.

For boosts, which act in real space according to

$$\mathsf{B}[\mathbf{v}]\mathbf{x} = \mathbf{x} + \mathbf{v}t \tag{406}$$

we proceed along the same lines,

$$F'(\mathbf{x}) = \mathsf{U}_B[\mathbf{v}]F(\mathbf{x}) = F(\mathsf{B}^{-1}[\mathbf{v}]\mathbf{x}) = F(\mathbf{x} - \mathbf{v}t). \tag{407}$$

Introducing the notation $\mathsf{K} = -it\boldsymbol{\nabla}$ leads to

$$\begin{aligned} \mathsf{U}_B[\delta\mathbf{v}] &= 1 - i\delta\mathbf{v}\cdot\mathsf{K}, & (408)\\ \mathsf{U}_B[\mathbf{v}] &= e^{-i\mathbf{v}\cdot\mathsf{K}}, \quad \mathsf{U}_B[\mathbf{v}]F(\mathbf{x}) = F(\mathbf{x}-\mathbf{v}t). & (409)\end{aligned}$$

Obviously time translations obey

$$F'(\mathbf{x},t) = \mathsf{U}_t[\tau]F(\mathbf{x},t) = F(\mathbf{x}, t-\tau) \tag{410}$$

and

$$\mathsf{U}_t[\delta\tau] = 1 + i\delta\tau\mathsf{H}, \tag{411}$$
$$\mathsf{U}_t[\tau] = e^{i\tau\mathsf{H}}, \quad \mathsf{U}_t[\tau]F(\mathbf{x},t) = F(\mathbf{x}-\tau). \tag{412}$$

with $\mathsf{H} = i\frac{\partial}{\partial t}$, with sign modifications due to the standard definition of the Hamiltonian in QM. This has not to be confused with the evolution operator in QM which defines the evolution of a quantum state $\mathsf{U}(t,t_0)\varphi(t_0) = \varphi(t)$ with $\mathsf{U}(t,t_0) = \exp -i\mathsf{H}(t-t_0)$ for a time independent Hamiltonian.

– Diego: There is something that I don't understand. For translations, you write $\mathsf{T}[\delta\mathbf{a}]\mathbf{x} = \mathbf{x} + \delta\mathbf{a} = (1 + i\delta\mathbf{a}\cdot\mathbf{P})\mathbf{x}$ *and*

$$\mathsf{U}_T[\delta\mathbf{a}]F(\mathbf{x}) = F(x-\delta\mathbf{a}) = (1 - i\delta\mathbf{a}\cdot\mathbf{P})F(\mathbf{x}), \tag{413}$$

but for rotations you write something different. You have indeed

$$\mathsf{U}_R[\delta\boldsymbol{\alpha}]F(\mathbf{x}) = F(x-\boldsymbol{\alpha}\times\mathbf{x}) = (1 - i\delta\boldsymbol{\alpha}\cdot\mathbf{L})F(\mathbf{x}), \tag{414}$$

but you *do not have* something like $\mathsf{R}[\delta\boldsymbol{\alpha}]\mathbf{x} = \mathbf{x} + \delta\boldsymbol{\alpha}\times\mathbf{x} = (1 + i\delta\mathbf{a}\cdot\mathbf{L})\mathbf{x}$ when it acts on vectors, since it would be in contradiction with equation (317) for finite dimensional representations.

– Aïssata: You are perfectly right Diego. These are different representations, since they act on different kind of mathematical objects. In fact, (317) and (414) agree with each other in the sense that they produce the same Lie algebra for the angular momentum, while the plus sign in (414) would lead to a minus sign in the Lie algebra. If we follow Laloë, (414) and (413) can be used to *define* the linear and angular momenta[62].

Note also that the Galilean algebra is now written in terms of the correspondences

$$\mathsf{Q}_i \to -i\mathsf{P}_i, \quad \mathcal{L}_t \to i\mathsf{H}, \quad \mathsf{M}_i \to -i\mathsf{L}_i, \quad \text{and} \quad \mathsf{N}_i \to -i\mathsf{K}_i, \tag{415}$$

and the group elements take the form

$$\text{space translation:} \quad \exp(-ia^a\mathsf{P}_a), \tag{416}$$
$$\text{space rotation:} \quad \exp(-i\alpha^a\mathsf{L}_a), \tag{417}$$
$$\text{boost:} \quad \exp(-iv^a\mathsf{K}_a), \tag{418}$$
$$\text{time translation:} \quad \exp(i\tau\mathsf{H}). \tag{419}$$

[62] F. Laloë, Cours de DEA sur les symetries, cel-00092953, https://cel.archives-ouvertes.fr/cel-00092953/document

▷ 12.5 Infinite dimensional ray unitary representations and the Bargmann group

– Aïssata: You see Diego, your professor applied the formalism to real scalar fields, which could describe for example an ordinary field. There is no need to be quantum to see $-i\boldsymbol{\nabla}$ as the generator of translations. On the other hand, when you really want to find the generators in quantum mechanics, there are subtleties in which concerns the boosts.

This is what we do now. We would like to find a unitary representation (thus it will be an infinite dimensional one) of the Galilean group, which acts on wave functions in (non relativistic) quantum mechanics. What happens there is that the algebra must be modified. The mathematical reason is pretty sophisticated. This is connected to the fact that physical states in quantum mechanics are not represented by vectors in Hilbert space, but rather by *rays* , that is vectors up to a phase factor of modulus 1. It means that $|\varphi\rangle$ and $e^{i\alpha}|\varphi\rangle$ describe the same physical state [63]. I am sure that you remember that physical properties in quantum mechanics are obtained by the *modulus* of the wave functions or of matrix elements.

– Diego: Of course! The probability to find a particle lacated between x and $x + dx$ at time t for example is given in $1D$ by $dP(x,t) = |\varphi(x,t)|^2 dx$.

– Aïssata: Right. A ray can be considered as a direction, or a line in the Hilbert space, and this is why mathematicians use the term projective representation. When the generators act on rays, the group relations are modified accordingly by these phase factors,

$$\mathsf{U}(g_j)\mathsf{U}(g_i) = e^{i\omega(g_j,g_i)}\mathsf{U}(g_j g_i) \tag{420}$$

with $i\omega(g_j, g_i)$ some function of the group elements and as a consequence, the algebra is *extended*. This has been shown by Bargmann[64] and the corresponding algebra is called a *central extension* of the Galilean algebra or a Bargmann algebra. I must confess that I ignore the mathematics behind all this stuff. What I know was told to me by a mathematician at the university[65].

– Diego: Look, Aïssata, I can't believe that you don't feel secure with your knowledge!

– Aïssata: And there is a lot more that I ignore!

Anyway, we're gonna generalize your last lecture, but acting now on rays. For translations and rotations, we still have

$$\mathsf{U}_T[\mathbf{a}] = e^{-i\mathbf{a}\cdot\mathsf{P}}, \quad \mathsf{P} = -i\boldsymbol{\nabla} \tag{421}$$

$$\mathsf{U}_T[\mathbf{a}]\varphi(\mathbf{x}) = \varphi(\mathbf{x} - \mathbf{a}), \tag{422}$$

$$\mathsf{U}_R[\boldsymbol{\alpha}] = e^{-i\boldsymbol{\alpha}\cdot\mathsf{L}}, \quad \mathsf{L} = -i\mathbf{x}\times\boldsymbol{\nabla} \tag{423}$$

$$\mathsf{U}_R[\boldsymbol{\alpha}]\varphi(\mathbf{x}) = \varphi(\mathbf{x}\cos\alpha + \mathbf{n}\,(\mathbf{n}\cdot\mathbf{x})(1-\cos\alpha) - (\mathbf{n}\times\mathbf{x})\sin\alpha). \tag{424}$$

This is a unitary infinite dimensional representation given in terms of a differential operator and acting on rays. A generalization acting on more general fields (e.g. 4−component spinors) is obvious by multiplication by the appropriate unity matrix, e.g. 1_4.

[63] This is for example at p. 2 already of P.A.M. Dirac's Lectures on Quantum Mechanics and Relativistic Field Theory, Martino Publishing, Mansfield Center, 2012.

[64] V. Bargmann, On unitary ray representations of continuous groups, Ann. Maths. **59**, 1, 1954. See also L.E. Ballentine, Quantum mechanics, World Scientific, Singapore, 1998.

[65] I acknowledge Julien Maubon for his explanations. Anything wrong or inaccurate here is due to my misunderstanding of what he told me.

Then, we have to contemplate again the case of boosts. And this is where the central extension will appear. We will develop a reasoning which is maybe not fully rigorous, but which justifies the need to generalize the boost generator introduced in (408). Let us consider a boost in the $x-$direction, i.e. a modification of the velocity of an inertial frame of reference by a constant amount $\delta\mathbf{v} = \delta v_x \mathbf{u}_x$ and denote the unitary transformation acting on a quantum ray as

$$\mathsf{U}_B[\delta v_x]|\varphi(\mathbf{x},t)\rangle = (1 - i\delta v_x \mathsf{K}_x)|\varphi(\mathbf{x},t)\rangle = |\varphi'(\mathbf{x},t)\rangle. \tag{425}$$

This defines the generator K_x in the $x-$direction. Writing that the average of the momentum P_x and position X in the state φ' are given in terms of those in the state φ by

$$\langle \mathsf{P}_x \rangle_{\varphi'} = \langle \varphi(t)|\mathsf{U}_B^{-1}[\delta v_x]\mathsf{P}_x \mathsf{U}_B[\delta v_x]|\varphi(t)\rangle = \langle \mathsf{P}_x\rangle_\varphi + m\delta v_x, \tag{426}$$

$$\langle \mathsf{X} \rangle_{\varphi'} = \langle \varphi(t)|\mathsf{U}_B^{-1}[\delta v_x]\mathsf{X} \mathsf{U}_B[\delta v_x]|\varphi(t)\rangle = \langle \mathsf{X}\rangle_\varphi + \delta v_x t, \tag{427}$$

the first equation leads to

$$(1 + i\delta v_x \mathsf{K}_x)\mathsf{P}_x(1 - i\delta v_x \mathsf{K}_x) = \mathsf{P}_x - i\delta v_x[\mathsf{P}_x, \mathsf{K}_x] = \mathsf{P}_x + m\delta v_x, \tag{428}$$

$$\text{or} \quad [\mathsf{P}_x, \mathsf{K}_x] = im = -m[\mathsf{P}_x, \mathsf{X}] \tag{429}$$

which demands that $\mathsf{K}_x = -m\mathsf{X} + f(\mathsf{P}_x)$. In a similar manner, the expression of $\langle \mathsf{X}\rangle_{\varphi'}$ leads to

$$\mathsf{X} - i\delta v_x[\mathsf{X}, \mathsf{K}_x] = \mathsf{X} + \delta v_x t \tag{430}$$

$$\text{or} \quad [\mathsf{X}, \mathsf{K}_x] = it = t[\mathsf{X}, \mathsf{P}_x] \tag{431}$$

which now requires that $\mathsf{K}_x = t\mathsf{P}_x + g(\mathsf{X})$. Both expressions are reconciled if

$$\mathsf{K}_x = t\mathsf{P}_x - m\mathsf{X} \tag{432}$$

where one should remember that $\hbar = 1$. All three generators can be unified in vector notation

$$\mathsf{U}_B[\delta\mathbf{v}] = 1 - i\delta\mathbf{v}\cdot\mathbf{K} = 1 - i\delta\mathbf{v}\cdot(t\mathbf{P} - m\mathbf{X}). \tag{433}$$

– Diego: And these generators are hermitian this time!

– Aïssata: Right! You can now build the Lie algebra of the Bargmann group [66],

$$\mathrm{Barg}(3,1) = (\mathbb{R}^4 \ltimes \mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes O(3)) \tag{434}$$

using

$$\mathsf{H} = i\partial_t, \tag{435}$$

$$\mathsf{P}_i = -i\partial_i, \tag{436}$$

$$\mathsf{L}_i = -i\epsilon_i{}^{jk}x_j\partial_k, \tag{437}$$

$$\mathsf{K}_i = -it\partial_i - mx_i, \tag{438}$$

[66] $\mathbb{R}^4$ is for the three space translations and one time translation, $\mathbb{R}$ for the mass, $\mathbb{R}^3$ for the three boosts and $O(3)$ for the three rotations.

and calculate the commutators

$$
\begin{aligned}
&[\mathsf{P}_i, \mathsf{P}_j] = 0, && (439)\\
&[\mathsf{P}_i, \mathsf{H}] = 0, && (440)\\
&[\mathsf{P}_i, \mathsf{L}_j] = -\epsilon_j{}^{kl}[\partial_i, x_k\partial_l] = \epsilon_{ij}{}^k\partial_k = i\epsilon_{ij}{}^k\mathsf{P}_k, && (441)\\
&[\mathsf{P}_i, \mathsf{K}_j] = im[\partial_i, x_j] = im\delta_{ij} \equiv i\mathsf{M}, && (442)\\
&[\mathsf{H}, \mathsf{L}_i] = 0, && (443)\\
&[\mathsf{H}, \mathsf{K}_i] = i[\partial_t, t\mathsf{P}_i] = i\mathsf{P}_i, && (444)\\
&[\mathsf{L}_i, \mathsf{L}_j] = i\epsilon_{ij}{}^k\mathsf{L}_k, && (445)\\
&[\mathsf{L}_i, \mathsf{K}_j] = -i\epsilon_i{}^{lm}[x_l\partial_m, t\mathsf{P}_j - mx_j] = -i\epsilon_i{}^{lm}(it\delta_{jl}\partial_m - mx_l\delta_{mj}) &&\\
&\qquad\qquad = \epsilon_{ij}{}^k(t\partial_k - imx_k) = i\epsilon_{ij}{}^k\mathsf{K}_k, && (446)\\
&[\mathsf{K}_i, \mathsf{K}_j] = 0, && (447)\\
&[\mathsf{M}, \mathsf{P}_i] = [\mathsf{M}, \mathsf{H}] = [\mathsf{M}, \mathsf{L}_i] = [\mathsf{M}, \mathsf{K}_i] = 0. && (448)
\end{aligned}
$$

The extension from Galilean algebra to Bargmann algebra is the fact that the translations P_i's do no longer commute with the boosts K_i's (compare (388) and (442), and the appearance of an *eleventh* generator, $\mathsf{M} = m\delta_{ij}$ which appears in (442) and (448) and which commutes with all the other generators of the group. This is a Casimir invariant.

▷ 12.6 A consequence of the form of the boost generator

– Aïssata: There is a pretty interesting outcome of the existence of Bargmann generators on the most general form of the non relativistic Hamiltonian in quantum mechanics. Together with

$$\mathsf{U}_B^{-1}[\delta v_x]\mathsf{P}_x\mathsf{U}_B[\delta v_x] = \mathsf{P}_x + m\delta v_x, \tag{449}$$

you can also write

$$\mathsf{U}_B^{-1}[\delta v_x]\mathsf{V}_x\mathsf{U}_B[\delta v_x] = \mathsf{V}_x + \delta v_x \tag{450}$$

from which you can also deduce that $\mathsf{U}_B^{-1}[\delta v_x](\mathsf{P}_x - m\mathsf{V}_x)\mathsf{U}_B[\delta v_x] = \mathsf{P}_x - mV_x$. It follows that $\mathsf{P}_x - m\mathsf{V}_x$ commutes with U_B, or with $\mathsf{K}_x = t\mathsf{P}_x - m\mathsf{X}$. We can therefore conclude that $[\mathsf{P}_x - m\mathsf{V}_x, \mathsf{X}] = 0$ which proves that $\mathsf{P}_x - m\mathsf{V}_x$ is a function of X only.

Call $A(\mathsf{X})$ this function, or better, $eA_x(\mathsf{X})$ for reasons which will become clear in a moment. The kinetic energy writes as $\mathsf{K} = \frac{1}{2}m\mathsf{V}_x^2 = \frac{1}{2m}(\mathsf{P}_x - eA_x(\mathsf{X}))^2$. The Hamiltonian differs from the kinetic energy by the potential energy only, so $[\mathsf{H}, \mathsf{X}] = [\mathsf{K}, \mathsf{X}]$ or $[\mathsf{H} - \mathsf{K}, \mathsf{X}] = 0$. This implies that $\mathsf{H} - \mathsf{K}$ is a function of X only.

Call it $e\phi(\mathsf{X})$. Therefore we have shown that Galilean invariance implies that the general form of the Hamiltonian can only be

$$\mathsf{H} = \frac{1}{2m}(\mathsf{P}_x - eA_x(\mathsf{X}))^2 + e\phi(\mathsf{X}) \tag{451}$$

which can obviously be generalized to the $3D$ case.

– Diego: Incredible! This is the expression of the Schrödinger Hamiltonian of a charge particle in an EM field! You mean that this expression follows from Galilean invariance?

– Aïssata: Correct. This is a nice result, indeed. Another obvious thing to mention is that the generators are not always conserved quantities. You note that $[\mathsf{H}, \mathsf{K}_x]$ is not vanishing. The Galilean boost doesn't conserve energy, of course!

►EXERCISE 13 – **Galilean transformation of the wave function** –

– Aïssata: Another interesting result is the Galilean invariance, or covariance, of the free-particle wave function obeying Schrödinger equation. Assume a free particle (in $1D$) described in a coordinate system (x, t) in which Schrödinger equation thus takes the form

$$\frac{-\hbar^2}{2m}\partial_x^2\varphi(x,t) = i\hbar\partial_t\varphi(x,t). \tag{452}$$

We would like to know the form of the wave function in another frame of reference, say with coordinates (x', t') boosted according to

$$x' = x - vt, \quad t' = t. \tag{453}$$

– Diego: This is a Galilean transformation. The free particle Schrödinger equation displays Galilean invariance (the kinetic energy has the Newtonian form). Therefore, one would expect that the wave function in the unprimed and the primed variables can at most differ by a phase factor,

$$\varphi(x,t) = \varphi(x',t')e^{i\phi(x',t')}, \tag{454}$$

in such a way that they correspond to the same probability distributions.

– Aïssata: Correct Diego. Now, using the coordinate transformations, you can proceed to the explicite calculation.

– Diego: We have

$$\partial_x = \partial_{x'}(\partial x'/\partial x) + \partial_{t'}(\partial t'/\partial x) = \partial_{x'} \tag{455}$$

and

$$\partial_t = \partial_{x'}(\partial x'/\partial t) + \partial_{t'}(\partial t'/\partial t) = -v\partial_{x'} + \partial_{t'}. \tag{456}$$

Then we just apply the chain rule for the derivatives, e.g.

$$\partial_x\varphi(x,t) = [i(\partial_{x'}\phi(x',t'))\varphi(x',t') + (\partial_{x'}\varphi(x',t'))]e^{i\phi(x',t')}, \tag{457}$$

and an analogous form for the time derivative

$$\begin{aligned}\partial_t\varphi(x,t) \quad = \quad & [i(\partial_{t'}\phi(x',t'))\varphi(x',t') + (\partial_{t'}\varphi(x',t')) \\ & -iv(\partial_{x'}\phi(x',t'))\varphi(x',t') - v(\partial_{x'}\varphi(x',t'))]e^{i\phi(x',t')}.\end{aligned} \tag{458}$$

Now, we use equation (452) and we demand that the same equation also holds in terms of primed coordinates. This implies the condition

$$\begin{aligned}&\varphi(x',t')\left(\frac{-\hbar^2}{2m}[i(\partial_{x'}^2\phi) - (\partial_{x'}\phi)^2] - i\hbar[i(\partial_{t'}\phi) - iv(\partial_{x'}\phi)]\right) \\ &\quad +\partial_{x'}\varphi(x',t')\left(\frac{-\hbar^2}{2m}2i\partial_{x'}\phi + i\hbar v\right) = 0.\end{aligned} \tag{459}$$

What should I do then?

– Aïssata: You have done the essential job. You want to get this equation satisfied for arbitrary $\varphi(x', t')$ and $\partial_{x'}\varphi(x', t')$, so you have two conditions. The second one demands

$$\frac{\hbar}{m}\partial_{x'}\phi = v \tag{460}$$

i.e. $\phi(x', t') = mvx'/\hbar + f(t')$ and, after a few simplifications, the condition given by the first equation is

$$\partial_{t'}\phi = \frac{mv^2}{2\hbar} \tag{461}$$

which gives $\phi(x', t') = g(x') + mv^2 t'/2\hbar$. Eventually you get

$$\phi(x', t') = (mv^2 t'/2 + mvx')/\hbar \tag{462}$$

or

$$\varphi(x - vt, t) = \varphi(x, t) \exp\left[i\left(\frac{mv^2 t}{2\hbar} - \frac{mvx}{\hbar}\right)\right] \tag{463}$$

which is the wanted expression. ◄

Diego keeps silence for a while. Trying to record the consequences of what he learned today. After such a brainstorming session on group theory, both Aïssata and Diego are tired and they decide to leave their stuff at the library and go for a dinner downtown.

Spacetime symmetries in Minkowski manifold

13. Day 9 – The geometry of Minkowski spacetime

□ 13.1 A short reminder of the geometry of Special Relativity

In Special Relativity, spacetime is a pseudo-Eulidean (Minkowskian) four-dimensional manifod $\mathbb{M}^4$. Points are located by a set of global Cartesian coordinates x^α, $\alpha = 0, \ldots 3 = 0, i$ and $i = 1, 2, 3$ for x, y, z, such that the infinitesimal line element can be denoted as a contravariant vector

$$dx^\alpha = \begin{pmatrix} dt \\ d\mathbf{x} \end{pmatrix} \tag{464}$$

with $d\mathbf{x}$ the standard Euclidean space line element – this is because we have chosen Cartesian coordinates. To this vector, there also corresponds a covector (covariant)

$$dx_\alpha = \eta_{\alpha\beta} dx^\beta \tag{465}$$

There, $dx^i = dx$, dy, dz and the covector is $dx_\alpha = (dt, -d\mathbf{x})$ – this is also because we have chosen Cartesian coordinates. We have fixed $c = 1$, $\hbar = 1$. The Minkowski metric tensor is chosen such as $\eta_{00} = -\eta_{ii} = 1$, $\eta_{ij} = 0$ for $i \neq j$ with the signature choice $(+, -, -, -)$.

The line element is denoted as

$$ds^2 = dx_\alpha dx^\alpha = \eta_{\alpha\beta} dx^\alpha dx^\beta = dt^2 - |d\mathbf{x}|^2 \tag{466}$$

Special relativity can be built as a geometrical theory on $\mathbb{M}^4$ in which one is mainly interested in the way various objects (scalars, vectors, tensors, spinors, etc) transform under change of inertial frames, i.e. Lorentz transformations

$$x \to x' = \Lambda x, \tag{467}$$

or more precisely in components form

$$x^\alpha \quad \to x'^\alpha \quad = \Lambda^\alpha{}_\beta x^\beta. \tag{468}$$

Here we consider *passive transformations*, i.e. those in which the physical system is not "touched", but the coordinate system is moved. In Cartesian coordinates, a boost at velocity

u in the x direction corresponds to the Lorentz transformation described in matrix form by

$$(\Lambda^\alpha{}_\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{469}$$

where the line index is α and the column index is β, and where *the other* $\beta = u/c$ (no confusion allowed!), and $\gamma = (1-\beta^2)^{-1/2}$. The notation $\cosh\phi = \gamma$, $\sinh\phi = \beta\gamma$ can be found in the literature. It gives the boost matrix a form closer to a rotation matrix, sometimes called hyperbolic rotation

$$(\Lambda^\alpha{}_\beta) = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{470}$$

A proper (passive) rotation of angle α around the x axis (or more rigorously in the yz plane) leaves the time coordinate unchanged and corresponds to the Lorentz transformation described in matrix form by

$$(\Lambda^\alpha{}_\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{471}$$

Equations (470) and (471) are $4-$dimensional matrices which tell how vectors are transformed under Lorentz transformations. These are four dimensional representations of the Lorentz group. More about that later. There are three rotation matrices and three boost matrices. They correspond to finite *proper orthochronous* Lorentz transformations. The group of such transformations is called $SO^+(3,1)$. This is the group of rotations in $\mathbb{M}^4$. Under such a transformation, the line element, being a scalar is unchanged,

$$ds^2 \to ds'^2 = ds^2. \tag{472}$$

We say that it is a *Lorentz scalar*. More generally, scalar fields are unchanged under this group of transformations

$$\phi(x) \to \phi'(x') = \phi(x'), \tag{473}$$

hence, the scalar field has the same value at a given spacetime point when it is expressed in terms of x or of x' coordinates. Vectors transform like the coordinates,

$$v^\alpha(x) \to v'^\alpha(x') = \Lambda^\alpha{}_\beta v^\beta(x), \tag{474}$$

and more generally, contravariant tensors obey (extension to cotensors is obvious)

$$w^{\alpha\beta\dots}(x) \to w'^{\alpha\beta\dots}(x') = \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \dots w^{\gamma\delta\dots}(x) \tag{475}$$

where there should be no confusion (again) with the meaning of the index γ here. Eventually, spinors transform in the spin representation of the Lorentz group

$$\psi(x) \to \psi'(x') = \exp\left(-\frac{i}{4}\epsilon^{\alpha\beta}\sigma_{\alpha\beta}\right)\psi(x) \tag{476}$$

with $\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$ in terms of the Dirac matrices γ_α and $\epsilon^{\alpha\beta}$ antisymmetric parameters, analogous to ϕ in (470). The transformations (474-476) are written in Cartesian coordinates ($\alpha\beta$ indices).

▷ 13.2 Special vs general coordinate transformations

– Diego: If I have understood properly, in Special Relativity, we contemplate only special kinds of coordinate transformations (maybe this is the origin of the name *Special* relativity), symbolically denoted as $x \to \Lambda x = x'$ and not the general ones that we can consider in differential geometry.

As a consequence, tensors transform like

$$w^{\alpha\beta\cdots}(x) \to w'^{\alpha\beta\cdots}(x') = \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \ldots w^{\gamma\delta\cdots}(x) \tag{477}$$

$$w^\alpha{}_{\beta\ldots}(x) \to w'^\alpha{}_{\beta\ldots}(x') = \Lambda^\alpha{}_\gamma \Lambda_\beta{}^\delta \ldots w^\gamma{}_{\delta\ldots}(x) \tag{478}$$

with $\Lambda^\alpha{}_\gamma$ given by (469) or (471) and $\Lambda_\beta{}^\delta$ the inverse matrices.

What would happen for arbitrary coordinates?

– Aïssata: The natural coordinates used in Special Relativity are the Cartesian ones, since there is a privileged coordinate along which boosts (which are particular Lorentz transformations) are performed, the remaining perpendicular coordinates being unchanged. So this chapter will essentially deal with $\alpha, \beta \ldots$ tensor indices[67]. But this is true that General Relativity considers *arbitrary* coordinate transformations. Then, the most general transformations which will define tensors take the form

$$w^{\mu\nu\cdots}(x) \to w'^{\rho\sigma\cdots}(x') = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} \ldots w^{\mu\nu\cdots}(x) \tag{479}$$

$$w^\mu{}_{\nu\ldots}(x) \to w'^\rho{}_{\sigma\ldots}(x') = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\sigma} \ldots w^\mu{}_{\nu\ldots}(x) \tag{480}$$

for a change of coordinates $\{x^\mu\} \to \{x'^\rho\}$. An important difference is also the fact that matrix elements of $\Lambda^\alpha{}_\gamma$ and $\Lambda_\beta{}^\delta$ are constant while those of $\frac{\partial x'^\rho}{\partial x^\mu}$ and $\frac{\partial x^\nu}{\partial x'^\sigma}$ are functions which depend on spacetime coordinates.

– Diego: Up to now I feel comfortable. I had an introductory course in Special Relativity during my undergrad studies, but I must confess that I am a bit lost with the vocabulary, proper, orthochronous, . . . etc.

– Aïssata: I will give you a short reminder on the Lorentz group and the Poincaré group. You remember that I told you that I am not a fan of Group Theory. But, as it happens very often in mathematics and physics, when you start studying seriously a new topic, you learn to appreciate it. This is a bit an illustration of what we discussed earlier: understanding something is getting used to it!

So, let us proceed. You may look in any book in quantum field theory, there is usually a chapter on the Lorentz group. For example if you want something brief, I recommend the *cursory* look in Ramond[68], or, if you are not afraid of gravitation, the first pages of

[67] Remember that Greek letters from the beginning of the alphabet are used for Cartesian coordinates.
[68] P. Ramond, Field theory: a modern primer, Westview Press, Boulder, 1990, chap. 1.

the book of Blagojević[69]. If you want to have more details, the recent texts of Maggiore[70] or of Padmanabhan[71] are excellent choices I think. An older authoritative reference is Schweber[72]. And there is obviously the inescapable first volume of Weinberg's classics[73].

Chapter 2

Spacetime symmetries

The physics of elementary particles and gravitation is successfully described by Lagrangian field theory. The dynamical variables in this theory are fields $\phi(x)$ and the dynamics is determined by a function of the fields and their derivatives, $\mathcal{L}(\phi, \partial_\mu \phi)$, called the Lagrangian. Equations of motion are given as the Euler–Lagrange equations of the variational problem $\delta_\phi I = 0$ for the action integral $I = \int \mathrm{d}^4 x\, \mathcal{L}$.

In physical processes at low energies the gravitational field does not play a significant role, since the gravitational interaction is extremely weak. The structure of spacetime without gravity is determined by the relativity principle and the existence of a finite, maximal velocity of propagation of physical signals. The unity of these two principles, sometimes called Einstein's relativity principle, represents the basis for special relativity theory. Spacetimes based on Einstein's relativity have the structure of the Minkowski space M_4. The equivalence of inertial reference frames is expressed by the *Poincaré symmetry* in M_4.

Figure 18. M. Blagojević, Gravitation and Gauge Symmetries, Institute of Physics Publishing, London, 2002, p. 20

▷ 13.3 "Classifying" the Lorentz group

– Aïssata: A *finite* Lorentz transformation maps spacetime coordinates x^α representing points $x \in \mathbb{M}^4$ to coordinates x'^α

$$x^\alpha \to x'^\alpha = \Lambda^\alpha{}_\beta x^\beta \tag{481}$$

leaving $ds^2 = dx_\alpha dx^\alpha$ invariant. When this is satisfied, the two sets of coordinates refer to inertial frames of reference. It is generically denoted as $T[\Lambda]$, with T for "transformation"

[69] M. Blagojević, Gravitation and Gauge Symmetries, Institute of Physics Publishing, London, 2002, chap. 2.

[70] M. Maggiore, A modern introduction to quantum field theory, Oxford University Press, New-York, 2005, chap. 5.

[71] T. Padmanabhan, Quantum Field Theory, The Why, What and How, Springer, Cham, 2016, chap. 5.

[72] S.S. Schweber, Relativistic Quantum Field Theory, Harper, New-York, 1961, chap. 2.

[73] S. Weinberg, The quantum theory of fields, vol.I, Cambridge University Press, Cambridge, 1995, chap. 2.

and Λ for "Lorentz". The x^α's are the components of (contra)vectors in $T_x\mathbb{M}^4$ and we denote x_α the components of the corresponding covector in $T_x^*\mathbb{M}^4$, and $\eta_{\alpha\beta}$ the Minkowski metric tensor in $\mathbb{M}^4$. The scalar product appearing in ds^2 can be written as the matrix operation $(dx^\alpha)^{\mathrm{T}} \cdot (\eta_{\alpha\beta}) \cdot (dx^\beta)$. The Poincaré group (to be discussed later) is the group of affine transformations preserving this scalar product. It is also called *inhomogeneous*[74] Lorentz group. The Lorentz group is the group of linear transformations which preserve the line element (466)

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = dt^2 - |d\mathbf{x}|^2. \tag{482}$$

This means that

$$\begin{aligned} \eta'_{\alpha\beta} dx'^\alpha dx'^\beta &= \eta'_{\alpha\beta} \Lambda^\alpha{}_\gamma dx^\gamma \Lambda^\beta{}_\delta dx^\delta \\ &= \eta_{\gamma\delta} dx^\gamma dx^\delta \end{aligned} \tag{483}$$

which implies

$$\eta'_{\alpha\beta} \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta = \eta_{\gamma\delta} \tag{484}$$

This form invariance of the Minkowski metric is the central property of the Lorentz group from which all the properties of the Lorentz transformations will be deduced. In particular, the form of the matrices in equations (470) and (471) are consequences of (484).

For the group composition operation, we know that $g_b g_c$ has the meaning that g_c acts first, then only g_b acts. Since the Lorentz transformations $T(\Lambda)$ form a group, the combination of any two Lorentz transformations is a Lorentz transformation,

$$T[\Lambda_b] T[\Lambda_c] = T[\Lambda_b \Lambda_c], \tag{485}$$

the product is associative,

$$T[\Lambda_b](T[\Lambda_c] T[\Lambda_d]) = (T[\Lambda_b] T[\Lambda_c]) T[\Lambda_d], \tag{486}$$

there is an identity transformation E with elements

$$\mathsf{E}^\alpha{}_\beta = \delta^\alpha_\beta, \tag{487}$$

and every Lorentz transformation Λ with elements $\Lambda^\alpha{}_\beta$ has an inverse Λ^{-1} with elements $(\Lambda^{-1})^\alpha{}_\beta$ which are found using (484), since this equation can be rewritten $\Lambda_{\beta\gamma} \Lambda^\beta{}_\delta = \eta_{\gamma\delta}$ where, raising the index γ, implies that

$$\Lambda_\beta{}^\gamma \Lambda^\beta{}_\delta = \delta^\gamma_\delta, \tag{488}$$

hence

$$(\Lambda^{-1})^\beta{}_\gamma = \Lambda_\gamma{}^\beta. \tag{489}$$

There is a clear analogy with the rotation group $O(3)$ in 3 dimensions. There, denoting 1_3 the metric tensor of the three-dimensional Euclidean space, one has

$$\mathsf{R}^{\mathrm{T}} 1_3 \mathsf{R} = 1_3 \tag{490}$$

which implies that $(\det \mathsf{R})^2 = 1$ and $\mathsf{R}^{-1} = \mathsf{R}^{\mathrm{T}}$, i.e. that R is an orthogonal matrix.

[74] Due to the presence of the translation.

Taking in the same manner the determinant of both sides of equation (484), we deduce that $\det\Lambda = \pm 1$. The sign $+$ corresponds to *proper* transformations while the sign $-$ corresponds to *improper* ones. For example the case of space inversion $x^0 \to x^0$, $x^i \to -x^i$, i.e. $\Lambda = \Pi = \mathrm{diag}(1,-1,-1,-1)$, is an improper transformation.

If we now take the 00 entry of (484), we get $1 = \Lambda^\alpha{}_0 \Lambda^\beta{}_0 \eta_{\alpha\beta}$ which, once expanded, leads to

$$1 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 \tag{491}$$

and requires that $|\Lambda^0{}_0| \geqslant 1$. The case $\Lambda^0{}_0 \geqslant 1$ corresponds to *orthochronous* and the other case, $\Lambda^0{}_0 \leqslant -1$ to *non-orthochronous* Lorentz transformations. The standard denomination is the following:

– Proper orthochronous Lorentz transformations, called restricted and denoted as belonging to $L_+^\uparrow$ with $\det\Lambda = +1$ and $\Lambda^0{}_0 \geqslant 1$. Proper rotations and boosts belong to $L_+^\uparrow$. The term orthochronous refers to the fact that these transformations preserve the time orientation.

– Proper non-orthochronous Lorentz transformations, ($L_+^\downarrow$), with $\det\Lambda = +1$ and $\Lambda^0{}_0 \leqslant -1$. These transformations reverse the time orientation.

– Improper orthochronous Lorentz transformations, ($L_-^\uparrow$), with $\det\Lambda = -1$ and $\Lambda^0{}_0 \geqslant 1$, As we have seen, space reflection $\Pi \in L_-^\uparrow$. Improper rotations also belong to $L_-^\uparrow$.

– Improper non-orthochronous Lorentz transformations, ($L_-^\downarrow$), with $\det\Lambda = -1$ and $\Lambda^0{}_0 \leqslant -1$, An example of Lorentz transformation in this subgroup is time reversal $x^0 \to -x^0$, $x^i \to x^i$, i.e. $\Lambda = \mathsf{T} = \mathrm{diag}(-1,1,1,1) \in L_-^\downarrow$.

– Diego: Usually, we only speak about the rotations and the boosts when we discuss Lorentz transformations.

– Aïssata: You are right, and there is a good reason for that. Every element of $L_-^\uparrow$ can be written as the product of an element of $L_+^\uparrow$ with Π, every element of $L_-^\downarrow$ as the product of an element of $L_+^\uparrow$ with T and every element of $L_+^\downarrow$ as the product of an element of $L_+^\uparrow$ with $\Pi\,\mathsf{T}$. So one usually studies only $L_+^\uparrow$ which gives an access to the whole Lorentz group. The transformations (469) and (471) belong to this subgroup which is also called $SO(3,1)$ or, more rigorously $SO^+(3,1)$, while the whole Lorentz group is the rotation group in $3+1$ dimensions, $O(3,1)$. Note that $O(3,1)$ is not simply connected, because there is no continuous manner to evolve between transformations in the four parts of the group, due to the discrete transformations T and Π.

– Diego: This clarifies the discussion, but this is a technical vocabulary! This doesn't tell me why we spend time to elaborate representations of groups?

– Aïssata: You saw that representations are operators or matrices which satisfy the group algebra. And you saw that there can be many different representations of a group or an algebra. In physics, they can correspond to the way the symmetry acts on various objects of the theory and this is an essential point.

For example in your lecture notes, you have seen the Lorentz transformations written in terms of 4×4 matrices, which tell you how *vectors* transform (and by extension tensors), but these matrices don't tell you anything about the transformation of spinors which are other objects of a physical theory. You probably noticed indeed that in equation

([476]) there is no $\Lambda^\alpha{}_\beta$ hidden, although you might know that some spinors have four components, like vectors in $4-$dimensional spacetime. This is in fact a sort of coincidence, a spinor is a different kind of object. It doesn't live in the same space $T_x\mathbb{M}^4$ than $4-$vectors and as a consequence, it obeys different transformation properties under change of inertial frame of reference. This is what you learn by studying the representations of the group.

14. Day 10 – Representations of the Lorentz group

▷ 14.1 Active and passive transformations

The following day, Aïssata and Diego meet in the morning, ready for a new working party!

– Aïssata: Today, we will extend our presentation of Lorentz transformations in $\mathbb{M}^4$, the Minkowski manifold.

But there is another essential feature before we start. I think that the difference between active and passive transformations is something important. In the spirit of Special Relativity, one is usually interested in the description of physical systems by different observers, attached to different frames, and who have their own coordinate frames x and x' to describe the same physical system. This is the case considered in the notes with equations (470) and (471) for a boost in the x direction and a rotation around that x axis, respectively.

In the spirit of what we have done with the study of the Galilean group, in QM we usually prefer active transformations e.g.

$$\varphi(x) \to \varphi'(x) = \mathsf{U}[G]\varphi(\mathsf{G}^{-1}x). \tag{492}$$

We denote with a prime the "object" which is Lorentz transformed. Imagine a scalar field φ which, under the action of an active Lorentz transformation Λ (say a rotation) is mapped onto φ'. At point M, denoted as x, $\varphi'(x)$ takes the value that φ had at the point $\tilde{M}$ (or $\tilde{x}$) which was mapped onto x, i.e. $x = \Lambda\tilde{x}$, or $\tilde{x} = \Lambda^{-1}x$. So we have $\varphi'(x) = \varphi(\tilde{x})$, or

$$\varphi(x) \to 1\varphi(\Lambda^{-1}x) \tag{493}$$

where 1 is there because this is a scalar field and we use the identity representation of the group element associated to the Lorentz transformation considered. The corresponding reasoning for a vector field $\mathbf{v} = v^\alpha\mathbf{e}_\alpha$ which is mapped onto $\mathbf{v}' = v'^\alpha\mathbf{e}_\alpha$ is the following: at point x, the component $v'^\alpha(x)$ is the transformed (rotated) vector component at the original component $\tilde{x} = \Lambda^{-1}x$, i.e. $\mathbf{v}'(x) = \Lambda^\alpha{}_\beta v^\beta(\tilde{x})\mathbf{e}_\alpha$, or, as summarized in the table below:

ACTIVE transformation Λ		
$\tilde{M}(\tilde{x}) \in \mathbb{M}^4$	$\to$	$M(x) \in \mathbb{M}^4$
$\tilde{x}$	$\to$	x
$\mathbf{v}(\tilde{x}) = v^\alpha(\tilde{x})\mathbf{e}_\alpha$	$\to$	$\mathbf{v}'(x) = v'^\alpha(x)\mathbf{e}_\alpha$
		$v'^\alpha(x) = \Lambda^\alpha{}_\beta v^\beta(\Lambda^{-1}x)$

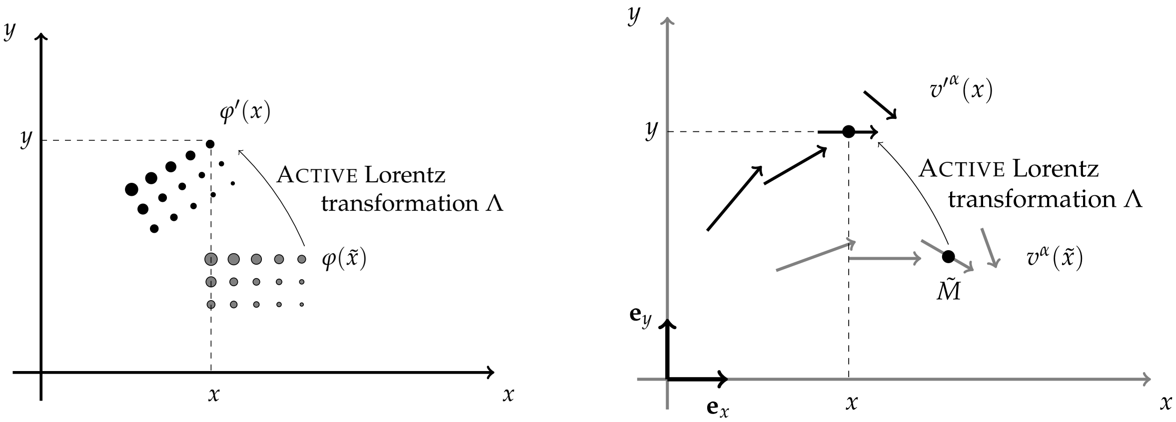


Figure 19. Active Lorentz transformation: a scalar field (left) φ is mapped onto φ', and a vector field (right) v^α is mapped onto v'^α.

This leads to the Lorentz transformation of the vector components

$$v^\alpha(x) \to v'^\alpha(x) = \Lambda^\alpha{}_\beta v^\beta(\Lambda^{-1}x) \tag{494}$$

with $\Lambda^\alpha{}_\beta$ the group elements in the vector representation of the group. This is important to understand that in the active point of view, we relate the components of the vector field at *different* spacetime points. Only the system (e.g. here a vector field) is transformed, not the basis $\{\mathbf{e}_\alpha\}$ w.r.t. which it is measured.

The case of a passive transformation is *a priori* very different. There, the system is not moved, but the reference frame (i.e. the coordinate vector basis) w.r.t. which it is measured is modified by the action of the inverse transformation Λ^{-1} in order for the operation to produce the same global effect:

PASSIVE transformation Λ^{-1}		
$M(x) \in \mathbb{M}^4$	$\to$	$M(x') = M(x)$
x	$\to$	x'
$\mathbf{v}(x) = v^\alpha(x)\mathbf{e}_\alpha$	$\to$	$\mathbf{v}(x) = v'^\alpha(x')\mathbf{e}'_\alpha$

The basis vectors are transformed covariantly by Λ^{-1}, $\mathbf{e}'_\alpha = (\Lambda^{-1})_\alpha{}^\beta \mathbf{e}_\beta$ (remember that $(\Lambda^{-1})_\alpha{}^\beta = \Lambda^\beta{}_\alpha$). Since the vector $\mathbf{v}$ is unchanged, its components when referred to the prime frame, denoted as v'^α, are such that $v'^\alpha \mathbf{e}'_\alpha = v'^\alpha (\Lambda^{-1})_\alpha{}^\beta \mathbf{e}_\beta = v^\alpha \mathbf{e}_\alpha$ which demands $v'^\alpha = (\Lambda^{-1})^\alpha{}_\gamma v^\gamma$ for $\Lambda_\gamma{}^\alpha \Lambda^\beta{}_\alpha = \delta^\beta_\gamma$ to ensure $\mathbf{v}' = \mathbf{v}$. It follows automatically $v'^\alpha(x') = \Lambda_\beta{}^\alpha v^\beta(x)$ which, after multiplication by $\Lambda^\gamma{}_\alpha$ and substitution of x' by $\Lambda^{-1}x$ to

$$\Lambda^\gamma{}_\alpha v'^\alpha(\Lambda^{-1}x) = v^\gamma(x), \tag{495}$$

which is the same as (494) except for the prime which has a different meaning. The case of a scalar field also leads automatically to the same expression than (493), hence, most of the time, people even don't specify whether they consider active of passive transformations. But you have to understand that in the passive point of view, you relate the components of the vector field *at the same spacetime point* in two different coordinate systems.

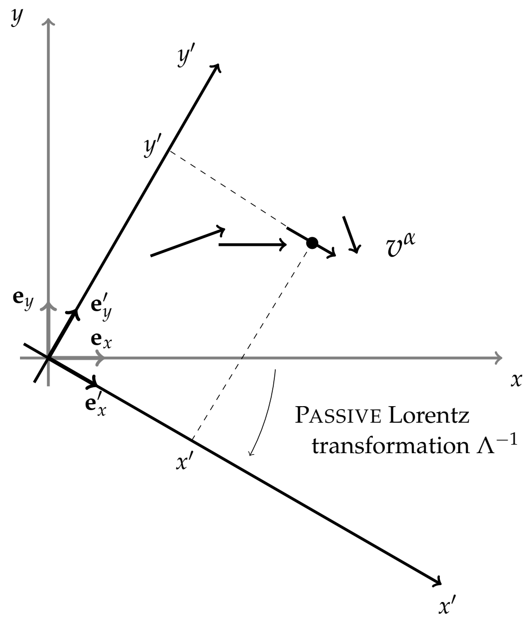


Figure 20. Passive Lorentz transformation: the reference frame is moved from $\mathbf{e}_\alpha$ to $\mathbf{e}'_\alpha$.

– Diego: This is still a bit obscure Aïssata. In my professor's notes, this is the passive point of view, but the transformation is called Λ, not Λ^{-1}.

– Aïssata: Correct, but the Lorentz group *is a group*, hence both Λ and Λ^{-1} are equally acceptable Lorentz transformations and you are free to call any of them Λ.

