A 3d Ising model with a weakly-coupled string dual?

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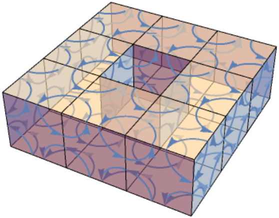
based on work with
Nabil Iqbal (Durham) in progress



2019-05-03

Testimonials.

... better than CATS.

– Anonymous

... a surprising confluence of physics, programming and arts & crafts.

– N. I.

Motivation

Landau was even more right than we thought.

Landau paradigm part 1:

Phases of matter are classified by how they represent their symmetries.

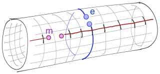
(Phases of matter are classified by the symmetries they break.)

Gapless excitations or degeneracy (in a phase) are Goldstone modes for spontaneously broken symmetries.

Some apparent exceptions:

- topological order [Wegner, Wen]

e.g. deconfined phase of $\mathbb{Z}_2$ lattice gauge theory, fractional quantum Hall states.



- other deconfined states of gauge theory (*e.g.* Coulomb phase of E&M).
- (Landau) Fermi liquid.
- topological insulator and integer quantum Hall states.
- CFTs with no (symmetric) relevant operators.

Higher-form symmetries.

[Willett et al 14, Hofman-Iqbal, Lake...]

0-form symmetry:

$\partial^\mu j_\mu = 0$ (*i.e.* $d \star j = 0$)
$\implies Q = \int_{\Sigma_{D-1}} \star j$ is independent of time-slice Σ,
i.e. **is topological.** [Thorngren]

Charged objects are local operators
$\delta \mathcal{O}(x) = \mathbf{i}[Q, \mathcal{O}(x)] = \mathbf{i} q \mathcal{O}(x)$.

Finite transformation:
$U_{g=e^{\mathbf{i}\alpha}} = e^{\mathbf{i}\alpha Q} = e^{\mathbf{i}\alpha \int_{\Sigma_{D-1}} \star j}$.

Charged particle worldlines can't end.

Discrete ($\mathbb{Z}_k$) version: particles can disappear in groups of k.

1-form symmetry:

$J_{\mu\nu} = -J_{\nu\mu}$ with $\partial^\mu J_{\mu\nu} = 0$
(*i.e.* $d \star J = 0$)
$\implies Q_\Sigma = \int_{\Sigma_{D-2}} \star J$ depends only on the topological class of Σ.

Charged objects are loop operators:
$\delta W(C) = \mathbf{i}[Q_\Sigma, W(C)] = \mathbf{i} q W(C)$
e.g. in free Maxwell theory:
$J^M = F$, $W^M(C) = e^{\mathbf{i}\oint_C A}$ and
$J^E = \star F, W^E(C) = e^{\mathbf{i}\oint_C \tilde{A}}$ $(dA \equiv \star d\tilde{A})$.

Finite transformation:
$U_{g=e^{\mathbf{i}\alpha}}(\Sigma_{D-p-1}) = e^{\mathbf{i}\alpha Q_\Sigma} = e^{\mathbf{i}\alpha \int_{\Sigma_{D-p-1}} \star J}$.

Charged string worldsheets can't disappear or end.

Discrete ($\mathbb{Z}_k$) version: strings can disappear or end in groups of k.

Higher-form symmetries.

[Willett et al, Hofman-Iqbal, Lake]

0-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object ($S^0 = \text{two points}$) grows.

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle \sim e^{-m|x|}$

($|x| = \text{Area}(S^0(x))$.)

1-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object grows.

$\langle W(C) \rangle \sim e^{-T_{p+1}\text{Area}(C)}$

For E&M, area law for $\langle W^E(C) \rangle$ is the superconducting phase.

Landau was even more right than we thought.

- The gaplessness of the photon can be understood as required by spontaneously broken $\mathsf{U}(1)$ 1-form symmetry.