– Diego: Let me do it explicitly and consider first the passive point of view. So I imagine two coincident reference frames at the synchronized $t' = t = 0$, and x' moves in the positive x direction at a constant velocity $\boldsymbol{\beta} = |\beta|\mathbf{e}_x$ w.r.t x (figure 21). Clearly, one has

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -|\beta|\gamma \\ -|\beta|\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{496}$$

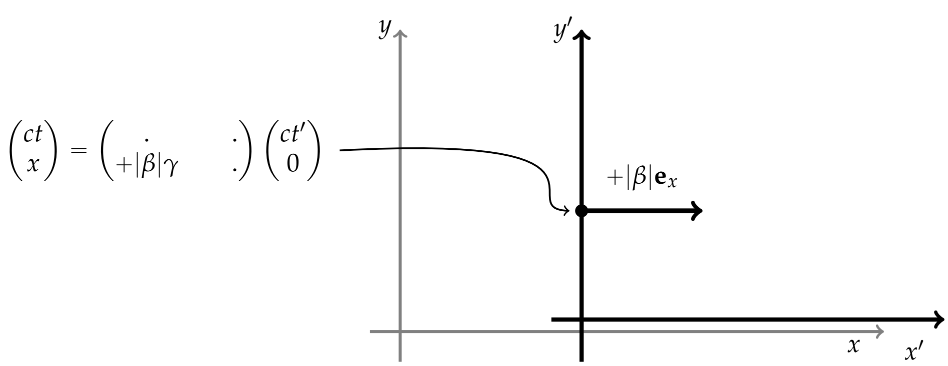


Figure 21. The expression of the Lorentz transformation matrix for a passive boost follows from one matrix element as shown here, x is positive when $x' = 0$.

The case of an active boost in the reverse direction is illustrated in figure 22, it leads to

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & -|\beta|\gamma \\ -|\beta|\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix}. \tag{497}$$

You are right, obviously, the two points of views are equivalent. Once it is done, it becomes pretty clear.

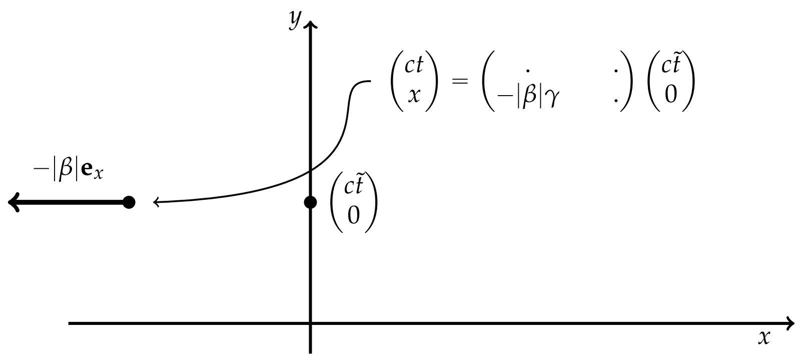


Figure 22. The expression of the Lorentz transformation matrix for an active boost follows from one matrix element as shown here, x, is negative when $\tilde{x} = 0$.

Now I understand equations (470) and (471) which, for the sake of convenience, I write again,

Passive transformations:

$$(\Lambda[\mathsf{B}_{\beta\mathbf{u}_x}]) = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\Lambda[\mathsf{R}_{\alpha\mathbf{u}_x}]) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{498}$$

as the passive Lorentz transformation called here Λ, exactly as I have done in Fig. 21. I went though several books yesterday, and I think that some of them adopt the passive point of view, e.g. Burgess[75], but many adopt the alternative active point of view, e.g. Peshkin and Schroeder[76], or Maggiore[77], and of course many others. In order to be consistent, the expression for the matrix representation corresponding to (498) *in the case of active transformations* should be written also:

Active transformation:

$$(\Lambda[\mathsf{B}_{\beta\mathbf{u}_x}]) = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\Lambda[\mathsf{R}_{\alpha\mathbf{u}_x}]) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix}. \tag{499}$$

[75] M. Burgess, Classical Covariant Fields, Cambridge University Press, Cambridge, 2003.

[76] M.E. Peshkin and D.V. Schroeder, An introduction to Quantum Field Theory, Perseus Books Publishing, Reading, Massachusetts, 1995, p. 35.

[77] M. Maggiore, A modern introduction to quantum field theory, Oxford University Press, New-York, 2005, p. 20.

These are the matrices $\Lambda^{\alpha}{}_{\beta}$ used by these authors.

– Aïssata: This is a good point Diego. You are right, and we will go on with the active point of view now.

▷ 14.2 The parameters of the Lorentz group and the spacetime finite dimensional representations

Let us denote x^α the coordinates in an inertial reference frame with Minkowski metric tensor $\eta_{\alpha\beta}$ and an active coordinate transformation[78] maps x onto x' written as

$$x^\alpha \to x'^\alpha = x^\alpha + \xi^\alpha(x) \tag{500}$$

under which the corresponding transformation of the metric tensor obeys

$$g'_{\alpha\beta}(x) = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}\eta_{\gamma\delta}. \tag{501}$$

With the transformation (500), we can write $\frac{\partial x'^\alpha}{\partial x^\gamma} = \delta^\alpha_\gamma + \partial_\gamma \xi^\alpha$, hence

$$\frac{\partial x^\gamma}{\partial x'^\alpha} = (\delta^\alpha_\gamma + \partial_\gamma \xi^\alpha)^{-1} \simeq \delta^\gamma_\alpha - \partial^\gamma \xi_\alpha \tag{502}$$

and the transformation of the metric tensor, to linear order in ξ, reads as

$$\begin{aligned} g'_{\alpha\beta}(x) &= \delta^\gamma_\alpha \delta^\delta_\beta \eta_{\gamma\delta} - (\partial^\gamma \xi_\alpha \delta^\delta_\beta + \partial^\delta \xi_\beta \delta^\gamma_\alpha)\eta_{\gamma\delta} \\ &= \eta_{\alpha\beta} - (\partial_\beta \xi_\alpha + \partial_\alpha \xi_\beta) \end{aligned} \tag{503}$$

The invariance of the metric, when the prime coordinates also refer to an inertial reference frame, demand that the above expression (503) also equals to $\eta_{\alpha\beta}$ hence

$$\partial_\beta \xi_\alpha + \partial_\alpha \xi_\beta = 0. \tag{504}$$

A general expansion of the functions $\xi^\alpha(x)$ would be

$$\xi^\alpha(x) = \varepsilon^\alpha + \omega^\alpha{}_\beta x^\beta + \omega^\alpha{}_{\beta\gamma} x^\beta x^\gamma + \dots \tag{505}$$

with ε^α, $\omega^\alpha{}_\beta$, $\omega^\alpha{}_{\beta\gamma}$, …constant parameters[79]. There, ε^α would describe the translations, which are irrelevant for the Lorentz group and thus fixed to zero for the moment, $\omega^\alpha{}_\beta$ parametrizes rotations and boosts and the next order term is also included for completeness but will play no role here.

Equation (504) then requires $\partial_\beta(\omega_{\alpha\gamma}x^\gamma) = -\partial_\alpha(\omega_{\beta\gamma}x^\gamma)$ or

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}. \tag{506}$$

[78]which possibly depends on the spacetime location (hence not a priori limited to Galilean transformations), to be general.

[79]We use a single notation ω for the second and third order parameters, but there should be no confusion, since the number of indices specifies the order of the corresponding terms. In your professor's notes, the second order parameter is denoted as $\epsilon_{\alpha\beta}$, but we prefer the more common use of ω instead of ϵ because multi-indices ϵ symbols usually refer to antisymmetric Levi-Civita symbols. There should be no confusion neither between the third order parameter, that we will essentially forget from now on, and the Lorentz connection.

Eventually, the infinitesimal global Lorentz transformations are parametrized by

$$\xi^\alpha(x) = \omega^\alpha{}_\beta x^\beta \tag{507}$$

with 6 antisymmetric rotation parameters $\omega^{\alpha\beta}$ [80]. In terms of the previous notations, in particular the infinitesimal form of equations (470) and (471), one has[81]

$$(\omega^\alpha{}_\beta) = \begin{pmatrix} 1 & \delta\phi_x & \delta\phi_y & \delta\phi_z \\ \delta\phi_x & 1 & -\delta\alpha_z & \delta\alpha_y \\ \delta\phi_y & \delta\alpha_z & 1 & -\delta\alpha_x \\ \delta\phi_z & -\delta\alpha_y & \delta\alpha_x & 1 \end{pmatrix} \tag{509}$$

and we can identify the parameters of the Lorentz group. For the boosts, they are defined by[82] $\delta\phi_x = \omega^0{}_1 = \omega^1{}_0$ or $\delta\phi_x = -\omega^{01} = \omega^{10}$. For rotations, we have for example the identification $\frac{1}{2}\epsilon_{1jk}\omega^{jk} = \epsilon_{123}\omega^{23} = \omega^{23} = -\omega^2{}_3 = \omega^3{}_2 = \delta\alpha_x$.

Here, there is an important caveat. When we use the angles or boosts variables, these are purely $3D$ Euclidean parameters and this is irrelevant to use upper or lower indices, e.g. $\delta\alpha_x = \delta\alpha^x$, while the space indices in the $4D$ notation are raised or lowered with η_{ij}, e.g. $\omega^{0i} = -\omega_{0i}$. In order to avoid confusion, in this chapter we will use a, b, c indices for Euclidean indices, $\delta\alpha_a = \delta\alpha^a$ to distinguish them from ijk indices which are pseudo-Euclidean. The complete identification for Lorentz transformations parameters is eventually given by

$$\delta\phi_a = \delta^i_a\omega_{0i}, \quad \delta\phi^a = -\delta^a_i\omega^{0i}, \tag{510}$$

$$\delta\alpha_a = \tfrac{1}{2}\epsilon_{ajk}\omega^{jk}, \quad \delta\alpha^a = \tfrac{1}{2}\epsilon^{ajk}\omega_{jk} \tag{511}$$

The inverse transformation for $\delta\alpha_a$ is $\epsilon^{ajk}\delta\alpha_a = -\omega^{jk}$.

►EXERCISE 14 – **Commutation relations from the spacetime representation of the Lorentz group** –

This being said, we can now elaborate on the spacetime representations. A generic Lorentz transformation is such that

$$\begin{pmatrix} 1 & \delta\phi_x & \delta\phi_y & \delta\phi_z \\ \delta\phi_x & 1 & -\delta\alpha_z & \delta\alpha_y \\ \delta\phi_y & \delta\alpha_z & 1 & -\delta\alpha_x \\ \delta\phi_z & -\delta\alpha_y & \delta\alpha_x & 1 \end{pmatrix}\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \tag{512}$$

where the 4×4 matrix defines 6 generators and writes as

$$1_4 + \delta\alpha_x \mathsf{M}_x + \delta\alpha_y \mathsf{M}_y + \delta\alpha_z \mathsf{M}_z + \delta\phi_x \mathsf{N}_x + \delta\phi_y \mathsf{N}_y + \delta\phi_z \mathsf{N}_z = 1_4 + \delta\alpha^a \mathsf{M}_a + \delta\phi^a \mathsf{N}_a \tag{513}$$

with, by identification,

$$\mathsf{M}_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \mathsf{M}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ \mathsf{M}_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

[80] This is the infinitesimal counterpart of the finite global Lorentz transformation

$$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta. \tag{508}$$

[81] With opposite signs for the α_i's and ϕ_i's, since we consider active Lorentz transformations here.

[82] Take attention to the fact that the antisymmetry $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ or $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ does not generally hold when the indices have opposite up/down positions.

$$\mathsf{N}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{N}_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{N}_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{514}$$

This representation is not the most interesting physically, and one prefers usually other representations which act on fields of the theory, rather than on spacetime coordinates. It is nevertheless instructive for at least two reasons. First this representation tells us how a generalization of the coordinates, namely *vectors* transform under the Lorentz group, and secondly, it is now easy to write down the commutators of the Lorentz algebra using the above matrices. You should do it Diego.

– Diego: Look Aïssata! You want to make me die on commutators and matrix products?

– Aïssata: There are plenty of zeros, look

$$\begin{aligned} [\mathsf{M}_x, \mathsf{M}_y] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathsf{M}_z \end{aligned} \tag{515}$$

– Diego: OK, this is not very difficult indeed. I will do all the matrix products at home. Now I prefer to have the benefit of your knowledge while you can give some of your time, Diego says, smiling.

– Aïssata: So we may go on. All calculations done, you get the following algebra

$$[\mathsf{M}_a, \mathsf{M}_b] = \epsilon_{abc}\mathsf{M}_c, \tag{516}$$
$$[\mathsf{M}_a, \mathsf{N}_b] = \epsilon_{abc}\mathsf{N}_c \tag{517}$$
$$[\mathsf{N}_a, \mathsf{N}_b] = -\epsilon_{abc}\mathsf{M}_c \tag{518}$$

with, as usual, $\epsilon_{123} = +1$. ◄

– Diego: Equation (513) is the infinitesimal version of

$$\Lambda = \exp(\delta\boldsymbol{\alpha}\cdot\mathbf{M} + \delta\boldsymbol{\phi}\cdot\mathbf{N}) \simeq 1_4 + \delta\boldsymbol{\alpha}\cdot\mathbf{M} + \delta\boldsymbol{\phi}\cdot\mathbf{N} \tag{519}$$

but I imagine that we can also define unitary matrices instead of $\mathbf{M}$ and $\mathbf{N}$, like we have done for the Galilean transformations (417) and (418).

– Aïssata: You are right Diego. The transformation (519) is suitably written as

$$\Lambda = \exp\left(-\tfrac{1}{2}i\omega_{\alpha\beta}\mathsf{L}^{\alpha\beta}\right) \simeq 1_4 - \tfrac{1}{2}i\omega_{\alpha\beta}\mathsf{L}^{\alpha\beta} \tag{520}$$

with

$$\mathsf{M}_a = -i\epsilon_{ajk}\mathsf{L}^{jk} = -i\mathsf{J}_a, \quad \mathsf{N}^a = -i\delta^a_k\mathsf{L}^{0k} = -i\mathsf{K}^a, \tag{521}$$

e.g. $\mathsf{M}_3 = -i\mathsf{L}^{12}$ (M_3 is of course M_z) and $\mathsf{N}_3 = -i\mathsf{L}^{03}$. Using (509), you can now check for example that $-\frac{1}{2}i\omega_{12}\mathsf{L}^{12} = \frac{1}{2}\omega_{12}\mathsf{M}_3 = -\frac{1}{2}\omega^1{}_2\mathsf{M}_3 = +\frac{1}{2}\delta\alpha_z\mathsf{M}_z$ and $-\frac{1}{2}i\omega_{03}\mathsf{L}^{03} = \frac{1}{2}\omega_{03}\mathsf{N}_3 = \frac{1}{2}\omega^0{}_3\mathsf{N}_3 = +\frac{1}{2}\delta\phi_z\mathsf{N}_z$, i.e.

$$-\tfrac{1}{2}i\omega_{\alpha\beta}\mathsf{L}^{\alpha\beta} = \delta\boldsymbol{\alpha}\cdot\mathbf{M} + \delta\boldsymbol{\phi}\cdot\mathbf{N} \tag{522}$$

as announced.

▷ 14.3 The link between finite-dimensional representations of the Lorentz group and representations of $SU(2)$

– Aïssata: Let us concentrate on the homogeneous part of the group, the Lorentz group, and more specifically on $L_+^\uparrow$, that is proper rotations and boosts generated by the J_a's and the K_a's in (521). The K_a's are non compact boost generators. In terms of these generators, a finite Lorentz transformation, acting on state vectors, is represented by the unitary operators (see (417) and (418) for Galilean transformations and (520) and (521) above in the Lorentz case)

$$\Lambda = e^{-i\boldsymbol{\alpha}\cdot\mathbf{J}-i\boldsymbol{\phi}\cdot\mathbf{K}}. \tag{523}$$

But more can be obtained once we observe that the commutation relations among the J_a's and the K_a's can be simplified by the introduction of

$$\mathsf{A}_a = \tfrac{1}{2}(\mathsf{J}_a + i\mathsf{K}_a). \tag{524}$$

These new operators are not hermitian, $\mathsf{A}_a^\dagger \neq \mathsf{A}_a$, but they lead to two copies of the $SU(2)$ algebra,

$$\begin{aligned} [\mathsf{A}_a, \mathsf{A}_b] &= i\epsilon_{abc}\mathsf{A}_c, & (525)\\ \left[\mathsf{A}_a^\dagger, \mathsf{A}_b^\dagger\right] &= i\epsilon_{abc}\mathsf{A}_c^\dagger, & (526)\\ \left[\mathsf{A}_a, \mathsf{A}_b^\dagger\right] &= 0, & (527)\end{aligned}$$

and new results can be obtained from those, already known, of the representation theory for $SU(2)$.

We write the content of the above commutation relations as $SO(3,1) = SU(2)\otimes SU(2)$. We have two Casimir operators, $\sum_{a=1}^3(\mathsf{A}_a)^2$ with eigenvalues $n(n+1)$ and $\sum_{a=1}^3(\mathsf{A}_a^\dagger)^2$ with eigenvalues $m(m+1)$ where $n, m = 0, \frac{1}{2}, 1, \frac{3}{2}, 2\ldots$. The pairs (n,m) are used to label the finite-dimensional representations of the Lorentz group, while the eigenvalues of A_3 ad $\mathsf{A}_3^\dagger$ are further used to distinguish the states within a representation.

– Diego: Are we now in position to elucidate the mysterious $(\frac{1}{2},0)$, $(1,1)$, $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ notations found in so many texts on relativistic theory?

– Aïssata: Right! The notation, often in bold font $(\mathbf{n},\mathbf{m})$, is used to emphasize the composition of angular momentum, so the $(\mathbf{n},\mathbf{m})$ representation contains spins $|n-m|$, $|n-m|+1,\ldots n+m$. We have for example $(\mathbf{\frac{1}{2}},\mathbf{0}) = \frac{1}{2}$, $(\frac{1}{2},\frac{1}{2}) = \mathbf{1}\oplus\mathbf{0}$, $(\mathbf{1},\mathbf{1}) = \mathbf{2}\oplus\mathbf{1}\oplus\mathbf{0}$. The quantity $n+m$ is sometimes called the spin of the representation, meaning in this case the maximum value of the angular momentum. Therefore, the $(\mathbf{0},\mathbf{0})$ representation is $1-$dimensional for the two $SU(2)$ copies. The generators are 1×1 matrices, and the only ones which obey (525 - 527) are trivially 0. $\mathsf{A}_a = \mathsf{A}_a^\dagger = 0$. It follows that $\mathsf{J}_a = \mathsf{K}_a = 0$ and

$$\Lambda = 1. \tag{528}$$

The $(\mathbf{0},\mathbf{0})$ representation of the Lorentz group, of spin 0, acts on objects that are Lorentz invariant, i.e. scalar fields. This is called the Lorentz scalar representation.

The $(\mathbf{\frac{1}{2}},\mathbf{0})$ representation has spin $\frac{1}{2}$. For the † sector we have again $\mathsf{A}_a^\dagger = 0$, therefore $\mathsf{J}_a = i\mathsf{K}_a$. For A_a, 2×2 matrices obeying $\mathfrak{su}(2)$ algebra are obviously $\mathsf{A}_a = \frac{1}{2}\sigma_a$ with σ_a

the Pauli matrices. It follows that $\mathsf{J}_a = \frac{1}{2}\sigma_a$ and $\mathsf{K}_a = -\frac{i}{2}\sigma_a$. In the $(\mathbf{\frac{1}{2}}, \mathbf{0})$ representation, Lorentz rotations and Lorentz boosts are given by the operators

$$\mathsf{R}_{\boldsymbol{\alpha}} = e^{-i\boldsymbol{\alpha}\cdot\mathbf{J}} = e^{-i\frac{\boldsymbol{\alpha}}{2}\cdot\boldsymbol{\sigma}}, \tag{529}$$

$$\mathsf{B}_{\boldsymbol{\phi}} = e^{-i\boldsymbol{\phi}\cdot\mathbf{K}} = e^{-\frac{\boldsymbol{\phi}}{2}\cdot\boldsymbol{\sigma}}. \tag{530}$$

The two-component objects of the theory on which this two-dimensional representation acts are called left chiral spinors. Note that they change sign under a 2π rotation. This is a remarkable feature of spinors. We also speak about Spin representation of the Lorentz group and similarly $(\mathbf{0}, \mathbf{\frac{1}{2}})$ is another two-dimensional representation which acts on two-components objects called right chiral spinors. Chiral spinors are also called Weyl spinors. The linear combination $(\mathbf{\frac{1}{2}}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{\frac{1}{2}})$ yields Dirac spinors.

Then, you can build further representations from these fundamental ones. For example $(\mathbf{\frac{1}{2}}, \mathbf{0}) \otimes (\mathbf{0}, \mathbf{\frac{1}{2}}) = (\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}})$ gives a spin 1 representation with four components. This is a 4−vector representation. We also have $(\mathbf{\frac{1}{2}}, \mathbf{0}) \otimes (\mathbf{\frac{1}{2}}, \mathbf{0}) = (\mathbf{1}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{0})$

□ 14.4 Infinite dimensional representations and Poincaré and Lorentz algebras

Let us now come back to the infinitesimal spacetime transformation

$$x^\alpha \mapsto x'^\alpha = x^\alpha + \xi^\alpha(x) \tag{531}$$

for which one can build the following representations acting on quantum states

$$\mathsf{U}(\xi)\psi(x) \mapsto \psi'(x) = \psi(x - \xi(x)) = \psi(x) - \xi^\alpha(x)\partial_\alpha\psi(x) \tag{532}$$

like we have done when we were studying the infinite dimensional representations of the Galileo group.

In $3+1$ dimensions, translations are promoted to spacetime translations

$$x^\alpha \mapsto x'^\alpha = x^\alpha + \varepsilon^\alpha. \tag{533}$$

Using the fact that $\varepsilon^\alpha = -i\varepsilon^\beta(i\partial_\beta)x^\alpha = -i\varepsilon^\beta \mathsf{P}_\beta x^\alpha$ with

$$\mathsf{P}_\beta = i\partial_\beta, \tag{534}$$

one can build the following representations, respectively for infinitesimal and finite translations

$$\psi(x) \quad\mapsto\quad \psi'(x) = \mathsf{U}_T(\varepsilon)\psi(x) = (1 + i\varepsilon^\alpha\mathsf{P}_\alpha)\psi(x), \tag{535}$$

$$\psi(x) \quad\mapsto\quad \psi'(x) = \mathsf{U}_T(a)\psi(x) = e^{ia^\alpha\mathsf{P}_\alpha}\psi(x). \tag{536}$$

The sign in the exponent is consistent with the space part of $a^\alpha\mathsf{P}_\alpha$ being $-\mathbf{a}\cdot\mathbf{P}$ (compare (399) and (536)), since $(\mathsf{P}_\beta) = (E, -\mathbf{P}) = i\partial_\beta = (i\partial_t, i\boldsymbol{\nabla})$.

We now want to write the case of a Lorentz transformation

$$x^\alpha \mapsto x'^\alpha = x^\alpha + \omega^\alpha{}_\beta x^\beta. \tag{537}$$

The transformed ψ is $\psi(x) - \omega^{\alpha}{}_{\beta}x^{\beta}\partial_{\alpha}\psi(x)$ and the second term of the r.h.s. can be expanded, using the antisymmetry of $\omega^{\alpha\beta}$ as $\omega^{\alpha}{}_{\beta}x^{\beta}\partial_{\alpha} = \omega^{\alpha\beta}x_{\beta}\partial_{\alpha} = -\frac{1}{2}\omega^{\beta\alpha}(x_{\beta}\partial_{\alpha} - x_{\alpha}\partial_{\beta})$. Therefore we can introduce the operator

$$\mathsf{L}_{\alpha\beta} = i(x_{\alpha}\partial_{\beta} - x_{\beta}\partial_{\alpha}) \tag{538}$$

that we recognize as the $4D-$generalization of the angular momentum (in Cartesian coordinates) $\mathbf{L} = \mathbf{r} \times (-i\boldsymbol{\nabla})$ of quantum mechanics.

For example, the 12 covariant component is

$$\mathsf{L}_{12} = i(x_1\partial_2 - x_2\partial_1) = i((-x)\partial_y - (-y)\partial_x) = -i(x\partial_y - y\partial_x) \tag{539}$$

which we recognize as L_z, the $z-$component of $\mathbf{L}$. The 01 component gives a boost, indeed

$$\mathsf{L}_{01} = i(x_0\partial_1 - x_1\partial_0) = i(t\partial_x + x\partial_t) = -t\mathsf{P}_x + x\mathsf{E} = -\mathsf{K}_x \tag{540}$$

if we compare with the generator of the boost in the $x-$direction given in (432) and remember that $c = 1$, hence for a non relativistic particle, $E = m$.

We can write, for an infinitesimal Lorentz transformations,

$$\psi(x) \quad \mapsto \quad \psi'(x) = \mathsf{U}_R(\omega)\psi(x) = (1 - i\tfrac{1}{2}\omega^{\alpha\beta}\mathsf{L}_{\alpha\beta})\psi(x), \tag{541}$$

and for finite ones,

$$\psi(x) \quad \mapsto \quad \psi'(x) = \mathsf{U}_R(\Lambda)\psi(x) = e^{-i\frac{1}{2}\omega^{\alpha\beta}\mathsf{L}_{\alpha\beta}}\psi(x). \tag{542}$$

For a general Poincaré infinitesimal transformation, the unitary representation acting on wavefunctions thus reads as follows

$$\begin{aligned}
\mathsf{U}(\varepsilon,\omega) &= 1 + i\varepsilon^{\alpha}\mathsf{P}_{\alpha} - \tfrac{1}{2}i\omega^{\alpha\beta}\mathsf{L}_{\alpha\beta}, & (543)\\
\mathsf{P}_{\alpha} &= i\partial_{\alpha}, & (544)\\
\mathsf{L}_{\alpha\beta} &= i(x_{\alpha}\partial_{\beta} - x_{\beta}\partial_{\alpha}). & (545)
\end{aligned}$$

The factor $\frac{1}{2}$ in (541) or (543) enables to count only once each pair $\alpha\beta$. Note in particular the sign in the two terms comprising generators at the r.h.s. of (543). This was (seemingly) different in equations (413), (414) and (425) which all had a minus sign. This is due to the Minkowski metric which hides these sign complications.

Using the differential expressions for the generators, we can compute the commutation relations among them. For comparison with the literature, we have to take care about the possible sign conflicts between various authors who may for example use different choices in the metric signature, or define the generators without the imaginary i's if they don't require hermitian generators.

$$\begin{aligned}
\left[\mathsf{L}_{\alpha\beta}, \mathsf{L}_{\gamma\delta}\right] &= i(\eta_{\beta\gamma}\mathsf{L}_{\alpha\delta} - \eta_{\alpha\gamma}\mathsf{L}_{\beta\delta} + \eta_{\alpha\delta}\mathsf{L}_{\beta\gamma} - \eta_{\beta\delta}\mathsf{L}_{\alpha\gamma}), & (546)\\
\left[\mathsf{P}_{\alpha}, \mathsf{L}_{\gamma\delta}\right] &= i(\eta_{\alpha\gamma}\mathsf{P}_{\delta} - \eta_{\alpha\delta}\mathsf{P}_{\gamma}), & (547)\\
\left[\mathsf{P}_{\alpha}, \mathsf{P}_{\beta}\right] &= 0. & (548)
\end{aligned}$$

This is the Lie algebra of the Poincaré group. You can find a detailed analysis of the Poincaré group in the first volume of Steven Weinberg's trilogy on quantum theory of fields (S. Weinberg, The quantum theory of fields, vol.I, Cambridge University Press, Cambridge, 1995) . If we only retain the algebra defined by (546), we have the Lie algebra of the Lorentz group which is also called $SO(3,1)$ ($SO(4)$ is the group of rotations in 4−dimensional Euclidean space. The notation $SO(3,1)$ refers to the specific signature of the Minkowski spacetime). I should also emphasize that the most general representation of $SO(3,1)$ is defined in terms of generators which obey the same commutation relations (546), but are given by

$$\mathsf{M}_{\alpha\beta} = \mathsf{L}_{\alpha\beta} + \mathsf{S}_{\alpha\beta} \tag{549}$$

with the hermitian $\mathsf{S}_{\alpha\beta}$ also obeying the algebra (546) and commuting with the $\mathsf{L}_{\alpha\beta}$'s. The $\mathsf{S}_{\alpha\beta}$'s will play an important role later in spinor theory.

– Diego: I noticed that the literature contains various signs which do not always appear to be self-consistent in the same book. Could we check this once for all?

– Aïssata: This is true that you can find more or less fanciful notations. This is instructive to understand how it works.

►EXERCISE 15 – **More on the signs of the generators of the Poincaré group** –

There is nothing difficult for translations, since $\varepsilon^\alpha \mathsf{P}_\alpha = \delta t \mathsf{E} - \boldsymbol{\varepsilon}\cdot\mathbf{P}$ and the sign there is consistent with (413) for space translations. In the case of rotations, we calculate the *space part* of $\frac{1}{2}\omega^{\alpha\beta}\mathsf{L}_{\alpha\beta}$,

$$\tfrac{1}{2}\omega^{jk}\mathsf{L}_{jk} = \omega^{12}\mathsf{L}_{12} + \omega^{23}\mathsf{L}_{23} + \omega^{31}\mathsf{L}_{31} = \alpha_z\mathsf{L}_z + \alpha_x\mathsf{L}_x + \alpha_y\mathsf{L}_y = \boldsymbol{\alpha}\cdot\mathbf{L}, \tag{550}$$

where we have used the identification of parameters in (511) and of the components of the angular momentum in (539) and equivalent formulas. The sign is consistent with (414). Eventually, the case of the boosts leads to

$$\begin{aligned}\tfrac{1}{2}(\omega^{0i}\mathsf{L}_{0i} + \omega^{i0}\mathsf{L}_{i0}) &= \omega^{01}\mathsf{L}_{01} + \omega^{02}\mathsf{L}_{02} + \omega^{03}\mathsf{L}_{03} \\ &= -\phi_x(-\mathsf{K}_x) - \phi_y(-\mathsf{K}_y) - \phi_z(-\mathsf{K}_z) = \boldsymbol{\phi}\cdot\mathbf{K}.\end{aligned} \tag{551}$$

Here again we have used the identification of parameters in (510) and that of boosts generators in (540). The sign is also consistent with (425). ◄

An expression such as (546) seems hard to remember, but this is not so difficult in fact. You start from the indices in the order $\alpha\beta\gamma\delta$ and the signs at the r.h.s. are given by the odd/even number of permutations, e.g. $-\eta_{\alpha\gamma}\mathsf{L}_{\beta\delta}$ because one permutation of $\alpha\beta\gamma\delta$ is performed to get $\alpha\gamma\beta\delta$. The same is true with the other terms.

In Hamiltonian quantum mechanics, we usually like specifying separately space and time components. Some of these generators play special roles, since, commuting with the Hamiltonian $\mathsf{H} = \mathsf{P}^0$, they are associated to conserved quantities. This is the case of the 3 components of the canonical momentum which are given by

$$\mathbf{P} = (\mathsf{P}^1, \mathsf{P}^2, \mathsf{P}^3)^{\mathrm{T}}, \tag{552}$$

and those of the angular momentum which are

$$\mathbf{J} = (\mathsf{M}^{23}, \mathsf{M}^{31}, \mathsf{M}^{12})^{\mathrm{T}}. \tag{553}$$

But there are still the three other generators associated to the boosts,

$$\mathbf{K} = (\mathsf{M}^{01}, \mathsf{M}^{02}, \mathsf{M}^{03})^{\mathrm{T}} \tag{554}$$

which, like in the non relativistic case, are not conserved (a boost obviously does not conserve the energy of a particle, since its velocity is modified). In the more standard $3D-$notations, the Lie algebra of the Poincaré group is described by

$$\begin{aligned}
[\mathsf{J}_a, \mathsf{J}_b] &= i\epsilon_{abc}\mathsf{J}_c, && (555)\\
[\mathsf{J}_a, \mathsf{K}_b] &= i\epsilon_{abc}\mathsf{K}_c && (556)\\
[\mathsf{K}_a, \mathsf{K}_b] &= -i\epsilon_{abc}\mathsf{J}_c && (557)\\
[\mathsf{J}_a, \mathsf{P}_b] &= i\epsilon_{abc}\mathsf{P}_c && (558)\\
[\mathsf{K}_a, \mathsf{P}_b] &= i\mathsf{H}\delta_{ab} && (559)\\
[\mathsf{J}_a, \mathsf{H}] &= 0 && (560)\\
[\mathsf{P}_a, \mathsf{H}] &= 0 && (561)\\
[\mathsf{K}_a, \mathsf{H}] &= i\mathsf{P}_a. && (562)
\end{aligned}$$

This is an alternative to (546)-(548).

▷ 14.5 Scalar field representation of the Lorentz group

– Aïssata: We are now in position to study the transformations properties of various physically important fields. Let us come back to the $4D$ notation. Remember that the $\omega_{\alpha\beta}$ have six independent parameters, corresponding to the six antisymmetric generators $\mathsf{J}^{\alpha\beta}$. A generic element of the group writes as (520):

$$\Lambda = e^{-\frac{1}{2}i\omega_{\alpha\beta}\mathsf{J}^{\alpha\beta}}. \tag{563}$$

A set of objects of the theory ϕ^A, $A = 1, \dots N$ transforms according to

$$\phi^A \to (e^{-\frac{1}{2}i\omega_{\alpha\beta}\mathsf{J}^{\alpha\beta}})^A{}_B\phi^B \tag{564}$$

in a given representation of the group generators by $N \times N$ matrices. The physical objects are classified/defined according to the representation under which they do actually transform.

For scalar fields the index A takes only one value, we thus look for a $1-$dimensional representation of the Lorentz group, i.e. 1×1 matrices. $\mathsf{J}^{\alpha\beta} = 0$ obey trivially the algebra (546) and leads to $\delta\phi = 0$. This leaves the scalar fields unchanged. The $1-$dimensional representation is sometimes called the *identity representation* or the *scalar representation* and a scalar field can be defined as a field which transforms according to the trivial identity representation.

Klein-Gordon fields are examples of complex scalar fields which transform in the identity representation of the Lorentz group. We met the equation of motion of the free particle already,

$$\partial_t^2\varphi - \boldsymbol{\nabla}^2\varphi + m_\varphi^2\varphi = 0. \tag{565}$$

There are other peculiarities to mention concerning scalar fields. We may ask the question about the associated electric charge. As you will see when studying gauge theory, the electric charge conservation is intimately connected with the complex character of the field. Klein-Gordon particles which carry an electric charge are described by complex scalar fields while neutral particles are associated to real scalar fields.

For 4−vectors, writing the infinitesimal Lorentz transformation

$$V^\gamma \to \Lambda^\gamma{}_\delta V^\delta \simeq (\delta^\gamma_\delta + \omega^\gamma{}_\delta)V^\delta \simeq (1 - \tfrac{1}{2}i\omega_{\alpha\beta}\mathsf{L}^{\alpha\beta})V^\gamma = (\delta^\gamma_\delta - \tfrac{1}{2}i\omega_{\alpha\beta}(\mathsf{L}^{\alpha\beta})^\gamma{}_\delta)V^\delta, \tag{566}$$

i.e.

$$\delta V^\gamma = \omega^\gamma{}_\delta V^\delta = -\tfrac{1}{2}i\omega_{\alpha\beta}(\mathsf{L}^{\alpha\beta})^\gamma{}_\delta V^\delta \tag{567}$$

or

$$-\tfrac{1}{2}i\omega_{\alpha\beta}(\mathsf{L}^{\alpha\beta})^\gamma{}_\delta = \omega^\gamma{}_\delta = \eta^{\alpha\gamma}\omega_{\alpha\delta} = \eta^{\alpha\gamma}\delta^\beta_\delta\omega_{\alpha\beta}, \tag{568}$$

leads to the expression for the matrix elements of $\mathsf{L}^{\alpha\beta}$, $\frac{1}{2}(\mathsf{L}^{\alpha\beta})^\gamma{}_\delta = i\eta^{\alpha\gamma}\delta^\beta_\delta$ and, since $\mathsf{L}^{\alpha\beta}$ is an antisymmetric tensor, it also equals $\mathsf{L}^{\alpha\beta} = \frac{1}{2}(\mathsf{L}^{\alpha\beta} - \mathsf{L}^{\beta\alpha})$ or

$$(\mathsf{L}^{\alpha\beta})^\gamma{}_\delta = i(\eta^{\alpha\gamma}\delta^\beta_\delta - \eta^{\beta\gamma}\delta^\alpha_\delta). \tag{569}$$

These 4×4 matrices obey the $SO(3,1)$ algebra (546). This is a 4−dimensional representation of the Lorentz group, also called a vector representation. Like we said before for scalar fields, Lorentz vector fields can be defined as objects which transform under the *vector representation* of the Lorentz group.

15. Day 11 – Spin representation of the Lorentz group and the Dirac action

– Diego: My knowledge in spinor theory is. . . very limited, and this is an euphemism. I only know that these are the objects which appear in Dirac equation and that they generalize the notion of Schrödinger wave function.

– Aïssata: You are right, spinors are mathematical objects which have a richer structure than just single complex scalar fields of Schrödinger equation. For the moment let me just tell you that when Dirac tried to write down a differential equation which is of first order both in time and space derivatives, in order to be consistent with Special Relativity, he noticed that it was impossible to do it for a single complex scalar field. Instead, he had to generalize and he found that the differential equation took the form of a matrix equation acting at least on 4 coupled complex scalar fields (in 4 spacetime dimensions). This collection of 4 complex scalar fields is usually denoted in a vector form (the components of which live in an *internal* space) and is called a spinor. There are 4 matrices entering this matrix equation denoted as γ^α and called Dirac matrices. They obey a specific algebra.

– Diego: Could you elaborate a bit on this Aïssata?

– Aïssata: You may remember that we have established the Klein-Gordon equation in order to find a "wave equation" compatible with the famous relativistic equation

$$p_\alpha p^\alpha = E^2 - |\mathbf{p}|^2 = m^2, \tag{570}$$

and that it led to second order derivatives w.r.t. space and time. Dirac noticed that the compatible conserved current density then doesn't impose a positive probability density and this "obstruction" is due to the second order time derivative. As a consequence, Dirac suggested to look for a kind of square root equation, something like "$p_\alpha = m$", but in order to make sense with tensor balance of indices you need to invent a "kind of 1" which carries a contravariant index, let call it γ^α, such that $\gamma^\alpha p_\alpha = m$ when both sides act on a generalization of the scalar wave function, ψ,

$$\gamma^\alpha p_\alpha \psi = m\psi. \tag{571}$$

Obviously, the number 1 doesn't make the job and γ^α is a bit more elaborate, at least a 2×2 matrix. The γ^α are thus in principle non commuting objects. To put constraints on these objects, we demand to recover Klein-Gordon equation via the square of equation (571). Again, we use the "some kind of" or the "something like" to speak about the square, $(\gamma^\alpha p_\alpha)(\gamma^\beta p_\beta)\psi = m^2\psi$, because as indicated above, the γ^α's don't commute with the γ^β's, a proper constraint is obtained after symmetrization[83],

$$\tfrac{1}{2}\{\gamma^\alpha, \gamma^\beta\} p_\alpha p_\beta \psi = m^2\psi \tag{572}$$

which requires two things:

- this is a matrix equation (say with $N \times N$ matrices), hence the r.h.s. has to be understood in the sense of $m^2 1_N \psi$,
- at the l.h.s., we must impose the Dirac algebra or the Clifford algebra $\mathrm{Cl}_{1,3}(\mathbb{C})$

$$\tfrac{1}{2}\{\gamma^\alpha, \gamma^\beta\} = \eta^{\alpha\beta} 1_N. \tag{573}$$

– Diego: Aren't the matrices 4×4 to fit with the metric tensor $\eta^{\alpha\beta}$ which is also 4-dimensional?

– Aïssata: There is a dangerous confusion there. This is true that the gamma matrices will appear to be four-dimensional, but this has nothing to do with the four spacetime dimensions. The $\eta^{\alpha\beta}$ in equation (573) are taken as numbers. For example for $\alpha = \beta = 1$, you have

$$(\gamma^1)^2 = \eta^{11} 1_N = -1_N \tag{574}$$

which implies $(\gamma^1)^A{}_B (\gamma^1)^B{}_C = -(1_n)^A{}_C$, e.g. for $A = C = 1$, $(\gamma^1)^1{}_B (\gamma^1)^B{}_1 = -(1_n)^1{}_1 = -1$, or for $A = 1$ and $C = 2$, $(\gamma^1)^1{}_B (\gamma^1)^B{}_2 = -(1_n)^1{}_2 = 0$.

The next thing is to find representations of the Dirac algebra (573). We saw that numbers (1×1 matrices) don't do the job. You can easily convince yourself that two- or three-dimensional matrices neither don't realise (573) and you have to go at least up to 4×4

[83] $\{\gamma^\alpha, \gamma^\beta\}$ is the anticommutator $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha$.

matrices. There are various representations, the most common being the Weyl (or chiral) representation

$$(\gamma^0) = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad (\gamma^i) = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (\gamma^5) = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}, \tag{575}$$

and the Dirac representation

$$(\gamma^0) = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad (\gamma^i) = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (\gamma^5) = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \tag{576}$$

given here in terms of Pauli matrices. I also give you above the so-called

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{577}$$

matrix which is useful for some applications.

– Diego: I have an objection Aïssata! I saw that there are applications of Dirac equation in condensed matter physics, in graphene for example. And there, the Dirac equation is a 2×2 matrix, with a representation of Dirac algebra which is directly in terms of 2×2 Pauli matrices!

– Aïssata: This is a good observation Diego. But the objection doesn't work. This is true that I didn't say that we still work in $\mathbb{M}^4$, i.e. in four spacetime dimensions, but that was implicit. In graphene physics, you have only two space dimensions, hence $\alpha, \beta = 0, 1, 2$ and you have to find 3 matrices only, which obey the algebra (573) instead of 4. And this is true that in $2+1$ spacetime, as well as in the $1+1$ spacetime, one can show that 2×2 matrices can realize the Dirac algebra. This has the advantage to show that the dimensions of Dirac matrices is indeed not the spacetime dimension!

– Diego: I still have another comment Aïssata. Dirac equation is often written in terms of matrices denoted as $\boldsymbol{\alpha}$ and β. Why did you use the notation γ^α?

– Aïssata: You are right. Historically, Dirac used a different notation for matrices which are related to, but not identical to the gamma matrices. If you expand (571), using $p_\alpha = i\partial_\alpha$:

$$\begin{aligned} i\gamma^\alpha \partial_\alpha \psi &= m 1_4 \psi, \\ i\gamma^0 \partial_t \psi &= (-i\gamma^i \partial_i + m 1_4)\psi, \end{aligned} \tag{578}$$

and, multiplying by γ^0 (remember that $(\gamma^0)^2 = 1_4$),

$$i\partial_t \psi = (-i\gamma^0\gamma^i\partial_i + \gamma^0 m)\psi \tag{579}$$

where the identification is

$$\gamma^0 = \beta, \quad \gamma^0\gamma^i = \alpha^i, \quad \{\alpha^i, \alpha^j\} = 0, \quad \{\alpha^i, \beta\} = 0, \quad (\alpha^i)^2 = (\beta)^2 = 1_4, \tag{580}$$

with a Dirac equation which takes the form

$$i\partial_t \psi = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi, \quad \mathbf{p} = -i\boldsymbol{\nabla}. \tag{581}$$

– Aïssata: Now we have these new objects ψ with four components (in 4-spacetime dimensions) and we have to look at their transformation properties under the Lorentz group. Being 4-component objects, they could transform like 4-vectors in $T_x(\mathbb{M}^4)$, i.e. with $\Lambda^\alpha{}_\beta$, but we will see that they don't. Look at equation (476) which *is not* (474)! This is the reason why these 4-component objects are called *spinors* , to distinguish them from ordinary vectors, although spinors obey the usual rules applying in vector spaces and would probably be called *vectors* by mathematicians. There is obviously a profusion of texts on Dirac equation, for example the celebrated Peshkin and Schroeder[84], but we will follow here the excellent lecture notes of Tong [85], although we use slightly different notations[86].

Let us first define the 4×4 matrices[87]

$$\mathsf{S}_{\alpha\beta} = \frac{1}{2}\sigma_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta] = \frac{i}{2}(\gamma_\alpha\gamma_\beta - \eta_{\alpha\beta}) = \begin{cases} 0 & \text{if } \alpha = \beta \\ \frac{i}{2}\gamma_\alpha\gamma_\beta & \text{if } \alpha \neq \beta \end{cases} \tag{582}$$

They obey the Lorentz algebra as we will show.

For that purpose, we have to evaluate the commutator

$$[\mathsf{S}_{\alpha\beta}, \mathsf{S}_{\gamma\delta}] = \frac{i}{2}([\mathsf{S}_{\alpha\beta}, \gamma_\gamma]\gamma_\delta + \gamma_\gamma[\mathsf{S}_{\alpha\beta}, \gamma_\delta]). \tag{583}$$

We need first

$$[\mathsf{S}_{\alpha\beta}, \gamma_\gamma] = i(\gamma_\alpha\eta_{\beta\gamma} - \gamma_\beta\eta_{\gamma\alpha}) \tag{584}$$

which is obtained, ..., just by writing it. Then, expanding the commutator $[\mathsf{S}_{\alpha\beta}, \mathsf{S}_{\gamma\delta}]$ in (583) and using expressions like $\gamma_\alpha\gamma_\beta = \eta_{\alpha\beta} - 2i\mathsf{S}_{\alpha\beta}$ we can prove the important relation

$$[\mathsf{S}_{\alpha\beta}, \mathsf{S}_{\gamma\delta}] = i(\eta_{\beta\gamma}\mathsf{S}_{\alpha\delta} - \eta_{\alpha\gamma}\mathsf{S}_{\beta\delta} + \eta_{\alpha\delta}\mathsf{S}_{\beta\gamma} - \eta_{\beta\delta}\mathsf{S}_{\alpha\gamma}). \tag{585}$$

This is exactly the Lorentz algebra (546) and this was anticipated in your professor's notes in equation (549), where the option of a complementary piece to the angular momentum was introduced. The $\mathsf{S}_{\alpha\beta}$ being 4-dimensional matrices, they act on 4-components "vectors" $\psi^A(x)$ in such a way that they obey, under Lorentz transformations, to the linear relations

$$\psi^A(x) \to \psi'^A(\Lambda^{-1}x) = (\mathsf{S}[\Lambda])^A{}_B\psi^B(\Lambda^{-1}x). \tag{586}$$

This is tempting to consider the ψ vectors as ordinary vectors and the 4×4 transformation matrix $\mathsf{S}[\Lambda]$ as the Lorentz transformation matrix, with A and B indices being the spacetime indices α and β, but we have seen already that the Dirac matrices act on objects which live in a different space, and we have to study more carefully (586).

Let us consider specific examples of Lorentz transformations and compare with the transformation of ordinary 4-vectors. Using equation (509), that we rewrite below for

[84] M.E. Peshkin and D.V. Schroeder, An introduction to Quantum Field Theory, Perseus Books Publishing, Reading, Massachusetts ,1995.

[85] D. Tong, The Dirac Equation, `https://www.damtp.cam.ac.uk/user/tong/qft/four.pdf`

[86] The difference is essentially in the absence of i in the group representation in Tong's notes.

[87] Obviously, an identity matrix 1_4 is expected in factor of $\eta_{\alpha\beta}$.

convenience,

$$
(\omega^{\alpha}{}_{\beta}) = \begin{pmatrix} 1 & \delta\phi_x & \delta\phi_y & \delta\phi_z \\ \delta\phi_x & 1 & -\delta\alpha_z & \delta\alpha_y \\ \delta\phi_y & \delta\alpha_z & 1 & -\delta\alpha_x \\ \delta\phi_z & -\delta\alpha_y & \delta\alpha_x & 1 \end{pmatrix} \tag{587}
$$

and (520) applied to the generators $\mathsf{S}^{\alpha\beta}$, we write

$$
\mathsf{S}[\Lambda] = \exp\left(-\tfrac{1}{2} i\omega_{\alpha\beta}\mathsf{S}^{\alpha\beta}\right). \tag{588}
$$

►EXERCISE 16 – **Spinor representation of rotations** –

In the case of a rotation of angles $(\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}} = \boldsymbol{\alpha}$, we need an expression for the purely space part of the generators S^{ij}. In the chiral representation (575), we obtain

$$
\mathsf{S}^{ij} = \tfrac{i}{2}(\gamma^i\gamma^j - \eta^{ij}) = \tfrac{1}{2}\epsilon^{ijc}\begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{pmatrix}. \tag{589}
$$

It follows that

$$
\begin{aligned}
\mathsf{S}[\Lambda] &= \exp\left(-\tfrac{1}{2} i\omega_{ij}\mathsf{S}^{ij}\right) = \exp\left[-\tfrac{1}{2} i\alpha^a \begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}\right] \\
&= \begin{pmatrix} e^{-\frac{1}{2} i\alpha^a\sigma_a} & 0 \\ 0 & e^{-\frac{1}{2} i\alpha^a\sigma_a} \end{pmatrix}.
\end{aligned} \tag{590}
$$

A rotation of a spinor, say by an angle α^3 around the z-axis, is described by the matrix

$$
(\mathsf{S}[\mathsf{R}_{\alpha^3\mathbf{u}_z}]) = \begin{pmatrix} e^{-i\frac{\alpha^3}{2}} & 0 & & \\ 0 & e^{i\frac{\alpha^3}{2}} & & \\ & & e^{-i\frac{\alpha^3}{2}} & 0 \\ & & 0 & e^{i\frac{\alpha^3}{2}} \end{pmatrix}. \tag{591}
$$

In the case of a simple $2\pi-$rotation, around the Oz axis, i.e. $\alpha^a = (0, 0, 2\pi)^{\mathrm{T}}$ we get

$$
S[\mathsf{R}_{2\pi\mathbf{u}_z}] = \begin{pmatrix} e^{-i\pi\sigma_3} & 0 \\ 0 & e^{-i\pi\sigma_3} \end{pmatrix} \equiv -1_4. \tag{592}
$$

This means that the vector ψ^A, $A = 1, 2, 3, 4$ is not an ordinary vector, since it changes sign under a 2π rotation. It follows that $\mathsf{S}[\Lambda]$ *is not* the usual 4-dimensional vector representation of Lorentz algebra. Indeed, an ordinary vector would transform according to $\Lambda[\mathsf{R}] = \exp(\alpha^3\mathsf{M}_3)$, see (499) to be specified in the case of a rotation around the z-axis. You can work this out by yourself, this is a good exercise. ◄

►EXERCISE 17 – **Vector representation of rotations** –

– Diego: Let's go. We have to expand the exponential

$$
\Lambda[\mathsf{R}_{\alpha^3\mathbf{u}_z}] = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha^3 & 0 \\ 0 & \alpha^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{593}
$$

in power series of $\alpha^3 M_3$. This is easily evaluated using

$$(M_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad ((M_3)^2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$((M_3)^3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad ((M_3)^4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{594}$$

and expanding

$$\begin{aligned} \exp(\alpha^3 M_3) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left(1 - \frac{1}{2!}(\alpha^3)^2 + \frac{1}{4!}(\alpha^3)^4 + \ldots\right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &+ \left(\alpha^3 - \frac{1}{3!}(\alpha^3)^3 + \frac{1}{5!}(\alpha^3)^5 + \ldots\right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{595}$$

After identification of $\cos(\alpha^3)$ and $\sin(\alpha^3)$,

$$(\Lambda[R_{\alpha^3 \mathbf{u}_z}]) = \exp(\alpha^3 M_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(\alpha^3) & -\sin(\alpha^3) & 0 \\ 0 & \sin(\alpha^3) & \cos(\alpha^3) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{596}$$

This expression differs from (591) and in particular we find that a 2π rotation is

$$\Lambda[R_{2\pi \mathbf{u}_z}] = 1_4 \tag{597}$$

which differs from (592) and leaves ordinary vectors unchanged.◄

►EXERCISE 18 – **Spinor representation of boosts** –

– Aïssata: Very good Diego. We can proceed with boosts now, in order to complement the comparison between the two representations. First, we give an expression for S^{0i},

$$(S^{0i}) = \tfrac{i}{2}\gamma^0\gamma^i = \tfrac{i}{2}\delta^i_a \begin{pmatrix} -\sigma^a & 0 \\ 0 & \sigma^a \end{pmatrix}. \tag{598}$$

With $\omega_{0i} = \delta\phi_a \delta^a_i$, we form

$$\omega_{0i} S^{0i} = \tfrac{i}{2}\delta\phi_a \begin{pmatrix} -\sigma^a & 0 \\ 0 & \sigma^a \end{pmatrix} \tag{599}$$

and obtain

$$S[B] = \exp\left[\tfrac{1}{2}\delta\phi_a \begin{pmatrix} -\sigma^a & 0 \\ 0 & \sigma^a \end{pmatrix}\right] = \begin{pmatrix} e^{-\frac{1}{2}\delta\phi_a \sigma^a} & 0 \\ 0 & e^{\frac{1}{2}\delta\phi_a \sigma^a} \end{pmatrix}. \tag{600}$$

A boost along Oz is thus described by

$$(S[B_{\beta \mathbf{u}_z}]) = \begin{pmatrix} e^{-\frac{\phi^3}{2}} & 0 & & \\ 0 & e^{\frac{\phi^3}{2}} & & \\ & & e^{\frac{\phi^3}{2}} & 0 \\ & & 0 & e^{-\frac{\phi^3}{2}} \end{pmatrix}. \tag{601}$$

Again, this has nothing to do with (499). ◄

►**Exercise 19** – 4**-vector representation of a boost along** Oz –

– Diego: Let me check! The 4-vector representation is described by

$$(\Lambda[\mathsf{B}_{\beta\mathbf{u}_z}]) = \exp(\phi^3 \mathsf{N}_3) \tag{602}$$

where the expansion requires the expression of the successive powers of N_3:

$$(\mathsf{N}_3) = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix}, \quad ((\mathsf{N}_3)^2) = \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & 1 \end{pmatrix}, \quad ((\mathsf{N}_3)^3) = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix}, \ldots \tag{603}$$

then,

$$\begin{aligned} \exp(\phi^3 \mathsf{N}_3) \quad &= \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} + \left(1 + \frac{1}{2!}(\phi^3)^2 + \frac{1}{4!}(\phi^3)^4 + \ldots\right) \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & 1 \end{pmatrix} \\ &+ \left(\phi^3 + \frac{1}{3!}(\phi^3)^3 + \ldots\right) \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix} \end{aligned} \tag{604}$$

and eventually, collecting all terms in a single matrix, we get

$$(\Lambda[\mathsf{B}_{\beta\mathbf{u}_z}]) = \begin{pmatrix} \cosh(\phi^3) & 0 & 0 & \sinh(\phi^3) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\phi^3) & 0 & 0 & \cosh(\phi^3) \end{pmatrix} \tag{605}$$

which indeed, *is not* (601)! ◄

▷ 15.3 Dirac action

– Aïssata: We just understood that Dirac spinors obey specific transformation laws under Lorentz transformations. In order to write an action, we now need to build Lorentz scalars which remain invariant and can appear in a Lagrangian density. A first trial is to generalize an expression like $\varphi^*\varphi$ for scalar fields.

– Diego: What about $\psi^\dagger\psi$?

– Aïssata: This is a natural guess. Let us see whether it is a Lorentz scalar. We have

$$\psi(x) \to \mathsf{S}[\Lambda]\psi(\Lambda^{-1}x) \tag{606}$$

and, since $\psi^\dagger(x) = (\psi^*(x))^{\mathrm{T}}$,

$$\psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)\mathsf{S}[\Lambda]^\dagger. \tag{607}$$

It follows that $\psi^\dagger\psi$ doesn't make the job. Indeed,

$$\psi^\dagger(x)\psi(x) \to \psi^\dagger(\Lambda^{-1}x)\mathsf{S}[\Lambda]^\dagger\mathsf{S}[\Lambda]\psi(\Lambda^{-1}x) \tag{608}$$

is not a Lorentz scalar, because $\mathsf{S}[\Lambda]$ *is not a unitary transformation*, hence $\mathsf{S}[\Lambda]^\dagger\mathsf{S}[\Lambda] \neq 1_4$. To make it clear, remember that $\mathsf{S}[\Lambda] = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}\mathsf{S}^{\alpha\beta}\right)$ would be unitary, if $\mathsf{S}^{\alpha\beta}$ were hermitian. But

$$(\mathsf{S}^{\alpha\beta})^\dagger = \left(\frac{i}{4}[\gamma^\alpha, \gamma^\beta]\right)^\dagger = -\frac{i}{4}[(\gamma^\beta)^\dagger, (\gamma^\alpha)^\dagger] \tag{609}$$

and, for exemple in the chiral representation, the explicit expressions

$$(\gamma^0)^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^0, \tag{610}$$

$$(\gamma^i)^\dagger = \delta^i_a \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}^\dagger = \delta^i_a \begin{pmatrix} 0 & (-\sigma^a)^\dagger \\ (\sigma^a)^\dagger & 0 \end{pmatrix} = \delta^i_a \begin{pmatrix} 0 & -\sigma^a \\ \sigma^a & 0 \end{pmatrix} = -\gamma^i \tag{611}$$

$$\tag{612}$$

show that it leads e.g. to

$$(S^{0i})^\dagger = \tfrac{i}{4}[\gamma^0, \gamma^i] = -S^{0i}, \tag{613}$$

hence $S[\Lambda]$ is not unitary.

To proceed further, we have to manipulate the gamma matrices. First note that

$$\gamma^0 \gamma^\alpha \gamma^0 = (\gamma^\alpha)^\dagger \tag{614}$$

as it can be shown, e.g. again in the chiral representation.

►EXERCISE 20 – **Calculation of $(\gamma^\alpha)^\dagger$** –

– Diego: Let me do it to train myself. For $\alpha = 0$, I can build

$$(\gamma^0\gamma^0\gamma^0)^\dagger = \left(\gamma^0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)^\dagger = (\gamma^0)^\dagger = \gamma^0. \tag{615}$$

For $\alpha = i$, proceeding the same way,

$$\begin{aligned} (\gamma^0\gamma^i\gamma^0)^\dagger &= (\gamma^0)^\dagger(\gamma^i)^\dagger(\gamma^0)^\dagger = \gamma^0(-\gamma^i)\gamma^0 \\ &= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \delta^i_a \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \delta^i_a \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} = \gamma^i \end{aligned} \tag{616}$$

which proves (614). ◄

– Aïssata: This being done, we can now form

$$\begin{aligned} \gamma^0 S^{\alpha\beta} \gamma^0 &= \tfrac{i}{4}\gamma^0[\gamma^\alpha, \gamma^\beta]\gamma^0 = \tfrac{i}{4}(\gamma^0\gamma^\alpha\gamma^0 - \gamma^0\gamma^\beta\gamma^0) \\ &= \tfrac{i}{4}[(\gamma^\alpha)^\dagger - (\gamma^\beta)^\dagger] = -\tfrac{i}{4}[(\gamma^\beta)^\dagger, (\gamma^\alpha)^\dagger] = (S^{\alpha\beta})^\dagger \end{aligned} \tag{617}$$

where (609) was used. We are now in position to find the adjoint of $S[\Lambda]^\dagger$:

$$\begin{aligned} S[\Lambda]^\dagger &= \exp\left[+\tfrac{i}{2}(S^{\alpha\beta})^\dagger\right] \\ &= \exp\left[+\tfrac{i}{2}\gamma^0 S^{\alpha\beta}\gamma^0\right] \\ &= \gamma^0\left(1_4 + \tfrac{i}{2}S^{\alpha\beta} + \ldots\right)\gamma^0 \\ &= \gamma^0 \exp\left[+\tfrac{i}{2}S^{\alpha\beta}\right]\gamma^0 \\ &= \gamma^0 S[\Lambda]^{-1}\gamma^0 \end{aligned} \tag{618}$$

►EXERCISE 21 – **Show that $\bar\psi\psi$, where $\bar\psi = \psi^\dagger\gamma^0$, is a Lorentz scalar** –

We can thus define the *Dirac conjugate*

$$\bar\psi(x) = \psi^\dagger(x)\gamma^0 \tag{619}$$

which transforms as $\bar{\psi}(x) \to (\psi^\dagger(\Lambda^{-1}x)\mathsf{S}[\Lambda]^\dagger)\gamma^0$

$$\begin{aligned}
\bar{\psi}(x)\psi(x) \quad \to \quad & \psi^\dagger(\Lambda^{-1}x)\mathsf{S}[\Lambda]^\dagger\gamma^0\mathsf{S}[\Lambda]\psi(\Lambda^{-1}x) \\
& = \psi^\dagger(\Lambda^{-1}x)(\gamma^0\mathsf{S}[\Lambda]^{-1}\gamma^0)\gamma^0\mathsf{S}[\Lambda]\psi(\Lambda^{-1}x) \\
& = \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}x) \\
& = \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x)
\end{aligned} \tag{620}$$

– Diego: Look Aïssata, this proves that $\bar{\psi}\psi$ is a Lorentz scalar, right?

– Aïssata: Yes Diego. This is the first important piece to build an invariant action. ◄

►EXERCISE 22 – **Show that $\bar{\psi}\gamma^\varepsilon\psi$ is a Lorentz vector** –

Next piece is to prove that

$$\bar{\psi}(x)\gamma^\varepsilon\psi(x) \tag{621}$$

is a Lorentz vector, i.e. an ordinary spacetime 4-vector, and thus that α in γ^α can properly be treated as an ordinary Lorentz index. We have the transformation law

$$\bar{\psi}(x)\gamma^\varepsilon\psi(x) \quad \to \quad \psi^\dagger(\Lambda^{-1}x)\mathsf{S}[\Lambda]^\dagger\gamma^0\gamma^\varepsilon\mathsf{S}[\Lambda]\psi(\Lambda^{-1}x) \tag{622}$$

Using (618) and simplifying $(\gamma^0)^2 = 1_4$, we have to evaluate

$$\begin{aligned}
\mathsf{S}[\Lambda]^{-1}\gamma^\varepsilon\mathsf{S}[\Lambda] \quad &= \quad (1_4 + \tfrac{1}{2}i\omega_{\alpha\beta}\mathsf{S}^{\alpha\beta} + \ldots)\gamma^\varepsilon(1_4 - \tfrac{1}{2}i\omega_{\gamma\delta}\mathsf{S}^{\gamma\delta} + \ldots) \\
&= \quad \gamma^\varepsilon + \tfrac{1}{2}i[\mathsf{S}^{\alpha\beta}, \gamma^\varepsilon] + \ldots
\end{aligned} \tag{623}$$

where (584) can be used and leads to

$$\begin{aligned}
[\mathsf{S}^{\alpha\beta}, \gamma^\varepsilon] \quad &= \quad i(\gamma^\alpha\eta^{\beta\varepsilon} - \gamma^\beta\eta^{\alpha\varepsilon}) \\
&= \quad i(\delta^\alpha_\delta\eta^{\beta\varepsilon} - \delta^\beta_\delta\eta^{\alpha\varepsilon})\gamma^\delta \\
&= \quad -(\mathsf{L}^{\alpha\beta})^\varepsilon{}_\delta\gamma^\delta.
\end{aligned} \tag{624}$$

Here, the definition of the Lorentz generator in the spacetime vector representation $\mathsf{L}^{\alpha\beta}$ in (569) has been introduced. It yields

$$\mathsf{S}[\Lambda]^{-1}\gamma^\varepsilon\mathsf{S}[\Lambda] = \gamma^\varepsilon - \delta\gamma^\varepsilon \tag{625}$$

where the variation is

$$\delta\gamma^\varepsilon = -\tfrac{1}{2}i\omega_{\alpha\beta}(\mathsf{L}^{\alpha\beta})^\varepsilon{}_\delta\gamma^\delta \tag{626}$$

and appears to be Lorentz 4-vector.

Hence, (622) becomes

$$\begin{aligned}
\bar{\psi}(x)\gamma^\varepsilon\psi(x) \quad \to \quad & \psi^\dagger(\Lambda^{-1}x)\gamma^0(\gamma^\varepsilon - \tfrac{1}{2}i\omega_{\alpha\beta}(\mathsf{L}^{\alpha\beta})^\varepsilon{}_\delta\gamma^\delta + \ldots)\psi(\Lambda^{-1}x) \\
= \quad & \bar{\psi}(\Lambda^{-1}x)\left[e^{-\frac{1}{2}i\omega_{\alpha\beta}\mathsf{L}^{\alpha\beta}}\right]^\varepsilon{}_\delta\psi(\Lambda^{-1}x) \\
= \quad & \Lambda^\varepsilon{}_\delta\bar{\psi}(\Lambda^{-1}x)\gamma^\varepsilon\psi(\Lambda^{-1}x)
\end{aligned} \tag{627}$$

which closes the proof. ◄

– Diego: If I understand, although the components ψ^A transform like spinors, i.e. with $\mathsf{S}[\Lambda]^A{}_B$, the components of the combination $\bar{\psi}\gamma^\alpha\psi$ transform like vectors, i.e. with $\Lambda^\alpha{}_\beta$.

– Aïssata: This is perfectly formulated Diego. We can draw the following table to summarize the transformation properties of various objects made from spinors:

Lorentz transformation Field	Representation	Exemple
Scalar Field	Identity 1 $\varphi(x) \to \varphi'(x) = 1\varphi(x') = \varphi(\Lambda^{-1}x)$	$\bar{\psi}(x)\psi(x)$
Vector Field	Spacetime 4-vector $\Lambda^{\alpha}{}_{\beta} = \left[e^{-\frac{1}{2}i\omega_{\gamma\delta}\mathsf{L}^{\gamma\delta}}\right]^{A}{}_{B}$ with $(\mathsf{L}^{\alpha\beta})^{\gamma}{}_{\delta} = i(\eta^{\alpha\gamma}\delta^{\beta}_{\delta} - \eta^{\beta\gamma}\delta^{\alpha}_{\delta})$ $v^{\alpha}(x) \to v'^{\alpha}(x) = \Lambda^{\alpha}{}_{\beta}v^{\beta}(x') = \Lambda^{\alpha}{}_{\beta}v^{\beta}(\Lambda^{-1}x)$	$\bar{\psi}(x)\gamma^{\alpha}\psi(x)$
Spinor Field	Spin $\mathsf{S}[\Lambda]^{\alpha}{}_{\beta} = \left[e^{-\frac{1}{2}i\omega_{\gamma\delta}\mathsf{S}^{\gamma\delta}}\right]^{A}{}_{B}$ with $\mathsf{S}^{\alpha\beta} = \frac{i}{4}[\gamma^{\alpha},\gamma^{\beta}]$ $\psi^{A}(x) \to \psi'^{A}(x) = \mathsf{S}[\Lambda]^{A}{}_{B}\psi^{B}(x') = \mathsf{S}[\Lambda]^{A}{}_{B}\psi^{B}(\Lambda^{-1}x)$	$\psi(x)$

– Diego: And I suppose that you can show a covariant form of Dirac equation (581)?

– Aïssata: Yes Diego, this is almost automatic now to build a Lorentz invariant action from which equation (581) follows by the use of Euler-Lagrange formalism, although in a slightly different form, since (581) is still pretty "unsymmetric" in the use of symbols like ∂_t and $\boldsymbol{\nabla}$. I am sure that you will learn how do do that in your next lectures.

▷ 15.4 Dirac, Weyl and Majorana fermions

– Aïssata: Here we can discuss various types of relativistic spin $\frac{1}{2}$ fermions, depending on the structure of the corresponding spinors and on the associated representation. Dirac spinors described above are 4-component spinors which transform according to a *reducible* representation of the Lorentz group (see p. 129):

$$\psi^{A}(x) \to \psi'^{A}(\Lambda^{-1}x) = (\mathsf{S}[\Lambda])^{A}{}_{B}\psi^{B}(\Lambda^{-1}x). \tag{628}$$

where

$$\mathsf{S}[\Lambda] = \exp\left(-\tfrac{1}{2}i\omega_{\alpha\beta}\mathsf{S}^{\alpha\beta}\right). \tag{629}$$

and the $S^{\alpha\beta}$ *are* 4×4 *matrices* which obey the algebra

$$[\mathsf{S}_{\alpha\beta},\mathsf{S}_{\gamma\delta}] = i(\eta_{\beta\gamma}\mathsf{S}_{\alpha\delta} - \eta_{\alpha\gamma}\mathsf{S}_{\beta\delta} + \eta_{\alpha\delta}\mathsf{S}_{\beta\gamma} - \eta_{\beta\delta}\mathsf{S}_{\alpha\gamma}). \tag{630}$$

There are *two inequivalent irreducible* 2×2 *representations*, these are the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations, called chiral, and Dirac spinors described so far transform in the $(\frac{1}{2},0)+(0,\frac{1}{2})$ reducible representation. I can tell you a bit more on that if you are interested.

– Diego: Of course, and I can anticipate that you will refer to the distinction between Dirac, Weyl and Majorana fermions!

– Aïssata: Correct! I will mainly follow the discussion of Pal[88] which I strongly recommend, but you can also refer to other reviews[89]. Pal starts with the remark that real

[88] P.B. Pal, Dirac, Majorana, and Weyl fermions, Am. J. Phys. **79**, 485 2011.

[89] e.g. L. Bonora, R. Soldati, S. Zalel, Dirac, Majorana, Weyl in 4D, Universe **6**, 111 2000

scalar solutions of Klein-Gordon equation describe neutral particles and asks the equivalent question concerning solutions of Dirac equation. Considering the form of Dirac equation

$$(i\gamma^\alpha \partial_\alpha - m)\psi = 0, \tag{631}$$

one can easily understand that with appropriate initial conditions, spinors ψ with only real components will be found if all the non-vanishing elements of Dirac matrices are purely imaginary. This happens in the Majorana representation

$$(\tilde{\gamma}^0) = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad (\tilde{\gamma}^1) = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad (\tilde{\gamma}^2) = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad (\tilde{\gamma}^3) = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \tag{632}$$

which guarantees the equality $\tilde{\gamma}^*_\alpha = -\tilde{\gamma}_\alpha$. Following Pal, we denote the gamma matrices in the Majorana representation with a tilde. Any other representation can be deduced from this one through a unitary transformation U such that

$$\gamma^\alpha = \mathsf{U}\tilde{\gamma}^\alpha \mathsf{U}^\dagger \tag{633}$$

and, if $\tilde{\psi}$ is a solution of Dirac equation in the Majorana representation $\{\tilde{\gamma}^\alpha\}$, the spinor $\psi = \mathsf{U}\tilde{\psi}$ is a solution in the new $\{\gamma^\alpha\}$ representation. A Majorana fermion field obeys the reality condition

$$\tilde{\psi}^* = \tilde{\psi}. \tag{634}$$

Like real scalar Klein-Gordon fields are associated to neutral particles, Majorana fermions are also neutral. To translate this condition in the representation $\{\gamma^\alpha\}$, we write $\mathsf{U}^\dagger\psi = (\mathsf{U}^\dagger\psi)^*$ and we define the charge conjugation matrix C by $\gamma_0\mathsf{C} = \mathsf{U}\mathsf{U}^T$ such that $\psi = \mathsf{U}\mathsf{U}^T\psi^* = \gamma_0\mathsf{C}\psi^*$. The reality condition is thus

$$\hat{\psi} = \psi, \quad \text{where} \quad \hat{\psi} \equiv \gamma_0\mathsf{C}\psi^*. \tag{635}$$

The matrix C obeys various properties. Among them,

$$\mathsf{C}^{-1}\gamma_\alpha\mathsf{C} = -\gamma_\alpha^T \tag{636}$$

can be taken as the definition of C and this does not refer to the particular representation used. One also has $\mathsf{C}^* = -\mathsf{C}^{-1}$, $\mathsf{C}^T = -C$. The reality condition (635) is robust through Lorentz transformation, which justifies the denomination of $\hat{\psi}$ as the Lorentz covariant conjugate of ψ.

Then, we can introduce the concepts of helicity and of chirality. The helicity operator acting on Dirac fields is defined as

$$h_\mathbf{p} = \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{|\mathbf{p}|} \tag{637}$$

where $\frac{1}{2}\mathbf{\Sigma}$ are the spin matrices, $\Sigma^i = \frac{1}{2}\epsilon^{ijk}\sigma_{jk}$ with $\sigma_{\alpha\beta} = (i/2)[\gamma_\alpha, \gamma_\beta]$. This can be understood as the projection of spin on the direction of motion. For a free particle, the helicity, the eigenvalues of which are ± 1[90], is a conserved quantity. It is also invariant under rotations, but not under boosts. The chirality relies on the matrix

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{638}$$

[90] -1 for left-handed fields and $+1$ for right-handed ones.

which commutes will all other gamma matrices, $[\gamma_5, \gamma_\alpha] = 0$. It also obeys $\gamma_5^\dagger = \gamma_5$ and $(\gamma_5)^2 = 1$. Two projectors are then defined,

$$\mathsf{L} = \tfrac{1}{2}(1 - \gamma_5), \tag{639}$$
$$\mathsf{R} = \tfrac{1}{2}(1 + \gamma_5). \tag{640}$$

For any Dirac spinor (solution of Dirac equation), we call chiral fields the spinors

$$\psi_L \equiv \mathsf{L}\psi, \tag{641}$$
$$\psi_R \equiv \mathsf{R}\psi \tag{642}$$

and $\psi = \psi_L + \psi_R$.

We can show that chirality is preserved by Lorentz transformations, but that it is not conserved, even for massless particles. On the other hand, helicity is conserved for a free particle, but not Lorentz invariant. None of these properties is adequate to describe properly massive fermions.

We are now able to introduce the notion of Weyl fermions and of Majorana fermions. If the fermions are massless, there is no longer any problem in assigning them a well defined frame-independent helicity. The meaning of positive or negative helicity, as well as that of left or right chirality are unambiguous in that case. A left-handed chiral fermion is in the $(\frac{1}{2}, 0)$ irreducible representation. It is sometimes called fundamental spinor. A right-handed chiral fermion is in $(0, \frac{1}{2})$. It is also called anti-fundamental spinor. Both describe Weyl fermions, and both have definite helicity, since for massless particles helicity and chirality coincide. Weyl fermions being the irreducible representations of the proper Lorentz group, they can be used to build any fermion state.

Majorana fermions have both left- and right-handed chiral components arranged in such a way as to satisfy the reality condition. They also have non zero mass. Let us consider any Dirac fermion ψ from which we build a left-handed chiral Weyl fermion $\psi_L = \mathsf{L}\psi$. Then, the R operator annihilates this left-state, $(1+\gamma_5)\psi_L = 0$. Taking the complex conjugate expression and acting on the left with $\gamma_0\mathsf{C}$ leads to the condition $\gamma_0\mathsf{C}(1+\gamma_5^*)\psi_L^* = 0$. With $\gamma_5^* = \gamma_5^T$ and $\gamma_5^T = \mathsf{C}^{-1}\gamma_5\mathsf{C}$ which also leads to $\mathsf{C}\gamma_5^T = \gamma_5\mathsf{C}$, we can rewrite the previous combination as

$$0 = \gamma_0\mathsf{C}\psi_L^* + \gamma_0\gamma_5\mathsf{C}\psi_L^* = (1-\gamma_5)\gamma_0\mathsf{C}\psi_L^* = (1-\gamma_5)\hat{\psi}_L. \tag{643}$$

This proves that $\hat{\psi}_L$ is a right-handed Weyl spinor.

We can also easily show that the hat operation in equation (635) applied twice leads to the identity operation. Therefore, if we form the quantity $\chi = \psi_L + \hat{\psi}_L$, then $\hat{\chi} = \chi$ which is the reality condition and proves that the combination

$$\tilde{\psi} = \psi_L + \hat{\psi}_L, \quad \text{with} \quad \psi_L = \mathsf{L}\psi, \quad \hat{\psi}_L = \gamma_0\mathsf{C}(\mathsf{L}\psi)^* \tag{644}$$

is a Majorana field. By adding two massless "opposite" charged chiral fermions, one generates a massive neutral Majorana fermion.

A Dirac field is any general combination

$$\psi = \chi_L + \varphi_L, \quad \text{with} \quad \chi_L = \mathsf{L}\psi, \quad \varphi_R = \mathsf{R}\psi \tag{645}$$

with any chiral components.

– Aïssata: Usually, one does not consider a single particle equation for the photon. Quantum field theories propose a scheme to quantize "classical" theories, substituting field operators (creation and annihilation) to wave functions. This is what is done for Klein-Gordon particles or for spin $\frac{1}{2}$ fermions. In the case of light, the initial classical theory is Maxwell theory. From this perspective, one does not need a single-particle equation for the photon. Nevertheless, soon after Dirac had obtained the equation for spin $\frac{1}{2}$ fermions, Majorana[91] and Openheimer independently obtained a Dirac-like equation equivalent to Maxwell equations. You may wish to build such an equation. For that, a hint is given by the first order derivatives w.r.t. time in Maxwell equations.

►EXERCISE 23 – **Single particle wave equation for the photon** –

– Diego: Let me try. Time-dependent Maxwell equations in the vacuum (in the absence of electric sources) read as

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{646}$$
$$\nabla \times \mathbf{B} = \frac{1}{c^2}\partial_t \mathbf{E}. \tag{647}$$

They are complemented by the two divergences

$$\nabla \cdot \mathbf{E} = 0, \tag{648}$$
$$\nabla \cdot \mathbf{B} = 0. \tag{649}$$

Seeking for a first order differential equation, it is natural to consider the combination

$$\mathbf{F} = \mathbf{E} + ic\mathbf{B} \tag{650}$$

and its derivative w.r.t. time,

$$\begin{aligned}
\partial_t \mathbf{F} &= c^2\nabla \times \mathbf{B} - ic\nabla \times \mathbf{E} \\
&= \frac{c}{\hbar}(-i\hbar\nabla) \times \mathbf{B} + \frac{c}{\hbar}(-i\hbar\nabla) \times \mathbf{E} \\
&= \frac{c}{\hbar}\mathsf{p} \times (\mathbf{E} + ic\mathbf{B})
\end{aligned} \tag{651}$$

from where one deduces the first order partial derivative equation

$$i\hbar\partial_t \mathbf{F} = ic\mathsf{p} \times \mathbf{F} \tag{652}$$

with the transversality equation following from the divergenceless conditions

$$\mathsf{p} \cdot \mathbf{F} = 0. \tag{653}$$

The r.h.s. can be written $\mathsf{H}\mathbf{F}$ where H can be called Majorana Hamiltonian. In matrix form in Cartesian coordinates one has

$$i\hbar\partial_t \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = ic \begin{pmatrix} 0 & -\mathsf{p}_z & \mathsf{p}_y \\ \mathsf{p}_z & 0 & -\mathsf{p}_x \\ -\mathsf{p}_y & \mathsf{p}_x & 0 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}. \tag{654}$$

Squaring this equation results in the ordinary wave equation,

$$-\hbar^2\partial_t^2 \mathbf{F} = \mathsf{H}^2\mathbf{F} = c^2\hbar^2\nabla \times (\nabla \times \mathbf{F}) = -c^2\hbar^2\nabla^2\mathbf{F}. \tag{655}$$

[91] E. Mignani, E. Recami and M. Baldo, About a Dirac-like equation for the photon according to Ettore Majorana, Lettere al Nuovo Cimento **11** 568 1974.

Acting on a plane wave solution $\mathbf{F} = \mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ (with $\mathbf{A}$ a vector with three constant complex components), one obtains

$$\mathsf{p}\mathbf{F} = -i\hbar\nabla\cdot\mathbf{F} = \hbar\mathbf{k}\cdot\mathbf{F}, \tag{656}$$
$$\mathsf{H}\mathbf{F} = i\hbar\partial_t\mathbf{F} = \hbar\omega\mathbf{F}. \tag{657}$$

The "wave function" $\mathbf{F}$ is normalized to the electromagnetic energy. ◄

– Aïssata: The relation to Dirac equation $i\hbar\partial_t\psi = (c\alpha\cdot\mathsf{p} + \beta mc^2)\psi$ can be made more transparent if we write (652) as

$$i\hbar\partial_t\mathbf{F} = c(\tau\cdot\mathsf{p})\mathbf{F} \tag{658}$$

where the three tau matrices are the analogous to Dirac matrices. This is clearly a massless equation. Here they take the form

$$(\tau_x) = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad (\tau_y) = \begin{pmatrix}0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{pmatrix}, \quad (\tau_z) = \begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{659}$$

These matrices don't obey Dirac algebra. They have the following commutators instead,

$$[\tau_i, \tau_j] = i\epsilon_{ijk}\tau^k. \tag{660}$$

There has been many attempts to get similar relativistic equations for various types of particles and specialized books (see e.g. Ramond[92]) present the cases of spin $\frac{3}{2}$ and spin 2 particles.

[92] P. Ramond, Field theory: a modern primer, Westview Press, Boulder, 1990.

Relativistic field theory

We will consider here field theories in a Minkowski manifold. This is a "fixed" background spacetime which is not coupled to the various fields living there. By "relativistic" we understand Special Relativity. Some extensions incorporating the effects of gravitation will be considered in a section dedicated to curved spacetimes.

16. Day 12 – Special relativity and dynamics of particles

☐ 16.1 Lagrangian formulation of relativistic dynamics for free particles

In classical dymanics, a free particle in an inertial frame follows a least action trajectory which is the shortest line in Euclidean space

$$\delta \int \tfrac{1}{2} m |\mathbf{v}|^2 dt = 0 \quad \text{or} \quad \delta \int \mathbf{v} \cdot d\mathbf{x} = 0 \tag{661}$$

with $d\mathbf{x} = \mathbf{v}dt$. The natural extension to Special Relativity is

$$\delta \int v_\alpha dx^\alpha = \delta \int \eta_{\alpha\beta} v^\alpha dx^\beta = 0 \tag{662}$$

with

$$v^\alpha = \frac{dx^\alpha}{ds} \tag{663}$$

the 4-velocity. It follows that $v_\alpha dx^\alpha = ds$ (analogous to $\mathbf{v}dt = d\mathbf{x}$ in classical kinematics) and the action expression takes an alternative simple expression

$$\delta \int ds = 0 \tag{664}$$

up to a multiplicative constant which is chosen in order to recover the correct Newtonian limit (661):

$$\kappa ds = \kappa (dt^2 - |d\mathbf{x}|^2)^{1/2} \simeq \kappa dt (1 - (1/2)|\mathbf{v}|^2 + \dots) = \kappa dt - \tfrac{1}{2}\kappa |\mathbf{v}|^2 dt + \dots \tag{665}$$

The first term contributes to the action by an irrelevant additive constant and the second term demands that $\kappa = -m$ in order to recover the well-known kinematic energy. Therefore,

$$\delta \left(-m \int ds \right) = 0 \tag{666}$$

is the least action principle for a relativistic free particle in Minkowski spacetime. This is the geometrical content of the principle of inertia. It is also automatically true in any inertial frame, since the action is built on a Lorentz scalar which remains unchanged in a Lorentz transformation.

An observer at rest in the inertial frame can write this action as an integral over their own time

$$\delta \int \left(-m\sqrt{1-|\mathbf{v}|^2}\right) dt \tag{667}$$

where the Lagrangian for this observer is

$$L = -m\sqrt{1-|\mathbf{v}|^2}. \tag{668}$$

This observer defines the canonical momentum and energy for the free particle,

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1-|\mathbf{v}|^2}} = \gamma(\mathbf{v})m\mathbf{v}, \tag{669}$$

$$E = \mathbf{p}\cdot\mathbf{v} - L = \frac{m}{\sqrt{1-|\mathbf{v}|^2}} = \gamma(\mathbf{v})m. \tag{670}$$

One usually denotes γ instead of $\gamma(\mathbf{v})$ when there is no risk of confusion, but a dependence on the velocity is understood. Restoring c, one would have $E = \gamma mc^2$ in the last equation. Using (668), the (expected) equation of motion is derived as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) = \frac{d\mathbf{p}}{dt} = 0. \tag{671}$$

It is important to take care about the definition of the momentum. We use the same notation $\mathbf{p}$ than in classical dynamics, but while the *classical* $\mathbf{p} = m\mathbf{v}$, the relativistic $\mathbf{p} = \gamma m\mathbf{v}$ instead. This is the reason why some authors have used for long time the notion of a velocity dependent mass. In our opinion, this is a misleading conception, since the velocity dependence appears for purely kinematical reasons and not because of any dynamical content.

In 4-vector notation, we write

$$p^\alpha = mv^\alpha. \tag{672}$$

Since $v^\alpha = \frac{dx^\alpha}{ds}$ and $x^\alpha = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$, the 4-velocity components can be written $v^0 = dt/ds$, $v^i = (d\mathbf{x}/dt)(dt/ds)$. The quantity dt/ds is evaluated from $ds = \sqrt{dt^2 - |d\mathbf{x}|^2}$, hence $\frac{ds}{dt} = \sqrt{1-|\mathbf{v}|^2} = \gamma^{-1}$ and the 4-velocity is

$$v^\alpha = \begin{pmatrix} \gamma \\ \gamma\mathbf{v} \end{pmatrix}. \tag{673}$$

Everywhere, as said before, γ has to be understood as $\gamma(\mathbf{v})$. The corresponding Lorentz-invariant scalar is easily obtained in the particle's rest frame where $v^0 = 1$, $v^i = 0$, hence $\eta_{\alpha\beta}v^\alpha v^\beta = 1$. It follows that

$$p_\alpha p^\alpha = (p^0)^2 - |\mathbf{p}|^2 = E^2 - |\mathbf{p}|^2 = m^2. \tag{674}$$

The expression of the Hamiltonian (the total energy of the free particle) follows when the energy is expressed in terms of the momentum, $H = E(\mathbf{p})$,

$$H = \sqrt{|\mathbf{p}|^2 + m^2}, \tag{675}$$

or, restoring c again, $H = c\sqrt{|\mathbf{p}|^2 + m^2c^2}$.

– Diego: At the end of the lecture, the professor told that there are many outcomes that we have no time to discuss in a field theory course, but which could be easily derived using the tensor formalism. She mentioned the Newtonian limit of the kinetic energy and linear momentum, the expression of the total energy, the case of a particle in an external potential, the Lorentz transformation of velocities, of the energy-momentum, the discussion of the interesting case of zero mass particles and probably other things which I had no time to write down. Let's discuss these things if you agree.

▷ 16.2 Newtonian limit

– Aïssata: OK, we can consider the items in the order of your list. If I remember well, the Newtonian limit first. This is something easy that you can do by yourself. This is just an expansion in powers of v/c.

– Diego: Yes, I have done it already. There is only to write the first terms of

$$\gamma = (1 - |\mathbf{v}|^2)^{-1/2} \simeq 1 + \tfrac{1}{2}|\mathbf{v}|^2 + O(v^4). \tag{676}$$

It follows the ordinary Newtonian momentum at lowest order while the energy comprises a constant (I mean independent of the velocity) term known as *rest energy*, then the Newtonian kinetic energy at the next order,

$$\begin{aligned} \mathbf{p} &= \gamma m\mathbf{v} \simeq m\mathbf{v}(1 + O(v^2)), & (677)\\ E &= \gamma m = m(1 + \tfrac{1}{2}|\mathbf{v}|^2 + O(v^4)). & (678) \end{aligned}$$

– Aïssata: Right! So, you can notice from the second equation that this is wise to define the *kinetic* energy after removing the rest energy,

$$K = E - m \simeq \tfrac{1}{2}m|\mathbf{v}|^2. \tag{679}$$

– Diego: We can also take the next terms into account to estimate the deviation from the Newtonian behaviour, again, restoring c,

$$\begin{aligned} \mathbf{p} &\simeq m\mathbf{v} + \tfrac{1}{2}\tfrac{v^2}{c^2}m\mathbf{v}, & (680)\\ E &\simeq mc^2 + \tfrac{1}{2}mv^2 - \tfrac{1}{8}\tfrac{v^2}{c^2}mv^2. & (681) \end{aligned}$$

– Aïssata: This is a good observation. There are historical experiments which were devoted to measure these deviations. The last expression in particular is often written in terms of the *Newtonian* momentum $\mathbf{p}_\mathrm{N} = m\mathbf{v}$ as

$$K = \frac{|\mathbf{p}_\mathrm{N}|^2}{2m} - \frac{|\mathbf{p}_\mathrm{N}|^4}{8m^3c^2}. \tag{682}$$

Another interesting aspect is via the Hamilton formalism. If you write

$$H = c\sqrt{|\mathbf{p}|^2 + m^2c^2} \simeq mc^2 + \tfrac{1}{2}m|\mathbf{v}|^2 \tag{683}$$

you get the same expansion than γmc^2, and this suggests to take into account external forces via a potential term V

$$H = c\sqrt{|\mathbf{p}|^2 + m^2c^2} + V. \tag{684}$$

The total energy in the classical sense (without the rest energy) would be $H - mc^2$.

▷ 16.3 Lorentz transformation of the 4-momentum

– Diego: The last question, I can do it by myself. The Lorentz transformation of the energy-momentum is something that we can find in any book on Special Relativity. First, we remember that the definition of the 4-momentum is

$$p^\alpha = mv^\alpha = m\frac{dx^\alpha}{ds} = \begin{pmatrix} p^0 \\ \mathbf{p} \end{pmatrix} \tag{685}$$

and the Lorentz scalar that we can built from squaring is

$$p_\alpha p^\alpha = (p^0)^2 - |\mathbf{p}|^2 = m^2\frac{dx_\alpha}{ds}\frac{dx^\alpha}{ds} = m^2. \tag{686}$$

This shows, like in the lecture notes, that $p^0 = \sqrt{|\mathbf{p}|^2 + m^2}$ is nothing but the energy E (or E/c). Now, p^α being a 4-vector, it does transform like in (494),

$$p^\alpha \quad \rightarrow p'^\alpha \quad = \Lambda^\alpha{}_\beta p^\beta, \tag{687}$$

or, explicitly,

$$\begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}. \tag{688}$$

– Aïssata: Very good Diego, but you have to take care about the factors of β and γ here.

– Diego: What do you mean? I know that they depend on the velocity.

– Aïssata: Yes, but you have for example $p^0 = \gamma m$ and $\mathbf{p} = \gamma m\mathbf{v}$, and you have the γ's in the matrix above *which are not the same!* The coefficients entering the Lorentz transformation matrix should be $\beta(\mathbf{u})$ and $\gamma(\mathbf{u})$ with $\mathbf{u}$ the relative velocity between the two reference frames. Say $\mathscr{R}'$ moves at a constant velocity $\mathbf{u}$ along the common x-axis w.r.t $\mathscr{R}$. But $\mathbf{p}$ for example is the linear momentum of a particle, which can have an arbitrary velocity, say $\mathbf{v}$ (not necessarily constant) with respect to $\mathscr{R}$, and thus $\mathbf{p} = \gamma(\mathbf{v})m\mathbf{v}$. The first line of your matrix equation could be written

$$p'^0 = \gamma(\mathbf{u})p^0 + \beta(\mathbf{u})\gamma(\mathbf{u})p^1 + p^2 + p^3, \tag{689}$$

and, once translated in terms of a velocity equation,

$$\gamma(\mathbf{v}')m = \gamma(\mathbf{u})\gamma(\mathbf{v})m + \beta(\mathbf{u})\gamma(\mathbf{u})\gamma(\mathbf{v})mv_x + \gamma(\mathbf{v})mv_y + \gamma(\mathbf{v})mv_z. \tag{690}$$

This, and similar equations given by the three remaining lines of the matrix equations, is what you would need if you were to look at the Lorentz transformation of velocities.

– Diego: I see. Now I understand why we can find in the literature the use of capital letters, B and Γ in the Lorentz transformation matrix, in order to emphasize this distinction.

– Aïssata: Eventually, a point that you noticed already I suppose, the 4-velocity squared equals 1, and the velocity components have no dimension with the notations of your professor. Some authors like to define the 4-velocity as $v^\alpha = \frac{dx^\alpha}{d\tau}$ with the proper time $d\tau = ds/c$. This gives usual dimensions to the components, but still a constant 4-velocity squared which now equals c^2. Anyway, here we have $c = 1$, so there is no difference between τ and s!

□ 16.4 Relativistic dynamics of charged particles

The case of a charged particle in an electromagnetic field is built by simple extension of (666). Anticipating just a bit on the next sections, we describe an electromagnetic field in terms of a 4-vector $A^\alpha = \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}$ with $\phi(\mathbf{x}, t)$ the scalar potential and $\mathbf{A}(\mathbf{x}, t)$ the vector potential. This field couples to the particle via the charge e, and in order to build a Lorentz scalar, we have to contract with another 4-vector which contains the relevant information on the position and dynamical state of the particle. A good candidate is x^α, so we can try with $eA_\alpha dx^\alpha$ and propose the action

$$S = \int \left(-m(dx_\alpha dx^\alpha)^{1/2} - eA_\alpha dx^\alpha \right). \tag{691}$$

The sign is chosen so as to agree with the Newtonian limit.