[Willett et al, Hofman-Iqbal, Lake]

Broken phase for 0-form sym:
$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \langle \mathcal{O}^\dagger \rangle \langle \mathcal{O} \rangle + ...$

If we couple to a bg field $\Delta L = j_\mu \mathcal{A}^\mu$,

$$\mathcal{L}_{\text{eff}} = \frac{\kappa}{2} \left(\underbrace{d\phi}_{\text{Goldstone}} + \mathcal{A} \right)^2 .$$

The goldstone transforms nonlinearly $\phi \to \phi + \lambda, \mathcal{A} \to \mathcal{A} - d\lambda$. This is a global symmetry if $d\lambda = 0$.
(By (form)^2 I mean (form) $\wedge \star$(form).)

Broken phase for 1-form sym:
$\langle W(C) \rangle = e^{-T_p \text{Perimeter}(C)} + ...$
(set to 1 by counterterms local to C: large loop has a vev)

If we couple to a bg field $\Delta L = J_{\mu\nu} \mathcal{B}^{\mu\nu}$,

$$\mathcal{L}_{\text{eff}} = \frac{g^2}{4} \left(\underbrace{d\tilde{A}}_{\text{Goldstone}} + \mathcal{B} \right)^2 .$$

The goldstone transforms nonlinearly $\tilde{A} \to \tilde{A} + \lambda, \mathcal{B} \to \mathcal{B} - d\lambda$. This is a global symmetry if $d\lambda = 0$.
Maxwell term for A.

Landau was even more right than we thought.

- topological order $\stackrel{?}{=}$ SSB of *discrete* higher-form symmetry.

SSB of 0-form discrete symmetry $\Longrightarrow$ domain wall excitations.

SSB of $(q > 1)$-form discrete symmetry implies topological order, since the algebra of loop (or surface) operators must be realized on the vacuum.

- eg 1 ($\mathbb{Z}_k$ gauge theory): in D spacetime dimensions with $\mathbb{Z}_k^{(1)} \times \mathbb{Z}_k^{(D-2)}$ 1-form and $(D-2)$-form symmetries, represented by $U^m(C_1), V^n(M_{D-2}), m, n = 1..k$

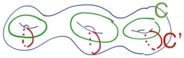
$$U^m(C)V^n(M) = e^{\frac{2\pi \mathbf{i} mn\#(C,M)}{k}} V^n(M)U^m(C). \quad (\#(C,M) \equiv \text{intersection } \#)$$

This is the algebra of electric and magnetic flux surfaces in $\mathbb{Z}_k$ gauge theory. Simple realization is BF theory:

$$S = \frac{k}{2\pi}\int_D B_{D-2} \wedge dA, \qquad U^m(C) = e^{\mathbf{i}m\int_C A}, \quad V^n(M) = e^{\mathbf{i}n\int_M B_{D-2}}$$

- eg 2 (FQHE): in $D = 2+1$, $\mathbb{Z}_k^{(1)}$ 1-form symmetry with an 't Hooft anomaly

$$U^m(C)U^n(C') = e^{\frac{2\pi \mathbf{i} mn\#(C,C')}{k}} U^n(C')U^m(C).$$



(the flux carries charge) gives k groundstates on T^2.

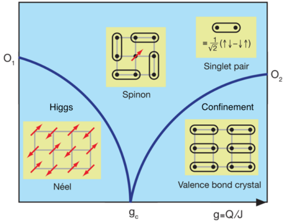
(Whether the most general topologically ordered state can be understood in this way is an open question [Wen 18].)

‘Beyond-Landau’ critical points?

Landau paradigm part 2:
At a critical point, the critical dofs are the fluctuations of the order parameter.

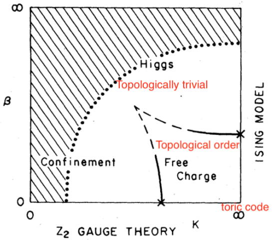
Apparent exceptions:

- Direct transitions between states which break *different symmetries* (deconfined quantum critical points), *e.g.* Neel to VBS in $D = 2 + 1$.



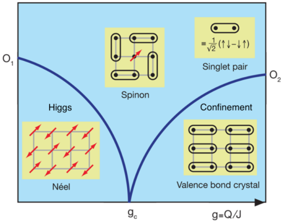
[Image: Alan Stonebraker]

- Transitions out of topologically-ordered states (no local order parameter).



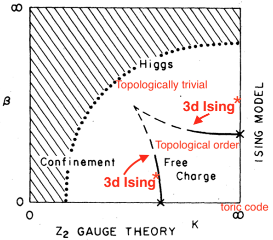
[Image: Fradkin-Shenker]

‘Beyond-Landau’ critical points?

- 

Can be understood in terms of mixed ’t Hooft anomalies [Metlitski-Thorngren 18]

$\implies$ WZW terms coupling the order parameters on both sides. (Not today’s focus.)

- 

Can we understand the critical theory in terms of fluctuations of the string order parameter $W(C)$? But by Wegner’s duality, this theory (up to global data) is in the same universality class as the 3d Ising model.

This suggests that the near-critical 3d Ising model should have a description as a string theory.

3d Ising model as a string theory

3d Ising model as a string theory.

This is something which has been suggested before, from other points of view. [Fradkin-Srednicki-Susskind 80, Polyakov 81, Dotsenko, Itzykson 82, Casher-Foerster-Windey 85, Kavalov, Sedrakyan, Distler 92]

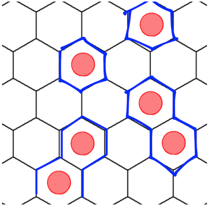
Reasons to hope for progress here:

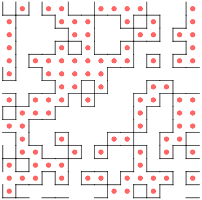
- We're going to propose a modification to the Ising model, which we think may have a better string theory description.
- We've learned a lot about non-perturbative string theory since 1992!

Fermions from 2d Ising model. [Jordan-Wigner, Lieb-Mattis, ..., Polyakov]

$$Z_{\triangle}(\beta) = \sum_{\sigma} e^{-\beta \sum_{\langle ij \rangle}(1-\sigma_i \sigma_j)}$$

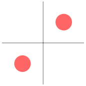
$$= 2 \sum_{\gamma} e^{-2\beta L[\gamma]}$$ (● = spin up)



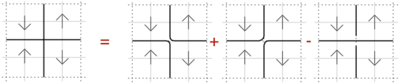


On the square lattice, this can happen:

(This is an avoidable, non-universal technicality, but its resolution is instructive.)



Resolution:



$$Z_{\square}(\beta) = 2 \sum_{\gamma} (-1)^{n[\gamma]} e^{-2\beta L[\gamma]}$$ $n[\gamma] \equiv$ # of self-intersections

Fermions from 2d Ising model.

[Jordan-Wigner, Lieb-Mattis, ..., Polyakov]

$$Z_\square(\beta) = 2\sum_\gamma (-1)^{n[\gamma]} e^{-2\beta L[\gamma]} = 2\exp\left(\underbrace{\sum_{\gamma,\text{connected}} (-1)^{n[\gamma]} e^{-2\beta L[\gamma]}}_{\text{worldline sum for real fermion}} \right)$$

$n[\gamma] \equiv$ # of self-intersections

w/ PBC: only even winding configs $w_{x,y}[\gamma] \in 2\mathbb{Z}$ correspond to spins

$$Z_{T^2} = \sum_\gamma \frac{1}{2}\left(1 + (-1)^{w_x(\gamma)}\right)\frac{1}{2}\left(1 + (-1)^{w_y(\gamma)}\right)(-1)^{n[\gamma]} e^{-2\beta L[\gamma]}$$

$$= Z_{++} + Z_{+-} + Z_{-+} + Z_{--}.$$

This sum over spin structures says $(-1)^F$ is gauged.

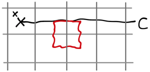
Fermions from 2d Ising model. [Jordan-Wigner, Lieb-Mattis, ..., Polyakov]

More explicitly, we can make fermion operators:

Disorder operator: $\mu(x) \equiv \prod_{\langle ij \rangle \perp C_x} e^{-2\beta\sigma_i\sigma_j}$.

$x \in$ dual lattice. (Flip sign of β along links crossed by C.)

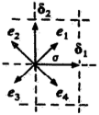
μ is independent of local changes in C by $\sigma_i \to -\sigma_i$ symmetry. C is a branch cut for σ_i.



Duality interchanges $\mu \leftrightarrow \sigma$.