Written by an observer who parametrizes the trajectory by time t, the action is

$$S = \int \left(-m\sqrt{1 - |\mathbf{v}|^2} - e\phi + e\mathbf{A} \cdot \mathbf{v} \right) dt. \tag{692}$$

and the corresponding Lagrangian

$$L = -m\sqrt{1 - |\mathbf{v}|^2} - e\phi + e\mathbf{A} \cdot \mathbf{v} \tag{693}$$

from where the canonical momentum and the energy follow,

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1 - |\mathbf{v}|^2}} + e\mathbf{A} = \gamma m\mathbf{v} + e\mathbf{A} \tag{694}$$

$$E = \mathbf{p} \cdot \mathbf{v} - L = \frac{m}{\sqrt{1 - |\mathbf{v}|^2}} + e\phi = \gamma m + e\phi. \tag{695}$$

In tensor notation this is

$$p^\alpha = mv^\alpha + eA^\alpha. \tag{696}$$

The kinematic momentum is often denoted as $\pi^\alpha = mv^\alpha$, so one can write

$$\pi^\alpha = p^\alpha - eA^\alpha = \begin{pmatrix} H - e\phi \\ \mathbf{p} - e\mathbf{A} \end{pmatrix} \tag{697}$$

and the associated square invariant is $m^2 = (H - e\phi)^2 - |\mathbf{p} - e\mathbf{A}|^2$ which leads to the expression of the Hamiltonian for the charged particle in an electromagnetic field

$$H = \sqrt{|\mathbf{p} - e\mathbf{A}|^2 + m^2} + e\phi. \tag{698}$$

Restoring the factor c gives $H = c\sqrt{|\mathbf{p} - e\mathbf{A}|^2 + m^2c^2} + e\phi$.

The principle of least action in tensor form follows from the functional variation of equation (691). The action comprises two terms. For the first term one can write

$$\delta S_1 = \delta \left(-m \int_a^b \delta (dx_\alpha dx^\alpha)^{1/2} \right) = -m \int_a^b v_\alpha \delta dx^\alpha. \tag{699}$$

Using the fact that $\delta dx^\alpha = d\delta x^\alpha$ (δx^α is the variation), integration by parts leads to

$$\delta S_1 = -m \left([v_\alpha \delta x^\alpha]_a^b - \int_a^b dv_\alpha \delta x^\alpha \right) \tag{700}$$

The integrated term vanishes due to the standard boundary conditions in variational calculus and it follows that

$$\delta S_1 = m \int_a^b dv_\alpha \delta x^\alpha. \tag{701}$$

For the second term, we develop

$$\delta S_2 = -e \int_a^b \delta (A_\alpha dx^\alpha) \tag{702}$$

using $\delta(A_\alpha dx^\alpha) = A_\alpha \delta dx^\alpha + \delta A_\alpha dx^\alpha$ to integrate by parts the first term,

$$\delta S_2 = -e \left([A_\alpha \delta x^\alpha]_a^b - \int_a^b (dA_\alpha \delta x^\alpha - \delta A_\alpha dx^\alpha) \right). \tag{703}$$

The integrated term vanishes and we incorporate $dA_\alpha = \partial_\beta A_\alpha dx^\beta$ and $\delta A_\alpha = \partial_\beta A_\alpha \delta x^\beta$ under the remaining integral to get

$$\delta S_2 = e \int_a^b F_{\beta\alpha} dx^\beta \delta x^\alpha \tag{704}$$

where we have defined the antisymmetric *Faraday tensor*

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{705}$$

The action variation vanishes when the sum of the two terms under the integrals vanishes

$$mdv_\alpha = eF_{\alpha\beta} dx^\beta \tag{706}$$

or, divinding by ds

$$m \frac{dv_\alpha}{ds} = eF_{\alpha\beta} v^\beta \tag{707}$$

which is the tensor form of the equation of motion of a point charger under a Lorentz force.

17. Day 13 – Maxwell equations and the dynamics of fields in the vacuum

Here, with the term vacuum we don't mean that we study the empty space. There can be charges and currents, which act as sources for electromagnetic fields, but apart from the presence of these charges, there is no intermediate material medium with dielectric or magnetic properties.

□ 17.1 Covariant formulation of electrodynamics

The introduction of a 4-vector

$$A^\alpha = \begin{pmatrix} \phi(\mathbf{x}, t) \\ \mathbf{A}(\mathbf{x}, t) \end{pmatrix} \tag{708}$$

to describe mathematically the scalar and vector potentials is something natural. An argument in favor of this mathematical object is that it allows a compact expression for the Lorenz-Lorentz gauge condition for example,

$$\partial_\alpha A^\alpha = \partial_t \phi + \boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{709}$$

This gauge condition is thus Lorentz invariant. For the EM fields, another "natural" attempt could be to look for a 4-vector, the space components of which would be identified to those of the electric field components (and similarly for the magnetic field components). The identification of the time component would be left to further investigations. This strategy nevertheless fails, and the previous section has shown the advantage to introduce the Faraday tensor instead, an antisymmetric second-rank tensor,

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \tag{710}$$

for a compact notation of the equation of motion of a point charge in the presence of electromagnetic fields. An alternative justification of this definition easily follows from the search of a covariant expression which corresponds to the standard definition of the magnetic and electric fields in terms of potentials. For example, the expression in Cartesian coordinates

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{711}$$

strongly suggests Eq. (710), the 32 component of which is indeed B_x. In a similar manner, the 01 component delivers E_x. Written in matrix form, we have

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{712}$$

and the corresponding contravariant expression is $F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta}$, i.e.

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{713}$$

Note that if factors c were to be reintroduced, everywhere when E_i appears, it should become E_i/c. An important caveat is in order here. The quantities E_i for $i = 1, 2, 3$ *are not* the components of a 1st-rank tensor. To emphasize this distinction, we use $(\mathbf{E} \cdot \mathbf{u}_i)$ instead. For example, the electric and magnetic field components follow from

$$\mathbf{E} \cdot \mathbf{u}_i = F_{0i} = -F^{0i}, \tag{714}$$

$$\mathbf{B} \cdot \mathbf{u}_i = -\tfrac{1}{2}\epsilon^{ijk} F_{jk} = -\tfrac{1}{2}\epsilon_{ijk} F^{jk}. \tag{715}$$

Once the two pieces A^α and $F^{\alpha\beta}$ are available to describe the EM field, one still needs a similar description of the sources. We know that $\rho(\mathbf{x}, t)$, the electric charge density and $\mathbf{j}(\mathbf{x}, t)$, the electric current density obey the continuity equation

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \mathbf{j} = 0. \tag{716}$$

This strongly suggests the introduction of the 4-vector charge density

$$j^\alpha = \begin{pmatrix} \rho \\ \mathbf{j} \end{pmatrix} \tag{717}$$

in terms of which the charge conservation is simply

$$\partial_\alpha j^\alpha = 0. \tag{718}$$

Note that factors of c would just change $j^\alpha = \begin{pmatrix} \rho c \\ \mathbf{j} \end{pmatrix}$ and $\frac{1}{c}\partial_t(\rho c) + \boldsymbol{\nabla} \cdot \mathbf{j} = 0$ would remain unaffected.

Maxwell equations take a very compact form once expressed in terms of tensors. For the equations of motion, which determine the fields space and time dependences in terms of the sources, we look for expressions such at $\partial_\alpha F^{\alpha 0}$ and $\partial_\alpha F^{\alpha i}$. Indeed, Maxwell equations are first order differential equations in terms of the field components. Performing the calculation in Cartesian components yields

$$\partial_\alpha F^{\alpha 0} = \boldsymbol{\nabla} \cdot \mathbf{E}, \tag{719}$$

$$\partial_\alpha F^{\alpha i} = -\partial_t(\mathbf{E} \cdot \mathbf{u}_i) + (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{u}_i. \tag{720}$$

According to the known Maxwell equations, the first of these equation equals ρ and the second is identified to $\mathbf{j} \cdot \mathbf{u}_i$. They are thus gathered into the single and compact expression $\partial_\alpha F^{\alpha\beta} = j^\beta$.

The two remaining sourceless equations are automatically satisfied as soon as the expression (710) holds, but we may wish to write them in terms of derivatives of the Faraday tensor as well.

We form the combinations $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}$, e.g.

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{23} = -\boldsymbol{\nabla} \cdot \mathbf{B}, \tag{721}$$

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = -\partial_t(\mathbf{B} \cdot \mathbf{u}_z) - (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{u}_z, \tag{722}$$

which shows that Maxwell equations demand that the above combination vanishes. Eventually, covariant Maxwell equations are given by

$$\partial_\alpha F^{\alpha\beta} = j^\beta, \tag{723}$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{724}$$

The second equation is also written $\epsilon^{\delta\alpha\beta\gamma}\partial_\alpha F_{\beta\gamma} = 0$ with $\epsilon^{\delta\alpha\beta\gamma}$ the antisymmetric Levi-Civita symbol and is called *Bianchi idendity*. The name covariant equation is clear. These equations take the same form in any inertial frame of reference in terms of quantities *covariantly* transformed. Lorentz covariance of Maxwell equations is thus an automatic property which doesn't need further elaboration!

– Aïssata: We also say that these equations are *manifestly covariant* while standard Maxwell equations in terms of vectors **E** and **B** are only covariant. This means that these latter equations have the right transformation properties under a Lorentz change of reference frame, but this is hidden in the form of the equations, while it is *built in* in a tensor form.

Another comment which I would do is the following. If you write the usual Maxwell equations in their $3D$-vector form (restoring dimensional constants),

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \tag{725}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\partial_t \mathbf{E}, \tag{726}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{727}$$
$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{728}$$

the l.h.s and r.h.s of the four equations have a kind of historical meaning. Special Relativity on the other hand tells you to write instead

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \tag{729}$$
$$\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = \mu_0 \mathbf{j}, \tag{730}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{731}$$
$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0. \tag{732}$$

really showing that you write equations which have to do with the *dynamics* of the fields at the l.h.s in terms of sources (for the first two equations) at the r.h.s.

– Diego: This is an interesting comment Diego says.

Then, he continues: At the end of the class, the professor said that we should be aware of the sign problems which depend on the choice of signature for the Minkowski metric, and that we shouldn't be surprised to find in the literature for example for the inhomogeneous Maxwell equation $\partial_\alpha F^{\alpha\beta} = -j^\beta$. We should also know that the homogeneous Maxwell equation can be written in terms of a *dual* Faraday tensor.

– Aïssata: These are important things indeed. You know that Minkowski spacetime is a pseudo-Euclidean space with one timelike direction, which means that we have two choices for the metric tensor that we characterize by their signature, either $(+ - --)$ or $(- + ++)$. The literature refers to these conventions with various names, mostly minus versus mostly plus, timelike vs spacelike, particle physics vs relativity, West-coast (or Feynman) vs East-coast (or Schwinger), Landau vs Pauli. Your professor has chosen the first option, which you will find in most of the standard textbooks in Quantum Field Theory, so it makes sense in the present context. A notable exception is Weinberg's trilogy for whom I have the greatest respect, which would possibly lead me to the other convention. But Landau and Lifshitz in their second volume make the first choice, so I wouldn't know how to decide Aïssata adds, smiling.

The question is apparently not just a matter of personal taste, and Peter Woit for example has a list of arguments in favor of the mostly plus sign convention in a provocative post[93]. Some authors[94] even provide versions of their published papers with the two options for the signature convention ("with a line in the tex that lets you choose which Coast.")!

If you want to build a quantity which doesn't depend on the signature, this is the mixed tensor $F^{\alpha}{}_{\beta}$. You can learn it and then raise or lower indices at will with the chosen convention! Here it is,

$$(F^{\alpha}{}_{\beta}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0. \end{pmatrix} \tag{733}$$

But I will confess that I am unable to remember where the minus signs are in such an expression!

Anyway, there are indeed consequences of the choice made. With the mostly minus sign convention, the covariant 4-potential would be $A_\alpha = (-\phi, \mathbf{A})$ but ∂_α wouldn't be modified. Therefore, $\partial_\alpha A_\beta$ changes sign and then $F_{\alpha\beta}$ and $F^{\alpha\beta}$ also change signs. It follows that indeed, $\partial_\alpha F^{\alpha\beta}$ changes while j^β doesn't and the inhomogeneous Maxwell equation (723) takes a minus sign on one side.

Concerning the dual Faraday tensor, this is an alternative to the homogeneous equation which, by the way, is also known as a Bianchi identity. You can define

$$\mathscr{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \tag{734}$$

and call it the dual tensor. With our metric, it takes the matrix form

$$(\mathscr{F}^{\alpha\beta}) = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} \tag{735}$$

and essentially interchanges $\mathbf{E}$ and $\mathbf{B}$ components. If you form the derivative

$$\partial_\alpha \mathscr{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha F_{\gamma\delta} \tag{736}$$

and expand, you get also

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} \equiv \tfrac{1}{2}\partial_{[\alpha}F_{\beta\gamma]} \tag{737}$$

which is just the l.h.s. of equation (724) that you can thus write

$$\partial_\alpha \mathscr{F}^{\alpha\beta} = 0. \tag{738}$$

The brackets appearing in $\partial_{[\alpha}F_{\beta\gamma]}$ are a very common notation (in General Relativity in particular) to say that you take the fully antisymmetric expression of the quantity allowing for all permutations of the indices between the brackets. For example $\partial_{[\alpha}A_{\beta]} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ with only two indices.

[93] P. Woit, The West Coast Metric is the Wrong One, https://www.math.columbia.edu/~woit/wordpress/?p=7773&cpage=1

[94] https://www.niu.edu/spmartin/spinors/ and https://www.niu.edu/spmartin/TASI11/ This reference has a nice appendix A on the signs of various quantities in both coasts.

►EXERCISE 24 – **Calculation of $\partial_\alpha \mathscr{F}^{\alpha 0}$** –
When we fix $\beta = 0$, the explicit calculation leads to

$$\begin{aligned}
\partial_\alpha \mathscr{F}^{\alpha 0} &= \partial_1 \mathscr{F}^{10} + \partial_2 \mathscr{F}^{20} + \partial_3 \mathscr{F}^{30} \\
&= \tfrac{1}{2}(\varepsilon^{10\gamma\delta}\partial_1 + \varepsilon^{20\gamma\delta}\partial_2 + \varepsilon^{30\gamma\delta}\partial_3)F_{\gamma\delta} \\
&= \tfrac{1}{2}(\varepsilon^{1023}\partial_1 F_{23} + \varepsilon^{1032}\partial_1 F_{32} + \dots) \\
&= \varepsilon_{0123}(\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12})
\end{aligned} \tag{739}$$

where we have used the properties of the Levi-Civita symbol in Minkovski space like $\varepsilon^{1023} = -\varepsilon^{0123} = +\varepsilon_{0123}$ and the antisymmetry of $F_{\alpha\beta}$. ◄

– Diego: Look Aïssata, Diego says, I found a different form (actually with the opposite sign) for the dual electromagnetic tensor in Tong's lecture notes[95], although he uses the same metric signature than we do!

– Aïssata: Correct Diego. Not only the metric signature is important to fix the dual tensor components. You also have to make a choice for the antisymmetric symbol. Here we use the convention that the Levi-Civita symbol equals +1 when all its indices 0123 are lowered,

$$\varepsilon_{0123} = +1, \tag{740}$$

while Tong uses the other choice $\varepsilon^{0123} = +1$. You are right, this is also something that we have to say to be accurate.

– Diego: There is still something else to discuss. The professor mentioned Lorenz gauge. We discussed earlier together about gauge freedom in the definition of an electric field. This gauge freedom property of electric and magnetic fields is something which I heard about already during my undergraduate studies. I remember that the electric and magnetic fields, given in vector notation in terms of the potentials

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial_t \mathbf{A}, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{741}$$

are unchanged if the scalar and vector potentials are simultaneously modified by a gauge transformation

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\chi, \quad \phi \to \phi' = \phi + \partial_t \chi \tag{742}$$

where χ is an arbitrary function…

– Aïssata: …an arbitrary non singular function I would say. There is a discussion in the excellent American Journal of Physics[96] of what happens for singular gauge transformations that we should discuss later maybe.

– Diego: Okay, χ is an arbitrary (non singular) scalar function. Then, since the **E** and **B** fields are unchanged (we say that they are gauge invariant), the equations of motion are unchanged. In the lecture notes, this is the Lorenz-Lorentz gauge which is specified. What special is hidden in this gauge?

– Aïssata: This is a specific choice that you can write as $\Box\chi = 0$. I mean that you consider the gauge transformation (742) with χ obeying the additional property

$$\Box\chi = (\partial_t^2 - \boldsymbol{\nabla}^2)\chi = 0. \tag{743}$$

[95] D. Tong, Lectures on Electromagnetism, https://www.damtp.cam.ac.uk/user/tong/em.html
[96] ☺ B. Berche, D. Malterre and E. Medina, Am. J. Phys. **84**, 616, 2016.

The operator $\Box$ is the d'Alembertian operator. The advantage, as it is written in your professor's notes, is that this condition guarantees that if the condition

$$\partial_t \phi + \nabla \cdot \mathbf{A} = 0 \tag{744}$$

is satisfied, then it is also satisfied for the gauged-transformed potentials. This is the condition called *Lorenz-Lorentz gauge* condition. It is useful in the fact that it simplifies the equations of motion written in terms of potentials. Indeed,

$$\nabla \cdot \mathbf{E} = \rho \tag{745}$$

becomes

$$\nabla(-\nabla\phi - \partial_t \mathbf{A}) = -\nabla^2\phi + \partial_t^2\phi \equiv \Box\phi = \rho, \tag{746}$$

when we use $\nabla\partial_t \mathbf{A} = -\partial_t^2\phi$ with (744), while

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{j} \tag{747}$$

leads to

$$-\nabla^2\mathbf{A} + \partial_t^2\mathbf{A} = \Box\mathbf{A} = \mathbf{j}. \tag{748}$$

Equations (746) and (748) are wave equations. In any other gauge than the Lorenz-Lorentz gauge, the equations satisfied by the potentials are much more complicated.

Now, just a historical remark. Your professor uses the denomination of Lorenz-Lorentz gauge. Equation (744) was first written by Ludvig Valentin Lorenz, a Danish physicist of the XIXth century, but later attributed to Lorentz, the famous Hendrik Antoon Lorentz, a Dutch physicist. The story around equation (744) can be found in a paper by Jackson and Okun[97]:

> This equation, now almost universally called the "Lorentz condition", is seen to originate with Lorenz more than 25 years before Lorentz.

►EXERCISE 25 – **Maxwell equation of motion for the 4-potential in the Lorenz gauge** –

– Diego: Let me now follow the same lines of reasoning using the tensor formalism. First we know that $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. This is clearly gauge invariant in the sense that in the transformation

$$A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha \chi, \tag{749}$$

$F'_{\alpha\beta} = F_{\alpha\beta}$ because $\partial_\alpha\partial_\beta\chi = \partial_\beta\partial_\alpha\chi$. Now, if we fix the condition

$$\partial_\alpha A^\alpha = 0, \tag{750}$$

equivalent to equation (744), it also holds in terms of gauged quantities if

$$\partial_\alpha \partial^\alpha \chi = \Box\chi = 0, \tag{751}$$

and the equation of motion takes the simple form

$$\partial_\alpha F^{\alpha\beta} = j^\beta = \partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = \partial_\alpha\partial^\alpha A^\beta = \Box A^\beta. \tag{752}$$

This contains the wave equations for ϕ and $\mathbf{A}$.

– Aïssata: Perfect! ◄

[97] J.D. Jackson and L.B. Okun, Historical roots of gauge invariance, arXiv:hep-ph/0012061.

– Diego: So far so good! Now we can proceed further. I am used to some of the "almost automatic" interesting consequences of the tensor formalism, the transformation under a change of inertial reference frame and the construction of Lorentz invariants. The Lorentz transformation of the fields components is something delicate, since they appear in a *second-rank tensor*, so we have to write that

$$F'^{\alpha\beta} = \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta F^{\gamma\delta}. \tag{753}$$

– Aïssata: You should consider an exemple to see how things work.

– Diego: Right. Consider e.g. $F^{10} = E_x$. The above transformation is $F'^{10} = \Lambda^1{}_\gamma \Lambda^0{}_\delta F^{\gamma\delta}$. Keeping in mind that the Faraday tensor is antisymmetric, and that only the 0 and 1 components mix among themselves in the Lorentz matrix (there is no $\Lambda^0{}_2$ for example), among the 16 possible terms, only two are non vanishing and it yields $F'^{10} = \Lambda^1{}_0\Lambda^0{}_1 F^{01} + \Lambda^1{}_1\Lambda^0{}_0 F^{10}$ and all calculations done it leads to $E'_x = \gamma^2(1-\beta^2)E_x = E_x$. Proceeding along the same lines with the 20 and 13 components leads to $E'_y = \gamma(E_y + \beta B_z)$ and $B'_y = \gamma(B_y - \beta E_z)$. After having done the whole calculation for the six components of the electric and magnetic fields, we obtain more compact vector notations in terms of components along or perpendicular to the relative velocity $\mathbf{u}$,

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel, \tag{754}$$

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel, \tag{755}$$

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp - \boldsymbol{\beta} \times \mathbf{B}_\perp), \tag{756}$$

$$\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp + \boldsymbol{\beta} \times \mathbf{E}_\perp). \tag{757}$$

– Aïssata: Correct. And don't forget to change $E \to E/c$ if you want to restore the dimensions with c-factors. The important message here is that the electric and magnetic field components *are not those of Minkowski vectors*, since they do not transform according to $v'^\alpha = \Lambda^\alpha{}_\beta v^\beta$.

– Diego: In which concerns the field invariants, from $F^{\alpha\beta}$ and $\mathscr{F}^{\alpha\beta}$, we can form two Lorentz scalars by full contraction of all indices,

$$-\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta} = \mathbf{E}^2 - \mathbf{B}^2, \tag{758}$$

$$-\frac{1}{4}\mathscr{F}^{\alpha\beta}F_{\alpha\beta} = \mathbf{E}\cdot\mathbf{B}. \tag{759}$$

To get the explicit form, we have to develop the calculation,

$$\begin{aligned}
-\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta} &= -\frac{1}{2}\Big(\underbrace{F^{00}F_{00}}_{(F^{00})^2} + \underbrace{\sum_i\sum_j F^{ij}F_{ij}}_{\sum_i\sum_j (F^{ij})^2} + \underbrace{\sum_i F^{0i}F_{0i} + \sum_i F^{i0}F_{i0}}_{-2\sum_i (F^{0i})^2}\Big) \\
&= -\frac{1}{2}(2(B_x^2 + B_y^2 + B_z^2) - 2(E_x^2 + E_y^2 + E_z^2)) \\
&= \mathbf{E}^2 - \mathbf{B}^2.
\end{aligned} \tag{760}$$

For the second invariant, it leads to

$$\begin{aligned}\frac{1}{4}\mathscr{F}^{\alpha\beta}F_{\alpha\beta} &= \frac{1}{4}\Big(\underbrace{\mathscr{F}^{00}F_{00}}_{0}+\underbrace{\sum_i\sum_j\mathscr{F}^{ij}F_{ij}}_{2(-B_zE_z+B_y(-E_y)-B_xE_x)}\underbrace{+\sum_i\mathscr{F}^{0i}F_{0i}+\sum_i\mathscr{F}^{i0}F_{i0}}_{-2(B_zE_z+B_yE_y-B_xE_x)}\Big)\\ &= \frac{1}{4}(-4)\mathbf{E}\cdot\mathbf{B}.\end{aligned}\tag{761}$$

Note that $\mathscr{F}_{\alpha\beta}\mathscr{F}^{\alpha\beta}$ doesn't bring any new information.

– Aïssata: And there are some properties which follow immediately:

i) If the equality $|\mathbf{B}| = |\mathbf{E}|$ holds in an inertial reference frame, then it holds in all inertial reference frames.

ii) If there is an inertial reference frame in which $\mathbf{B}$ is vanishing and $\mathbf{E}$ takes the value $\mathbf{E}_0$, then there cannot be an inertial reference frame in which $|\mathbf{E}|$ would be smaller than $|\mathbf{E}_0|$.

iii) If $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to each other in an inertial reference frame, then, the same property holds in all inertial reference frames.

– Diego: Well Aïssata, could you give me a typical example in which the manifestly covariant formalism is a real advantage?

▷ 17.3 Radiation by an accelerated charge

– Aïssata: The "canonical case" is that of the accelerated charge I think. This is a very hard problem in the general case, but let us first contemplate the case of an electric charge in uniform motion w.r.t. an inertial reference frame $\mathscr{R}$. Call e the charge and $\mathbf{v}$ its velocity w.r.t $\mathscr{R}$. We are interested in the calculation of the electric and magnetic fields. Another reference frame, called $\mathscr{R}'$, moves together with the charge. Assume that the velocity of the particle is along an axis called x, coincident with x'. In $\mathscr{R}'$, the field is purely of Coulomb form (in this section, we keep the constants of SI units),

$$\mathbf{E}'(\mathbf{r}') = \frac{e}{4\pi\varepsilon_0}\frac{\mathbf{r}'}{r'^3},\tag{762}$$

$$\mathbf{B}'(\mathbf{r}') = 0.\tag{763}$$

Using the Lorentz transformation of coordinates, and of the field components between $\mathscr{R}'$ and $\mathscr{R}$, and if we denote θ the angle between $\mathbf{v}$ and the observer, $x = r\cos\theta$, $y^2+z^2 = r^2\sin^2\theta$, or

$$\gamma^2x^2+y^2+z^2 = \gamma^2r^2(1-\beta^2\sin^2\theta)\tag{764}$$

we get in vector form

$$\mathbf{E}(\mathbf{r}) = \frac{e}{4\pi\varepsilon_0}\frac{(1-\beta^2)\mathbf{r}}{r^3(1-\beta^2\sin^2\theta)^{3/2}}.\tag{765}$$

This is an electric field which is centered at any moment on the *instantaneous position* of the charge, without any retardation. This is remarkable. It is also noticeable that it is of Coulomb form at the leading order. The magnetic field follows from

$$\mathbf{B} = (\boldsymbol{\beta}/c)\times\mathbf{E} = \frac{1}{c^2}\mathbf{v}\times\mathbf{E}.\tag{766}$$

We are now in position to derive the radiation field created by an accelerated charge using an approximate method due to Thomson[98]. Assume a charge e at rest at $t = 0$. It is accelerated with constant $\mathbf{a}$ during a short time Δt, acquires a velocity $\mathbf{v} = \mathbf{a}\Delta t$ and then moves at this constant velocity until the observation time t. We use the limit $\Delta t \ll t$ et $|\mathbf{a}|\Delta t \ll c$.

At distances shorter than $r = ct$, the electric field is Coulombic and centered on the instantaneous position of the charge as we have shown before. We write $\mathbf{u}_\parallel$ and $\mathbf{u}_\perp$ the unit vectors in the observation direction and perpendicular to it. One gets

$$E_\parallel \simeq \frac{r}{4\pi\varepsilon_0 r^2} \quad \text{short distances.} \tag{767}$$

At distances larger than $c(t + \Delta t) \simeq ct$, the field is also Coulombic, but the information that the charge has been accelerated has not yet arrived in this region and the field is centered on the initial rest position of the particle. This is illustrated in Fig. 23.

The geometric construction shows that there is a proportionality ratio

$$\frac{E_\perp}{E_\parallel} = \frac{a_\perp t \Delta t}{c\Delta t}, \tag{768}$$

which gives, to leading order,

$$E_\perp = \frac{e}{4\pi\varepsilon_0}\frac{a_\perp}{rc^2}, \tag{769}$$

which is usually written in terms of the retarded acceleration, because at t there is no longer any acceleration,

$$E_\perp(t) = \frac{e}{4\pi\varepsilon_0}\frac{a_\perp(t - r/c)}{rc^2}. \tag{770}$$

The electric field is polarized in the plane $(\mathbf{a}, \mathbf{n})$ (here $\mathbf{n} = \mathbf{r}/r$ r is the unit vector in the observation direction) and is perpendicular to $\mathbf{n}$. It is then along $\mathbf{n} \times (\mathbf{n} \times \mathbf{a})$, and it follows that

$$\mathbf{E}_R(t) = \frac{e}{4\pi\varepsilon_0}\frac{\mathbf{n} \times (\mathbf{n} \times \mathbf{a}(t - r/c))}{rc^2} = \frac{e}{4\pi\varepsilon_0}\frac{\mathbf{r} \times (\mathbf{r} \times \mathbf{a}(t - r/c))}{r^3c^2}. \tag{771}$$

In the case of an arbitrary motion, as I said, this becomes a very hard problem. There is a very nice calculation in Landau and Lifshitz books, in the Classical Field Theory volume actually[99]. Consider an electric charge e in arbitrary motion w.r.t. an observer sitting at $\mathbf{r}_0$. At t_0 (in the observer's rest frame), the electromagnetic field is that created by the charge when it was located at the retarded position $\mathbf{r}_e(t_r) = [\mathbf{r}_e]_{ret.}$ such that

$$|\mathbf{r}_e(t_r) - \mathbf{r}_0| = c(t_0 - t_r) \tag{772}$$

due to the finite velocity of propagation of electromagnetic interactions. For the rest of the calculation, it is worth introducing the notation $\mathbf{R}(t_r) = \mathbf{r}_e(t_r) - \mathbf{r}_0$ and its 4-dimensional counterpart $R_\alpha(t_r) = (c(t_0 - t_r), -\mathbf{R}(t_r))$. Being a lightlike vector, it has a zero square

$$R_\alpha(t_r)R^\alpha(t_r) = c^2(t_0 - t_r)^2 - |\mathbf{R}(t_r)|^2 = 0. \tag{773}$$

[98] M. Longair, Theoretical concepts in Physics, Cambridge University Press, Cambridge, 2003.

[99] L.D. Landau and E. M. Lifshitz, Classical Fiel Theory, Butterworth Heinemann, Fourth revised English edition, Oxford, 1975.

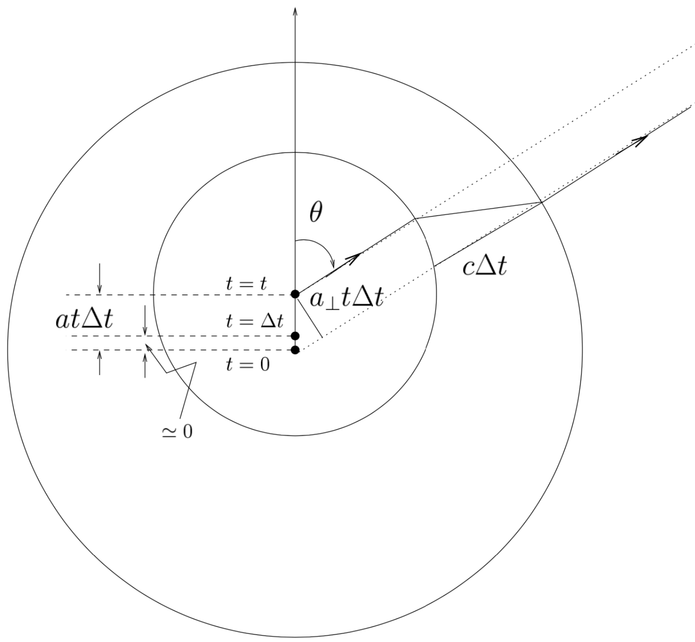


Figure 23. Thomson construction of the electric field radiated by an accelerated charge initially at rest.

In the rest frame of the charge, we use τ for the proper time and primes to denote the other quantities, e.g. $\mathbf{r}'_0$ is the observer's position and $\mathbf{R}'$ its relative distance to the charge. In this frame, the potential is purely Coulombian,

$$\phi'(\mathbf{r}'_0, \tau_0) = \frac{e}{4\pi\varepsilon_0} \frac{1}{|\mathbf{R}'(\tau_r)|}, \quad \mathbf{A}'(\mathbf{r}'_0, \tau_0) = 0. \tag{774}$$

The strategy is to find a tensor expression (a 4-vector here), the value of which equals

$$A'^{\alpha}(x_0) = \begin{pmatrix} \phi'(\mathbf{r}'_0, \tau_0)/c \\ \mathbf{A}'(\mathbf{r}'_0, \tau_0) \end{pmatrix} \tag{775}$$

in the charge's rest frame. Because of manifest covariance, such an expression will automatically be Lorentz covariant and will hold in the observer's frame with unprimed variables.

In the rest frame of the charge one has $v'^{\alpha} = (c, 0)^{\mathrm{T}}$, which allows to write the contraction $v'_{\beta}(\tau_r) R'^{\beta}(\tau_r) = c^2(\tau_0 - \tau_r)$ which appears at the denominator of the expression of the scalar potential. We can thus form $[v'^{\alpha}/v'_{\beta}R'^{\beta}]_{ret.}$ and get

$$A'^{\alpha}(x'_0) = \frac{e}{4\pi\varepsilon_0} \frac{v'^{\alpha}(\tau_r)}{v'_{\beta}(\tau_r) R'^{\beta}(\tau_r)} \tag{776}$$

from which $A^{\alpha}(x_0)$ follows, just discarding the primes and replacing τ_r by t_r. We can give

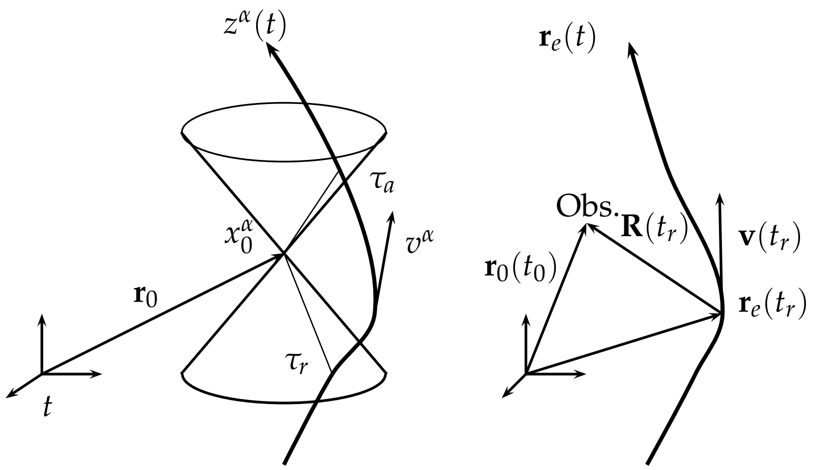


Figure 24. A charge e at z^α in an arbitrary motion w.r.t. an observer at x_0^α.

the full component expression, extracting time and space components,

$$A^\alpha(x_0) = \frac{e}{4\pi\varepsilon_0}\frac{v^\alpha(t_r)}{v_\beta(t_r)R^\beta(t_r)} \tag{777}$$

$$\phi(\mathbf{r}_0, t_0) = \frac{e}{4\pi\varepsilon_0|\mathbf{R}(t_r)]}\frac{1}{1-\left[\frac{\mathbf{R}\cdot\mathbf{v}}{|\mathbf{R}|c}\right]_{ret.}} \tag{778}$$

$$\mathbf{A}(\mathbf{r}_0, t_0) = \frac{e}{4\pi\varepsilon_0|\mathbf{R}(t_r)]}\frac{[\mathbf{v}]_{ret.}}{c^2}\frac{1}{1-\left[\frac{\mathbf{R}\cdot\mathbf{v}}{|\mathbf{R}|c}\right]_{ret.}} \tag{779}$$

These potentials are known as the Liénard-Wiechert potentials and were found independently, solving Maxwell equations (by Alfred Liénard in 1898 and by Emile Wiechert in 1900) before the advent of Special Relativity.

▷ 17.4 Introduction to differential forms and exterior calculus

– Diego: I have seen in Wikipedia that there is still another formulation of Maxwell equations in terms of *differential forms*. I asked the professor who said that this is a very interesting question which unfortunately goes a bit beyond what she wanted to teach us in this section, but that we will occasionally mention this formalism later in the course (See p. **??**). But she suggested that I take the lead and anticipate to work on the concept of differential forms and its application to electrodynamics. She even asked me to prepare a talk in front of the class when I would be ready! This is quite challenging, but I am sure that you can help me at least to introduce myself to the subject.

– Aïssata: Well. This is something that I studied recently. You remember my test last week? That was the topic.

Differential forms are part of a mathematical language which is particularly suited to deal with electrodynamics. I will just tell you a few basic notions, this is already some important material. People also say *exterior calculus* for this formalism.

Let us first discuss 1-, 2-, et 3-forms in $\mathbb{R}^3$. We thus consider $3+1$ dimensions where $\mathcal{M}$ is a 3-dimensional manifold with coordinates $\{x^j\}_{j=1,2,3}$ and time is an external parameter, like in classical mechanics. We call differential form of degree p or p-form the quantity denoted as $\overset{p}{\omega}$ which, upon integration over a p-dimensional domain $\underset{p}{\Omega} \in \mathcal{M}$, delivers a scalar quantity, say s,

$$\int_{\underset{p}{\Omega}} \overset{p}{\omega} = s. \tag{780}$$

This is a pretty vague definition, but reviewing a few exemples will help. A 1-form $\overset{1}{\omega}$ integrated along a curve $\mathcal{C} = \underset{1}{\Omega}$ is equivalent to the line integral of a vector field along the curve

$$\int_{\underset{1}{\Omega}} \overset{1}{\omega} = \int_{\mathcal{C}} \boldsymbol{\omega} \cdot d\mathbf{x} \tag{781}$$

with $\boldsymbol{\omega} = \omega_j \mathbf{e}^j$ a vector in the 3-dimensional cotangent space and $d\mathbf{x} = dx^k \mathbf{e}_k$ the line element in the tangent space. In $3D$, we write a 1-form as

$$\overset{1}{\omega} = \omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3 \mathrel{\hat{=}} \boldsymbol{\omega} \cdot d\mathbf{x}. \tag{782}$$

where the symbol $\hat{=}$ means "corresponds to".

A form is independent of the coordinate system (as a vector is). The invariance of $\overset{1}{\omega}$ under $x^j \to x^{j'}$ shows that ω_j transforms like a first rank covariant tensor

$$\omega_{j'} = \frac{\partial x^j}{\partial x^{j'}} \omega_j. \tag{783}$$

A 2-form and a 3-form, $\overset{2}{\omega}$ and $\overset{3}{\omega}$, respectively integrated over a surface $\mathcal{S} = \underset{2}{\Omega}$, or over a volume $\mathcal{V} = \underset{3}{\Omega}$, are equivalent to a surface integral,

$$\int_{\underset{2}{\Omega}} \overset{2}{\omega} = \int_{\mathcal{S}} \boldsymbol{\omega} \cdot d^2\mathbf{Surf}, \tag{784}$$

and a volume integral,

$$\int_{\underset{3}{\Omega}} \overset{3}{\omega} = \int_{\mathcal{V}} \omega \, d^3\mathrm{Vol}\,. \tag{785}$$

The 2- et 3-forms can be written as

$$\overset{2}{\omega} = \omega_{12} dx^1 dx^2 + \omega_{23} dx^2 dx^3 + \omega_{31} dx^3 dx^1, \tag{786}$$

$$\overset{3}{\omega} = \omega_{123} dx^1 dx^2 dx^3, \tag{787}$$

but in these expressions, an orientation must be chosen to fully specify the signs of the integrals. For example, the r.h.s. of (784) assumes an orientation for $d^2\mathbf{Surf}$. This is an example of a $2D$ oriented integral which transforms under a change of variables according to

$$\int_{\mathcal{S}(x,y)} f(x,y) dx dy = \int_{\mathcal{S}(u,v)} f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)} du dv \tag{788}$$

where the Jacobian dterminant is not taken in absolute value, precisely due to this reason of orientation. With $u = y$ and $v = x$ it yields $\int dxdy = -\int dydx$ and this is the motivation to introduce the exterior product (the wedge) $\wedge$, such that the antisymmetry property is satisfied,

$$dx^j \wedge dx^k = -dx^k \wedge dx^j, \quad \text{in particular} \quad dx^j \wedge dx^j = 0. \tag{789}$$

A 2-form may now be rewritten as

$$\overset{2}{\omega} = \frac{1}{2!}\omega_{jk}dx^j \wedge dx^k, \tag{790}$$

with antisymmetric components $\omega_{jk} = -\omega_{kj}$ and a 3-form as,

$$\overset{3}{\omega} = \frac{1}{3!}\omega_{ijk}dx^i \wedge dx^j \wedge dx^k, \tag{791}$$

with $\omega_{ijk} = -\omega_{jik}$ and additional changes of sign at each additional permutation. Their components transform like those of rank p covariant antisymmetric tensors,

$$\omega_{j'k'\dots} = \frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\cdots\omega_{jk\dots}. \tag{792}$$

As for the 1- form, we take the correspondances

$$\overset{2}{\omega} \quad \hat{=} \quad \boldsymbol{\omega}\cdot d^2\mathbf{Surf} \tag{793}$$

$$\overset{3}{\omega} \quad \hat{=} \quad \omega\, d^3\mathrm{Vol} \tag{794}$$

as equalities.

There exits better approaches, that you can find in books and papers, e.g. in Bertlmann or in Göckeler and Schücker[100] , or Baez and Muniain[101], see also quotation in Fig. 25 [102].

Various operations can be performed with forms. Among them, the exterior product, the exterior derivative and the Hodge product are central pieces.

– The exterior product builds, from two forms of ranks p and q a form of rank $p+q$ with the property

$$\overset{p}{\alpha} \wedge \overset{q}{\beta} = (-1)^{pq}\overset{q}{\beta} \wedge \overset{p}{\alpha}. \tag{795}$$

– The exterior derivative d increases the rank of a differential form by 1, and acting on

$$\overset{p}{\omega} = \omega_{1\dots p}dx^1 \wedge \cdots \wedge dx^p = \frac{1}{p!}\omega_{\mu_1\dots\mu_p}dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{796}$$

(no contraction on the explicit indices $1, \dots p$) it delivers

$$d\overset{p}{\omega} = \frac{1}{p!}\frac{\partial\omega_{\mu_1\dots\mu_p}}{\partial x^{\mu_j}}dx^{\mu_j} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{797}$$

[100] R.A. Bertlmann, Anomalies in Quantum Field Theory, Oxford University Press, Oxford, 1996; M. Göckeler and T. Schücker, Differential geometry, gauge theories, and gravity, Cambridge University Press, Cambridge, 1987.

[101] J. Baez and J.P. Muniain, Gauge fields, knots and gravity, World Scientific, Singapore, 1994.

[102] S. Fumeron, B. Berche and F. Moraes, Improving student understanding of electrodynamics: the case for differential forms, Am. J. Phys., **88**, 1083, 2020.

What is also essential to understand is that the formalism works in arbitrary coordinate systems in arbitrary manifolds. This is the reason why the μ, ν indices (and later $g_{\mu\nu}$) are used here, instead of the α, β ones (and $\eta_{\alpha\beta}$) according to our convention in Minkowski spacetime vs more general metrics. Nevertheless, in order not to be confused, we will revert most of the time in this chapter to our standard use of α, β indices and the above formulas, as well as that of the Hodge dual defined a bit later are just an exception to show the wide degree of generality of the exterior calculus.

courses. Given a Cartesian coordinate system on the Euclidean space $\mathbb{R}^3$, the total differential of any scalar function F writes as (Einstein's summation convention on repeated indices is used)

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = \frac{\partial F}{\partial x^a}dx^a. \tag{9}$$

When changing the function, only the partial derivatives will change: this means that anytime an exact differential is computed, one is working in a vector space whose basis elements are the $\{dx^a, a = 1..3\} = \{dx, dy, dz\}$.

More generally, any object (not necessarily a total differential) that is written as a linear combination of dx^j is called a 1-form and it belongs to the 1-form vector space (or cotangent space) denoted by $\Lambda^1(\mathbb{R}^3)$. Once given the 1-form vector space, more general objects can be built in a natural way: this is the idea underlying the more general concept of tensors (technically, a (m,p)-tensor is simply a multilinear map acting on a collection of m 1-forms and p vectors to produce a real number - for more details, see for example[30]). Taking the antisymmetrized tensor product (denoted for short by $\wedge$) for each pair of 1-form basis elements gives

$$dx^a \otimes dx^b - dx^b \otimes dx^a = dx^a \wedge dx^b. \tag{10}$$

Here, the regular tensor product $\otimes$ is defined as the ordered product of pairs of 1-forms (and of vectors) and it is associative.

Now, it appears that one can generate only a finite number of non-zero terms, which are a linearly independent and spanning subset of a new vector space: $\Lambda^2(\mathbb{R}^3)$, the space of 2-forms. For example, in Cartesian coordinates, the 2-form basis written in the right cyclic order is the set: $dy \wedge dz, dz \wedge dx, dx \wedge dy$ (no element dx^a is repeated as (10) would return 0). That process can be iterated for p-uples basis elements $\{dx^{a_1} \wedge .. \wedge dx^{a_p}, a_1 \neq .. \neq a_p\}$ and generates forms of degree p (or p-forms) that belong to a vector space $\Lambda^p(\mathbb{R}^3)$ of dimension $C_3^p = 3\,!/(p\,!(3-p)\,!)$. By construction, a p-form is a completely antisymmetric $(0,p)$ tensor. Concretely, for Cartesian coordinates in $\mathbb{R}^3$, the general expression for a form of degree

$$\begin{array}{lll} 0 & \text{is} & f(x,y,z) \\ 1 & \text{is} & f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz \\ 2 & \text{is} & g_1(x,y,z)dy \wedge dz + g_2(x,y,z)dz \wedge dx \\ & & +g_3(x,y,z)dx \wedge dy \\ 3 & \text{is} & g(x,y,z)dx \wedge dy \wedge dz \end{array}$$

Figure 25. From: S. Fumeron, B. Berche and F. Moraes, Improving student understanding of electrodynamics: the case for differential forms, Am. J. Phys., **88**, 1083, 2020 (taken here from the arXiv version (arXiv:2009.10356).

The operator d acting on a product of forms obeys a generalized Leibniz rule,

$$d(\overset{p}{\omega} \wedge \overset{q}{\theta}) = (d\overset{p}{\omega}) \wedge \overset{q}{\theta} + (-1)^p \overset{p}{\omega} \wedge d\overset{q}{\theta}. \tag{798}$$

Note that the degree q of the second differential form doesn't play any role in the sign modification.

You can easily check that $d\overset{0}{\omega}$ produces a gradient, $d\overset{1}{\omega}$ gives a curl, and $d\overset{2}{\omega}$ a divergence. This is a kind of "unification" of these operators of vector algebra.

– Diego: You mean that a single operation produces the three standard operators of vector analysis? May I try this? But what is a zero form?