The self dual object $\psi_a(x) \equiv \sigma(x)\mu(x + e_a)$ is a fermion

$$R_{2\pi}(\psi(x)) = \psi_{a+4}(x) = -\psi_a(x)$$

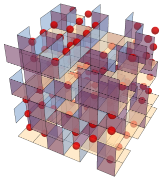


and satisfies

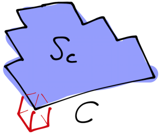
$$\langle \psi_a(x) \rangle = \cosh(2\beta)\langle \psi_{a+1}(x) \rangle - \sinh(2\beta)\langle \psi_{a+2}(x + \delta_{a+1}) \rangle$$

In the continuum limit, this is the Dirac equation, with $m \propto \beta - \beta_c$.

Fermionic strings from 3d Ising model.



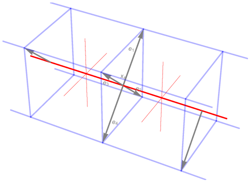
[Polyakov, 80s]



Disorder operator:

$\mu(C) \equiv \prod_{\langle ij \rangle \perp S_C, \partial S_C = C} e^{-2\beta \sigma_i \sigma_j}$.

μ is independent of local changes in S_C by $\sigma_i \to -\sigma_i$ symmetry. S_C is a branch cut for σ_i.



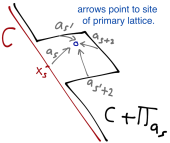
$\Psi_{a_1 \cdots a_L}(C) \equiv \mu(C) \prod_{s=1}^{L} \sigma(x_s + e_{a_s})$

(x_s = center of link s) satisfies

$\Psi_{a_1 \cdots a_L}(C) = \cosh(2\beta) \Psi_{a_1 \cdots a_s+1, a_{s+1}, \cdots a_L}(C)$

$- \sinh(2\beta) \Psi_{a_1 \cdots a_{s-1}, a'_s, a_s+2, a'_s+2, a_{s+1}, \cdots a_L}(C + \Pi_{a_s})$

Links like free Dirac particles, connected by unbreakability of domain wall.



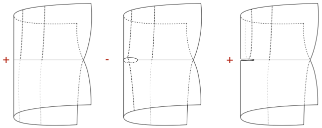
This description is shared by the RNS superstring.

$\psi^\mu \left(\dot{x}_\mu - x'_\mu \right) |\text{phys}\rangle = 0$.

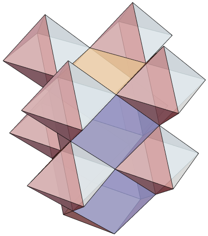
Strong coupling problem.

Distler (1992) argued that the analog of self-intersection number term in the 3d case is the Euler character

$$Z_{3d}(\beta) = 2\sum_{\Sigma}(-1)^{\chi[\Sigma]}e^{-2\beta \text{Area}[\Sigma]}$$



Just as in the 2d case, we can avoid this issue by working on a lattice where each edge touches only 3 faces, such as this one: (corner-sharing octahedra)



But this highlights the fact that $|g_s| = 1$.

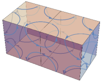
Appeal to universality.

Q: can we modify the Ising model so that the dual string theory is weakly coupled?

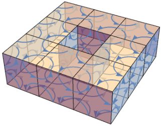
($i.e.$ decrease the weight of domain walls with higher genus in the sum)

$$Z_{3d}(\beta, g_s) = 2 \sum_{\Sigma} (g_s)^{\chi[\Sigma]} e^{-2\beta \text{Area}[\Sigma]}$$

$\chi = 2$:



$\chi = 0$:



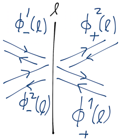
Possible outcomes, assuming there is still a continuous transition (there is):
(1) Finite $g_s < 1$ leads to a new universality class, where spherical domain walls dominate.

(2) This changes T_c, but stays in the same 3d Ising universality class.

The planar 3d Ising model

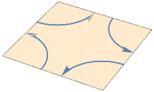
How to change g_s?