►EXERCISE 26 – $d\overset{0}{\phi}$ **and the gradient** –

– Aïssata: A 0-form is just a scalar function, say $\phi(x)$. Then, acting with d, for example in spherical coordinates, yields

$$d\overset{0}{\phi} = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta + \frac{\partial\phi}{\partial\varphi}d\varphi \tag{799}$$

and if I decide to call it

$$\begin{aligned} d\overset{0}{\phi} &= \mathbf{grad}\,\phi \cdot d\boldsymbol{\ell} \\ &= (\mathbf{grad}\,\phi \cdot \mathbf{u}_r)dr + (\mathbf{grad}\,\phi \cdot \mathbf{u}_\theta)rd\theta + (\mathbf{grad}\,\phi \cdot \mathbf{u}_\varphi)r\sin\theta d\varphi \end{aligned} \tag{800}$$

this is a 1-form and after identification you recognize

$$\mathbf{grad}\,\phi = \frac{\partial\phi}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\mathbf{u}_\varphi \tag{801}$$

which is the right object for a gradient in spherical coordinates. ◄

►EXERCISE 27 – $d\overset{1}{E}$ **and the curl** –

– Diego: Now for a 1-form $\overset{1}{E}$, one has

$$\overset{1}{E} = E_r dr + E_\theta d\theta + E_\varphi d\varphi \tag{802}$$

The exterior derivative is

$$\begin{aligned} d\overset{1}{E} = & \frac{\partial E_r}{\partial\theta}d\theta \wedge dr + \frac{\partial E_r}{\partial\varphi}d\varphi \wedge dr \\ & + \frac{\partial E_\theta}{\partial r}dr \wedge d\theta + \frac{\partial E_\theta}{\partial\varphi}d\varphi \wedge d\theta \\ & + \frac{\partial E_\varphi}{\partial r}dr \wedge d\varphi + \frac{\partial E_\varphi}{\partial\theta}d\theta \wedge d\varphi \end{aligned} \tag{803}$$

which I decide to call, as you suggested,

$$\begin{aligned} d\overset{1}{E} &= \mathbf{rot}\,\mathbf{E} \cdot d^2\mathbf{Surf} \\ &= (\mathbf{rot}\,\mathbf{E} \cdot \mathbf{u}_r)rd\theta r\sin\theta d\varphi + (\mathbf{rot}\,\mathbf{E} \cdot \mathbf{u}_\theta)r\sin\theta d\varphi dr + (\mathbf{rot}\,\mathbf{E} \cdot \mathbf{u}_\varphi)drrd\theta. \end{aligned} \tag{804}$$

This is a 2- form and I identify term by term, for example for the first term

$$(\mathbf{rot}\,\mathbf{E} \cdot \mathbf{u}_r) = \frac{1}{r^2\sin\theta}\left(\frac{\partial E_\varphi}{\partial\theta} - \frac{\partial E_\theta}{\partial\varphi}\right). \tag{805}$$

Look Aïssata, it looks like a component of the curl operator applied to a vector field, but *it isn't*!

– Aïssata: Well spotted Diego, but in fact there is no mistake here, the expression is correct. But the thing which I forgot to tell you is that the basis vectors in local coordinates $\{\mathbf{e}_i\}$ are generally *non normalized vectors*. In differential geometry, the metric tensor elements are given in terms of the basis vectors, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. This means that the vector field that you have called $\mathbf{E}$ writes in this basis

$$\mathbf{E} = E_r \mathbf{e}_r + E_\theta \mathbf{e}_\theta + E_\varphi \mathbf{e}_\varphi, \tag{806}$$

while the normalized basis $\{\mathbf{u}_i\}$ is such that

$$||\mathbf{u}_i||^2 = |g_{ii}|^{-1} ||\mathbf{e}_i||^2 \tag{807}$$

and the same vector field is thus

$$\mathbf{E} = E_r |g_{rr}|^{-1/2} \mathbf{u}_r + E_\theta |g_{\theta\theta}|^{-1/2} \mathbf{u}_\theta + E_\varphi |g_{\varphi\varphi}|^{-1/2} \mathbf{u}_\varphi. \tag{808}$$

You can now identify the components in the normalized basis and call them with a tilda for example,

$$\tilde{E}_i = E_i |g_{ii}|^{-1/2} \tag{809}$$

You can easily remember this from the expression $\overset{1}{E} = \mathbf{E} \cdot d\boldsymbol{\ell}$ in spherical coordinates. In terms of these tilda components you have the correct component of the curl operator as you know it.

$$(\mathbf{rot}\, \mathbf{E} \cdot \mathbf{u}_r) = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \tilde{E}_\varphi) - \frac{1}{r \sin\theta} \frac{\partial}{\partial\varphi} \tilde{E}_\theta. \tag{810}$$

And you can proceed similarly for the two remaining components. ◄

►**Exercise 28** – $d\overset{2}{B}$ **and the divergence** –

– Diego: Not so easy at the end! OK, I consider now a 2-form $\overset{2}{B}$ and I write the *non normalized basis components* B_{ij} and the normalized ones $\tilde{B}_k$ with a cyclic permutation of the indices, e.g. $B_{\theta\varphi} = r^2 \sin\theta \tilde{B}_r$.

$$\begin{aligned} \overset{2}{B} &= B_{r\theta} dr \wedge d\theta + B_{\theta\varphi} d\theta \wedge d\varphi + B_{\varphi r} d\varphi \wedge dr && (811) \\ &= \tilde{B}_r r d\theta r \sin\theta d\varphi + \tilde{B}_\theta r \sin\theta d\varphi dr + \tilde{B}_\varphi dr r d\theta && (812) \end{aligned}$$

with wedges omitted in the last line. This is indeed easy to go from the B_{ij}'s to the $\tilde{B}_k$'s when you remember the expression of $\mathbf{B} \cdot d^2\mathbf{Surf}$ in the corresponding coordinate system. Now we calculate $d\overset{2}{B}$,

$$\begin{aligned} d\overset{2}{B} &= \frac{\partial B_{r\theta}}{\partial\varphi} \underbrace{d\varphi \wedge dr \wedge d\theta}_{dr \wedge d\theta \wedge d\varphi} + \frac{\partial B_{\theta\varphi}}{\partial r} dr \wedge d\theta \wedge d\varphi + \frac{\partial B_{\varphi r}}{\partial\theta} d\theta \wedge d\varphi \wedge dr \\ &= \left[\frac{\partial}{\partial\varphi} (r \tilde{B}_\varphi) + \frac{\partial}{\partial r} (r^2 \sin\theta \tilde{B}_r) + \frac{\partial}{\partial\theta} (r \sin\theta \tilde{B}_\theta) \right] dr \wedge d\theta \wedge d\varphi. && (813) \end{aligned}$$

I call it

$$\begin{aligned} d\overset{2}{B} &= \operatorname{div} \mathbf{B}\, d^3\mathrm{Vol} && (814) \\ &= \operatorname{div} \mathbf{B}\, r^2 \sin\theta dr d\theta d\varphi && (815) \end{aligned}$$

and the expression for the divergence follows:

$$\operatorname{div} \mathbf{B} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tilde{B}_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \tilde{B}_\theta) + \frac{1}{r \sin\theta} \frac{\partial \tilde{B}_\varphi}{\partial\varphi}. \tag{816}$$

I think that I have understood. And it works in arbitrary coordinates? I can have the form of these operators, say in elliptic coordinates?

– Aïssata: Of course! But I must confess that I have never done it. This would be a nice exercise I suppose. ◄

– Now let us elaborate on the next piece, the Hodge product. Let us denote it as $\star_3$ and act on a p-form $\overset{p}{\omega}$ (with $p \leqslant 3$ here). It delivers a $(3-p)$-form that we write $\overset{3-p}{\underset{\sim}{\omega}}$. In a metric space with metric tensor,

$$g_{jk} = \mathbf{e}_j \cdot \mathbf{e}_k, \quad g_{ij}g^{jk} = \delta_i^k, \tag{817}$$

one defines successively

$$\begin{aligned} \text{1-form} \quad \overset{1}{\omega} &= \omega_i dx^i \\ \star_3 \overset{1}{\omega} &= \overset{2}{\underset{\sim}{\omega}} = \frac{1}{2!} \underset{\sim}{\omega}_{jk}\, dx^j \wedge dx^k \\ &= \frac{1}{2} \sqrt{g} \varepsilon_{ijk} g^{il} \omega_l dx^j \wedge dx^k, \end{aligned} \tag{818}$$

$$\begin{aligned} \text{2-form} \quad \overset{2}{\omega} &= \frac{1}{2!} \omega_{ij} dx^i \wedge dx^j \\ \star_3 \overset{2}{\omega} &= \overset{1}{\underset{\sim}{\omega}} = \underset{\sim}{\omega}_k\, dx^k \\ &= \frac{1}{2!} \sqrt{g} \varepsilon_{ijk} g^{il} g^{jm} \omega_{lm} dx^k, \end{aligned} \tag{819}$$

$$\begin{aligned} \text{3-form} \quad \overset{3}{\omega} &= \frac{1}{3!} \omega_{ijk} dx^i \wedge dx^j \wedge dx^k \\ \star_3 \overset{3}{\omega} &= \overset{0}{\underset{\sim}{\omega}} = \underset{\sim}{\omega} \\ &= \sqrt{g} \varepsilon_{ijk} g^{il} g^{jm} g^{kn} \omega_{lmn}. \end{aligned} \tag{820}$$

– Diego: This is awful, I cannot remember that.

– Aïssata: I understand. And I must confess that I can hardly remember these expressions myself. There is a general expression for arbitrary manifold dimension,

$$\star \overset{p}{v} = \frac{1}{p!(D-p)!} v^{\mu_1 \dots \mu_p} \sqrt{|\det g_{\mu\nu}|}\, \epsilon_{\mu_1 \dots \mu_D} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_D}, \tag{821}$$

where

$$\overset{p}{v} = \frac{1}{p!} v_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \tag{822}$$

and the totally anti-symmetric D-rank Levi-Civita symbol is defined as

$$\epsilon_{\mu_1 \dots \mu_D} = \begin{cases} +1 & \text{if } \mu_1, ..\mu_D \text{ is an even permutation of } 1, ..D \\ -1 & \text{if } \mu_1, ..\mu_D \text{ is an odd permutation of } 1, ..D \\ 0 & \text{otherwise.} \end{cases}$$

This is not really much more pleasant. What you have to understand is essentially that the Hodge star of a given p-form is the form which complements to the volume form (of rank equal to the manifold dimension). Again, the use μ,ν indices (and $g_{\mu\nu}$) instead of the α,β ones (and $\eta_{\alpha\beta}$) reminds the reader of the independence w.r.t. the choice of coordinates.

There are many properties associated to the use of Levi-Civita symbols, products and contractions of such symbols, etc. These properties depend on the metric signature and will differ in Euclidean and Minkowskian manifolds. I have to emphasize also that there are two conventions. I choose the one where $\epsilon_{12\ldots D} = +1$, but you will find in the literature the option $\epsilon^{12\ldots D} = +1$, which makes a difference in Minkowski spacetimes for which $\epsilon^{12\ldots D} = -\epsilon_{12\ldots D}$[103]. In pseudo-Riemannian manifolds which are required for the study of General Relativity, one has to define a Levi-civita *tensor* or pseudo-tensor, but I won't discuss it here.

►EXERCISE 29 – **Volume form** –

The first thing is the orientation of the volume form. We define a D-rank form, called volume form, and we fix its sign by the choice

$$\begin{aligned} \overset{D}{d\mathrm{Vol}} &= dx^0 \wedge dx^1 \wedge \cdots \wedge dx^d, \\ &= \frac{1}{D!}\epsilon_{\alpha_1\alpha_2\ldots\alpha_D} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_D} \quad D = d+1 \end{aligned} \tag{823}$$

This is valid both in Euclidean and Minkowskian manifolds.

– Diego: I can check it in $\mathbb{M}^4$: let me write explicitly all terms starting with $dx^0\wedge$:

$$\begin{aligned} \epsilon_{0\beta\gamma\delta} dx^0 \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta &= \epsilon_{0123} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &\quad + \epsilon_{0132} dx^0 \wedge dx^1 \wedge dx^3 \wedge dx^2 \\ &\quad + \epsilon_{0213} dx^0 \wedge dx^2 \wedge dx^1 \wedge dx^3 \\ &\quad + \epsilon_{0231} dx^0 \wedge dx^2 \wedge dx^3 \wedge dx^1 \\ &\quad + \epsilon_{0312} dx^0 \wedge dx^3 \wedge dx^1 \wedge dx^2 \\ &\quad + \epsilon_{0321} dx^0 \wedge dx^3 \wedge dx^2 \wedge dx^1 \\ &= 6 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{824}$$

playing with the permutations. There are 4 equivalent terms for the 4 values of α and eventually

$$\overset{4}{d\mathrm{Vol}} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta. \tag{825}$$

– Aïssata: Correct. Taking care at the minus sign in pseudo-Euclidean manifolds (because of the property $\epsilon^{12\ldots D} = -\epsilon_{12\ldots D}$ there), you also have the simpler "inverse" relation

$$dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta = -\epsilon^{\alpha\beta\gamma\delta}\overset{4}{d\mathrm{Vol}} \quad \text{in } \mathbb{M}^4 \tag{826}$$

which is sometimes useful. Other interesting properties are $\star\overset{4}{d\mathrm{Vol}} = 1$ and $\star 1 = \overset{4}{d\mathrm{Vol}}$.

In arbitrary spacetimes, the square root of the determinant of the metric tensor matrix is needed to make the volume form an invariant quantity under a change of coordinates, and we define,

$$\overset{D}{d\mathrm{Vol}} = \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^d \tag{827}$$

instead of (823). ◄

[103] At least with $D = 1+d$ even, or with a signature $(-,+,+,\ldots)$

►EXERCISE 30 – **Gymnastics with Levi-Civita symbols** –

Another important property is the partial contraction of Levi-Civita symbols,

$$\epsilon_{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_D}\epsilon^{\beta_1\ldots\beta_p\alpha_{p+1}\ldots\alpha_D} = -(D-p)!\delta^{\beta_1\ldots\beta_p}_{\alpha_1\ldots\alpha_p} \quad \text{in } \mathbb{M}^4. \tag{828}$$

Here the contraction is on the last indices $\alpha_{p+1}\ldots\alpha_D$, the minus sign appears in Minkowski space, *but is absent in the Euclidean case* and the generalized Kronecker delta is defined according to

$$\delta^{\beta_1\ldots\beta_p}_{\alpha_1\ldots\alpha_p} = \begin{array}{l}\text{signature of the permutation which sends } \alpha_1\ldots\alpha_p \text{ to } \beta_1\ldots\beta_p \text{ if it exists,}\\ 0 \text{ otherwise.}\end{array} \tag{829}$$

A good exercise again is to check this identity.

– Diego: Let me go to the blackboard. I will just write down specific examples to see how things work. I stay in $\mathbb{M}^4$, assuming that the Euclidean case just contains marginal modifications.

In the case $p = 1$, I can write for example

$$\epsilon_{0\alpha\beta\gamma}\epsilon^{1\alpha\beta\gamma} = 0 \tag{830}$$

because the sequence $\alpha\beta\gamma$ must be different from 0 for the first epsilon not to vanish, but then the second epsilon is zero! If I compare to the r.h.s. of (828), $-(4-1)!\delta^1_0$ is also zero. Now, if I look at

$$\epsilon_{0\alpha\beta\gamma}\epsilon^{0\alpha\beta\gamma} = \epsilon_{0123}\epsilon^{0123} + \epsilon_{0132}\epsilon^{0132} + \cdots = -3! \tag{831}$$

because there are 3 choices for α, then 2 for β, then γ is fixed, and in all these $3!$ identical terms, the “all superscripts epsilon” is minus the “all subscripts epsilon”, hence the result -6. Comparison with (828) leads to $-(4-1)!\delta^0_0 = -3!$.

When $p = 2$, I can try e.g. with

$$\epsilon_{01\alpha\beta}\epsilon^{12\alpha\beta} = \epsilon_{0123}\epsilon^{1223} + \epsilon_{0132}\epsilon^{1232} = 0 \tag{832}$$

and compare to $-(4+2)!\delta^{12}_{01} = 0$. Another example is

$$\epsilon_{01\alpha\beta}\epsilon^{10\alpha\beta} = \epsilon_{0123}\epsilon^{1023} + \epsilon_{0132}\epsilon^{1032} = (+1)^2 + (-1)^2 = +2 \tag{833}$$

which agrees with $-(4-2)!\delta^{10}_{01} = +2$.

Just to be sure, I also check the case $p = 3$:

$$\epsilon_{012\alpha}\epsilon^{021\alpha} = \epsilon_{0123}\epsilon^{0213} = 1 \tag{834}$$

and $(4-3)!\delta^{021}_{012} = 1$, while

$$\epsilon_{012\alpha}\epsilon^{123\alpha} = \epsilon_{0123}\epsilon^{1233} = 0 \tag{835}$$

and $(4-3)!\delta^{123}_{012} = 0$. Obviously, I haven't exhausted all possible cases, but this really helps to understand how these contractions operate.

– Aïssata: Very good Diego. We now come to one of the central relations between forms, very useful to produce acceptable Lagrangians for relativistic theories. For that purpose, we still need a bit of gymnastic. Diego, could you check the following identity,

$$u_{\alpha_1\ldots\alpha_p}v^{\beta_1\ldots\beta_p}\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p} = p!\,u_{\alpha_1\ldots\alpha_p}v^{\alpha_1\ldots\alpha_p}. \tag{836}$$

where $u_{\alpha_1\ldots\alpha_p}$ and $v_{\beta_1\ldots\beta_p}$ are the components of two p-rank forms (hence with the appropriate antisymmetry properties).

– Diego: OK. I will do as I know, just take examples and check that it works. If $p = 1$ this is pretty easy:

$$u_\alpha v^{\alpha'}\delta^\alpha_{\alpha'} = u_\alpha v^\alpha. \tag{837}$$

For $p = 2$, we have

$$u_{\alpha\beta} v^{\alpha'\beta'} \delta^{\alpha\beta}_{\alpha'\beta'} = u_{\alpha\beta} v^{\alpha\beta}(+1) + u_{\alpha\beta} v^{\beta\alpha}(-1) = u_{\alpha\beta}(v^{\alpha\beta} - v^{\beta\alpha}) = 2! u_{\alpha\beta} v^{\alpha\beta} \tag{838}$$

because of the antisymmetry $v^{\beta\alpha} = -v^{\alpha\beta}$. If I go on, I can check that it works also for $p = 3$,

$$u_{\alpha\beta\gamma} v^{\alpha'\beta'\gamma'} \delta^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'} = u_{\alpha\beta\gamma}(v^{\alpha\beta\gamma} - v^{\alpha\gamma\beta} + v^{\beta\gamma\alpha} - v^{\beta\alpha\gamma} + v^{\gamma\alpha\beta} - v^{\gamma\beta\alpha}) = 3! u_{\alpha\beta\gamma} v^{\alpha\beta\gamma}. \tag{839}$$

I am satisfied with the iterations! ◄

►EXERCISE 31 – **Calculation of** $\overset{p}{u} \wedge \star\overset{p}{v}$ –

– Aïssata: The central result that I was mentioning is the following: between two forms of the same rank p, we have the property in $\mathbb{M}^4$

$$\overset{p}{u} \wedge \star\overset{p}{v} = \frac{1}{p!} u_{\alpha_1\ldots\alpha_p} v^{\alpha_1\ldots\alpha_p} d\overset{4}{\mathrm{Vol}} \tag{840}$$

– Diego: Let me check this. I will use (822) and (821) in $\mathbb{M}^4$ to write ($D = 4$)

$$\begin{aligned}
\overset{p}{u} \wedge \star\overset{p}{v} &= \frac{1}{p!p!(D-p)!} u_{\alpha_1\ldots\alpha_p} v^{\beta_1\ldots\beta_p} \epsilon_{\beta_1\ldots\beta_p\alpha_{p+1}\ldots\alpha_D} dx^{\alpha_1} \wedge \ldots dx^{\alpha_p} \wedge dx^{\alpha_{p+1}} \wedge \ldots dx^{\alpha_D} \\
&= \frac{-1}{p!p!(D-p)!} u_{\alpha_1\ldots\alpha_p} v^{\beta_1\ldots\beta_p} \epsilon_{\beta_1\ldots\beta_p\alpha_{p+1}\ldots\alpha_D} \epsilon^{\alpha_1\ldots\alpha_p\alpha_{p+1}\ldots\alpha_D} d\overset{4}{\mathrm{Vol}} \quad \text{with (826)} \\
&= \frac{1}{p!p!} u_{\alpha_1\ldots\alpha_p} v^{\beta_1\ldots\beta_p} \delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p} \quad \text{with (828)} \\
&= \frac{1}{p!} u_{\alpha_1\ldots\alpha_p} v^{\alpha_1\ldots\alpha_p} d\overset{4}{\mathrm{Vol}}. \quad \text{with (836)}
\end{aligned} \tag{841}$$

I understand that this is the meaning of your previous statement that the Hodge star complement a form to the volume form. ◄

– Aïssata: The other important thing is that there appear metric factors in the definition: the Hodge product is an operation which requires the existence of a metric structure on the manifold $\mathcal{M}$. This is not a problem for physicists. Measuring lengths (and durations) are part of the basis, maybe the most fundamental things that physicists do and the metric structure is generally assumed by default. The case of non metric manifolds may be a question of interest to mathematicians.

There are additional general properties to which the Hodge star product obeys. Several conventions are used in the literature and here we stick to those of the excellent book that I already mentioned,explicit *Differential geometry, gauge theories, and gravity* by Göckeler and Schücker[104]. We thus have the properties:

$$\star(\star v) = (-1)^{s+p(D-1)}\, v, \tag{842}$$

$$u \wedge (\star v) = v \wedge (\star u), \tag{843}$$

$$\text{if} \quad u \wedge (\star u) = 0, \quad \text{then} \quad u = 0, \tag{844}$$

where s is the number of $-$ signs in the metric, D is the manifold dimension, and p the degree of the form as usual. The Hodge product also enters the definition of the coderivative $d^\dagger$ which acts on a p-form and produces a $(p-1)$-form,

$$d^\dagger \overset{p}{\omega} = (-1)^{D(p+1)+1+s} \star d \star. \tag{845}$$

[104] M. Göckeler and T. Schücker, Differential geometry, gauge theories, and gravity, Cambridge University Press, Cambridge, 1987.

Acting on both even and odd degree forms in $\mathbb{M}^4$ it yields $d^\dagger = \star d \star$, but the property (842) discriminates the forms according to their rank as $\star\star \overset{2k}{u} = -\overset{2k}{u}$ while $\star\star \overset{2k+1}{u} = \overset{2k+1}{u}$.

There are still two central pieces to discuss in order to proceed with electrodynamics in this formalism. First one is the Poincaré lemma $d(d\overset{p}{\omega}) = 0$ for any form, or

$$d^2 = 0. \tag{846}$$

It follows that a closed form, i.e. which obeys $d\overset{p}{\theta} = 0$, can be written as $\overset{p}{\theta} = d(\overset{p-1}{\omega})$ and it is said *exact*[105]. The form $\overset{p-1}{\omega}$ is a potentiel, since it is itself defined up to a $(p-2)$-form. An interesting case is that of a D- form, for which, automatically

$$d\overset{D}{\omega} = 0. \tag{847}$$

As a consequence of (846), we also have

$$d^{\dagger 2} = 0. \tag{848}$$

The operators d and $d^\dagger$ allow for the construction of the Laplace-de Rham operator

$$\Delta_{\rm LdR} = dd^\dagger + d^\dagger d = (d + d^\dagger)^2. \tag{849}$$

It is the Laplace-de Rham operator when given in terms of differential forms, but is called Laplace-Beltrami operator, $\Delta_{\rm LB}$, when it acts on an arbitrary manifold and is expressed in terms of the metric tensor components. We will mainly use the denomination Laplace-Beltrami. In $\mathbb{E}^d$, it is the ordinary Laplacian and in $\mathbb{M}^4$ it is identified to the opposite of the d'Alembertian. You can check that if you wish. We write also

$$\Delta_{\rm LB} = d \star d \star + \star d \star d. \tag{850}$$

►EXERCISE 32 – **Laplace-Beltrami operator and d'Alembertian in $\mathbb{M}^4$** –

– Diego: Let me act with $dd^\dagger + d^\dagger d$ on a scalar function to get the familiar expression of the d'Alembertian operator in $\mathbb{M}^4$. I assume here a zero form $\overset{0}{\phi} = \phi$ with ϕ a simple function. I have obviously $d^\dagger \overset{0}{\phi} = 0$ and just have to consider $\Delta_{\rm LB}\overset{0}{\phi} = d^\dagger d\overset{0}{\phi}$. First I consider

$$d\overset{0}{\phi} = \partial_\alpha \phi \, dx^\alpha. \tag{851}$$

Its Hodge dual is a 3-form,

$$\star\, d\overset{0}{\phi} = \frac{1}{3!}\partial^\alpha \phi \epsilon_{\alpha\beta\gamma\delta}\, dx^\beta \wedge dx^\gamma \wedge dx^\delta. \tag{852}$$

Acting again with the exterior derivative leads to a 4-form

$$d \star d\overset{0}{\phi} = \frac{1}{3!}\partial_{\alpha'}(\partial^\alpha \phi \epsilon_{\alpha\beta\gamma\delta})\, dx^{\alpha'} \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta. \tag{853}$$

[105] Note that we don't consider here possible problems of topology, so what is said here is generally correct only locally.

Then we use the properties of the volume form, $dx^{\alpha'} \wedge dx^{\beta} \wedge dx^{\gamma} \wedge dx^{\delta} = -\epsilon^{\alpha'\beta\gamma\delta} \overset{4}{d\mathrm{Vol}}$ and equation (828) to write

$$d \star d\overset{0}{\phi} = \frac{1}{3!}\partial_{\alpha'}(\partial^{\alpha}\phi)(-(4-1)!)\delta_{\alpha}^{\alpha'} \overset{4}{d\mathrm{Vol}} = \partial_{\alpha}\partial^{\alpha}\phi \overset{4}{d\mathrm{Vol}}. \tag{854}$$

We just have to use then $\star \overset{4}{d\mathrm{Vol}} = 1$ to have

$$\Delta_{\mathrm{LB}}\overset{0}{\phi} = d^{\dagger}d\overset{0}{\phi} = -\partial_{\alpha}\partial^{\alpha}\phi. \tag{855}$$

In $\mathbb{M}^4$, this is just $-\partial_{\alpha}\partial^{\alpha}\phi = (-\partial_t^2 + \boldsymbol{\nabla}^2)\phi = -\square\phi$. ◄

– Aïssata: Yes, and the central interest of differential forms being their independence with respect to coordinates, the coordinate-free expressions like equations (849) or (850) hold in any manifold. If you wish to have a general explicit expression, you can use

$$\Delta_{\mathrm{LB}} = \frac{\mathrm{sign}}{\sqrt{|g|}}\partial_{\mu}(\sqrt{|g|}g^{\mu\nu}\partial_{\nu}) \tag{856}$$

where sign $= 1$ for Riemann manifolds and sign $= -1$ in the case of pseudo-Riemann manifolds and g is the determinant of the metric tensor $g_{\mu\nu}$.

Eventually, the differential forms formalism allows a very compact expression for the Stokes theorem which takes the form

$$\int_{\underset{p+1}{\Omega}} d\overset{p}{\omega} = \int_{\partial \underset{p+1}{\Omega}} \overset{p}{\omega} \tag{857}$$

where $\partial \underset{p+1}{\Omega}$ is the boundary of the $(p+1)$-dimensional submanifold $\underset{p+1}{\Omega}$.

▷ 17.5 Differential forms formulation of Maxwell equations

– Aïssata: Let us now follow Hehl and Obukhov[106], they have a nice way to introduce the equations which are relevant to electrodynamics. They first make the *obvious* observation that the total charge Q in a volume $\mathcal{V}$ is a scalar quantity which can be written as the space integral of a density ρ,

$$Q = \int_{\mathcal{V}} \rho\, d^3\mathrm{Vol}\,. \tag{858}$$

This is suggesting to call this latter density a 3-form (in the ordinary three dimensional space $\mathbb{R}^3$), and denote it as $\overset{3}{\rho}$ such that

$$Q = \int_{\mathcal{V}} \overset{3}{\rho}. \tag{859}$$

It follows that in $\mathbb{R}^3$

$$d\overset{3}{\rho} = 0 \tag{860}$$

[106] F.W. Hehl and Y.N. Obukhov, Foundations of classical electrodynamics: Charge, flux, and metric, Birhäuser, Boston, 2003.

which requires that $\overset{3}{\rho}$ being, *at least locally*, the exterior derivative of a 2-form denoted as $\overset{2}{D}$,

$$d\overset{2}{D} = \overset{3}{\rho}. \tag{861}$$

This is just Maxwell-Gauss equations and justifies the notation for $\overset{2}{D}$, the *electric displacement*. One can also consider the time variation of Q. The derivative $\frac{dQ}{dt}$ is non vanishing if charges can enter or leave the volume $\mathcal{V}$, hence there is a well known conservation property

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, d^3\mathrm{Vol} = -\oint_{\mathcal{S}} \mathbf{j}\cdot d^2\mathbf{Surf} \tag{862}$$

where $\mathcal{S} = \partial\mathcal{V}$ is the boundary of the volume under consideration and $\mathbf{j}$ is a current density. Its surface integral (flux) through a surface Σ is the (oriented) current intensity I (a scalar quantity) crossing the surface, from where one deduces that a 2-form $\overset{2}{j}$ is defined via

$$I = \int_\Sigma \overset{2}{j}. \tag{863}$$

Using Stokes theorem, the conservation equation (862) becomes

$$\int_{\mathcal{V}} \partial_t\overset{3}{\rho} + \int_{\partial\mathcal{V}} \overset{2}{j} = \int_{\mathcal{V}} (\partial_t\overset{3}{\rho} + d\overset{2}{j}) = 0 \tag{864}$$

and since $\mathcal{V}$ is arbitrary,

$$\partial_t\overset{3}{\rho} + d\overset{2}{j} = 0 \tag{865}$$

which is the conservation of electric charge written in differential forms. Combining this equation with Gauss law (861) one observes that $\overset{2}{j} + \partial_t\overset{2}{D}$ is an exact 2-form,

$$d(\overset{2}{j} + \partial_t\overset{2}{D}) = 0, \tag{866}$$

which demands the introduction of a 1-form as a potentiel,

$$d\overset{1}{H} = \overset{2}{j} + \partial_t\overset{2}{D}. \tag{867}$$

This is Maxwell-Ampère law with the notation $\overset{1}{H}$ for the auxiliary magnetic field or *magnetic excitation*.

The two remaining Maxwell equations can be obtained along similar lines of reasoning. The magnetic flux Φ and the potential difference V are both scalar quantities defined e.g. as

$$\Phi = \int_\Sigma \mathbf{B}\cdot d^2\mathbf{Surf}, \tag{868}$$

$$V = \int_{\mathcal{L}} \mathbf{E}\cdot d\mathbf{Line}\,. \tag{869}$$

This suggests the introduction of a 2-form $\overset{2}{B}$ and a 1-form $\overset{1}{E}$ such that the above scalars are

$$\Phi = \int_{\Sigma} \overset{2}{B}, \tag{870}$$

$$V = \int_{\mathcal{L}} \overset{1}{E}. \tag{871}$$

One can also assume that there exists a linear flux current density $\mathbf{j}^{\Phi}$ which can describe the variation of magnetic flux across a surface,

$$\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot d^2\mathbf{Surf} = -\oint_{\mathcal{L}=\partial\mathcal{S}} \mathbf{j}^{\Phi}\cdot d\mathbf{Line}, \tag{872}$$

or, in terms of differential forms,

$$\int_{\mathcal{S}} \partial_t \overset{2}{B} + \int_{\partial\mathcal{S}} \overset{1}{j}{}^{\Phi} = \int_{\mathcal{S}} (\partial_t \overset{2}{B} + d\overset{1}{j}{}^{\Phi}) = 0. \tag{873}$$

The linear flux current density is a 1-form, like the electric field and you can check that it also has the dimensions of the electric field. Given that it has the same geometrical and dimensional properties, it is a natural assumption to consider the two quantities as equal, up to a possible dimensionless coefficient. We can then deduce that $\partial_t \overset{2}{B} + \alpha d\overset{1}{E} = 0$, that we recognize as Faraday induction law, provided that $\alpha = 1$, in agreement with experiments. It yields Maxwell-Faraday law

$$\partial_t \overset{2}{B} + d\overset{1}{E} = 0. \tag{874}$$

Applying d to this latter equation and using Poincaré lemma, we get $\partial_t(d\overset{2}{B}) = 0$. Upon time integration, the integration constant must vanish, otherwise there would exist magnetic monopoles which would compromise the identification of $\overset{1}{j}{}^{\Phi}$ with $\overset{1}{E}$, but there is no experimental evidence of such monopoles up to now. The last Maxwell equation (also called Maxwell-flux) follows:

$$d\overset{2}{B} = 0. \tag{875}$$

An important comment is that equations (861), (867), (874) et (875) are quite general. They were obtained on the basis of conservation equations and of geometrical arguments in a language which does not depend on coordinates. Their form is true in any 3-dimensional manifold, even non Euclidean manifolds. We say that these are *topological* expressions.

Up to this point, this topological formalism is called *pre-metric*, since no metric assumption has been done and there are 4 fundamental fields, described by 1-forms $\overset{1}{H}$ and $\overset{1}{E}$, and by 2-forms $\overset{2}{D}$ and $\overset{2}{B}$. These 4 fundamental fields are possibly independent of each other, but relations among them appear as soon as one introduces a medium and its *constitutive relations*. For example in a linear material, one has

$$\mathbf{D} = \varepsilon\mathbf{E}, \tag{876}$$
$$\mathbf{H} = \mu^{-1}\mathbf{B}. \tag{877}$$

with possibly a tensor extension. In the vacuum, these expressions are valid with the vacuum permittivity and permeability. Nevertheless, these two equations are not correct from the differential forms point of view, because they suppose proportionality between forms of different ranks. One has to translate these relations into

$$\overset{2}{D} = \varepsilon(\star_3\overset{1}{E}), \tag{878}$$
$$\overset{1}{H} = \mu^{-1}(\star_3\overset{2}{B}), \tag{879}$$

where the Hodge product is required to repair the rank mismatch. Since the Hodge product is involved, the relations (878) and (879) require the existence of a metric structure on the manifold. From this point of view, the vacuum is such a medium. In a certain sense, electrodynamics requires a metric structure, which is maybe not surprising, since one needs an interaction to be able to perform length and time measurements.

– Diego: This is a very instructive formalism Aïssata. But there should be a relativistic counterpart I suppose, with the central role given to the Faraday tensor, which is antisymmetric, then almost ready for a differential form description.

– Aïssata: This is correct. In Minkowski spacetime, you have to consider t as a fourth coordinate on the manifold. The Faraday 2-form is defined as[107]

$$\overset{2}{F} = -\overset{1}{E} \wedge dt - \overset{2}{B} \tag{880}$$

To get that expression, you have to remember that the E_i's and B_i's are the components of vectors in $\mathbb{E}^3$, hence $E_i = \delta_{ij}E^j$ and the same for B_i, without minus sign! Collecting the components of $\overset{2}{F}$ in a matrix, $(\overset{2}{F})_{\alpha\beta}$, produces the same matrix than the cotensor $(F_{\alpha\beta})$ in equation (712)

$$(\overset{2}{F})_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{881}$$

and it is instructive to consider its exterior derivative.

The exterior derivative d_4 takes the fourth component into account, e.g. acting on a zero form

$$d_4\phi = \partial_i\phi dx^i + \partial_t\phi dt = d_3\phi + \partial_t\phi dt \tag{882}$$

with the summation over the three space coordinates.

If we want to make it carefully, it is probably better to restore the dimensions, since it facilitates some of the identifications. The Faraday 2-form (880) takes the form

$$\overset{2}{F} = -\frac{\overset{1}{E}}{c} \wedge c dt - \overset{2}{B}. \tag{883}$$

[107] Here the signs depend, as usual, on the metric signature.

►**EXERCISE 33 – Calculation of $d_4\overset{2}{F}$ –**

– Aïssata: The first thing is to calculate $d_4\overset{2}{F}$. For that purpose, we decompose d_4 into $d_3 + \partial_t \wedge dt$:

$$d_4\overset{2}{F} = -\frac{1}{c}d_4(\overset{1}{E}\wedge cdt) - d_4\overset{2}{B}, \quad (884)$$

$$d_4(\overset{1}{E}\wedge cdt) = (d_3\overset{1}{E})\wedge cdt, \quad (885)$$

$$d_4\overset{2}{B} = d_3\overset{2}{B} + \frac{1}{c}(\partial_t\overset{2}{B})\wedge cdt. \quad (886)$$

The analysis of the three terms at the r.h.s. shows that $d_4(\overset{1}{E}\wedge cdt)$ contains terms of the form $dx^0 \wedge dx^i \wedge dx^j$, like $(\partial_t\overset{2}{B})\wedge cdt$, while $d_3\overset{2}{B}$ has purely space terms in $dx^i \wedge dx^j \wedge dx^k$, which suggests to rewrite

$$d_4\overset{2}{F} = -\frac{1}{c}(d_3\overset{1}{E} + \partial_t\overset{2}{B})\wedge cdt - d_3\overset{2}{B} \equiv 0 \quad (887)$$

because equations (874) and (875) are satisfied. This single equation comprises the two homogeneous Maxwell equations. ◄

►**EXERCISE 34 – Calculation of $\star_4\overset{2}{F}$ –**

– Aïssata: Like in the $3D$ version, we can infer that a specific role is played by the Hodge star operator acting on $\overset{2}{F}$. It is therefore useful to calculate first

$$\begin{aligned}\frac{\overset{1}{E}}{c}\wedge cdt &= \frac{E_x}{c}dx\wedge cdt + \frac{E_y}{c}dy\wedge cdt + \frac{E_z}{c}dz\wedge cdt\\ &= \Big(\frac{\overset{1}{E}}{c}\wedge cdt\Big)_{01}dx^0\wedge dx^1 + \Big(\frac{\overset{1}{E}}{c}\wedge cdt\Big)_{02}dx^0\wedge dx^2 + \Big(\frac{\overset{1}{E}}{c}\wedge cdt\Big)_{03}dx^0\wedge dx^3\end{aligned} \quad (888)$$

from where we deduce that $(\frac{\overset{1}{E}}{c}\wedge cdt)_{01} \equiv -E_x/c$, hence $(\frac{\overset{1}{E}}{c}\wedge cdt)^{01} \equiv +E_x/c$. The $4D$ Hodge star applied to $\frac{\overset{1}{E}}{c}\wedge cdt$ now leads to

$$\begin{aligned}\star_4\frac{\overset{1}{E}}{c}\wedge cdt &= \Big(\frac{\overset{1}{E}}{c}\wedge cdt\Big)^{01}\epsilon_{0123}\,dx^2\wedge dx^3 + \Big(\frac{\overset{1}{E}}{c}\wedge cdt\Big)^{02}\epsilon_{0231}\,dx^3\wedge dx^1\\ &\quad + \Big(\frac{\overset{1}{E}}{c}\wedge cdt\Big)^{03}\epsilon_{0312}\,dx^1\wedge dx^2\\ &= \frac{E_x}{c}dy\wedge dz + \frac{E_y}{c}dz\wedge dx + \frac{E_z}{c}dx\wedge dy.\end{aligned} \quad (889)$$

Applied to $\overset{2}{B}$ it yields directly

$$\begin{aligned}\star_4\overset{2}{B} &= B^{12}\epsilon_{1230}\,dx^3\wedge dx^0 + B^{23}\epsilon_{2301}\,dx^0\wedge dx^1 + B^{31}\epsilon_{3120}\,dx^2\wedge dx^0\\ &= B_z cdt\wedge dz + B_x cdt\wedge dx + B_y cdt\wedge dy.\end{aligned} \quad (890)$$

We can now write $\star_4\overset{2}{F}$ in matrix form,

$$(\star_4\overset{2}{F})_{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z\\ B_x & 0 & -E_z/c & E_y/c\\ B_y & E_z/c & 0 & -E_x/c\\ B_z & -E_y/c & E_x/c & 0\end{pmatrix} \quad (891)$$

which corresponds to the covariant form (with c factors where appropriate) of equation (735), and thus suggests to denote $\star_4 \overset{2}{F} \equiv \overset{2}{\mathscr{F}}$ i.e. the tensor dual or the Hodge dual describe the same object (the contrary would cause many troubles!).

An interesting observation is that

$$\star_4 \left(\frac{\overset{1}{E}}{c} \wedge cdt\right) = \star_3 \frac{\overset{1}{E}}{c} \quad \text{and} \quad \star_4 \overset{2}{B} = -(\star_3 \overset{2}{B}) \wedge cdt \tag{892}$$

because $\star_3 \overset{1}{E} = E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy$ and $\star_3 \overset{2}{B} = B_z dz + B_x dx + B_y dy$ where we use $E_i = \delta_{ij} E^j$, $B_k = \epsilon_{ijk} B^{ij}$ and $B_{ij} = \delta_{il}\delta_{jm} B^{lm}$ in $\mathbb{E}^3$. Using the $3D$ constitutive equations (878)and (878), we can deduce the important results

$$\star_4 \left(\frac{\overset{1}{E}}{c} \wedge cdt\right) = \frac{1}{\varepsilon_0 c}\overset{2}{D} \quad \text{and} \quad \star_4 \overset{2}{B} = -\mu_0 \overset{1}{H} \wedge cdt \tag{893}$$

from where it follows that

$$\star_4 \overset{2}{F} = -\frac{1}{\varepsilon_0 c}\overset{2}{D} + \mu_0 \overset{1}{H} \wedge cdt \tag{894}$$

We obtain this result using (878) and (879), but as we have argued earlier, at the pre-metric level, before the introduction of the constitutive relations, $\overset{2}{D}$ and $\overset{1}{H}$ were considered as quantities not depending on $\overset{1}{E}$ and $\overset{2}{B}$. If we want to preserve such an independence, we define the r.h.s. of equation (894) as a new 2-form, called $\overset{2}{G}$,

$$\overset{2}{G} = -c\overset{2}{D} + \overset{1}{H} \wedge cdt \tag{895}$$

and then only, we close the system of equation with the supplementary condition

$$\overset{2}{G} = \mu_0^{-1} \star_4 \overset{2}{F}. \tag{896}$$

The matrix expression of (895) in terms of components of the electric displacement et and the magnetic excitation is therefore

$$(\overset{2}{G})_{\alpha\beta} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & -cD_z & cD_y \\ H_y & cD_z & 0 & -cD_x \\ H_z & -cD_y & cD_x & 0 \end{pmatrix}. \tag{897}$$

– Diego: I think that this is not an extraordinary property, because inserting the components of the vector form of the constitutive equations, (876) and (877) in the vacuum with ε_0 and μ_0 for the dielectric permittivity and magnetic permeability inside (891) automatically leads to (897). ◄

– Aïssata: This is correct Diego, but I think that equation (897) in a sense is more general than equation (891) and its form gives the intuition that this is $\overset{2}{G}$ rather than $\overset{2}{F}$ which is associated, in exterior calculus, to the charged sources. It also has an interesting consequence in the fact that it shows that $\mathbf{H}$ has the same Lorentz transformation properties than $\mathbf{E}$, and $\mathbf{D}$ has those of $\mathbf{B}$, which is another argument against equations (876) and (877).

Next, we want to act on $\overset{2}{G}$ with the $4D$ exterior derivative:

$$d_4\overset{2}{G} \quad = \quad -cd_4\overset{2}{D} + d_4\overset{1}{H} \wedge cdt \tag{898}$$

where $d_4\overset{2}{D} = d_3\overset{2}{D} + \frac{1}{c}\partial_t\overset{2}{D} \wedge cdt = \overset{3}{\rho} + \partial_t\overset{2}{D} \wedge dt$ and $d_4(\overset{1}{H} \wedge cdt) = (d_3\overset{1}{H}) \wedge cdt = (\overset{2}{j} + \partial_t\overset{2}{D}) \wedge cdt$. As a consequence, after simplification, one has

$$d_4\overset{2}{G} = -\overset{3}{\rho}c + \overset{2}{j} \wedge cdt. \tag{899}$$

On can thus define a current 3-form

$$\overset{3}{\mathscr{J}} = -\overset{3}{\rho}c + \overset{2}{j} \wedge cdt \tag{900}$$

which gives the inhomogeneous Maxwell equations the compact form

$$d_4\overset{2}{G} = \overset{3}{\mathscr{J}} \tag{901}$$

and, when we take the constitutive relation into account,

$$d_4 \star_4 \overset{2}{F} = \mu_0 \overset{3}{\mathscr{J}}. \tag{902}$$

You can notice, firstly, that like in the $3D$ formalism if you want to close the system of equations, the metric structure is required via the Hodge star as soon as sources are present and you have to work with the inhomogeneous equations. Secondly the current 3-form is automatically conserved, since $d_4(d_4\overset{2}{G}) = 0$, hence

$$d_4\overset{3}{\mathscr{J}} = 0. \tag{903}$$

18. Day 14 – Lagrangian formulation for electromagnetic fields

□ 18.1 Lagrangian density of the electromagnetic field

A simple and natural candidate for the description of the electromagnetic field is the 4-vector A^α. Looking then for a free field Lagrangian density in terms of a quadratic expression in the field's derivatives leads to consider first as a candidate the quantity $\frac{1}{2}\partial_\alpha A_\beta \partial^\alpha A^\beta$ sometimes referred to as Fermi electrodynamics. Nevertheless, there is an obstruction, since this is not a gauge invariant object.

The two invariants (760) and (761) are thus obvious candidates to build the Lagrangian density of the free electromagnetic field. The second invariant, $\mathscr{F}^{\alpha\beta}F_{\alpha\beta}$, is nevertheless not acceptable due to symmetry properties which are not those of electrodynamics. Indeed, $\mathbf{E}\cdot\mathbf{B}$ does not exhibit time reversal symmetry for example, while all experiments support such symmetry in electrodynamics. We therefore assume the following Lagrangian density

$$\mathscr{L}_{\mathrm{EM}} \propto F_{\alpha\beta}F^{\alpha\beta}. \tag{904}$$

At this point, nothing can constrain a possible coefficient in front of this expression. The equation of motion takes the simplified form

$$\partial_\alpha \frac{\partial \mathscr{L}_{\text{EM}}}{\partial(\partial_\alpha A_\beta)} = 2\partial_\alpha \left(\frac{\partial F_{\gamma\delta}}{\partial(\partial_\alpha A_\beta)} F^{\gamma\delta} \right) = 4\partial_\alpha F^{\alpha\beta} = 0. \tag{905}$$

This is the correct *source-free* inhomogeneous Maxwell equation.

The introduction of sources naturally couples the 4-current j^α to the 4-potential A^α, and the simplest term to consider is $A_\alpha j^\alpha$, such that

$$\mathscr{L}_{\text{EM}} = -\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - A_\alpha j^\alpha, \tag{906}$$

and the corresponding action,

$$S_{\text{EM}}(A, \partial A) = -\int d^4x \left(\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + A_\alpha j^\alpha \right), \tag{907}$$

where the numerical prefactors are chosen in order to be consistent with the known Maxwell equations. We could indeed restore the dimensional constants and write the $3D$ version (with our signature convention $+---$), known as the Schwarzschild Lagrangian,

$$\mathscr{L}_{\text{Schw.}} = \tfrac{1}{2}\varepsilon_0 |\mathbf{E}(\mathbf{r},t)|^2 - \tfrac{1}{2\mu_0} |\mathbf{B}(\mathbf{r},t)|^2 - \rho\phi(\mathbf{r},t) + \mathbf{j}(\mathbf{r},t)\mathbf{A}(\mathbf{r},t). \tag{908}$$

This is the justification for the minus signs in (906) which otherwise is purely optional. With (907), Lagrange equations now read as

$$\frac{\partial \mathscr{L}_{\text{EM}}}{\partial A_\beta} - \partial_\alpha \frac{\partial \mathscr{L}_{\text{EM}}}{\partial(\partial_\alpha A_\beta)} = -j^\beta + \partial_\alpha F^{\alpha\beta} = 0. \tag{909}$$

– Diego: I didn't understand the argument on gauge invariance. What happens with a Lagrangian like $\mathscr{L}_F = \frac{1}{2}\partial_\alpha A_\beta \partial^\alpha A^\beta$, and why is it forbidden?