First idea: On each link of dual lattice (= face of the primal lattice), place four $N \times N$-matrix-valued real variables $\phi^{1,2}_{\pm}$, associated with the four faces incident on the link:



$$\Delta S[\phi, z] = \sum_{\langle ij \rangle}(1+\sigma_i\sigma_j)\Gamma \sum_{\alpha,\ell \in \partial\langle ij \rangle} \phi^2_\alpha(\ell) + \sum_{\langle ij \rangle}(1-\sigma_i\sigma_j) g \mathrm{tr}\phi^4 + \sum_{\ell,\alpha,\beta} \phi_\alpha(\ell)\phi_\beta(\ell)$$

The $\mathrm{tr}\phi^4$ interaction connects the indices of the matrices on the links bounded by the plaquette like this:



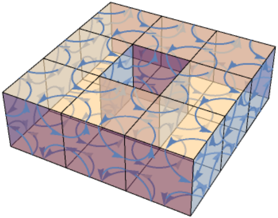
It costs a factor of $g \sim \frac{1}{N}$.

Configurations where the indices are not contracted contribute zero because of the angular integral over the ϕs.

The contribution of a spin configuration acquires a factor of

$$g^{\#\text{ of faces}} N^{\#\text{ of index loops}} = \lambda^{\#\text{ of faces}} N^{2-2g}$$

with $\lambda \equiv gN$.



But this model is difficult to simulate and has an extra $\mathsf{O}(N)$ symmetry.

The planar 3d Ising model.

But there's a much easier way to change the relative weighting of the domain walls depending on their topology: just modify the Boltzmann weights!

$$Z = \sum_s g_s^{-\chi(s)} W_0(s)$$

where $W_0(s) = e^{-\beta \sum_{\langle ij \rangle} Z_i Z_j}$ is the usual Ising model Boltzmann weight, $\chi(s) \equiv F(s) - E(s) + V(s)$.

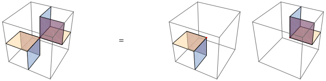
$F, E, V = \#$ of faces, edges and vertices of the dual lattice participating in a domain wall.

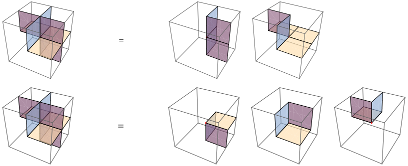
A local Hamiltonian!

This statement requires some refinement.

Ambiguity & Resolution.

In how many DWs does a vertex participate?

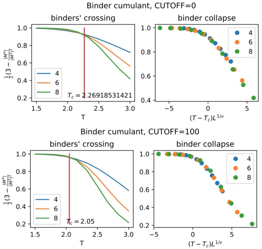




One possibility: add an energetic penalty to exclude the (9) ambiguous configurations.

Doing the analogous thing to the 2d Ising model ($\Delta E(+) = \mathrm{CUTOFF}$) does not change the critical behavior (it merely moves T_c, but $\nu = 1$ still).

Bad for the MC acceptance rate.

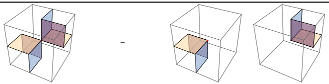


Alternative: decide on a decomposition into elementary constituents.

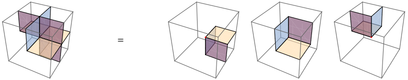
There are 2^8 possible configs α of the 8 spins adjacent to a vertex of $\hat{\Gamma}$

$\longrightarrow$ Binary vectors, $p_\alpha \in \mathbb{Z}_2^{12}$.

Order them by # of faces = Hamming weight (0 to 12). Choose a basis of lowest weight.

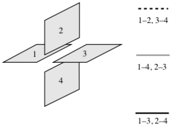


$$p_{77} = p_2 + p_9$$

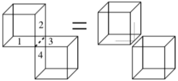


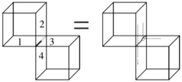
$$p_{126} = p_6 + p_7 + p_8$$

Ambiguity & Resolution.

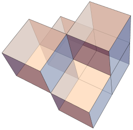


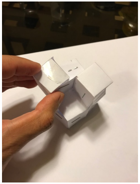
[images: Distler]





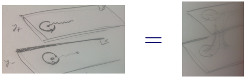
But: not all vertex resolutions are mutually compatible.
e.g. These two touching S^2s would be assigned $\chi = 5$:





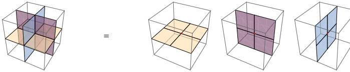
(1) For each vertex of $\hat{\Gamma}$, record face connections implied by the vertex decomposition.

(2) For each edge, check for compatibility between these face connections. If not, that edge carries a 4π branch point, $\Delta\chi = -1$.



Note: This prescription eliminates unoriented configurations. (An *unoriented* immersed surface must have an odd number of triple points: $\chi = \#$ of triple points, mod 2. [Banchoff, 74])

a triple point:



Numerical implementation: cluster updates.

Critical slow-down: Near a critical point, correlation lengths grow, and for local Monte Carlo dynamics, so do correlation times.
Remedy: non-local MC dynamics [Sweeny, Wolff, Swendsen-Wang 80s]: propose moves which update an order-1 fraction of spins at once.

Happily, because our modification of the Ising interactions depends on the domain wall configuration, we can adapt these methods to our model.

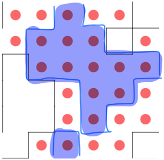
Detailed balance

$$\pi(a)\mathcal{A}\left(a \to b\right) P\left(a \to b\right) \overset{!}{=} \pi(b)\mathcal{A}\left(b \to a\right) P\left(b \to a\right)$$

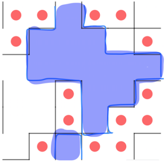
($\pi =$ Boltzmann wt, $\mathcal{A} =$ construction prob,
$P =$ acceptance prob) determines

$$P\left(a \to b\right) = \min\left(1, g_s^{\Delta\chi}\right).$$

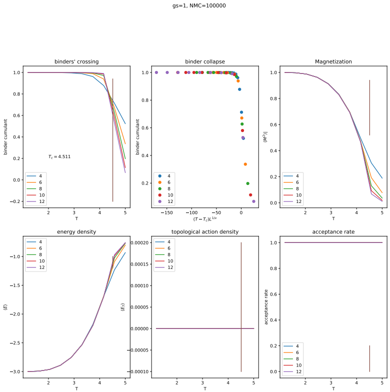
a:



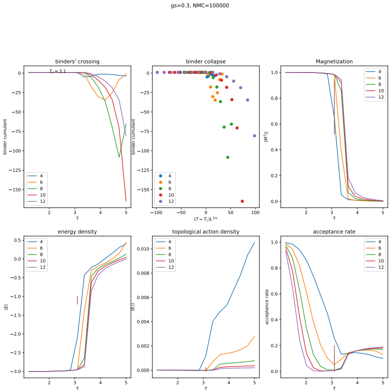
b:



Simulation results.

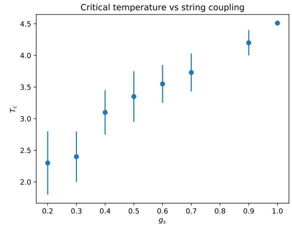


$g_s = 1$



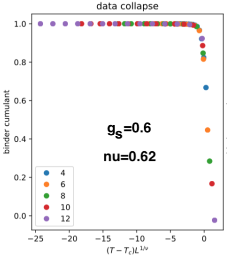
$g_s = 0.3$

Simulation results.



T_c does change with g_s.

Low-temperature AFM phase?



We can infer the correlation-length critical exponent ν from the collapse of the Binder cumulant. We find that the 3d Ising value $\nu = 0.62$ gives the best data collapse for all values of g_s (option (2) above).

Comment on universality: the 3d Ising fixed point has a fixed point value of g_s, which we cannot change (and do not know yet).

We are merely trying to make the dual string theory weakly coupled *on the way* to the fixed point.

Speculations about the worldsheet

Comments on worldsheet theory.

Important Q: how does the Ising $\mathbb{Z}_2$ act in the string theory??

Hint 1: The string worldsheet is a branch cut for the spin.

Hint 2: Ising gauge theory has fermions in its spectrum – the boundstate of e (end of string) and m (vison) is a fermion.

RNS superstring spectra:

$$\begin{array}{c} \text{RR} \\ \text{NS-R R-NS} \\ \text{NS-NS} \end{array} \overset{\text{mod }\Omega}{\rightarrow} \begin{array}{c} \text{RR} \\ \text{NS-R} \\ \text{NS-NS} \end{array} \overset{?}{=} \begin{array}{c} e\text{-particle?} \\ \text{dyon} \\ \text{glueballs} \end{array}$$

has a spacetime fermion number symmetry.