– Aïssata: The first part of your question is something easy. If you perform the gauge transformation $A_\alpha \to A_\alpha - \partial_\alpha\theta$, the Lagrangian candidate gets modified according to $\mathscr{L}_F \to \mathscr{L}_F + \frac{1}{2}(\partial_\alpha A_\beta \partial^\alpha\partial^\beta\theta + \partial^\alpha A^\beta \partial_\alpha\partial_\beta\theta) + \frac{1}{4}\partial_\alpha\partial_\beta\theta\partial^\alpha\partial^\beta\theta$ and it doesn't remain unchanged. In which concerns the second part of your question, I can give some arguments which should become clear later. Gauge invariance probably first appeared as a pleasant mathematical property before it became a demand that a theory has to satisfy. Although there is no definitive argument in favor of such a gauge invariance requirement, as far as I know, the consequences are just great, since this is a guide for the construction of theories of the fundamental interactions of the Standard Model. You will probably elaborate more on that later during your courses. There are also questions of renormalizability of gauge theories when they are quantized, but this is far beyond our current concerns.

►EXERCISE 35 – **Free electromagnetic field equation of motion** –

– Diego: Now there is a technical question. In the derivation of (905), although I see that this is essentially the derivative of the square of F, hence a factor of 2 appearing, one of the F's has covariant indices and the other has contravariant ones and this doesn't look very rigorous.

– Aïssata: You are right, but this is easy to make it more precise. Just write $F_{\alpha\beta}F^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}$

and then make the derivative as usual,

$$\begin{aligned}
\partial_\zeta\left(\frac{F_{\alpha\beta}F^{\alpha\beta}}{\partial(\partial_\zeta A_\eta)}\right) &= \partial_\zeta\left[g^{\alpha\gamma}g^{\beta\delta}[(\delta^\zeta_\alpha\delta^\eta_\beta-\delta^\eta_\alpha\delta^\zeta_\beta)F_{\gamma\delta}+F_{\alpha\beta}(\delta^\zeta_\gamma\delta^\eta_\delta-\delta^\eta_\gamma\delta^\zeta_\delta)]\right] \\
&= \partial_\zeta\left[(g^{\zeta\gamma}g^{\eta\delta}-g^{\eta\gamma}g^{\zeta\delta})F_{\gamma\delta}+F_{\alpha\beta}(g^{\alpha\zeta}g^{\beta\eta}-g^{\alpha\eta}g^{\beta\zeta})\right] \\
&= \partial_\zeta(F^{\zeta\eta}-F^{\eta\zeta}+F^{\zeta\eta}-F^{\eta\zeta}) \\
&= 4F^{\zeta\eta}. \qquad (910)
\end{aligned}$$

And this is done! ◄

▷ 18.2 Coupling between scalar fields and electrodynamics

– Aïssata: An interesting, and important question, is that of the coupling between the complex scalar field and electrodynamics. This is the reverse question than the coupling of the EM field to matter charge current as it was addressed in your course in equation (906). What I propose here is a *preliminary* approach, which will be completed (and corrected) later when we will address gauge theory.

We start from the free field

$$\mathscr{L}_{\rm KG}(\varphi,\partial\varphi)=\eta^{\alpha\beta}\partial_\alpha\varphi^*\partial_\beta\varphi-m_\varphi^2\varphi^*\varphi \qquad (911)$$

and just add the coupling to the 4-potential,

$$\mathscr{L}(\varphi,\partial\varphi,A)=\eta^{\alpha\beta}\partial_\alpha\varphi^*\partial_\beta\varphi-m_\varphi^2\varphi^*\varphi-A_\alpha j^\alpha. \qquad (912)$$

The sign of the coupling term is compatible with $A_\alpha j^\alpha=\rho\phi-\mathbf{j}\cdot\mathbf{A}$. The current density associated to the KG field is obtained here from the free field equation of motion

$$\partial_\alpha\partial^\alpha\varphi+m_\varphi^2\varphi=0 \qquad (913)$$

and its complex conjugate. The two equations show that the combination $\varphi^*\partial^\alpha\varphi-(\partial^\alpha\varphi^*)\varphi$ has a vanishing 4-divergence. It is therefore natural to call current density the quantity

$$j^\alpha=ie(\varphi^*\partial^\alpha\varphi-(\partial^\alpha\varphi^*)\varphi) \qquad (914)$$

where i makes it a real quantity (by virtue of the hermiticity of $i\partial_\alpha$) and e is a constant that we call the electric charge. We thus have the conservation equation

$$\partial_\alpha j^\alpha=0. \qquad (915)$$

– Diego: Playing with the equation of motion, this is the same method that we have applied to Schrödinger equation earlier (see p. 77).

– Aïssata: Right! Now, expanding the lagrangian (912), one obtains

$$\mathscr{L}(\varphi,\partial\varphi,A)=\partial_\alpha\varphi^*\partial^\alpha\varphi-m_\varphi^2\varphi^*\varphi+ieA^\alpha((\partial_\alpha\varphi^*)\varphi-\varphi^*\partial_\alpha\varphi), \qquad (916)$$

where you can see that the last term corresponds to the interaction, the matter field φ being now coupled to the 4-potential which describes the electromagnetic field.

The equation of motion obviously gets modified w.r.t. (913) and the Euler-Lagrange equation applied to (916) leads to

$$\partial_\alpha \partial^\alpha \varphi + m_\varphi^2 \varphi + 2ieA^\alpha \partial_\alpha \varphi + ie(\partial_\alpha A^\alpha)\varphi = 0 \tag{917}$$

and this is usually simplified using Lorenz gauge $\partial_\alpha A^\alpha = 0$ which cancels the last term.

– Diego: Is it the obvious way to add the coupling to electrodynamics? I mean, in QM, when we write Schrödinger equation in an EM field, there are three terms which appear and describe different interaction processes as we have seen earlier. First, there are two linear terms, one in ϕ and one in $\mathbf{A}$, the scalar and vector potentials. The term linear in $\mathbf{A}$ in particular is responsible for paramagnetic phenomena in the atom or in matter. There is then a third term which is quadratic in $\mathbf{A}$ and corresponds to diamagnetism of atoms. I have the idea that this is not what we get here.

– Aïssata: You are right Diego. The approach used here is empirical and will have to be completed later as I said, via gauge theory. In (916) we miss the quadratic term, this is correct. There is also another subtle point which I cannot explain here, this is the fact that the conserved current *is modified* by the presence of the EM field. You will study this when you will revisit Noether theorem in field theory (see also p. 77).

▷ 18.3 The relationship between the variational and the differential forms formulations of electrodynamics

– Diego: There is something which looks strange when I try to connect the tensor presentation with that which makes use of differential forms, although both formulations fundamentally deal with the same types of objects. Let me try to formulate my concern.

First, there is equation (887), $d\overset{2}{F} = 0$ which looks pretty much like $\partial_\alpha \mathscr{F}^{\alpha\beta} = 0$. Then (902), $d \star \overset{2}{F} = \overset{3}{\mathscr{J}}$ seems to correspond to $\partial_\alpha F^{\alpha\beta} = j^\beta$. The correspondence then seems to be between $\overset{2}{F}$ and $\mathscr{F}^{\alpha\beta}$ and between $\star\overset{2}{F}$ and $F^{\alpha\beta}$. I understand that the dual of the dual is the starting object (possibly up to a sign depending on the manifold dimension and signature), but this looks like if one had misplaced who is the tensor and who the dual?

– Aïssata: You raise a delicate point Diego, but as you see with the matrix representation, this doesn't work. For example the matrices in (712) and (881) coincide. But let me reformulate along my own understanding. The fundamental ingredient is probably the definition of the electromagnetic curvature[108] $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ (electromagnetic field strength, or Faraday tensor (710), which determines the forces acting on charged particles), and this definition automatically implies (724), $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$, which is called the *Bianchi identity*.

There is even a very elegant way to present electrodynamics from topological arguments only (almost). You just postulate primarily the existence of two "conserved forms", a 2-form $\overset{2}{F}$ and a 3-form $\overset{3}{\mathscr{J}}$ which obey

$$d\overset{2}{F} = 0 \quad \text{and} \quad d\overset{3}{\mathscr{J}} = 0, \tag{918}$$

[108]This terminology will be clarifed with Gauge theories.

which then demand, at least locally, the existence of a 1-form $\overset{1}{A}$ and of a 2-form $\overset{2}{G}$ such that

$$\overset{2}{F} = d\overset{1}{A} \quad \text{and} \quad \overset{3}{\mathscr{J}} = d\overset{2}{G}. \tag{919}$$

Of course, the constitutive equation $\overset{2}{G} = \star\overset{2}{F}$ is still a supplementary postulate needed to close the system of equations. Nevertheless, this line of reasoning again places Bianchi identity at the level of the foundations of the theory. Equations (710) and (724) are thus probably precursory to anything else. The action in particular comes later.

In my opinion, $d\overset{2}{F} = 0$ then implies firstly $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$ rather than $\partial_\alpha \mathscr{F}^{\alpha\beta} = 0$ as you suspect, although of course you have seen in your lecture notes that both identities, when expanded, lead to the same constraints. This is because $d\overset{2}{F} = 0$ is a third-rank tensor equation while $\partial_\alpha \mathscr{F}^{\alpha\beta} = 0$ is a first-rank tensor. Let me show you my lines of reasoning. Define the Faraday 2-form

$$\overset{2}{F} = \tfrac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta, \tag{920}$$

then, the 3-form $d_4\overset{2}{F}$ reads as

$$\begin{aligned} d_4\overset{2}{F} &= \tfrac{1}{3!}(d_4\overset{2}{F})_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \\ &= \tfrac{1}{2!}\partial_\gamma F_{\alpha\beta}\, dx^\gamma \wedge dx^\alpha \wedge dx^\beta \\ &= \tfrac{1}{3!}(\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta})\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma. \end{aligned} \tag{921}$$

The identity $d_4\overset{2}{F} = 0$ thus leads to the tensor form $(d_4\overset{2}{F})_{\alpha\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$. The correspondence in terms of matrices (712) and (881) as we said comes in support to this claim, since it is simply $(\overset{2}{F})_{\alpha\beta} = (F_{\alpha\beta})$, and thus to the identification between $\overset{2}{F}$ and $F_{\alpha\beta}$ rather than your hypothesis which associates $\overset{2}{F}$ to $\mathscr{F}_{\alpha\beta}$. The introduction of the dual components is second, and allows to write the same equation in a more compact form as we know.

In component form, the Faraday tensor is given, say by definition, in terms of components of the electric and magnetic fields E^i and B^i's, and if I restore the dimensional constants, we have[109]

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}, \quad (F^{\alpha\beta}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{922}$$

We have, *only then*, argued that this is economic to define a dual tensor $\mathscr{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ with components

$$(\mathscr{F}_{\alpha\beta}) = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z/c & E_y/c \\ B_y & E_z/c & 0 & -E_x/c \\ B_z & -E_y/c & E_x/c & 0 \end{pmatrix}, \quad (\mathscr{F}^{\alpha\beta}) = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}. \tag{923}$$

[109] Remember that we have $(\eta) = \mathrm{diag}(+,-,-,-)$.

such that $\partial_\alpha \mathscr{F}^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha F_{\gamma\delta} = 0$ is an alternative form of the Bianchi identity. It is clear that E^i/c and B^i are essentially interchanged[110] between $F_{\alpha\beta}$ and $\mathscr{F}_{\alpha\beta}$ and that $\mathscr{F}_{\alpha\beta}$ coincides with $(\star \overset{2}{F})_{\alpha\beta}$ in (891), hence the denomination *dual* for $\mathscr{F}$.

Now, we introduce a new piece with interactions via an action which is built on Lorentz invariants, themselves made from the 4-potential. The Faraday tensor again is a good guide for that, this is why equation (906), namely with a Lagrangian density (again with dimensions restored) $\mathscr{L}_{\text{EM}} = -\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} - A_\alpha j^\alpha$ is the natural candidate. The external sources are given by

$$j^\beta = -\frac{\partial \mathscr{L}_{\text{EM}}}{\partial A_\beta} \tag{924}$$

and the form of the equation of motion (Euler-Lagrange equations) therefore suggests the introduction of a second-rank tensor

$$G^{\alpha\beta} = -\frac{\partial \mathscr{L}_{\text{EM}}}{\partial(\partial_\alpha A_\beta)} \tag{925}$$

for $G^{\alpha\beta}$ to satisfy

$$\partial_\alpha G^{\alpha\beta} = j^\beta. \tag{926}$$

This tensor is called Maxwell tensor, or simply conjugate electromagnetic tensor, and its components are given, *by definition*, in terms of those of the electric displacement, D^i, and the magnetic excitation, H^i.

$$(G_{\alpha\beta}) = \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & -H_z & H_y \\ -cD_y & H_z & 0 & -H_x \\ -cD_z & -H_y & H_x & 0 \end{pmatrix}, \quad (G^{\alpha\beta}) = \begin{pmatrix} 0 & -cD_x & -cD_y & -cD_z \\ cD_x & 0 & -H_z & H_y \\ cD_y & H_z & 0 & -H_x \\ cD_z & -H_y & H_x & 0 \end{pmatrix}. \tag{927}$$

In terms of vector components, the equations of motion take the form

$$\nabla \cdot \mathbf{D} = \rho, \tag{928}$$
$$\nabla \times \mathbf{H} - \partial_t \mathbf{D} = \mathbf{j}. \tag{929}$$

The link between the conjugate tensor and $\overset{2}{G}$ is given by the same procedure that we have followed with Bianchi identity, i.e. introducing a dual, and the reason is the same, $d_4\overset{2}{G} = \overset{3}{\mathscr{J}}$ is a third-rank tensor equation. We define the tensorial dual of $\overset{2}{G}$ as

$$G^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\mathscr{G}_{\gamma\delta} \tag{930}$$

with the covariant and contravariant matrix forms[111]

$$(-\mathscr{G}_{\alpha\beta}) = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & -cD_z & cD_y \\ H_y & cD_z & 0 & -cD_x \\ H_z & -cD_y & cD_x & 0 \end{pmatrix}, \quad (-\mathscr{G}^{\alpha\beta}) = \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z/c & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{pmatrix}. \tag{931}$$

This identifies $(\overset{2}{G})_{\gamma\delta}$ to $(-\mathscr{G}_{\gamma\delta})$, rather than your hypothesis between $(\overset{2}{G})_{\gamma\delta}$ and $(F_{\gamma\delta})$, but still, the real meaning of that correspondence is missing. The explanation comes when

[110] i.e. the *duality* transformation $E^i/c \to B^i$ and $B^i \to -E^i/c$ performs $F_{\alpha\beta} \to \mathscr{F}_{\alpha\beta}$.
[111] Remember that $\epsilon^{0123} = -1$.

one writes the equation of motion (926) as a third-rank tensor expression. Inserting (930) into the equation of motion leads, to[112]

$$\partial_\alpha \mathscr{G}_{\beta\gamma} + \partial_\beta \mathscr{G}_{\gamma\alpha} + \partial_\gamma \mathscr{G}_{\alpha\beta} = \frac{1}{3!}\epsilon_{\delta\alpha\beta\gamma} j^\delta, \tag{932}$$

where the dual current makes the link between the sources in the two formulations,

$$\mathscr{J}_{\alpha\beta\gamma} = \frac{1}{3!}\epsilon_{\delta\alpha\beta\gamma} j^\delta. \tag{933}$$

Applied to the Schwarzschild Lagrangian $\mathscr{L}_{\mathrm{EM}}$, the definition of the Maxwell tensor leads to its relation with the Faraday tensor

$$G^{\alpha\beta} = \frac{1}{\mu_0} F^{\alpha\beta} = \varepsilon_0 c^2 F^{\alpha\beta}, \tag{934}$$

or, in vector notation,

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{935}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0}. \tag{936}$$

There is obviously a redundancy in the vacuum between the Faraday tensor and the excitation tensor, or between **E** and **D** on one hand and **B** and **H** on the other hand and it seems pretty artificial to introduce two names for a single object, but this is strongly connected to the structure of *Maxwell electrodynamics*. Situations in which the different roles of the two tensors will appear clearly will be discussed e.g. in section **??**.

▷ 18.4 Variational formulation in exterior differential calculus

– Diego: OK, I accept your argument. But something else now, you mentioned the lagrangian formalism in tensor notations, but can we build a variational description in the language of differential forms?

– Aïssata: This is a pretty good question Diego, and in the recent years, more and more research papers and books use such a description. Since we are now used to the formalism, I get rid of the $\star_4$, d_4, for shorter notations, $\star$, d, respectively, and we stay in $\mathbb{M}^4$ with obvious extensions to arbitrary pseudo-Riemannian manifolds.

As we have seen many times now, the free field action in tensor form is given by $S_0[A] = -\frac{1}{4}\int d^4x F_{\alpha\beta}F^{\alpha\beta}$ and you can convince yourself that it corresponds to

$$S_0[A] = \int -\tfrac{1}{2}\overset{2}{F} \wedge \star\overset{2}{F} \tag{937}$$

$$= \int \left(-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{938}$$

►**Exercise 36 – Computation of $\frac{1}{2}\overset{2}{F} \wedge \star\overset{2}{F}$ –**

[112] For example $\beta = 0$ leads to $\partial_1\mathscr{G}_{23} + \partial_2\mathscr{G}_{31} + \partial_3\mathscr{G}_{12} = j^0 = c\rho$ and $\beta = 1$ gives $-\partial_0\mathscr{G}_{23} - \partial_2\mathscr{G}_{30} - \partial_3\mathscr{G}_{02} = j^1 = j_x$.

– Diego: Let me try to do it. I remember that $\overset{2}{F}$ is a 2-form

$$\overset{2}{F} = \tfrac{1}{2} F_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}, \tag{939}$$

then according to the definition of the Hodge written in component form (821) one has

$$\star \overset{2}{F} = \tfrac{1}{4} \sqrt{-\eta}\, \varepsilon_{\alpha'\beta'\gamma'\delta'} F^{\alpha'\beta'} dx^{\gamma'} \wedge dx^{\delta'} \tag{940}$$

with $\eta = -1$ the determinant of the Minkowski metric tensor. Let us look for the terms in $F_{01}F^{01}$, $F_{01}F^{10}$, etc, which appear in $F \wedge \star F$. Due to the presence of $dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma'} \wedge dx^{\delta'}$, such terms can only appear if γ', δ' is 2, 3 or 3, 2. There are 8 corresponding terms:

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{1}{4}(\varepsilon_{0123} F_{01} F^{01}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&\quad + \varepsilon_{0123} F_{10} F^{01}\, dx^1 \wedge dx^0 \wedge dx^2 \wedge dx^3 \\
&\quad + \varepsilon_{1023} F_{01} F^{10}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&\quad + \varepsilon_{1023} F_{10} F^{10}\, dx^1 \wedge dx^0 \wedge dx^2 \wedge dx^3 \\
&\quad + \varepsilon_{0132} F_{01} F^{01}\, dx^0 \wedge dx^1 \wedge dx^3 \wedge dx^2 \\
&\quad + \varepsilon_{0132} F_{10} F^{01}\, dx^1 \wedge dx^0 \wedge dx^3 \wedge dx^2 \\
&\quad + \varepsilon_{1032} F_{01} F^{10}\, dx^0 \wedge dx^1 \wedge dx^3 \wedge dx^2 \\
&\quad + \varepsilon_{1032} F_{10} F^{10}\, dx^1 \wedge dx^0 \wedge dx^3 \wedge dx^2) \\
&\quad = F_{01} F^{01}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&\quad = \tfrac{1}{2}(F_{01} F^{01} + F_{10} F^{10})\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.
\end{aligned} \tag{941}$$

We can proceed along the same lines for the other combinations γ', δ' and get eventually[113]

$$-\tfrac{1}{2} \overset{2}{F} \wedge \star \overset{2}{F} = -\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{942}$$

which is equation (938), a special case of (841). ◄

– Aïssata: Of course, we also have to add the coupling to the sources, $-\int d^4x A_\alpha j^\alpha$. Here there is an important caveat. When you consider Maxwell equations in the form of (899) that, for the sake of simplicity of the discussion we repeat here,

$$d \star \overset{2}{F} \quad = \quad -\overset{3}{\rho} + \overset{2}{j} \wedge dt, \tag{943}$$

it seems clear that a 3-form is the correct description of the sources, hence the introduction of $\overset{3}{\mathscr{J}}$. This is indeed very often the choice made in the literature[114], with the exception

[113] For a reason that we don't fully understand, we have here a sign difference with H. Năstase, Classical Field Theory, Cambridge University Press, Cambridge, 2019, p.123 and with nLab authors, https://ncatlab.org/nlab/show/Hodge+star+operator, Revision 28, Feb. 2021, but we agree with M. Göckeler and T. Schücker, Differential geometry, gauge theories, and gravity, Cambridge University Press, Cambridge, 1987 p.46 and with R.A. Bertlmann, Anomalies in Quantum Field Theory, Oxford University Press, Oxford, 1996, p. 292. We have the same sign conventions for the Minkowski metric tensor, Levi-Civita symbol and the volume form than Bertlmann, but Năstase and nLab have the opposite metric signature, and Năstase also has the opposite sign convention for the Levi-Civita symbol and for the volume form.

[114] e.g. A. Altland and J. von Delft, Mathematics for Physicists, Cambridge University Press, Cambridge, 2019; T. Frenkel, The Geometry of Physics, Cambridge University Press, Cambridge, 2012.

of people[115] who use the variational formulation[116]. There, a current 1-form is usually preferred (we call it $\overset{1}{J}$), since then (841) suggests to use the coupling to the potential 1-form as a term proportional to the volume form,

$$\overset{1}{A} \wedge \star\overset{1}{J} = A_\alpha J^\alpha \, d\mathrm{Vol}, \tag{944}$$

and this is also conform to the presence of a single tensor index for the current in the action. Resuming these arguments, one can propose the coupling of the EM field to the sources via

$$\int -\overset{1}{A} \wedge \star\overset{1}{J} = \int (-A_\alpha J^\alpha)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{945}$$

but the identification of $\star\overset{1}{J}$ is still an important step.

►EXERCISE 37 – **The 1-form current and its dual** –

– Diego: I want to give the complete expression of the 1-form current and its dual. Let me define

$$\overset{1}{J} = \rho dt - j_i dx^i = \rho dx^0 - j_1 dx^1 - j_2 dx^2 - j_3 dx^3 = J_\alpha dx^\alpha \tag{946}$$

with $J_0 = \rho$, $J_i = -j_i = -J^i$. The dual form is defined according to

$$\begin{aligned}
\star\overset{1}{J} &= \frac{1}{1!}\frac{1}{3!} J^\alpha \, \epsilon_{\alpha\beta\gamma\delta} \, dx^\beta \wedge dx^\gamma \wedge dx^\delta \\
&= \frac{1}{6}\Big(J^0 \, \epsilon_{0123} \, dx^1 \wedge dx^2 \wedge dx^3 + \ldots \\
&\quad + J^1 \, \epsilon_{1230} \, dx^2 \wedge dx^3 \wedge dx^0 + \ldots \\
&\quad + J^2 \, \epsilon_{2301} \, dx^3 \wedge dx^0 \wedge dx^1 + \ldots \\
&\quad + J^3 \, \epsilon_{3012} \, dx^0 \wedge dx^1 \wedge dx^2 + \ldots \Big) \\
&= J^0 dx^1 \wedge dx^2 \wedge dx^3 - J^1 dx^2 \wedge dx^3 \wedge dx^0 \\
&\quad + J^2 dx^3 \wedge dx^0 \wedge dx^1 - J^3 dx^0 \wedge dx^1 \wedge dx^2 \\
&= \rho dx^1 \wedge dx^2 \wedge dx^3 - j_1 dx^2 \wedge dx^3 \wedge dx^0 \\
&\quad + j_2 dx^3 \wedge dx^0 \wedge dx^1 - j_3 dx^0 \wedge dx^1 \wedge dx^2.
\end{aligned} \tag{947}$$

Comparison of signs with equation (900),

$$\overset{3}{\rho} - \overset{2}{j} \wedge dt = \rho dx^1 \wedge dx^2 \wedge dx^3 - j_1 dx^2 \wedge dx^3 \wedge dx^0 - j_2 dx^3 \wedge dx^1 \wedge dx^0 - j_3 dx^1 \wedge dx^2 \wedge dx^0, \tag{948}$$

magically works perfectly when we put the dx^α's in the correct order and we conclude that

$$\star\overset{1}{J} = \overset{3}{\rho} - \overset{2}{j} \wedge dt = -\overset{3}{\mathscr{J}}. \tag{949}$$

This is the exterior calculus version of equation (933). We can thus anticipate that the inhomogeneous Maxwell equation which will follow from the variational formalism will be written as $d \star \overset{2}{F} = - \star \overset{1}{J}$. ◄

[115] e.g. R.A. Bertlmann, Anomalies in Quantum Field Theory, Oxford University Press, Oxford, 1996; J. Baez and J.P. Muniain, Gauge fields, knots and gravity, World Scientific, Singapore, 1994; S. Carroll, Spacetime and Geometry, Pearson Education Ltd, Harlow, 2014; H. Năstase, Classical Field Theory, Cambridge University Press, Cambridge, 2019.

[116] Exceptions have their own exceptions, in M. Göckeler and T. Schücker, Differential geometry, gauge theories, and gravity, Cambridge University Press, Cambridge, 1987, the variational method is used, but the current is defined as a 3-form.

– Aïssata: Very good Diego, but your assumption for the form of $\overset{1}{J}$ is a bit premature. The only thing that I said up to now was that the action formalism *suggests* the introduction of a current 1-form instead of a 3-form, but in the end, the two formulations should be consistent and a relation among the 1-form and the 3-form should be given. With the relation that you claim, $\overset{1}{J} = -\star\overset{3}{\mathscr{J}}$, indeed consistency demands $d\star\overset{2}{F} = -\star\overset{1}{J}$, so let's see what comes out.

I relax again the dimensional constants which you can retrieve in the dictionary later. In terms of A directly, the action is

$$S_{\rm EM}[A] = \int -\tfrac{1}{2}d\overset{1}{A}\wedge\star d\overset{1}{A} - \overset{1}{A}\wedge\star\overset{1}{J}. \tag{950}$$

You can notice that the action being a 4-dimensional integral, only 4-forms ($\overset{2}{F}\wedge\star\overset{2}{F}$ and $\overset{1}{A}\wedge\star\overset{1}{J}$) enter the Lagrangian density, which is another interesting guide when we want to generalize to alternative theories. The variation is considered by adding a small piece[117] δA to A, and to first order $S_{\rm EM}[A+\delta A]-S_{\rm EM}[A]$ is thus given by

$$S_{\rm EM}[A+\delta A]-S_{\rm EM}[A] = \int -\tfrac{1}{2}d\delta\overset{1}{A}\wedge\star\overset{2}{F} - \tfrac{1}{2}\overset{2}{F}\wedge\star d\delta\overset{1}{A} - \delta\overset{1}{A}\wedge\star\overset{1}{J} \tag{951}$$

that we want to write in a form like $\int\delta\overset{1}{A}\wedge$ (something). Therefore, setting this to zero will imply the equations of motion (something) $=0$.

For that purpose, we need the properties of the wedge and star products, of the derivative and coderivative, that we have listed earlier and which I will remind you here for clarity (see also[118]) :

$$\text{star commutativity:}\quad \overset{p}{u}\wedge\star\overset{q}{v} = \overset{q}{v}\wedge\star\overset{p}{u}, \tag{952}$$

$$\text{derivative of a product}\quad d(\overset{p}{u}\wedge\star\overset{q}{v}) = d\overset{p}{u}\wedge\star\overset{q}{v} - \overset{p}{u}\wedge\star(d^\dagger\overset{q}{v}), \tag{953}$$

$$\text{coderivative}\quad d^\dagger\overset{p}{u} = (-1)^{D(p+1)+s+1}\star d\star\overset{p}{u}, \tag{954}$$

$$\text{double Hodge}\quad \star\star\overset{p}{u} = (-1)^{p(D-1)+s}\overset{p}{u}. \tag{955}$$

In a Minkowski manifold $\mathbb{M}^4$ with metric tensor $(\eta_{\alpha\beta}) = \mathrm{diag}(+,-,-,-)$, where $D=4$, $s=3$, we have seen already that the following holds, $d^\dagger\overset{p}{u} = \star d\star\overset{p}{u}$ (or just $d^\dagger = \star d\star$) for even and for odd-rank differential forms, while $\star\star\overset{2k}{u} = -\overset{2k}{u}$ for even rank forms, and $\star\star\overset{2k-1}{u} = \overset{2k-1}{u}$ for odd rank forms.

Using star commutativity, we reverse the order of $\overset{2}{F}\wedge\star d\delta\overset{1}{A} = d\delta\overset{1}{A}\wedge\star\overset{2}{F}$ and equation (951) becomes

$$S_{\rm EM}[A+\delta A]-S_{\rm EM}[A] = -\int d\delta\overset{1}{A}\wedge\star\overset{2}{F} + \delta\overset{1}{A}\wedge\star\overset{1}{J}, \tag{956}$$

[117] Take care here, δA is just a small variational piece which shouldn't be confused with the coderivative, since δ is often used for the coderivative in the literature, while we use $d^\dagger$ instead.

[118] M. Göckeler and T. Schücker, Differential geometry, gauge theories, and gravity, Cambridge University Press, Cambridge, 1987.

then, the derivative of a product leads to

$$\int d\delta\overset{1}{A} \wedge \star\overset{2}{F} = \int \delta\overset{1}{A} \wedge \star(d^{\dagger}\overset{2}{F}) + \text{Boundary terms} \tag{957}$$

from where it follows that

$$S_{\text{EM}}[A + \delta A] - S_{\text{EM}}[A] = -\int \delta\overset{1}{A} \wedge (\star(d^{\dagger}\overset{2}{F}) + \star\overset{1}{J}) + \text{Boundary terms} = 0, \tag{958}$$

hence $\star d^{\dagger}\overset{2}{F} = -\star\overset{1}{J}$, or, "dualizing" again,

$$d^{\dagger}\overset{2}{F} = -\overset{1}{J}. \tag{959}$$

This expression fits the ranks of the r.h.s and l.h.s, since $\overset{2}{F}$ being a 2-form, $d^{\dagger}\overset{2}{F}$ is a 1-form, like $\overset{1}{J}$ is. We could have equally written the analogous of Poisson equation in terms of the potential form,

$$d^{\dagger}d\overset{1}{A} = -\overset{1}{J}. \tag{960}$$

Now, if we use $d^{\dagger} = \star d\star$, the dual of equation (959) is also $\star(\star d\star)\overset{2}{F} = -\star\overset{1}{J}$, and since $\star\star\overset{3}{u} = \overset{3}{u}$ it yields

$$d\star\overset{2}{F} = -\star\overset{1}{J}. \tag{961}$$

– Diego: Yes! I had the good intuition! Now Aïssata, could you elaborate on gauge invariance and Lorenz-Lorentz gauge in the formalism of differential forms?

– Aïssata: This is another interesting question Diego. Let me try to do that properly. Like before, $\overset{2}{F} = d\overset{1}{A}$ doesn't fix $\overset{1}{A}$ completely, since the gauge transformed potential form

$$\overset{1}{A}{}' = \overset{1}{A} + d\chi \tag{962}$$

with χ a zero form leads to the same $\overset{2}{F}{}' = \overset{2}{F}$ because $d^2\chi = 0$.

The Lorenz-Lorentz condition is a specific gauge choice which simplifies the equation of motion when it is written in terms of the potential 1-form. Taking $d^{\dagger}\overset{2}{F} = -\overset{1}{J}$, you can develop in terms of $\overset{1}{A}$ and write $d^{\dagger}(d\overset{1}{A}) = -\overset{1}{J}$. Since the d'Alembertian operator in $\mathbb{M}^4$ is given in (849) by $\Box = dd^{\dagger} + d^{\dagger}d$, we thus have the equation of motion for the potential form

$$\Box\overset{1}{A} - dd^{\dagger}\overset{1}{A} = -\overset{1}{J}. \tag{963}$$

The gauge choice

$$d^{\dagger}\overset{1}{A} = 0 \tag{964}$$

simplifies this into

$$\Box\overset{1}{A} = -\overset{1}{J}. \tag{965}$$

We can check that we recover the wave equation in ordinary notations.

$$\begin{aligned}\Box \overset{1}{A} &= \partial_\alpha \partial^\alpha \overset{1}{A} \\ &= \partial_\alpha \partial^\alpha (\phi dt - A_1 dx^1 + A_2 dx^2 + A_3 dx^3) \\ &= \partial_\alpha \partial^\alpha (\phi dt - A_x dx - A_y dy - A_z dz) \\ &= \rho dt - j_x dx - j_y dy - j_z dz \end{aligned} \tag{966}$$

or

$$(\partial_t^2 - \boldsymbol{\nabla}^2)\phi = \rho, \tag{967}$$

$$(\partial_t^2 - \boldsymbol{\nabla}^2)\mathbf{A} = \mathbf{j}. \tag{968}$$

Then we must have $d^\dagger d\chi = 0$ to preserve the Lorenz gauge $d^\dagger \overset{1}{A}{}' = 0$. Since χ is a zero form, we also have $d^\dagger \chi = 0$ ($d^\dagger$ decreases the rank of a form by one unit). It follows that $\Box\chi = (dd^\dagger + d^\dagger d)\chi = d^\dagger d\chi = 0$ from the choice $d^\dagger d\chi = 0$ above.

►**EXERCISE 38 – The Lorenz gauge condition** $d^\dagger \overset{1}{A} = 0$ –
– Diego: Let me check that equation (964) is indeed the Lorenz gauge condition as I know it. For $\overset{1}{A} = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$, we form first the Hodge dual $\star \overset{1}{A} = A_0 dx^1 dx^2 dx^3 + A_1 dx^2 dx^3 dx^0 + A_2 dx^3 dx^0 dx^1 + A_3 dx^0 dx^1 dx^2$. We can then take the exterior derivative and get after an easy calculation

$$d \star \overset{1}{A} = \partial_\alpha A^\alpha \, d\mathrm{Vol} \tag{969}$$

and eventually

$$d^\dagger \overset{1}{A} = \star d \star \overset{1}{A} = \partial_\alpha A^\alpha \tag{970}$$

which answers my question! ◄

– Aïssata: I can now summarize the various approaches and correspondences among them in the following dictionary (table 1). The case which is considered here is that of external charges in free space. That of simple linear dielectric or magnetic media is obtained via the substitutions $\varepsilon_0 \to \varepsilon(\mathbf{r}, t)$ and $\mu_0 \to \mu(\mathbf{r}, t)$. For the sake of completeness, all dimensional quantities are restored and the differential forms notation incorporates the indication of forms ranks.

▷ 18.5 From Cartesian to spherical coordinates

– Diego: This is all formalism, but I would like to see how things work technically in specific examples, in order to compare the different approaches. Let me consider the simplest case possible, that of a point charge in free $3D$-space, and look for the electrostatic field. I am used to the vector approach. Due to spherical symmetry, I use spherical coordinates (r, θ, φ) and I know that rotations around the origin, where I assume that the point charge is located, leave the system unchanged. As a consequence, no physical quantity can depend on the angles θ and φ which describe such rotations. The potential only depends possibly on r, $\phi(r)$, and this leads to an electric field with only one component along the unit vector $\mathbf{u}_r$, $\mathbf{E} = E_r(r)\mathbf{u}_r = -(d\phi/dr)\,\mathbf{u}_r$. Now I can use the equation of motion $\boldsymbol{\nabla} \cdot \mathbf{E} = q\delta(\mathbf{r})/\varepsilon_0$ which, outside the origin, reads as

$$\frac{1}{r^2}\frac{d}{dr}(r^2 E_r(r)) = 0. \tag{971}$$

Vectors in $\mathbb{R}^3$	Tensors in $\mathbb{M}^4$	Differential forms
$\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial_t \mathbf{A}$, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$	$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$	$\overset{2}{F} = d\overset{1}{A} = -\overset{1}{E} \wedge dt - \overset{2}{B}$
$\boldsymbol{\nabla} \times \mathbf{E} + \partial_t \mathbf{B} = 0$, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$	$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$	$d\overset{2}{F} = 0$
$S_{\text{EM}}[\phi, \mathbf{A}] = \int dt d^3 r \big[\frac{1}{2}\varepsilon_0(\lvert\mathbf{E}\rvert^2$ $-c^2\lvert\mathbf{B}\rvert^2) - \rho\phi + \mathbf{j} \cdot \mathbf{A}\big]$	$S_{\text{EM}}[A] = -\int d^4 x \big[\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta}$ $+A_\alpha j^\alpha\big]$	$S_{\text{EM}}[A] = -\int \frac{1}{2\mu_0} \overset{2}{F} \wedge \star \overset{2}{F}$ $+\overset{1}{A} \wedge \star \overset{1}{J}$
$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho$, $\boldsymbol{\nabla} \times \mathbf{H} - \partial_t \mathbf{D} = \mathbf{j}$	$\partial_\alpha G^{\alpha\beta} = j^\beta$	$d\overset{2}{G} = \overset{3}{\mathscr{J}} = -\star \overset{1}{J}$
$\mathbf{D} = \frac{\partial \mathscr{L}_{\text{EM}}}{\partial \mathbf{E}} = \varepsilon_0 \mathbf{E}$ $\mathbf{H} = -\frac{\partial \mathscr{L}_{\text{EM}}}{\partial \mathbf{B}} = \frac{\mathbf{B}}{\mu_0}$	$G^{\alpha\beta} = -2\frac{\partial \mathscr{L}_{\text{EM}}}{\partial F_{\alpha\beta}} = \frac{F^{\alpha\beta}}{\mu_0}$ then $\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta$	$\overset{2}{G} = \star \overset{2}{F}$ then $d^\dagger \overset{2}{F} = -\mu_0 \overset{1}{J}$

Table 1. Comparison between various formulations of Maxwell electrodynamics.

This is solved immediately in $E_r(r) = \kappa/r^2$ and the constant for example follows from the integral form (Ostrogradski theorem)

$$\int \boldsymbol{\nabla} \cdot \mathbf{E} d^3 r = 4\pi r^2 E_r(r) = \frac{Q}{\varepsilon_0}. \tag{972}$$

Of course, here the integral form, Gauss theorem in electrostatics, does the job alone without resorting to the first integration, but I want to see the differential expressions at work.

So far, so good, I am happy with that. Now I want to train myself with the tensor formalism, but I am immediately faced with a problem. The equation of motion $\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta$, I think, works in Cartesian coordinates and the symmetry calls for a use of spherical symmetry.

– Aïssata: You are right Diego. This means that the tensor formalism has to be adapted to more general coordinate systems. I recommend the use of differential forms first, since it is especially designed to deal with arbitrary coordinates. We will then come back to tensors. The relevant equation of motion is $d\overset{2}{D} = \overset{3}{\rho}$ that we will write in integral form, using also Stokes theorem:

$$\int_{\underset{3}{\Omega}} d\overset{2}{D} = \int_{\partial \underset{3}{\Omega}} \overset{2}{D} = \int_{\underset{3}{\Omega}} \overset{3}{\rho} = Q \tag{973}$$

where $\underset{3}{\Omega}$ is a $3D$ ball and $\partial\underset{3}{\Omega}$ its surface. Intuition tells that symmetry here imposes that the 2-form has a single component

$$\overset{2}{D} = D_{\theta\varphi} d\theta \wedge d\varphi. \tag{974}$$

Of course, one knows that there is a surface integral over $\partial\underset{3}{\Omega}$ which appears naturally in spherical coordinates, this is the solid angle $\int_{\partial \underset{3}{\Omega}} \sin\theta d\theta d\varphi = 4\pi$ and we can write the trivial identity $Q = Q(4\pi)^{-1} \int_{\partial \underset{3}{\Omega}} \sin\theta d\theta d\varphi$ to identify

$$\int_{\partial \underset{3}{\Omega}} D_{\theta\varphi} d\theta \wedge d\varphi = Q(4\pi)^{-1} \int_{\partial \underset{3}{\Omega}} \sin\theta d\theta d\varphi \tag{975}$$

which demands that

$$D_{\theta\varphi} = \frac{Q}{4\pi}\sin\theta. \tag{976}$$

You can see that this is really counter-intuitive, because this is a function $D_{\theta\varphi}(\theta)$ where one could anticipate $D_{\theta\varphi}(r)$ with spherical symmetry! The explanation is in the meaning of the coordinates (r,θ,φ) and the associated basis vectors. The basis $\{dr, d\theta, d\varphi\}$ of the cotangent space *is not normalized*[119]. To make the things clear, we need to come back to the introduction of the tetrads (here the triads) at page **??**. The basis induced by the global coordinates (r,θ,φ) in the tangent space, $\{\partial_r, \partial_\theta, \partial_\varphi\}$, defines the metric tensor in spherical coordinates $g_{\mu\nu} = \partial_\mu \cdot \partial_\nu$. Linear combinations of these basis vectors $\mathbf{e}_\alpha = e_\alpha{}^\mu \partial_\mu$ can be chosen so that the metric becomes Cartesian, $\delta_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$, and the relation among the two metric tensors is $\delta_{\alpha\beta} = e_\alpha{}^\mu e_\beta{}^\nu g_{\mu\nu}$, with $\mu,\nu = r,\theta,\varphi$ and $\alpha,\beta = 1,2,3$. These relations, together with $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2\sin^2\theta$ are solved and lead to the expression of the triad basis vectors

$$\mathbf{e}_1 = \partial_r,\ \mathbf{e}_2 = \frac{1}{r}\partial_\theta,\ \mathbf{e}_\varphi = \frac{1}{r\sin\theta}\partial_\varphi. \tag{977}$$

In the cotangent space, where the components of forms live, the basis induced by the global spherical coordinates $\{dr, d\theta, d\varphi\}$ corresponds to the contravariant metric tensor $g^{\mu\nu} = dx^\mu \cdot dx^\nu$. Cotriad coefficients enable to build a Cartesian cobasis, $\mathbf{e}^\alpha = e^\alpha{}_\mu dx^\mu$ with $\delta^{\alpha\beta} = \mathbf{e}^\alpha \cdot \mathbf{e}^\beta$ and the relation $\delta^{\alpha\beta} = e^\alpha{}_\mu e^\beta{}_\nu g^{\mu\nu}$. The cotriad basis follows,

$$\mathbf{e}^1 = dr,\ \mathbf{e}^2 = r d\theta,\ \mathbf{e}^3 = r\sin\theta d\varphi. \tag{978}$$

The excitation 2-form can now be expanded in any of the two bases,

$$\overset{2}{D} = D_{\theta\varphi} d\theta \wedge d\varphi = D_{23}\mathbf{e}^2 \wedge \mathbf{e}^3. \tag{979}$$

This leads to the expression of the component of the 2-form in the normalized basis,

$$D_{23} = \frac{D_{\theta\varphi}}{r^2\sin\theta} = \frac{Q}{4\pi r^2}. \tag{980}$$

You can observe that in the normalized basis, we recover our intuition of what spherical symmetry is, with a component which does only depend on r.

The electric field follows from the expression of the Hodge dual (here in ordinary $3D$ space), $\star\overset{2}{D} = \varepsilon_0 E_\rho dx^\rho$

$$\begin{aligned} \star\overset{2}{D} &= \sqrt{g}\epsilon_{\mu\nu\rho} D^{\mu\nu} dx^\rho \\ &= r^2\sin\theta D^{\theta\varphi} dr \\ &= \frac{Q}{4\pi r^2} dr \\ &= \varepsilon_0 E_r dr. \end{aligned} \tag{981}$$

In the normalized basis, the elctric 1-form is then

$$\overset{1}{E} = E_r dr = E_1 \mathbf{e}^1 \tag{982}$$

[119]Remember that the metric tensor components correspond to the modulus squared of the basis vectors in the case of a diagonal metric as we have here with $g_{\mu\nu} = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$.

with the usual Coulombic expression

$$E_1 = \frac{Q}{4\pi\varepsilon_0 r^2}. \tag{983}$$

– Diego: You mentioned twice "our intuition of what spherical symmetry is" I believe. Why do you refer to intuition and not to something more solid?

– Aïssata: Our argument in equation (976) was correct, but maybe by chance and in my opinion, it is difficult to anticipate a symmetrical form except in an orthonormalized basis. From this perspective, one *should* start from $\overset{2}{D} = D_{23}(r)\,\mathbf{e}^2 \wedge \mathbf{e}^3$ for a spherically symmetric problem. This is important when one works in arbitrary coordinates in curved spacetimes.

The notion of spherical symmetry is discussed in more details by Carroll[120] or Zee[121] in terms of Killing vectors. This is more solid as you ask and I refer you to these books.

Now you are in position to go on the calculation in tensor form, with the proper notation for the basis chosen. There is still one point to be taken into account, but which does not present any major difficulty I think. You noticed that the equation of motion $\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta$ is valid in Cartesian coordinates and the natural symmetry here calls for the use of spherical coordinates. This equation of motion is deduced from the action (907) where the density under the integral is Lorentz invariant. This is correct here because the determinant in Cartesian coordinates equals to unity (minus unity actually), since time dilation and space contraction compensate, but in arbitrary coordinates, one needs to multiply the whole expression by $\sqrt{-g}$ as it will be explained in more details in the section on curved spacetimes. The action thus reads as

$$S_{\mathrm{EM}}[A] = -\int d^4x\sqrt{-g}\left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu\right), \tag{984}$$

and the corresponding equation of motion follows immediately

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = \mu_0 j^\nu. \tag{985}$$

– Diego: This is easy now. In our static problem of pointlike charge I can write this equation of motion for $r \neq 0$ as

$$\frac{1}{r^2\sin\theta}\partial_r(r^2\sin\theta F^{rt}) = 0, \tag{986}$$

the solution of which is simply $F^{rt} = \kappa/r^2$. The corresponding covariant component is $F_{rt} = -F^{rt}$ and the Faraday 2-form can be written in the local cobasis or in the normalized one, like you did for the excitation 2-form,

$$\overset{2}{F} = F_{rt}\,dr \wedge dt = F_{10}\,\mathbf{e}^1 \wedge \mathbf{e}^0 \tag{987}$$

with $\mathbf{e}^0 = cdt$, leading to

$$F_{10} = F_{rt} = -\frac{\hat{E}_r}{c} = -\frac{\kappa}{r^2} \tag{988}$$

if we denote $\hat{E}_r = E_1$. The result (983) is recovered if we adjust the constant to its standard value.

[120] S. Carroll, Spacetime and Geometry, Pearson Education Ltd, Harlow, 2014, section 5.5.
[121] A. Zee, Einstein gravity in a Nutshell, Princeton University Press, Princeton, 2013, section IX.6.

19. Day 15 – Lagrangian formulation for relativistic matter fields

☐ 19.1 Complex scalar Minkowskian fields

Assume a complex scalar field $\varphi(x)$ the dynamics of which is described by an action, written symbolically as

$$S = \int d^4x \mathscr{L}(\varphi, \partial\varphi, \ldots, \partial\partial\ldots\partial\varphi) \tag{989}$$

in terms of a Lagrangian density $\mathscr{L}(\varphi, \partial\varphi, \ldots, \partial\partial\ldots\partial\varphi)$. The Euler-Lagrange equations take the general form

$$\frac{\delta S}{\delta\varphi} = \frac{\partial\mathscr{L}}{\partial\varphi} - \partial_\alpha \frac{\partial\mathscr{L}}{\partial(\partial_\alpha\varphi)} + (-1)^m \partial_{\alpha_1}\partial_{\alpha_2}\ldots\partial_{\alpha_m} \frac{\partial\mathscr{L}}{\partial(\partial_{\alpha_1}\partial_{\alpha_2}\ldots\partial_{\alpha_m}\varphi)} = 0. \tag{990}$$

The key point is that in order to ensure that the action remains Lorentz invariant, leading to covariant field equations, the function $\mathscr{L}(\varphi, \partial\varphi, \ldots, \partial\partial\ldots\partial\varphi)$ has to be a Lorentz scalar. Terms like $\partial\varphi$ enter via $\partial^\alpha\varphi$ and a simple Lorentz invariant is therefore built from the contraction $\partial_\alpha\varphi\partial^\alpha\varphi = \eta^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi$. Requiring this to be real just demands a small modification to account for the complexity of the fields, $\partial_\alpha\varphi^*\partial^\alpha\varphi = \eta^{\alpha\beta}\partial_\alpha\varphi^*\partial_\beta\varphi$. Not only it is a natural prescription, but we will see later that this has to do also with gauge invariance. This term is often called the kinetic energy term because of its usual quadratic form, but once expanded it reads as $(\partial_t\varphi^*)(\partial_t\varphi) - (\boldsymbol{\nabla}\varphi^*)\cdot(\boldsymbol{\nabla}\varphi)$ and clearly will have to do with the wave equation. This is all for *massless free fields*.

In the presence of interactions, some complications arise. The simple case is that of interactions with an external source (another field for example) which can be described by a simple scalar function $\mathscr{V}(\varphi)$ acting on the field φ and that we are tempted to call potential energy for obvious reasons. Again, restricting ourselves to real potentials, it is better to denote it as $\mathscr{V}(\varphi^*\varphi)$. The Lagrangian density becomes

$$\mathscr{L}_\varphi(\varphi, \partial\varphi) = \eta^{\alpha\beta}\partial_\alpha\varphi^*\partial_\beta\varphi - \mathscr{V}(\varphi^*\varphi) \tag{991}$$

and we see that there is no need in general to consider higher order field derivatives like in equation (989).

The equations of motion follow from variation w.r.t. φ^* and φ, e.g.:

$$\frac{\partial\mathscr{L}_\varphi}{\partial\varphi^*} - \partial_\alpha\frac{\partial\mathscr{L}_\varphi}{\partial(\partial_\alpha\varphi^*)} = -\frac{\partial\mathscr{V}}{\partial\varphi^*} - \eta^{\alpha\beta}\partial_\alpha\partial_\beta\varphi = 0. \tag{992}$$

Assuming that we can expand the potential, up to an irrelevant constant, $\mathscr{V}(\varphi^*\varphi) = m_\varphi^2|\varphi|^2 + \frac{1}{2}\kappa|\varphi|^4$, the equation of motion takes the form

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\varphi + m_\varphi^2\varphi + \kappa|\varphi|^2\varphi = 0. \tag{993}$$

If we forget the last term in κ, it yields the free particle Klein-Gordon equation (228)

$$\partial_t^2\varphi - \boldsymbol{\nabla}^2\varphi + m_\varphi^2\varphi = 0 \tag{994}$$

and justifies the notation m^2_φ for the quadratic term (mass) in the potential energy. This is the reason why one calls free particle the quadratic potential. This is the integrable problem from which one can then consider much more delicate situations, described by non linear differential equations,

$$\nabla^2\varphi - \partial_t^2\varphi = -\Box\varphi = \frac{\partial\mathscr{V}}{\partial\varphi^*}. \tag{995}$$

– Diego: Aïssata, I have a sign problem here. When you have shown me the Klein-Gordon equation in (228) we had the opposite signs than here in (994). Similarly, the Lagrangian (991) differs from (229) by an overall sign. I understand that this doesn't change the result for the equation of motion, but still, we have obtained the Klein-Gordon equation from $E^2 - |\mathbf{p}|^2c^2 = m^2c^4$!

– Aïssata: You make a good point here, and this is true that there is a sign contradiction. The important thing is to take care about the sign of the energy which must be bounded from below. If you consider the *real scalar field* (this is simpler with only one degree of freedom), Coleman[122] discusses the free parameters in the Lagrangian, and among them the overall sign,

$$\mathscr{L}_\pm = \pm\tfrac{1}{2}(\partial_\alpha\varphi\partial^\alpha\varphi - m^2\varphi^2). \tag{996}$$

Keeping the plus sign is the only way to produce a positive Hamiltonian density. Indeed, when you develop the kinetic term you get in that case

$$\mathscr{L} = \tfrac{1}{2}((\partial_t\varphi)^2 - (\nabla\varphi)^2 - m^2\varphi) \tag{997}$$

from where the conjugate momentum follows,

$$\pi = \frac{\partial\mathscr{L}}{\partial(\partial_t\varphi)} = \partial_t\varphi. \tag{998}$$

The Hamiltonian density is therefore

$$\mathscr{H} = \pi(\partial_t\varphi) - \mathscr{L} = \tfrac{1}{2}(\pi^2 + (\nabla\varphi)^2 + m^2\varphi) \tag{999}$$

and it is indeed positive, while it would be negative and unbounded from below with the other sign choice. This is the reason why I used the expression "a possible solution is" for the KG Lagrangian density the first time we have discussed it. This was not the definitive choice!

Another subtlety, which is more semantic this time. Your professor noticed that the denomination of kinetic energy is perhaps a bit misleading (it contains both time and space derivatives). If you also consider the addition of a potential energy, this is usually denoted in the Lagrangian as

$$\mathscr{L} = \tfrac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi - \mathscr{V}(\varphi), \tag{1000}$$

as it is done in the complex scalar field case in your notes. But again, if you remember that the second derivative comes from the quadratic expression $E^2/c^2 - \mathbf{p}^2$, there is a

[122] S. Coleman, Quantum Field Theory, Lectures of Sidney Coleman, World Scientific, Singapore, 2019, p. 66.

kind of dimensional mismatch which goes beyond the dimensional constants c or $\hbar$. We essentially write something like $E^2 - \mathbf{p}^2 - \mathscr{V}$ which has little in common with the $(E - V)^2 - \mathbf{p}^2$ of (234)! But this is conventional to call $\mathscr{V}$ a density of potential energy, or simply a potential, and $\partial_\alpha \varphi \partial^\alpha \varphi$ a kinetic energy!

– Diego: And I suppose, to complete the confusion, that these expressions depend on the signature choice in the Minkowski metric.

– Aïssata: This is true. With the other convention $(-,+,+,+)$, you would have to define $\mathscr{L}_- = -\frac{1}{2}(\partial_\alpha \varphi \partial^\alpha \varphi + m^2 \varphi)$ to have the same equations (997) to (999).

□ 19.2 Relativistic spinor fields

We have seen how to build Lorentz scalars and Lorentz vectors from Dirac spinors $\psi(x)$ and their Dirac adjoints $\bar{\psi}(x)$. The candidates to build a Lagrangian density are thus terms like

$$\bar{\psi}\psi, \quad \bar{\psi}m\psi, \quad \bar{\psi}\gamma^\alpha p_\alpha \psi = \bar{\psi} i \gamma^\alpha \partial_\alpha \psi, \ \ldots \tag{1001}$$

If we remember the "empirical" approach to Dirac equation discussed earlier, the expected Dirac action (restoring for the occasion the dimensional constants) is

$$S_\mathrm{D}[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\hbar\gamma^\alpha \partial_\alpha - mc)\psi(x), \tag{1002}$$

and it is such that the equation of motion is Dirac equation

$$\frac{\delta S_\mathrm{D}}{\delta \bar{\psi}} = (i\hbar\gamma^\alpha \partial_\alpha - mc)\psi(x) = 0. \tag{1003}$$

An important difference with the case of the relativistic equation obeyed by scalar fields is that we now have a set (because spinors have several components) of first order equations.

Gauge field theory

20. Day 16 – Gauge invariance

□ 20.1 Gauge structure of a theory

Let us now consider briefly the essentials of the structure of a gauge theory (we will illustrate our purpose with the case of $U(1)$ symmetry). In Quantum Mechanics, the first postulate states that a physical system is represented by a *ray* in the Hilbert space. This means that $\varphi(x)$ and $e^{i\theta}\varphi(x)$ $(\theta = \text{const})$ both represent the same physical state. The global gauge transformation, here a phase modification,

$$\varphi(x) \to e^{ie\theta}\varphi(x) \tag{1004}$$

is therefore a symmetry (the constant e is introduced for later convenience), which means that it leaves unchanged the Lagrangian density which governs the dynamics.

In the case of a scalar theory, like that of Klein-Gordon, built from a complex scalar field $\varphi(x)$, the free field action reads as

$$\begin{aligned} S_0[\varphi, \varphi^*] &= \int d^4x \mathscr{L}_0 \\ &= \int d^4x (\partial_\alpha \varphi^* \partial^\alpha \varphi - m^2 \varphi^* \varphi). \end{aligned} \tag{1005}$$

Although not specified yet, we work in the Minkowskian manifold $\mathbb{M}^4$ and use Greek indices from the beginning of the alphabet to emphasize this point.

The dynamics is given by the equation of motion

$$\frac{\delta S_0}{\delta \varphi^*} = 0 = m^2\varphi + \partial_\alpha \partial^\alpha \varphi \tag{1006}$$

and under the gauge transformation, $\mathscr{L}_0$ doesn't change, up to a divergence,

$$\mathscr{L}_0 \to \mathscr{L}_0 + \partial_\alpha (.)^\alpha. \tag{1007}$$

The transformation being a symmetry, $\delta\mathscr{L}_0 = 0$, and this leads to the existence of a conserved Noether current, the thing between parenthesis above that we choose to call j^α,

$$\partial_\alpha j^\alpha = 0 \tag{1008}$$

the precise expression of which in the case of Klein-Gordon being simply

$$j^{\alpha} = ie(\varphi^* \partial^{\alpha} \varphi - (\partial^{\alpha} \varphi^*)\varphi). \tag{1009}$$

The constant e is clearly identified to the electric charge carried by the matter field φ. We observe here that a real scalar field would be associated to a neutral field.

The whole theory is based on the property that under a gauge transformation, which we may still write as

$$\varphi \to \varphi' = \mathsf{U}\varphi, \tag{1010}$$

the field derivative transforms like the field itself,

$$\partial_{\alpha}\varphi \to (\partial_{\alpha}\varphi)' = \mathsf{U}\partial_{\alpha}\varphi \tag{1011}$$

and this guarantees the invariance

$$\mathscr{L}_0 \to \mathscr{L}_0' = \mathscr{L}_0. \tag{1012}$$

The theory is then "gauged" by extending the symmetry group, i.e. demanding that local transformations $\theta = \theta(x)$, or $\mathsf{U} = \mathsf{U}(x)$, also become symmetries. One still has

$$\varphi \to \varphi' = \mathsf{U}(x)\varphi, \tag{1013}$$

but now

$$\partial_{\alpha}\varphi \to (\partial_{\alpha}\varphi)' = \mathsf{U}(x)\partial_{\alpha}\varphi + (\partial_{\alpha}\mathsf{U}(x))\varphi \neq \mathsf{U}(x)\partial_{\alpha}\varphi \tag{1014}$$

and as a consequence, $\delta\mathscr{L}_0 \neq 0$. The theory which exhibits the local gauge symmetry is thus a *different* theory, built from a different lagrangian, $\mathscr{L}$, still to determine. To repair the field derivative transformation law, one introduces a *covariant* derivative which generalizes ∂_{α} and thus requires a 4-vector A_{α} with appropriate properties

$$\mathcal{D}_{\alpha} = \partial_{\alpha} + ieA_{\alpha}(x). \tag{1015}$$

The factor i enables to have a Hermitian A_{α} (hence real here) and the constant e is incorporated for later convenience. One demands that the properties of A_{α}, called a gauge field, are such that

$$\mathcal{D}_{\alpha}\varphi \to (\mathcal{D}_{\alpha}\varphi)' = \mathsf{U}(x)\mathcal{D}_{\alpha}\varphi, \tag{1016}$$

or

$$A_{\alpha}(x) \to A_{\alpha}'(x) = A_{\alpha}(x) - \frac{1}{ie}\mathsf{U}^{-1}(x)\partial_{\alpha}\mathsf{U}(x) \tag{1017}$$

or, in the $U(1)$ case where

$$\mathsf{U}(x) = e^{ie\theta(x)}, \quad A_{\alpha}'(x) = A_{\alpha}(x) - \partial_{\alpha}\theta(x). \tag{1018}$$

We note that the introduction of the charge e in the covariant derivative gives the gauge field its usual meaning, since we recover the known gauge transformation of the gauge 4-vector of electrodynamics. This is a first clue that what we are doing has something to do with electrodynamics. Note that we call covariant derivative the object $\mathcal{D}_{\alpha}$ in the sense of a *gauge covariant* derivative, a derivative which remains covariant under a gauge transformation. This doesn't have to be confused with the denomination of covariant derivative for ∂_α which in this case means the (ordinary) derivative with a covariant index. The context is usually clear.

We are now in position to determine the theory which will admit the local gauge transformations as symmetries through the *minimal coupling prescription* $\partial_\alpha \to \mathcal{D}_\alpha$ in the expression of the action (or Lagrangian density). This prescription is not the only way to produce a *gauge covariant* theory, but at least this guarantees that this will work and the validity of the whole construction then comes from the fact that the theory produced that way is successfully tested experimentally. In the present case, the outcome is Maxwell theory of electrodynamics. Indeed, one gets the *matter plus gauge field* action

$$\begin{aligned} S_{\text{mat}}[\varphi, \varphi^*, A_\alpha] &= \int d^4x \mathscr{L}_{\text{mat}} \\ &= \int d^4x [(\mathcal{D}_\alpha \varphi)^* (\mathcal{D}^\alpha \varphi) - m^2 \varphi^* \varphi], \end{aligned} \tag{1019}$$

therefore described by the following Lagrangian

$$\mathscr{L}_{\text{mat}} = (\mathcal{D}_\alpha \varphi)^* (\mathcal{D}^\alpha \varphi) - m^2 \varphi^* \varphi = \mathscr{L}_0(\varphi, \partial\varphi) + \mathscr{L}_{\text{int}}(\varphi, \partial\varphi, A) \tag{1020}$$

where the original matter contribution (1005) was separated from the term which describes the interaction between the matter field φ and the gauge field A_α.

Since a new degree of freedom has been introduced in the theory with the gauge field, we also need to determine its own dynamics. For that purpose, one needs a *free gauge field* contribution $S_{\text{e.m.}}$ to the action. In its simpler form, this contribution has to be built from a quadratic form in the gauge field derivatives, in order to have something similar to a kinetic energy (look e.g. at the KG kinetic energy density). Note here that this is not the only option, but this is the one which will lead us to Maxwell theory! More will be said on that later. The expected contribution must also be Lorentz invariant, and a full contraction over Minkowski indices will be required. Eventually, this must obviously be gauge invariant otherwise the whole construction that we are elaborating would fall down (this excludes a simple form like $\partial_\alpha A_\beta \partial^\alpha A^\beta$).

The commutator of covariant derivatives is a nice tool to define a gauge invariant combination of the gauge field derivatives,

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta]\varphi = ieF_{\alpha\beta}\varphi, \qquad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{1021}$$

We call this $F_{\alpha\beta}$ the electromagnetic curvature tensor (in analogy with the curvature tensor in differential geometry), or the Faraday tensor.

The gauge vector A_α is also called a connection, or electromagnetic connection, again in analogy with differential geometry. Finally, a possible candidate for the free gauge field action is

$$S_{\text{e.m.}} = \int d^4x \left(-\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right). \tag{1022}$$

The factor $\frac{1}{4}$ (or $\frac{1}{4\mu_0}$ if we would restore all dimensional constants) appears for convenience. The complete action now stands as

$$S_{\text{tot}}[\varphi, \partial\varphi, A] = \int d^4x \left((\mathcal{D}_\alpha \varphi)^* (\mathcal{D}^\alpha \varphi) - m^2 \varphi^* \varphi - \tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right). \tag{1023}$$

A few results automatically follow:

– The Lagrangian for the new theory comprises additional terms,

$$\begin{aligned}
\mathscr{L}_{\text{tot}} &= (\mathcal{D}_\alpha\varphi)^*(\mathcal{D}^\alpha\varphi) - m^2\varphi^*\varphi - \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \\
&= \partial_\alpha\varphi^*\partial^\alpha\varphi - m^2\varphi^*\varphi + ie(\varphi\partial_\alpha\varphi^* - \varphi^*\partial_\alpha\varphi)A^\alpha \\
&\quad +e^2A_\alpha A^\alpha\varphi^*\varphi - \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \qquad (1024)
\end{aligned}$$

– The conserved current also takes a modified form, with a new contribution

$$\begin{aligned}
J^\alpha &= ie(\varphi^*(\mathcal{D}^\alpha\varphi) - (\mathcal{D}^\alpha\varphi^*)\varphi) \\
&= ie(\varphi^*\partial^\alpha\varphi - (\partial^\alpha\varphi^*)\varphi) - 2e^2A^\alpha\varphi^*\varphi, \qquad (1025)
\end{aligned}$$

– And the field equations for the matter field and for the gauge field become

$$\begin{aligned}
&\frac{\delta S_{\text{tot}}}{\delta\varphi^*} = 0, \\
&\qquad \partial_\alpha\partial^\alpha\varphi + m^2\varphi + ie(\partial_\alpha(A^\alpha\varphi) + A_\alpha\partial^\alpha\varphi) - e^2A_\alpha A^\alpha\varphi = 0, \qquad (1026) \\
&\frac{\delta S_{\text{tot}}}{\delta A_\alpha} = 0, \\
&\qquad \partial_\alpha F^{\alpha\beta} = J^\beta. \qquad (1027)
\end{aligned}$$

Hence, the outcome of the whole construction is the correct interaction of charged particles with the gauge field (1026) and the correct Maxwell equations (1027). It is remarquable that (1022) produces the correct Schwarzschild Lagrangian! This is an essentiel result which has been obtained solely via the simple assumption of (local) gauge invariance and this assumption was, in a certain sense, suggested to us by the simple analysis that quantum physical states are rays in Hilbert space.

– Aïssata: The conclusion given by your professor is very important Diego. You see, starting from a kind of obvious symmetry of the theory: *quantum states are rays in Hilbert space, hence the global phase of the fields is arbitrary and corresponds to a freedom of the theory*, you extend this to local phase freedom. The invariance of the initial theory is lost, due to the now incorrect transformation law of the field derivatives. This is repaired by the introduction of a covariant derivative involving new degrees of freedom via a gauge vector. This new object has to obey specific transformation properties in order to restore the correct invariance through the local phase transformations. Then, a new Lagrangian is easily built, using a kind of toy recipe, the minimal coupling prescription. And with no more effort, you get almost for free the correct theory for the electromagnetic interaction and the correct interaction with matter fields, here scalar matter fields.

Now we are in position to complete our earlier discussion on the coupling between scalar fields and electrodynamics. If you observe carefully, the Lagrangian (1024) contains a correction compared to (916) which was a linear approximation (the missing term is the term quadratic in the gauge field). There was indeed an inconsistency in (916), but we were not yet in position to detect it. This Lagrangian was indeed not gauge invariant. This is repaired in (1024) and that was precisely the purpose of the minimal coupling scheme.

– Diego: This appears as a very abstract way to prove Maxwell equations, don't you think so? And very artificial also. Why is this term "gauge theory" so popular?

– Aïssata: The power of gauge theory is that the whole construction is easily generalized to other gauge transformations. The phase modification introduced here is a Lie group. This is called $U(1)$. We can easily consider matter fields with a bit more structure and the whole construction can be repeated for arbitrary Lie groups describing internal symmetries and you will get *other possible theories* describing *other possible interactions*, but nothing guarantees that the resulting theory does indeed exist in Nature! This is a theoretical, free from inconsistencies scheme, but experiment once again has the final word and tells us which theory conveniently describe which aspect of Nature.

– Diego: Why do you say *possible* theories?

– Aïssata: This is always the same in Physics you know. At the end, you have to compare with Nature. There are Lie groups for which the procedure indicated above is fruitful in the sense that it leads to equations of motion which describe some physical processes observed in experiments, but for other Lie groups, the equations of motion produced have apparently no application in our world.

– Diego: You mean that gravitation, the weak interaction, or the strong interaction, are gauge theories?

– Aïssata: Right! Weak interaction is built from the $SU(2)$ gauge group and the strong interaction from $SU(3)$. The whole *Standard Model* is a gauge theory. This is one of the reasons of the incredible success of gauge theory. In which concerns gravitation, the situation is a bit different, but a similar construction, based on gauging Lorentz transformations, can be elaborated[123]. Then, the gauge group is not compact, which is maybe at the origin of the differences between gravitation and the Standard Model.

– Diego: Aïssata, our professor didn't explain us about these other gauge theories, although she insisted on the fact that the procedure elaborated was very general and had other applications. Could we discuss these examples together?

▷ **20.2 Noether $U(1)$ conserved currents**

– Aïssata: Yes we will do that, but I think that there is a point that you didn't seem to note. This concerns Noether theorem. I think that we should discuss this first. Do you remember how you have obtained the Klein-Gordon current density (914)?

– Diego: Yes, we have proceeded like in the case of Schrödinger current probability density in (298). We assume the equation of motion, then we manipulate a bit until we get a continuity equation, $\partial_\alpha j^\alpha = 0$ in the relativistic case.

– Aïssata: Correct. And was it what was done in order to obtain (1009)?

– Diego: I see what you mean Aïssata, (1009) is identical to (914), but it was not obtained via the equation of motion, but via the Lagrangian itself. Does this make a difference?

– Aïssata: In a certain sense yes. Of course, the Lagrangian is also the object which eventually leads to the equation of motion, but the second method relies on the gauge symmetry of the Lagrangian. This makes it the equivalent of other conservation equations associated to continuous symmetries, I mean conservations of the total energy, momentum and angular momentum that we have discussed earlier. This unifies the formalism and makes the electric charge conservation an application of Noether theorem as well!

[123] M. Blagojević and F.W. Hehl, Gauge theories of Gravitation, Imperial College Press, London, 2013, M. Blagojević, Gravitation and Gauge Symmetries, Institute of Physics Publishing, London, 2002.

Let me give you more details. I will assume a scalar field, denoted as φ. This might be the Klein-Gordon field or something else, this is not essential for the moment. Consider the Langrangian density

$$\mathscr{L}_0 = \partial_\alpha \varphi^* \partial^\alpha \varphi - V(\varphi^* \varphi). \tag{1028}$$

For any symmetry transformation, one has

$$\delta \mathscr{L}_0 = \delta\varphi^* \frac{\partial \mathscr{L}_0}{\partial \varphi^*} + \delta(\partial_\alpha \varphi^*) \frac{\partial \mathscr{L}_0}{\partial(\partial_\alpha \varphi^*)} + \varphi^* \to \varphi = 0. \tag{1029}$$

The notation $\varphi^* \to \varphi$ means that we add a similar term with all complex numbers replaced by their complex conjugate. The global phase transformation $\varphi \to \varphi' = \varphi e^{ie\theta}$ being such a symmetry, we use $\delta\varphi = ie\theta\varphi$ and $\delta(\partial_\alpha\varphi) = ie\theta\partial_\alpha\varphi$ (to linear order in the expansion of $e^{ie\theta}$) in the expression of $\delta\mathscr{L}_0$ to have

$$\begin{aligned} \delta \mathscr{L}_0 &= -ie\theta \left[\varphi^* \frac{\partial \mathscr{L}_0}{\partial \varphi^*} + \partial_\alpha \varphi^* \frac{\partial \mathscr{L}_0}{\partial(\partial_\alpha \varphi^*)} \right] + \varphi^* \to \varphi \\ &= -ie\theta \partial_\alpha \left[\varphi^* \frac{\partial \mathscr{L}_0}{\partial(\partial_\alpha \varphi^*)} \right] + \varphi^* \to \varphi \\ &= -\theta \partial_\alpha j^\alpha \end{aligned} \tag{1030}$$

where use has been made of the Euler-Lagrange equation $\frac{\delta \mathscr{L}_0}{\partial \varphi^*} = \partial_\alpha \frac{\delta \mathscr{L}_0}{\partial(\partial_\alpha \varphi^*)}$. It follows that the Noether current

$$j^\alpha = -ie \left[\frac{\partial \mathscr{L}_0}{\partial(\partial_\alpha \varphi)} \varphi - \varphi^* \frac{\partial \mathscr{L}_0}{\partial(\partial_\alpha \varphi^*)} \right] \tag{1031}$$

is conserved, $\partial_\alpha j^\alpha = 0$. *The electric charge conservation thus appears as the consequence of the invariance of the theory under global phase changes, this is called a global gauge symmetry*. Note that in the case of the Klein-Gordon Lagrangian, the conserved current indeed takes the form [124]

$$j^\alpha = -ie(\varphi \partial^\alpha \varphi^* - \varphi^* \partial^\alpha \varphi) \tag{1032}$$

identical to (1009).

This current is the Noether conserved current associated to the free matter field φ, the Lagrangian of which, $\mathscr{L}_0$, displays global gauge invariance.

Now, as your professor explained you, when gauge invariance is promoted to local gauge invariance, the theory is changed, and in which concerns the matter field $\mathscr{L}_0 \to \mathscr{L}_{\text{mat}} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}$. Hence, the conserved current is also subject to modifications. The new Lagrangian exhibits a larger symmetry group, but it obviously still possesses global gauge symmetry so that Noether theorem can still be applied. Therefore, equation (1031) applied to $\mathscr{L}_{\text{mat}}$ delivers the conserved electric current in the presence of gauge fields and one indeed obtains (1025):

$$J^\alpha = ie(\varphi^* \partial^\alpha \varphi - (\partial^\alpha \varphi^*)\varphi) - 2e^2 A^\alpha \varphi^* \varphi. \tag{1033}$$

▷ **20.3 Non Abelian currents**

– Diego: You mentioned possible generalization to other Lie groups Aïssata. Could you give an example?

– Aïssata: Yes. This is the result of the famous work of Yang and Mills[125] on non

[124]Note that at this point, the sign in front of the current density was arbitrary, but the present choice is imposed if we want to recover the usual expression of probability density current in quantum mechanics (after multiplication by e).

[125]C.N. Yang and R. Mills, Phys. Rev. **96** 191, 1954.

Abelian gauge theories. Before their celebrated paper, they published in the Physical Review an illuminating abstract, announcing their forthcoming paper. It is quoted in the review by Wu and Yang[126]:

> *The conservation of isotopic spin points to the existence of a fundamental invariance law similar to the conservation of electric charge. In the latter case, the electric charge serves as a source of electromagnetic field. An important concept in this case is gauge invariance which is closely connected with (1) the equation of motion of the electromagnetic field, (2) the existence of a current density, and (3) the possible interactions between a charged field and the electromagnetic field. We have tried to generalize this concept of gauge invariance to apply to isotopic spin conservation.*

As I said, we can contemplate cases where the matter field has an internal structure. Say a "vector" $\boldsymbol{\varphi}$ of complex scalar fields $\varphi_A(x)$ with $A = 1, \ldots N$. Equation (1004) generalizes to

$$\varphi_A(x) \to \exp\left(ig\theta^a (\mathrm{t}_a)_A{}^B\right) \varphi_B(x). \tag{1034}$$

In this equation, the t_a's are the n generators of the Lie algebra (see equation (302)). These are $N \times N$ matrices acting on the $(\varphi_A)_{A=1,\ldots N}$. The labels A and B are respectively the line and column indices in the matrix representation. The inessential constant g is some gauge charge and plays a role analogous to that of the electric charge e in the $U(1)$ case. The complex conjugate has to be generalized to the adjoint matter vector field $\boldsymbol{\varphi}^\dagger = (\varphi_A^*)^{\mathrm{T}}_{A=1,\ldots N}$.

– Diego: This is very abstract, again, all these indices. And you have not yet mentioned the spacetime α, β's!

– Aïssata: If you need an example that you know, you may imagine Pauli spinors which describe spins $\frac{1}{2}$. Define the two-component objects

$$\boldsymbol{\varphi}(x) = \begin{pmatrix} \varphi_\uparrow(x) \\ \varphi_\downarrow(x) \end{pmatrix}. \tag{1035}$$

Here, the dimension of the representation is $N = 2$. The adjoint field is

$$\boldsymbol{\varphi}^\dagger(x) = \left(\varphi_\uparrow^*(x)\ \varphi_\downarrow^*(x)\right). \tag{1036}$$

You might remember from your courses in QM, and from what we have done on symmetries earlier in our discussions, that under a rotation of an angle θ around an axis $\mathbf{n}$, the Pauli spinors transform according to

$$\boldsymbol{\varphi}(x) \to \exp\left(i\tfrac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right) \boldsymbol{\varphi}(x). \tag{1037}$$

With $\boldsymbol{\theta}\cdot\boldsymbol{\sigma} = \theta^a \sigma_a$, you recognize (1034) where there are three generators t_a which are related to the three 2×2 matrices $\mathrm{t}_a = \frac{1}{2}\sigma_a$, and where $\varphi_1 = \varphi_\uparrow$ and $\varphi_2 = \varphi_\downarrow$.

– Diego: Ok, it looks a bit more familiar. Go ahead please.

►EXERCISE 39 – **Conservation of the Non Abelian current** –

– Aïssata: With spinors the free matter field can possibly be described by a relativistic generalization of Klein-Gordon Lagrangian,

$$\mathscr{L}_0(\varphi_A, \partial\varphi_A) = \partial_\alpha \varphi_A^* \partial^\alpha \varphi^A - m^2 \varphi_A^* \varphi^A \tag{1038}$$

[126] A.C.T. Wu and C.N. Yang, Int. J. Mod. Phys. **21**, 3235, 2006.

where, like for other types of indices, there is a summation over the dummy index A but its position doesn't really matter: $\varphi^A = \varphi_A$[127].

In the transformation (1034) written for infinitesimal θ^a's, the field components variation obeys

$$\varphi_A \to (1 + ig\theta^a(\mathrm{t}_a)_A{}^B)\varphi_B, \tag{1039}$$

thus, $\delta\varphi_A = ig\theta^a(\mathrm{t}_a)_A{}^B\varphi_B$ and $\delta\partial_\alpha\varphi_A = ig\theta^a(\mathrm{t}_a)_A{}^B\partial_\alpha\varphi_B$ with opposite signs for the variations of the complex conjugate quantities. The variation of the Lagrangian density under the non Abelian gauge transformation then follows,

$$\begin{aligned}
\delta\mathscr{L}_0 &= \delta\varphi_A^*\frac{\partial\mathscr{L}_0}{\partial\varphi_A^*} + \delta(\partial_\alpha\varphi_A^*)\frac{\partial\mathscr{L}_0}{\partial(\partial\varphi_A^*)} + \text{c.c.} \\
&= -ig\theta^a\varphi_B^*(\mathrm{t}_a)_A{}^B\frac{\partial\mathscr{L}_0}{\partial\varphi_A^*} - g\theta^a\partial_\alpha\varphi_B^*(\mathrm{t}_a)_A{}^B\frac{\partial\mathscr{L}_0}{\partial(\partial_\alpha\varphi_A^*)} + \text{c.c.} \\
&= -ig\theta^a\partial_\alpha\left[\varphi_B^*(\mathrm{t}_a)_A{}^B\frac{\partial\mathscr{L}_0}{\partial(\partial_\alpha\varphi_A^*)} - \frac{\partial\mathscr{L}_0}{\partial(\partial_\alpha\varphi_A)}(\mathrm{t}_a)_A{}^B\varphi_B\right].
\end{aligned} \tag{1040}$$

The Euler-Lagrange equations of motion have been used to obtain the compact from in the last line. Now, writing that the gauge transformation is a symmetry, this variation is vanishing and, denoting

$$\delta\mathscr{L}_0 = -\theta^a\partial_\alpha j_a^\alpha = 0 \tag{1041}$$

leads to the identification of the non Abelian current density

$$j_a^\alpha = -ig\left[\frac{\partial\mathscr{L}_0}{\partial(\partial_\alpha\varphi_A)}(\mathrm{t}_a)_A{}^B\varphi_B - \varphi_B^*(\mathrm{t}_a)_A{}^B\frac{\partial\mathscr{L}_0}{\partial(\partial_\alpha\varphi_A^*)}\right]. \tag{1042}$$

This conserved Noether current carries an internal index a. Thus, there are n different components for this current (as many as there are symmetry generators). This is a result of the non Abelian character (i.e. multi-component) of the gauge group. ◄

– Diego: They are so many indices, this is a real mess!

– Aïssata: You might prefer a more compact, albeit not general, expression. In the case of Pauli spinors illustrated before, you can easily get rid of the ordinary spacetime indices and the vector space representation indices. The calculation in this specific case leads to

$$\rho_a = -ig\left[(\partial_t\boldsymbol{\varphi})^\dagger\tfrac{1}{2}\sigma_a\boldsymbol{\varphi} - \boldsymbol{\varphi}^\dagger\tfrac{1}{2}\sigma_a(\partial_t\boldsymbol{\varphi})\right]. \tag{1043}$$

$$\mathbf{j}_a = -ig\left[(\boldsymbol{\nabla}\boldsymbol{\varphi})^\dagger\tfrac{1}{2}\sigma_a\boldsymbol{\varphi} - \boldsymbol{\varphi}^\dagger\tfrac{1}{2}\sigma_a(\boldsymbol{\nabla}\boldsymbol{\varphi})\right]. \tag{1044}$$

Note that these expressions depend on the form of the Lagrangian, in particular the non Abelian charge density ρ_a associated to a Schrödinger like equation would be different (see e.g. Berche and Medina[128]). I also want to insist here on the fact that bold fonts here have two different meanings, because $\mathbf{j}_a$ are three vectors in the ordinary space, while $\boldsymbol{\varphi}$ is a vector in an internal space. This doesn't make life easy.

[127] This wouldn't bee true in van der Waerden formalism in which there the 2D Levi-Civita symbol plays the role of a kind of spinorial metric tensor to raise and lower spinorial undotted and dotted indices. More can be found on that in R.M. Wald, General Relativity, University of Chicago Press, 1984.

[128] B. Berche and E. Medina, Classical Yang-Mills theory in condensed matter physics, Eur. J. Phys. **34**, 161, 2013.

▷ 20.4 Noether theorem for ordinary spacetime continuous transformations

– Aïssata: The development of previous sections on Noether theorem applies to ordinary space continuous transformations as well and we can consider the example of spacetime translation

$$x^\alpha \to x'^\alpha = x^\alpha + \epsilon^\alpha \tag{1045}$$

for an illustration of our purpose. Here ϵ^α is an infinitesimal constant 4-vector.

– Diego: We have done that earlier. The result is the conservation of the energy and of the momentum.

– Aïssata: Correct. This time we do it with the covariant formalism. For the sake of simplicity we consider a real scalar field $\varphi(x)$, but something more complicated, like the $\boldsymbol{\varphi}(x)$ of the section on non-Abelian current would work in the same manner. Under translation (1045), the field and its derivatives get modified according to $\varphi(x) \to \varphi(x+\epsilon)$, hence

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \epsilon^\alpha \partial_\alpha \varphi(x), \tag{1046}$$

$$\partial_\beta \varphi(x) \to \partial_\beta \varphi'(x) = \partial_\beta \varphi(x) + \partial_\beta(\epsilon^\alpha \partial_\alpha \varphi(x)). \tag{1047}$$

The Lagrangian density associated to this scalar field is also subject to a transformation

$$\begin{aligned} \mathscr{L} \to \mathscr{L}' &= \mathscr{L} + \delta\mathscr{L} \\ &= \mathscr{L} + \frac{\partial \mathscr{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathscr{L}}{\partial(\partial_\beta \varphi)}\delta(\partial_\beta \varphi) \\ &= \mathscr{L} + \epsilon^\alpha \partial_\beta \left[\frac{\partial \mathscr{L}}{\partial(\partial_\beta \varphi)} \partial_\alpha \varphi \right] \end{aligned} \tag{1048}$$

where we have used (1046) and (1047) and the fact that ϵ^α has constant components. Writing $\delta\mathscr{L} = \epsilon^\alpha \partial_\alpha \mathscr{L}$, we thus obtain

$$\epsilon^\alpha \partial_\beta \left[\frac{\partial \mathscr{L}}{\partial(\partial_\beta \varphi)} \partial_\alpha \varphi \right] = \epsilon^\alpha \partial_\alpha \mathscr{L} \tag{1049}$$

which suggests the definition of a *stress-energy tensor*

$$T_{\alpha\beta} = -\eta_{\alpha\beta}\mathscr{L} + \frac{\partial \mathscr{L}}{\partial(\partial^\alpha \varphi)}\partial_\beta \varphi, \tag{1050}$$

$$T^{\alpha\beta} = -\eta^{\alpha\beta}\mathscr{L} + \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \varphi)}\partial^\beta \varphi, \tag{1051}$$

for which one deduces the conservation equations

$$\partial_\alpha T^{\alpha\beta} = -\eta^{\alpha\beta}\partial_\alpha \mathscr{L} + \partial_\alpha \left[\frac{\partial \mathscr{L}}{\partial(\partial_\alpha \varphi)} \partial^\beta \varphi \right] = 0. \tag{1052}$$

– Diego: This is the equivalent of the stress-energy tensor (291) that we have found in the case of Newtonian fields! How is it related to the conservation of the energy for example?

– Aïssata: Look, consider the 00 component,

$$T_{00} = -\mathscr{L} + \frac{\partial \mathscr{L}}{\partial(\partial_t \varphi)}\partial_t \varphi = -\mathscr{L} + \pi\dot{\varphi} = \mathscr{H} \tag{1053}$$

and $\partial_t T^{00} = 0$ is the local expression of the energy conservation.

– Aïssata: We can now generalize to the case of a *non Abelian* gauge theory. We will consider first the non Abelian gauge fields, the introduction of which is made necessary to define a covariant derivative with appropriate properties. This being done, we will have to generalize - via the minimal coupling scheme - the free matter Lagrangian in order to incorporate the interaction of matter field with the newly introduced gauge fields. The final step is the construction of a free gauge field contribution, which is not as easy as in the Abelian case. Once the programme is achieved, we are in good position to deduce the matter field, as well as the gauge field equations of motion.

Again, let us call $\boldsymbol{\varphi}(x) = (\varphi_A)_{A=1\ldots N}$ an N-component matter field which transforms in the t_a (unitary) representation of some non Abelian gauge group, i.e.

$$\delta\boldsymbol{\varphi} = ig\theta^a \mathsf{t}_a \boldsymbol{\varphi} \tag{1054}$$

with $\theta^a = \theta^a(x)$ some functions of spacetime location. The gauge covariant derivative is defined in terms of n vector fields ${A^a}_\alpha$, $a = 1\ldots n$,

$$\mathsf{D}_\alpha = \partial_\alpha 1_N + ig{A^a}_\alpha \mathsf{t}_a. \tag{1055}$$

The covariant derivative acting on the matter field being itself a field, its gauge transformation is of the form (1054), $\delta(\mathsf{D}_\alpha\boldsymbol{\varphi}) = ig\theta^a\mathsf{t}_a\mathsf{D}_\alpha\boldsymbol{\varphi}$. Once expanded, the l.h.s. reads as $\delta(\partial_\alpha\boldsymbol{\varphi} + ig{A^a}_\alpha\mathsf{t}_a\boldsymbol{\varphi})$ and leads to

$$\delta(\mathsf{D}_\alpha\boldsymbol{\varphi}) = ig[(\partial_\alpha\theta^a)\mathsf{t}_a\boldsymbol{\varphi} + \theta^a\mathsf{t}_a\partial_\alpha\boldsymbol{\varphi} + \delta{A^a}_\alpha\mathsf{t}_a\boldsymbol{\varphi}] - g^2{A^a}_\alpha\mathsf{t}_a\theta^b\mathsf{t}_b\boldsymbol{\varphi} \tag{1056}$$

while the r.h.s. gives

$$ig\theta^a\mathsf{t}_a\mathsf{D}_\alpha\boldsymbol{\varphi} = ig\theta^a\mathsf{t}_a\partial_\alpha\boldsymbol{\varphi} - g^2\theta^a{A^b}_\alpha\mathsf{t}_a\mathsf{t}_b\boldsymbol{\varphi}. \tag{1057}$$

The second term of the first expression cancels out with the first term of the second expression. It yields

$$[(\partial_\alpha\theta^a)\mathsf{t}_a + \delta{A^a}_\alpha\mathsf{t}_a]\boldsymbol{\varphi} = ig[\theta^a{A^b}_\alpha\mathsf{t}_a\mathsf{t}_b - {A^a}_\alpha\mathsf{t}_a\theta^b\mathsf{t}_b]\boldsymbol{\varphi}. \tag{1058}$$

"Simplifying" the $\boldsymbol{\varphi}$'s, we can keep an operator identity, then, exchanging a and b labels in the last term we arrive at

$$\delta{A^a}_\alpha\mathsf{t}_a = ig\theta^a{A^b}_\alpha[\mathsf{t}_a, \mathsf{t}_b] - \partial_\alpha\theta^a\mathsf{t}_a. \tag{1059}$$

Using the Lie algebra among the generators, $[\mathsf{t}_a, \mathsf{t}_b] = i{f_{ab}}^c\mathsf{t}_c$ one obtains

$$\delta{A^c}_\alpha = -g{f_{ab}}^c\theta^a{A^b}_\alpha - \partial_\alpha\theta^c, \tag{1060}$$

or

$${A'^a}_\alpha = {A^a}_\alpha + \delta{A^a}_\alpha = {A^a}_\alpha - g{f_{bc}}^a\theta^b{A^c}_\alpha - \partial_\alpha\theta^a. \tag{1061}$$

– Diego: Look Aïssata, in the Abelian case, the structure constants all vanish and it yields simply

$${A'}_\alpha = A_\alpha - \partial_\alpha\theta, \tag{1062}$$

like equation(1018) in the $U(1)$ case.

– Aïssata: Very good observation Diego.

– Diego: In Weinberg's second volume[129], I found a similar expression, with slightly different notations, but much more indices and I get a bit lost.

– Aïssata: This is true Diego, most of authors prefer a full index notation instead of an operator formulation like the one we have used here. Then, equation (1054) becomes

$$\delta\varphi_A = ig\theta^a(\mathsf{t}_a)_A{}^B\varphi_B, \tag{1063}$$

the action of the gauge covariant derivative on the gauge field writes as

$$\mathsf{D}_\alpha\varphi_A = \partial_\alpha\varphi_A + igA^a{}_\alpha(\mathsf{t}_a)_A{}^B\varphi_B, \tag{1064}$$

and the final results (1060) and (1061) keep the same form eventually.

But if you're interested in various notations, there is also a more compact, albeit maybe more rarely used. This is a full matrix notation that you can find e.g. in Ryder[130] or Aitchison and Hey[131]. You define $\mathsf{U} \simeq 1 + ig\Theta$ (we stay to the level of infinitesimal gauge transformations) with $\Theta = \theta^a\mathsf{t}_a$ an $N \times N$ matrix. Then, the gauge transfomation in vector notation reads as $\boldsymbol{\varphi}' = \mathsf{U}\boldsymbol{\varphi}$ and the gauge covariant as $\mathsf{D}_\alpha = \partial_\alpha + ig\mathsf{A}_\alpha$ with $\mathsf{A}_\alpha = A^a{}_\alpha\mathsf{t}_a$ being $N \times N$ matrices also. Then requiring $(\mathsf{D}_\alpha\boldsymbol{\varphi})' = \mathsf{U}(\mathsf{D}_\alpha\boldsymbol{\varphi})$ leads to

$$\mathsf{A}'_\alpha = \mathsf{U}\mathsf{A}_\alpha\mathsf{U}^{-1} - \frac{1}{ig}(\partial_\alpha\mathsf{U})\mathsf{U}^{-1} \tag{1065}$$

which corresponds to (1017) in the $U(1)$ case and, once expanded, leads to

$$\mathsf{A}'_\alpha = \mathsf{A}_\alpha + ig[\Theta, \mathsf{A}_\alpha] - \partial_\alpha\Theta + O(\Theta^2). \tag{1066}$$

Equations (1061), (1065) and (1066) are three equivalent formulations of the transformation of the non Abelian gauge fields. The non Abelian character is visible in the first of these equations by the appearance of structure constants of the associated Lie algebra, in the second equation via the operator structure and in the third equation by the presence of the commutator itself.

We are now in position to generalize the free particle Lagrangian

$$\mathscr{L}_0 = \partial_\alpha\boldsymbol{\varphi}^\dagger\partial^\alpha\boldsymbol{\varphi} - m^2\boldsymbol{\varphi}^\dagger\boldsymbol{\varphi} \tag{1067}$$

and the corresponding equation of motion

$$\partial_\alpha\partial^\alpha\boldsymbol{\varphi} + m^2\boldsymbol{\varphi} = 0. \tag{1068}$$

The minimal coupling hypothesis described in the case of $U(1)$ leads in the present case to

$$\mathscr{L}_{\text{tot}} = (\mathsf{D}_\alpha\boldsymbol{\varphi})^\dagger(\mathsf{D}^\alpha\boldsymbol{\varphi}) - m^2\boldsymbol{\varphi}^\dagger\boldsymbol{\varphi} - \tfrac{1}{4}F^a{}_{\alpha\beta}F_a{}^{\alpha\beta}. \tag{1069}$$

In the last term, all *three* indices are contracted to get a covariant expression.

– Diego: And how do you define this $F^a{}_{\alpha\beta}$, is it an obvious generalization of the Faraday tensor (1021) $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$?

[129] S. Weinberg, The quantum theory of fields, vol.II, Cambridge University Press, Cambridge, 1996.

[130] L.H. Ryder, Quantum Field Theory, Cambridge University Press, Cambridge, 1985.

[131] I.J.R. Aitchison and A.J.G Hey, Gauge Theories in Particle Physics, Vol. II, Taylor and Francis, New-York, 2004.

– Aïssata: This is a very good question Diego. You have a pretty good intuition, because in fact, this *is not* that simple. Yang wrote a lot on his theory with Mills and how they arrived at the correct form, but this question in particular resisted to their analysis for a while. This is quoted in the excellent *Dawning of Gauge Theory* by O'Raifeartaigh[132].