Orbifolding by $(-1)^{F_s}$ $\overset{\text{mod }(-1)^{F_s}}{\rightarrow}$ $\begin{array}{c} \text{RR}_L \oplus \text{RR}_R \\ -- \\ \text{NS-NS} \end{array} \overset{?}{=} \begin{array}{c} \text{spin} \oplus \text{neutral} \\ -- \\ \text{neutral} \end{array}$

This (unoriented) type 0 theory has two RR sectors, labelled by the chirality operator Γ.

Conjecture: Γ is the Ising $\mathbb{Z}_2$.

Comments on worldsheet theory.

It is tempting to interpret this as a holographic duality.

Adding one extra dimension ϕ doesn't solve the problem of making a critical string theory.

A spacelike linear dilaton (in the radial direction, $\Phi = Q\phi$) could be used to cancel the Weyl anomaly.

But linear dilaton and target-space conformal symmetry (required near the critical point) are not compatible:

At the critical point, we expect $ds^2 = ds^2_{AdS} = d\phi^2 + e^{-2\phi} d\vec{x}^2$.

If under a spacetime scale transformation $\phi \to \phi + \lambda$,

$S_{\text{worldsheet}} \ni \int Q\phi \frac{R}{2\pi} \to S_{\text{worldsheet}} + Q\lambda\chi$.

[Hellerman-Maeda-Maltz-Swanson 14]: 'composite linear dilaton'. add

$S_{\text{worldsheet}} \ni \int Q\varphi \frac{R}{2\pi}$ where $\varphi = \frac{1}{\Delta} \ln \mathcal{O}_\Delta$ is a composite operator which shifts under a worldsheet scale transformation.

We could choose $\mathcal{O}_2 = e^{2\phi} \partial_\alpha X^\mu \partial^\alpha X_\mu + \partial_\alpha \phi \partial^\alpha \phi$, the AdS_4 kinetic term, which is invariant under *target-space* scale transformations

$X^\mu \to e^\lambda X^\mu, \phi \to \phi + \lambda$.

And $\varphi = \frac{1}{\Delta} \ln \mathcal{O}_\Delta = \phi + \log |\partial X| + \log |\partial \phi|$.

What is $\log |\partial X|$?

Effective string theory.

[Polchinski-Strominger 91, ... Hellerman et al]

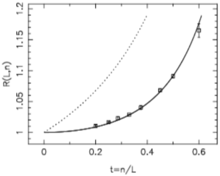
A less ambitious but more concrete connection with string theory governs the fluctuations of a large flat domain wall.

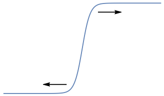
Worldsheet $X(\sigma, \tau)$ coordinate fields arise as Goldstones for breaking of translations by the wall.

'Large and flat' means $X(\sigma, \tau) = \sigma +$ fluctuations, so $\partial X \neq 0$, and $\log(\partial X)^2$ makes sense.

[Caselle-Fiore-Gliozzi-Hasenbusch-Provero 96]

Prediction for $R(L, n) \equiv \frac{\langle W(L+n, L-n)\rangle}{\langle W(L,L)\rangle} e^{-n^2\sigma}$, matches lattice simulation ($\beta = 0.752 < \beta_c = 0.761$):





[Kuti 05] find a gapped breathing mode on the worldsheet.

Closer to the critical point, we can expect this mode to become gapless: a goldstone for breaking of scale transformations by the profile of the wall. This should be the bulk radial coordinate.

Final comments.

It would be interesting to measure $\langle \chi \rangle$ at the critical point, and the fixed-point value of g_s. This requires measuring the *number* of connected components, which is not local information (but can be calculated [Sweeny 83, Hoshen-Kopelman]).

Large-N puzzle: String theory in flat space has Hagedorn growth of single-string states at high energy. In AdS/CFT, this is matched by the large-N growth of the number of words $\mathrm{tr}\,(XYXXY\cdots)$. But our weak-coupling limit did not involve large-N!

An unoriented string theory without space-filling D-branes? (In all examples I know, RR tadpole cancellation requires D-branes on top of the space-filling O-planes.)

A string theory with no dynamical D-branes?

The end.

Thanks for listening.