> While a graduate student in Kunming and in Chicago, I had thoroughly studied Pauli's review articles on field theory. I was very much impressed with the idea that charge conservation was related to the invariance of the theory under phase changes, an idea, I later found out, due originally to Weyl. I was even more impressed by the fact that gauge-invariance *determined* all the electromagnetic interactions. While in Chicago I tried to generalize this to istopic spin interactions by the procedure later written up in [4] [i.e., the accompanying Phys. Rev. **96** (1954) 191 article] equations (1) and (2). Starting from these it was easy to get equation (3). Then I tried to define the field strengths $F_{\mu\nu}$ by $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ which was a natural generalization of electromagnetism. This led to a mess, and I had to give up. But the basic idea remained attractive, and I came back to it several times in the next few years, always getting stuck at the same point .... As more and more mesons were discovered and all kinds of interactions were being considered, the necessity to have a *principle* for writing down interactions became more obvious to me. So while at Brookhaven [in the summer of 1953] I returned once more to the idea of generalizing gauge invariance. My office mate was R. L. Mills who was about to finish his Ph.D.... We worked on the problem and eventually produced [4]. We also wrote an Abstract for the April 1954 meeting of the AMS in Washington, which became [5]. Different motivations were emphasized in the two papers. The formal aspect of the work did not take long and was essentially finished by February 1954. But we found that we were unable to conclude what the mass of the gauge-particles should be. We

Figure 26. L. O'Raifeartaigh, The Dawning of Gauge Theory, Princeton University Press, Princeton, 1997, p182.

The correct expression for a *curvature*as we call it is through the commutator in equation (1021):

$$\mathsf{F}_{\alpha\beta} = F^a{}_{\alpha\beta}\mathsf{t}_a = \frac{1}{ig}[\mathsf{D}_\alpha, \mathsf{D}_\beta]. \tag{1070}$$

Explicit calculation leads to

$$\begin{aligned} \mathsf{F}_{\alpha\beta} &= \frac{1}{ig}[\partial_\alpha - igA^a{}_\alpha \mathsf{t}_a, \partial_\beta + igA^b{}_\beta \mathsf{t}_b] \\ &= \partial_\alpha(A^b{}_\beta \mathsf{t}_b) - \partial_\beta(A^a{}_\alpha \mathsf{t}_a) + igA^a{}_\alpha A^b{}_\beta[\mathsf{t}_a, \mathsf{t}_b] \\ &= (\partial_\alpha A^a{}_\beta - \partial_\beta A^a{}_\alpha - g f_{bc}{}^a A^b{}_\alpha A^c{}_\beta)\mathsf{t}_a \end{aligned} \tag{1071}$$

where we have made use of the Lie algebra commutation relations among the generators.

[132] L. O'Raifeartaigh, The Dawning of Gauge Theory, Princeton University Press, Princeton, 1997.

So eventually, the quantities appearing in the Lagrangian (1069) are defined as

$$F^a{}_{\alpha\beta} = \partial_\alpha A^a{}_\beta - \partial_\beta A^a{}_\alpha - g f_{bc}{}^a A^b{}_\alpha A^c{}_\beta, \tag{1072}$$

but some authors prefer a full matrix notation,

$$\mathsf{F}_{\alpha\beta} = \partial_\alpha \mathsf{A}_\beta - \partial_\beta \mathsf{A}_\alpha + ig[\mathsf{A}_\alpha, \mathsf{A}_\beta]. \tag{1073}$$

►EXERCISE 40 – **Non Abelian matter field equation of motion** –

The matter equation of motion for the field $\boldsymbol{\varphi}$ follows from

$$\frac{\delta}{\delta\boldsymbol{\varphi}^\dagger}\int d^4x \mathscr{L}_{\text{tot}} = 0. \tag{1074}$$

We develop

$$\begin{aligned}\frac{\partial\mathscr{L}_{\text{tot}}}{\partial\boldsymbol{\varphi}^\dagger} &= -m^2\boldsymbol{\varphi} + \left(\frac{\partial}{\partial\boldsymbol{\varphi}^\dagger}(\mathsf{D}_\alpha\boldsymbol{\varphi})^\dagger\right)(\mathsf{D}^\alpha\boldsymbol{\varphi}) \\ &= -m^2\boldsymbol{\varphi} - igA^a{}_\alpha \mathsf{t}_a(\partial^\alpha\boldsymbol{\varphi} + igA^{a\alpha}\mathsf{t}_a\boldsymbol{\varphi}), \end{aligned}\tag{1075}$$

$$\begin{aligned}\partial_\beta\left[\frac{\partial\mathscr{L}_{\text{tot}}}{\partial(\partial_\beta\boldsymbol{\varphi}^\dagger)}\right] &= \partial_\beta\left[\underbrace{\left(\frac{\partial}{\partial(\partial_\beta\boldsymbol{\varphi}^\dagger)}(\mathsf{D}_\alpha\boldsymbol{\varphi})^\dagger\right)}_{\delta^\beta_\alpha}(\mathsf{D}^\alpha\boldsymbol{\varphi}).\right] \\ &= \partial_\alpha(\mathsf{D}^\alpha\boldsymbol{\varphi}) \\ &= \partial_\alpha\partial^\alpha\boldsymbol{\varphi} + ig(\partial_\alpha A^{a\alpha})\mathsf{t}_a\boldsymbol{\varphi} + igA^{a\alpha}\mathsf{t}_a\partial_\alpha\boldsymbol{\varphi}. \end{aligned}\tag{1076}$$

The Euler-Lagrange equation follows:

$$\partial_\alpha\partial^\alpha\boldsymbol{\varphi} + 2igA^{a\alpha}\mathsf{t}_a\partial_\alpha\boldsymbol{\varphi} - g^2A^a{}_\alpha A^{b\alpha}\mathsf{t}_a\mathsf{t}_b\boldsymbol{\varphi} + m^2\boldsymbol{\varphi} = 0, \quad \text{with} \quad \partial_\alpha A^{a\alpha} = 0 \tag{1077}$$

where we have simplified a term, assuming a generalized Lorenz gauge. You can find, e.g. in Rubakov[133], that the equation of motion can be written in a compact form as

$$\mathsf{D}_\alpha\mathsf{D}^\alpha\boldsymbol{\varphi} + m^2\boldsymbol{\varphi} = 0, \tag{1078}$$

which is just the free particle case with the derivatives replaced by the covariant ones directly, a minimal coupling prescription at the level of the equation of motion. ◄

►EXERCISE 41 – **Non Abelian gauge field equation of motion** –

The Non Abelian gauge field equations (which we could call Maxwell-Yang-Mills equations) follow from the variation w.r.t. the gauge field components,

$$\frac{\delta}{\delta A^a{}_\alpha}\int d^4x \mathscr{L}_{\text{tot}} = 0. \tag{1079}$$

[133] V. Rubakov, Classical Theory of Gauge Fields, Princeton University Press, Princeton, 2002.

The calculation proceeds as follows. For the derivative w.r.t. $A^a{}_\alpha$, it comes

$$\begin{aligned}
\frac{\partial \mathscr{L}_{\rm tot}}{\partial A^a{}_\alpha} &= \frac{\partial}{\partial A^a{}_\alpha}\left[(\mathsf{D}_\beta\boldsymbol{\varphi})^\dagger(\mathsf{D}^\beta\boldsymbol{\varphi}) - \tfrac{1}{4}F^b{}_{\beta\gamma}F_b{}^{\beta\gamma}\right] \\
\frac{\partial}{\partial A^a{}_\beta}(\mathsf{D}_\beta\boldsymbol{\varphi})^\dagger(\mathsf{D}^\beta\boldsymbol{\varphi}) &= \frac{\partial(\mathsf{D}_\beta\boldsymbol{\varphi})^\dagger}{\partial A^a{}_\beta}(\mathsf{D}^\beta\boldsymbol{\varphi}) + (\mathsf{D}^\beta\boldsymbol{\varphi})^\dagger\frac{\partial(\mathsf{D}_\beta\boldsymbol{\varphi})}{\partial A^a{}_\beta} \\
&= -ig(\boldsymbol{\varphi}^\dagger \mathsf{t}_a(\mathsf{D}^\alpha\boldsymbol{\varphi}) - (\mathsf{D}^\alpha\boldsymbol{\varphi})^\dagger\mathsf{t}_a\boldsymbol{\varphi}) \\
-\frac{1}{4}\frac{\partial}{\partial A^a{}_\alpha}(F^b{}_{\beta\gamma}F_b{}^{\beta\gamma}) &= -\frac{1}{2}F_b{}^{\beta\gamma}\frac{\partial F^b{}_{\beta\gamma}}{\partial A^a{}_\alpha} \\
&= -\tfrac{1}{2}gF_b{}^{\beta\gamma}f_{cd}{}^b(\delta^c_a\delta^\alpha_\beta A^d{}_\gamma + \delta^d_a\delta^\alpha_\gamma A^c{}_\beta) \\
&= gf_a{}^{bc}F_b{}^{\alpha\beta}A_{c\beta}
\end{aligned} \tag{1080}$$

and for the derivative w.r.t. the gauge field derivatives,

$$\begin{aligned}
\partial_\beta\left[\frac{\partial\mathscr{L}_{\rm tot}}{\partial(\partial_\beta A^a{}_\alpha)}\right] &= -\tfrac{1}{4}\partial_\beta\left[\frac{\partial}{\partial(\partial_\beta A^a{}_\alpha)}(F^b{}_{\gamma\delta}F_b{}^{\gamma\delta})\right] \\
&= -\tfrac{1}{2}\partial_\beta\left[F_b{}^{\gamma\delta}\delta^b_a(\delta^\beta_\gamma\delta^\alpha_\delta - \delta^\alpha_\gamma\delta^\beta_\delta)\right] \\
&= -\partial_\beta F_a{}^{\beta\alpha}.
\end{aligned} \tag{1081}$$

The equation of motion is thus

$$\partial_\alpha F_a{}^{\alpha\beta} = ig(\boldsymbol{\varphi}^\dagger\mathsf{t}_a(\mathsf{D}^\beta\boldsymbol{\varphi}) - (\mathsf{D}^\beta\boldsymbol{\varphi})^\dagger\mathsf{t}_a\boldsymbol{\varphi}) + gf_a{}^{bc}F_b{}^{\alpha\beta}A_{c\alpha}. \tag{1082}$$

and comprises additional terms than in the Abelian case. ◄

– Diego: This is just crazy indices gymnastic!

– Aïssata: Yes, this is true. Nothing really difficult, except keeping concentration of the indices names and their order. And you may remember that the position of Latin indices doesn't really matter, although I try to keep them at the right position w.r.t. the summation convention.

But the important result is that we obtain a *non linear* theory. Already with the definition of the curvature, you certainly noticed that there was a term quadratic in the gauge field. This time, we get equations of motion which differ from those of Maxwell by a term, written here at the r.h.s., i.e. as a source term, where the matter field doesn't appear at all. This is a current contribution purely due to the gauge field itself. This means that the gauge field contributes to the charge of this interaction. Once quantized, the gauge fields excitations are associated to gauge bosons which are responsible for the mediation of the interaction. Hence, these gauge bosons carry the charge of the interaction that they mediate.

This is a strong difference with the case of $U(1)$. There, the gauge bosons (the photons) don't carry an electric charge. This is connected to the Abelian nature of electrodynamics for which the curvature tensor is linear in the gauge field, the equations of motion are linear and the gauge fields do not contribute to the charge current.

In the non Abelian situation, which we will see to describe weak and strong interactions, the gauge bosons are respectively the particles known as $\mathrm{W}^\pm$ and Z^0 and the gluons. The first ones carry the weak currents while the gluons carry the color charge which provides the interaction between quarks.

– Diego: Aïssata, there is another question, maybe a bit stupid, but I don't really see why this theory is called a gauge theory? This is probably more or less strange, depending on the language used, but in English (gauge), in Spanish (calibrador), in French (jauge), gauge refers for example to a level gauge indicating the oil level in a car engine. This is a length comparison. And the verb built on it gives the idea of a measure. I hardly see the length comparison in our phase factors.

– Aïssata: This is an interesting question Diego. The theory has similar names in various languages, Eichtheorie, gauge theory, théorie de jauge, teoría de calibres. You can find an interesting account in a paper by Wu and Yang[134],

> Weyl's papers on physics were rambling, discursive and philosophical. They were also extraordinarily original. He was exploring new ideas, so there was great fluidity in his style. He frequently changed names for key concepts. Proportionalität later became Strecke, so that when Schrödinger in 1922 referred to the Proportionalitätsfaktor, it became Streckenfaktor (. . .). When the concept of gauge invariance first came up, he called it Massstab-Invarianz. But later he settled on Eichinvarianz. English translation of this term is now gauge invariance, but had been at times calibration invariance and measure invariance.

Originally, as you see, the terminology came form German. In a celebrated paper dating from 1918, entitled Gravitation and electricity, (you can find an English translation in the excellent *Dawning of gauge theory* by O'Raifeartaigh [135]), H. Weyl was looking for a "geometrization" of electrodynamics like Einstein had done with gravitation in his General Theory of Relativity. In the case of GTR, gravitation essentially becomes a geometrical theory characterized primarily by a metric tensor (a second-rank tensor $g_{\alpha\beta}$, as you know) which allows for the introduction of a natural connection, the Levi-Civita connection. This connection is given in terms of derivatives of the metric tensor (Christoffel symbols) and is called a metric connection. The matter-energy content of spacetime governs its geometrical properties, essentially the curvature associated to the Levi-Civita connection through Einstein field equations. An essential feature of GTR is that $4D$ lengths are conserved. It means that when a vector is transported along a closed path in spacetime, its orientation may have been modified w.r.t. its initial orientation while its length remains unchanged. The orientation change is controlled by the curvature, i.e. essentially by the metric tensor. Although based on physical grounds (such a length can for example describe spectral lines emitted by atoms, and one may wish to keep atomic spectra independent of their possible history in terms of spacetime location), this is a strong constraint from the point of view of geometry and H. Weyl elaborated a theory in which this constraint is relaxed. There, a vector transported along a closed path in spacetime is possibly subject to both orientation and length variations. The theory needs a new ingredient to control the length variation, and for that purpose, Weyl had to introduce a 4-vector, say A^α. This "gauge" vector enters the definition of the Weyl connection which now comprises, together with the Christoffel symbols in terms of metric tensor derivatives, a new

[134] A.C.T. Wu and C.N. Yang, Int. J. Mod. Phys. A, **21**, 3235, 2006.
[135] L. O'Raifeartaigh, The Dawning of Gauge Theory, Princeton University Press, Princeton, 1997, p.24.

non metric piece where A_α appears. Changing this field modifies the *gauge* in terms of which lengths are measured and the *gauge transformation* of this new field is just that of the ordinary electromagnetic gauge potential $(\phi, \mathbf{A})^{\mathrm{T}}$ is standard electrodynamics. This is a rather attractive theory, where electrodynamics appears on the same footing as gravitation, i.e. as a geometrical property of spacetime. However, this theory hasn't known any popularity in the Physics community, since it was in contradiction with the experimental facts concerning spectral lines, as we said. This is reported in a very interesting paper by O'Raifeartaigh and Straumann in the Reviews of Modern Physics and that I strongly recommend[136],

> Einstein admired Weyl's theory as a coup of genius of the first rate . . . , but immediately realized that it was physically untenable: Although your idea is so beautiful, I have to declare frankly that, in my opinion, it is impossible that the theory corresponds to Nature.

– Diego: I understand the denomination of gauge in what you tell me, but there isn't any phase factor there.

– Aïssata: Let me go on our story! Right after the advent of Quantum Mechanics, authoritative physicists (Schrödinger, Fock, Klein, London) remarked that there was there a transformation which had some analogies with Weyl's gauge transformation, and they emphasized on the role of the phase transformations of the wave function. This is reported in the book of O'Raifeartaigh and in the review article that I mentioned. This had probably an influence on Weyl, who revisited his former intuition, but adapted it in 1929 in the context of Relativistic Quantum Mechanics, soon after Dirac's paper, in a paper which became one of the most famous in Physics. Again, this wasn't immediate. O'Raifeartaigh and Straumann tell us the story:

> In a letter by Pauli to Weyl of July 1, 1929, after he had seen a preliminary account of Weyls work: Before me lies the April edition of the Proc. Nat. Acad. (US). Not only does it contain an article from you under Physics but shows that you are now in a Physical Laboratory: from what I hear you have even been given a chair in Physics in America. I admire your courage; since the conclusion is inevitable that you wish to be judged, not for success in pure mathematics, but for your true but unhappy love for physics.
>
> When Pauli saw the full version of Weyl's paper he became more friendly and wrote (. . .): In contrast to the nasty things I said, the essential part of my last letter has since been overtaken, particularly by your paper in Z. f. Physik. For this reason I have afterward even regretted that I wrote to you. After studying your paper I believe that I have really understood what you wanted to do (this was not the case in respect of the little note in the Proc. Nat. Acad.). First let me emphasize that side of the matter concerning which I am in full agreement with you: your incorporation of spinor theory into gravitational theory. I am as dissatisfied as you are with distant parallelism and your proposal to let the tetrads rotate independently at different space-points is a true solution. In brackets Pauli adds: Here I must admit your ability in Physics. Your earlier theory with $g'_{ik} = \lambda g_{ik}$ was pure mathematics and unphysical. Einstein was justified in criticizing and scolding. Now the hour of your revenge has arrived.

[136] L. O'Raifeartaigh and N. Straumann, Rev. Mod. Phys. **72**, 1, 2000.

The work of Weyl then spread in the physics community through an influential paper of Pauli[137] himself.

– Diego: This is incredible how scientific theories have to reach a kind of maturity to be accepted!

– Aïssata: And also how this maturity often arises simultaneously in several minds. Going on with quotations of O'Raifeartaigh and Straumann:

> Independently of Weyl, Fock (1929) also incorporated the Dirac equation into general relativity using the same method. On the other hand, Tetrode (1928), Schrödinger (1932), and Bargmann (1932) reached this goal by starting with space-time-dependent γ^μ matrices, satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. A somewhat later work by Infeld and van der Waerden (1932) is based on spinor analysis.

– Diego: And the original 1918 theory of Weyl was completely abandoned?

– Aïssata: As far as I know, this is known as "Conformal gravity" and is still subject of interest in the context of cosmology, since it is a candidate to solve some of the cosmological problems, those of Dark matter and Dark energy in particular[138]. But I think that the theory has other inconveniences.

21. Day 17 – Spontaneous symmetry breaking

▷ 21.1 Spontaneous breaking of global discrete symmetries

– Aïssata: Let us first consider spontaneous breaking of a global *discrete* symmetry. It is illustrated here by the case of the real scalar field,

$$\mathscr{L} = \tfrac{1}{2}\partial_\alpha\phi\partial_\alpha\phi - V(\phi(x)), \tag{1083}$$

where $V(\phi(x))$ is a potential which depends on the field configuration. We will consider two cases

$$V(\phi) = \frac{\lambda}{4}\phi^4 \pm \frac{\mu^2}{2}\phi^2, \quad \lambda, \mu^2 > 0. \tag{1084}$$

Case 1 with the sign $+$ corresponds to a scalar field theory with square mass μ^2. Let us first build the Hamiltonian,

$$\mathscr{H} = \tfrac{1}{2}(\partial_0\phi)^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi). \tag{1085}$$

The field with the lowest energy (also called the ground state configuration is a constant field which minimizes the potential. In case 1, it corresponds to a vanishing field $\phi_0 = 0$. The discrete Z_2 symmetry $\phi \to -\phi$ of the Lagrangian is also a symmetry of the ground state. In case 2, with sign $-$ in the potential, the homogeneous ground state field is given by

$$\phi_0 = \pm\frac{\mu}{\sqrt{\lambda}} = \pm v. \tag{1086}$$

[137] W. Pauli, Rev. Mod. Phys. **13**, 203, 1941.

[138] Ph. Mannheim, Alternatives to Dark Matter and Dark Energy, arxiv.org/astro-ph/0505266.

While the Lagrangian still possesses the $\phi \to -\phi$ symmetry, in any of the two degenerate ground states, this Z_2 symmetry is broken. This situation occurs in the low temperature phase of second order phase transitions, when an ordered ground state emerges (for example a ferromagnetic ground state), which does not respect the full symmetry of the Hamiltonian (e.g. the "up-down" symmetry in an Ising model. You will see this later in Condensed Matter Physics applications.

It is instructive to study the field fluctuations around this ground state. For this purpose, we let

$$\phi = v + h, \quad h \ll v, \tag{1087}$$

(v is chosen positive without loss of generality) and we expand

$$V(\phi) \quad = \quad -\tfrac{1}{4}\mu^4/\lambda + 0 + \tfrac{1}{2}2\mu^2 h^2 + 2\sqrt{\lambda}\mu h^3 + \tfrac{1}{4}\lambda h^4. \tag{1088}$$

We note that the new field v has acquired a mass $\sqrt{2}\mu$ in the vicinity of the broken Z_2 symmetry (the coefficient of the quadratic term in h is now positive).

▷ 21.2 Spontaneous breaking of global continuous symmetries

– Aïssata: Spontaneous breaking of a global *continuous* symmetry can be encountered in the $SO(2)$ model (rotations in the plane), or in $O(3)$ rotation symmetry, e.g. in the Heisenberg model in Condensed Matter Physics. For the sake of simplicity, we consider now a theory with two real scalar fields $\phi_1(x)$ and $\phi_2(x)$, and with the potential

$$V(\phi_1, \phi_2) = \tfrac{1}{4}\lambda(\phi_1^2 + \phi_2^2 - v^2)^2 = \tfrac{1}{4}\lambda(|\boldsymbol{\phi}|^2 - v^2)^2. \tag{1089}$$

The fields ϕ_1 and ϕ_2 are massless (the coefficients of the quadratic terms are negative) and the theory is invariant under rotations in the plane,

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{1090}$$

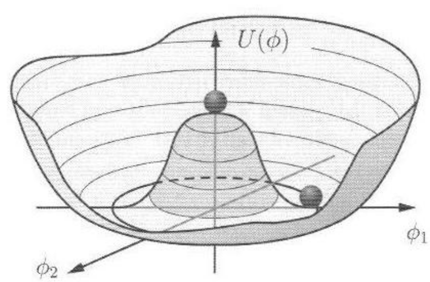


Figure 27. The "Mexican hat potential" (from E.A. Paschos, Electroweak theory, Cambridge University Press, Cambridge 2007).

The minima of the potential lie on the circle

$$|\boldsymbol{\phi}_0|^2 = \phi_{10}^2 + \phi_{20}^2 = v^2. \tag{1091}$$

As a result of the continuous symmetry, they are infinitely degenerate. In order to analyze the field fluctuations around the minimum, we choose a particular vacuum state $\phi_{10} = v$, $\phi_{20} = 0$ and denote the fluctuations by

$$\phi_1 = v + h_1, \quad \phi_2 = h_2 \tag{1092}$$

in terms of which the potential becomes

$$V(h_1, h_2) = \tfrac{1}{4}\lambda(h_1^2 + h_2^2 + 2vh_1)^2. \tag{1093}$$

Expansion of this potential shows that h_1 becomes massive while h_2 remains massless, and the appearance of cubic terms breaks the original $SO(2)$ symmetry. The massless field is called a Goldstone mode (or Nambu-Goldstone mode). It is easy to understand why h_2 remains massless while h_1 acquired a mass: close to the minimum which we have selected, ϕ_1 fluctuations have to survive to the potential growth, these are amplitude fluctuations in a polar representation of the model, while ϕ_2 fluctuations correspond to phase fluctuations which do not cost any energy.

▷ 21.3 Spontaneous breaking of local continuous symmetries

– Aïssata: A new phenomenon occurs with local gauge theories, where the selection of a particular minimum and the fluctuations around this minimum lead to massive gauge fields which would otherwise be forbidden, since mass terms for the gauge field would break gauge invariance. At the same time, the Goldstone mode disappears.

We consider the Lagrangian density of the $U(1)$ gauge theory,

$$\mathscr{L} = -\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + (\partial_\alpha\varphi)^*(\mathsf{D}_\alpha\varphi) - V(\varphi^*\varphi), \tag{1094}$$

with the potential

$$V(\varphi^*\varphi) = -\mu^2\varphi^*\varphi + \lambda(\varphi^*\varphi)^2. \tag{1095}$$

As we have seen before, the theory is invariant under the gauge transformation corresponding to a local rotation of the scalar field in the complex plane

$$\varphi(x) \quad \to \quad e^{ie\alpha(x)}\varphi(x) \tag{1096}$$

$$A_\alpha(x) \quad \to \quad A_\alpha(x) - \partial_\alpha\alpha(x). \tag{1097}$$

Let us define two real fields $\theta(x)$ and $h(x)$, associated to the phase and the amplitude fluctuations around a particular (chosen real) minimum $v = (\mu^2/2\lambda)^{1/2}$,

$$\varphi(x) = e^{i\theta(x)/v}\frac{1}{\sqrt{2}}(v + h(x)). \tag{1098}$$

The local gauge transformation defined by $e\alpha(x) = -\theta(x)/v$ eliminates $\theta(x)$, since

$$\varphi'(x) = e^{-i\theta(x)/v}\varphi(x) = \frac{1}{\sqrt{2}}(v + h(x)), \tag{1099}$$

$$A'_\alpha(x) = A_\alpha(x) + \frac{1}{ev}\partial_\alpha\theta(x). \tag{1100}$$

The net effect in the Lagrangian density is the following,

$$\mathscr{L} = -\frac{1}{4}F'_{\alpha\beta}F'^{\alpha\beta} + (\mathcal{D}'_\alpha\varphi')^*(\mathcal{D}'^\alpha\varphi') + \frac{1}{2}\mu^2(v+h^2(x))^2 - \frac{1}{4}\lambda(v+h(x))^4, \tag{1101}$$

with $\mathcal{D}'_\alpha = \partial_\alpha + ieA'_\alpha$. The kinetic energy term generates the mass for the gauge field A'_α:

$$(\mathcal{D}'_\alpha\varphi')^*(\mathcal{D}'^\alpha\varphi') = \frac{1}{2}\partial_\alpha h \mathrm{D}_\alpha h + \frac{1}{2}e^2 A'_\alpha A'^\alpha(v^2 + 2hv + h^2), \tag{1102}$$

and, as we announced, the Goldstone mode $\theta(x)$ was absorbed in the re-definition of the gauge field.

This mechanism is known in condensed matter physics as the Anderson mechanism, and it occurs in superconductivity, where the non-zero mass (which also defines a characteristic length scale) of the gauge field is responsible for the Meissner effect (the fact that the magnetic field is expelled from the bulk of the material). In particle physics, this mechanism enables to give a mass to the gauge bosons, as we discuss below. This is known in this context as the Higgs mechanism.

22. Day 18 – An introduction to the Standard Model and to its gauge symmetry breaking

▷ **22.1** $SU(3)_C$ **gauge theory of QCD**

– Aïssata: QCD (quantum chromodynamics) is the theory which describes the strong interactions among quarks. This is a gauge theory based on the Lie group $SU(3)$. The quarks possess colour quantum numbers and form the fundamental (triplet) representation of the $SU(3)$ group. There are six fundamental triplets (called the quarks flavours):

$$\begin{pmatrix} u_\mathrm{r} \\ u_\mathrm{g} \\ u_\mathrm{b} \end{pmatrix}, \quad \begin{pmatrix} d_\mathrm{r} \\ d_\mathrm{g} \\ d_\mathrm{b} \end{pmatrix}, \quad \begin{pmatrix} c_\mathrm{r} \\ c_\mathrm{g} \\ c_\mathrm{b} \end{pmatrix}, \quad \begin{pmatrix} s_\mathrm{r} \\ s_\mathrm{g} \\ s_\mathrm{b} \end{pmatrix}, \quad \begin{pmatrix} t_\mathrm{r} \\ t_\mathrm{g} \\ t_\mathrm{b} \end{pmatrix}, \quad \begin{pmatrix} b_\mathrm{r} \\ b_\mathrm{g} \\ b_\mathrm{b} \end{pmatrix}. \tag{1103}$$

In the fundamental triplet representation which is standard in the Physics literature, the 8 generators re defined by

$$\mathrm{t}_a = \frac{1}{2}\lambda_a \tag{1104}$$

with λ_a's the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1105}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{1106}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{1107}$$

They obey the algebra

$$\left[\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b\right] = \tfrac{1}{2} i f_{ab}{}^{c}\lambda_c \tag{1108}$$

where the antisymmetric structure constants f_{abc} vanish if the number of indices from the set $\{2,5,7\}$ is odd. The remaining structure constants are given by

$$f_{123} = 1, \tag{1109}$$

$$f_{147} = f_{246} = f_{257} = f_{345} = \tfrac{1}{2}, \tag{1110}$$

$$f_{156} = f_{367} = -\tfrac{1}{2}, \tag{1111}$$

$$f_{458} = f_{678} = \tfrac{\sqrt{3}}{2}. \tag{1112}$$

One usually introduces also the *anticommutation* relations

$$\left\{\tfrac{1}{2}\lambda_a, \tfrac{1}{2}\lambda_b\right\} = \tfrac{1}{3}\delta_{ab}1 + \tfrac{1}{2} d_{ab}{}^{c}\lambda_c, \tag{1113}$$

with *symmetric* coefficients d_{abc}.

This is in fact useful to introduce the *spherical representation* instead of the λ_a's representation:

$$\mathsf{T}_\pm = \mathsf{t}_1 \pm i\mathsf{t}_2, \quad \mathsf{T}_3 = \mathsf{t}_3, \tag{1114}$$

$$\mathsf{V}_\pm = \mathsf{t}_4 \pm i\mathsf{t}_5, \quad \mathsf{U}_\pm = \mathsf{t}_6 \pm i\mathsf{t}_7, \quad \mathsf{Y} = \frac{2}{\sqrt{3}}\mathsf{t}_8. \tag{1115}$$

– Diego: Where do you get these opreators Aïssata?

– Aïssata: The first three are just the same as in the case of $SU(2)$. You may have noticed that for $a = 1,2,3$ we have

$$\lambda_a = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix} \tag{1116}$$

and the operators $\mathsf{T}_\pm$ are just the equivalent as the ladder operators $\sigma_\pm$, while $\mathsf{T}_3 \sim \sigma_3$. The other $\mathsf{U}_\pm$, $\mathsf{V}_\pm$, Y are built in a similar manner (you can find the Pauli matrices still hidden in λ_a's with $a = 4,5,6,7$). There is now a bunch of commutation relations among these operators that you can find in the literature, e.g. in Guidry, or Greiner and Müller[139], and from where it appears that only two commuting operators can be chosen simultaneously. This is why the diagonal matrices λ_3 and λ_8 (or T_3 and Y) and their eigenvalues are used to label the components of an $SU(3)_C$ multiplet. Among the commutation relations, those which are of special interest for us are obviously

$$[\mathsf{T}_+, \mathsf{T}_-] = 2\mathsf{T}_3, \quad [\mathsf{T}_3, \mathsf{T}_\pm] = \pm\mathsf{T}_\pm \tag{1117}$$

that you deduce automatically from (1116) and which show that λ_1, λ_2,λ_3 form a stable $SU(2)$ subalgebra. The same comes out with the operators $\mathsf{U}_+, \mathsf{U}_-, \mathsf{U}_3$ and $\mathsf{V}_+, \mathsf{V}_-, \mathsf{V}_3$ for which one has

$$[\mathsf{U}_+, \mathsf{U}_-] = 2\mathsf{U}_3, \quad [\mathsf{U}_3, \mathsf{U}_\pm] = \pm\mathsf{U}_\pm, \tag{1118}$$

$$[\mathsf{V}_+, \mathsf{V}_-] = 2\mathsf{V}_3, \quad [\mathsf{V}_3, \mathsf{V}_\pm] = \pm\mathsf{V}_\pm. \tag{1119}$$

All three $SU(2)$ algebras are angular momentum algebras which show that $\mathsf{T}_\pm$, $\mathsf{U}_\pm$ and $\mathsf{V}_\pm$ are ladder operators. As we mentioned, T_3 and Y commute with all other generators and a convenient basis is given by their eigenstates, $|T_3, Y\rangle$, where one knows, from what was said above, that T_3 can take positive and negative half integer values.

[139] M. Guidry, Gauge Field Theory, Wiley,-VCH, Weinheim, 2004, p.183, W. Greiner and B. Müller, Mécanique Quantique, Symétries, Springer, Berlin, 1999, p.204.

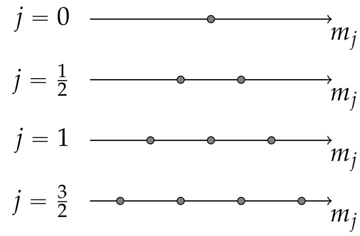


Figure 28. The $SU(2)$ multiplets classified for each value of j which characterize J^2 ($\mathsf{J}^2|j, m_j\rangle = j(j+1)|j, m_j\rangle$) vs m characterizing J_3 ($\mathsf{J}_3|j, m_j\rangle = m_j|j, m_j\rangle$).

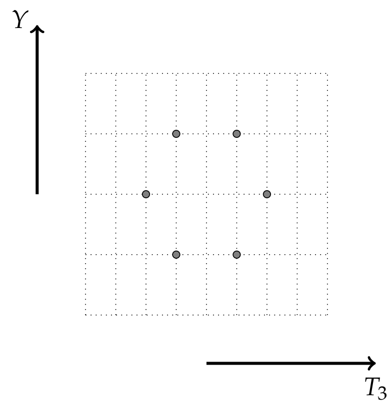


Figure 29. $SU(3)$ multiplets are represented in Y vs T_3 diagrams.

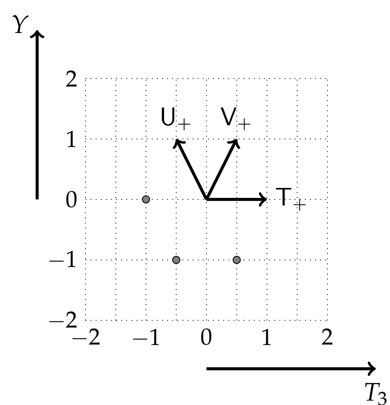


Figure 30. The values of Y are integer values while those of T_3 are half integers. T_+ (resp. T_-) acts in the horizontal direction while $\mathsf{V}_\pm$ and $\mathsf{U}_\pm$ acts according to the diagram.

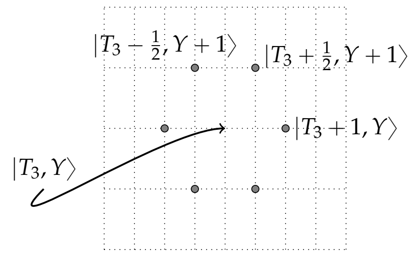


Figure 31. Starting from a state $|T_3, Y\rangle$ (e.g. here at the center of the diagram), $\mathsf{T}_\pm|T_3, Y\rangle$ produces a state at $|T_3 \pm 1, Y\rangle$, $\mathsf{V}_\pm|T_3, Y\rangle$ produces a state at $|T_3 \pm \frac{1}{2}, Y \pm 1\rangle$ and $\mathsf{U}_\pm|T_3, Y\rangle$ produces a state at $|T_3 \mp \frac{1}{2}, Y \pm 1\rangle$ (except if they annihilate the state).

▷ 22.2 Gauge symmetric electroweak $SU(2)_W \times U(1)_Y$ theory

– Aïssata: The electroweak symmetry breaking scenario discovered independently by Weinberg and Salam describes the emergence of the present structure of electromagnetic and weak interactions as the broken gauge symmetry phase of a symmetric (unbroken) phase $SU(2)_W \times U(1)_Y$ which existed in earlier times (higher energy scales) of the Universe[140]. With the spontaneous symmetry breaking scenario, some of the bosonic degrees of freedom (the gauge fields) acquire mass. In the symmetric phase, the relevant (non massive) fermionic particles (the electron and the neutrino) consist in a right-handed [141] electron $\mathtt{R} = e_R$ in an (weak) isospin singlet $I_W = 0$ and an isospin doublet $I_W = \frac{1}{2}$ made of the left-handed electron and the unique (left-handed) neutrino $\mathtt{L} = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$. The bosons are all non massive. The charges carried by the leptons follow from their weak isospin component I_W^3 and their hypercharge Y,

$$Q = I_W^3 + \frac{Y}{2}. \tag{1120}$$

The hypercharge of the doublet is thus $Y_L = -1$ and that of the singlet is $Y_R = -2$. Under the non-Abelian weak isospin gauge transformation $SU(2)_W$, the fields change according to

$$\begin{aligned} \mathtt{R} &\underset{SU(2)_W}{\longrightarrow} \mathtt{R}, \\ \mathtt{L} &\underset{SU(2)_W}{\longrightarrow} \exp\left(\tfrac{1}{2} i g \alpha^a \tau_a\right) \mathtt{L}, \end{aligned} \tag{1121}$$

with τ_a the Pauli matrices, and under the Abelian $U(1)_Y$ symmetry, they become

$$\mathtt{R} \underset{U(1)_Y}{\longrightarrow} \exp(-ig'\beta)\mathtt{R},$$

[140] Wikipedia tells the story that way: *In 1964, Salam, Ward and Weinberg had the same idea, but predicted a massless photon and three massive gauge bosons with a manually broken symmetry. Later around 1967, while investigating spontaneous symmetry breaking, Weinberg found a set of symmetries predicting a massless, neutral gauge boson. Initially rejecting such a particle as useless, he later realized his symmetries produced the electroweak force, and he proceeded to predict rough masses for the W and Z bosons. Significantly, he suggested this new theory was renormalizable. In 1971, Gerard 't Hooft proved that spontaneously broken gauge symmetries are renormalizable even with massive gauge bosons.*.

[141] In the Dirac Lagrangian $i\bar{\psi}\gamma^\alpha\partial_\alpha\psi - m\bar{\psi}\psi$, the right-handed and left-handed spinors are defined as $\mathtt{R} = \frac{1}{2}(1+\gamma_5)\psi$ and $\mathtt{L} = \frac{1}{2}(1-\gamma_5)\psi$. Since γ_5 and γ_α commute, it follows that $i\bar{\psi}\gamma^\alpha\partial_\alpha\psi = i\bar{\mathtt{L}}\gamma^\alpha\partial_\alpha\mathtt{L} + i\bar{\mathtt{R}}\gamma^\alpha\partial_\alpha\mathtt{R}$ (see p. 135).

$$\mathtt{L} \underset{U(1)_Y}{\longrightarrow} \exp(-ig'\beta/2)\mathtt{L}. \tag{1122}$$

Note that the isospin coupling is g while the hypercharge coupling is conventionally called $g'/2$.

$SU(2)_W \times U(1)_Y$ is made a local gauge symmetry through the introduction of gauge fields $W^a{}_\alpha$ and X_α with the covariant derivative

$$\begin{aligned} \mathsf{D}_\alpha \mathtt{L} &= \partial_\alpha \mathtt{L} + \tfrac{1}{2} i g W^a{}_\alpha \tau_a \mathtt{L} - \tfrac{1}{2} i g' X_\alpha \mathtt{L}, \\ \mathcal{D}_\alpha \mathtt{R} &= \partial_\alpha \mathtt{R} - i g' X_\alpha \mathtt{R}, \end{aligned} \tag{1123}$$

where $W^a{}_\alpha$ is a weak triplet gauge (non-massive) boson $I_W = 1$ with hypercharge zero and X_α is also a non-massive boson which has zero hypercharge, but is in an isospin singlet $I_W = 0$.

We consider non massive fermions, otherwise a term like $m^2\bar{\mathtt{L}}\mathtt{L}$ would assign the same mass to the electron and the neutrino (more precisely, if the electron would have a non zero mass in this theory, the corresponding neutrino would share the same mass, since it appears as the second component of an isospin doublet) [142]. If we forget about the pure gauge field contributions, the kinetic part of the Lagrangian in the minimal coupling is given by Dirac Lagrangian (the leptonic particles are fermions with spin $\frac{1}{2}$) i.e.

$$\mathscr{L} = i\bar{\mathtt{R}}\gamma^\alpha \left(\partial_\alpha - ig'X_\alpha\right)\mathtt{R} + i\bar{\mathtt{L}}\gamma^\alpha \left(\partial_\alpha + \tfrac{1}{2} i g W^a{}_\alpha \tau_a - \tfrac{1}{2} i g' X_\alpha\right)\mathtt{L}. \tag{1124}$$

The weakness of the gauge invariant formulation is obviously that it contains 4 massless gauge fields, while Nature (at the present energy scales) has only one, and that the fermions are similarly all non massive. The spontaneous symmetry breaking scenario leads to 3 massive gauge fields and at the same time, the electron acquires mass as well (but not the neutrino yet!).

▷ 22.3 Spontaneous Gauge Symmetry breaking and the Higgs mechanism in the electroweak theory

– Aïssata: The symmetry is broken by the introduction of a (bosonic) complex Higgs field

$$\boldsymbol{\phi} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta_1 + i\theta_2 \\ \theta_3 + i\theta_4 \end{pmatrix}. \tag{1125}$$

This is an isospin doublet $I_W = \frac{1}{2}$ with hypercharge unity $Y_\phi = 1$,

$$\mathsf{D}_\alpha \boldsymbol{\phi} = \left(\partial_\alpha + \tfrac{1}{2} i g W^a{}_\alpha \tau_a + \tfrac{1}{2} i g' X_\alpha\right)\boldsymbol{\phi}, \tag{1126}$$

and a Lagrangian of the form

$$\mathscr{L}_{\text{Higgs}} = (\mathsf{D}_\alpha\boldsymbol{\phi})^\dagger(\mathsf{D}^\alpha\boldsymbol{\phi}) - m^2\boldsymbol{\phi}^\dagger\boldsymbol{\phi} - \lambda(\boldsymbol{\phi}^\dagger\boldsymbol{\phi})^2 + \text{interaction with leptons}. \tag{1127}$$

The potential $V(|\boldsymbol{\phi}|) = m^2\boldsymbol{\phi}^\dagger\boldsymbol{\phi} + \lambda(\boldsymbol{\phi}^\dagger\boldsymbol{\phi})^2$ is chosen such that it gives rise to spontaneous symmetry breaking with $|\boldsymbol{\phi}|^2 = -m^2/2\lambda = v/\sqrt{2}$. For the classical field, the choice $\theta_3 = v$

[142] Particles in an isospin multiplet have approximately the same mass. This is the case for the strong isospin already, e.g. the proton and the neutron which form an isospin doublet, or the $\pi-$mesons which form an isoispin triplet.

is made and a local gauge transformation eliminates the other θ_i's. Fluctuations around v are introduced through

$$\boldsymbol{\phi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}. \tag{1128}$$

Acting with the covariant derivative gives

$$\mathsf{D}_\alpha \boldsymbol{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} ig(W_\alpha^1 - iW_\alpha^2)(v + h(x)) \\ \partial_\alpha h - \frac{1}{2} i(gW_\alpha^3 - g' X_\alpha)(v + h(x)) \end{pmatrix} \tag{1129}$$

and reported in the Lagrangian density, this leads to (up to cubic terms)

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &= \tfrac{1}{2} \Big[\partial_\alpha h \partial^\alpha h - \tfrac{1}{2} m^2 (v + h(x))^2 - \tfrac{1}{4} \lambda (v + h(x))^4 \\ &\quad + \tfrac{1}{4} g^2 v^2 (W_\alpha^1 W^{\alpha 1} + W_\alpha^2 W^{\alpha 2}) + \tfrac{1}{4} (gW_\alpha^3 - g' X_\alpha)(gW^{\alpha 3} - g' X^\alpha) v^2 \Big] \\ &= \tfrac{1}{2} \Big[\partial_\alpha h \partial^\alpha h - \tfrac{1}{2} m^2 (v + h(x))^2 - \tfrac{1}{4} \lambda (v + h(x))^4 \Big] \\ &\quad + M_W^2 W_\alpha^+ W^{-\alpha} + \tfrac{1}{2} M_Z^2 Z_\alpha Z^\alpha, \end{aligned} \tag{1130}$$

where the charged massive vector bosons are

$$W_\alpha^\pm = (W_\alpha^1 \mp iW_\alpha^2)/\sqrt{2} \tag{1131}$$

with masses $M_W^2 = \frac{1}{4} g^2 v^2$ and the neutral massive boson is such that [143]

$$\begin{aligned} \tfrac{1}{2} M_Z^2 Z_\alpha Z^\alpha &= \tfrac{1}{8} v^2 (gW_\alpha^3 - g' X_\alpha)(gW^{\alpha 3} - g' X^\alpha) \\ &= \tfrac{1}{8} v^2 (W_\alpha^{3*}, X_\alpha^*) \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^{3\alpha} \\ X^\alpha \end{pmatrix} \\ &= \tfrac{1}{2} (Z_\alpha{}^*, A_\alpha^*) \begin{pmatrix} M_Z^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z^{3\alpha} \\ A^\alpha \end{pmatrix}. \end{aligned} \tag{1132}$$

The last line is obtained by a diagonalization of the mass matrix by an orthogonal transformation

$$\begin{aligned} Z_\alpha &= \cos\theta_W W_\alpha^3 - \sin\theta_W X_\alpha \\ A_\alpha &= \sin\theta_W W_\alpha^3 + \cos\theta_W X_\alpha, \end{aligned} \tag{1133}$$

and the masses of the neutral fields are

$$\begin{aligned} M_Z^2 &= \tfrac{1}{4} v^2 (g^2 + g'^2) \\ M_A^2 &= 0. \end{aligned} \tag{1134}$$

The coupling constant of the (charged) leptons and the electromagnetic gauge field gets the value

$$e = g \sin\theta_W. \tag{1135}$$

With the symmetry breaking scenario, the coupling between the Higgs fields and the leptons of the theory (Yukawa term which forms a Lorentz scalar by the coupling between a Dirac spinor with a scalar field) in

$$-G_e(\bar{\mathtt{R}}\boldsymbol{\phi}^* \mathtt{L} + \bar{\mathtt{L}} \boldsymbol{\phi} \mathtt{R}) \tag{1136}$$

similarly leads to massive electrons [144]

$$m_e = G_e v/\sqrt{2}. \tag{1137}$$

[143] See T.-P. Cheng and L.-F. Li, Gauge Theory of Elementary Particle Physics, Clarendon Press, 1984, p.351.
[144] See C. Quigg, Gauge Theories of the Strong, Weak, and Electromagnetic Interactions, p.110.

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Le carré rouge sur le logo du LPCT se réfère à l'opposition à la loi indigne "Bienvenue en France" de 2019 qui a imposé une augmentation faramineuse des frais d'inscription à l'université pour les étudiants hors Union Européenne, brisant en cela l'équité entre étudiants et la tradition d'un enseignement public gratuit (ou en tout cas à prix modique comme l'a recommandé le Conseil Constitutionnel). Les premiers étudiants victimes de cette loi sont ceux qui viennent d'Afrique, les plus nombreux parmi les étudiants étrangers en France.