

Anderson Localization in One Dimension

Transfer matrices, Lyapunov exponents, and the localization length

Graduate Lecture

July 3, 2026

The physical question (Anderson, 1958)

- In a perfect crystal, eigenstates are Bloch waves — *extended* over the whole sample; an injected electron diffuses away forever.
- Add randomness (impurities, defects): can diffusion be *completely* suppressed?
- **Anderson's answer:** yes. Beyond a critical disorder the eigenstates become *exponentially localized*,

$$\psi(x) \sim e^{-|x-x_0|/\xi},$$

pinned to a region of size ξ (the *localization length*). The material is an insulator even where band theory predicts a metal.

- **Mechanism:** quantum interference of multiply back-scattered partial waves.
- In **1D** the interference is so effective that *any* disorder localizes *every* state. This is our clean laboratory.

Disorder on a 1D lattice

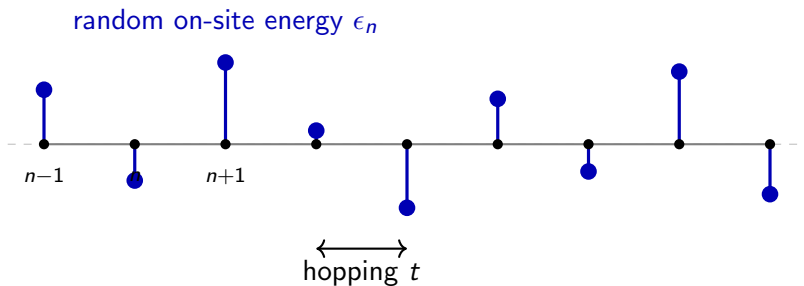


Figure 1: Single orbital per site, nearest-neighbor hopping t , i.i.d. random on-site energies ϵ_n .

The 1D tight-binding (Anderson) Hamiltonian

$$H = \sum_n \epsilon_n |n\rangle \langle n| + t \sum_n (|n\rangle \langle n+1| + |n+1\rangle \langle n|).$$

Disorder lives in $\{\epsilon_n\}$: independent, identically distributed, with

$$\langle \epsilon_n \rangle = 0, \quad \langle \epsilon_n \epsilon_m \rangle = \sigma^2 \delta_{nm}.$$

Box distribution: ϵ_n uniform on $[-W/2, W/2]$, so $\sigma^2 = W^2/12$.

With $|\psi\rangle = \sum_n \psi_n |n\rangle$, the equation $H|\psi\rangle = E|\psi\rangle$ becomes a three-term recurrence (a discrete Schrödinger equation):

$$\epsilon_n \psi_n + t(\psi_{n+1} + \psi_{n-1}) = E \psi_n$$

From recurrence to transfer matrix

The recurrence is *local*: two consecutive amplitudes fix all the rest. Solve for ψ_{n+1} :

$$\psi_{n+1} = \frac{E - \epsilon_n}{t} \psi_n - \psi_{n-1}.$$

Package neighbors into a vector:

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{E - \epsilon_n}{t} & -1 \\ 1 & 0 \end{pmatrix}}_{T_n} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}.$$

Iterate from the left boundary $\Rightarrow$ ordered product of **transfer matrices**:

$$\begin{pmatrix} \psi_{N+1} \\ \psi_N \end{pmatrix} = \mathcal{M}_N \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}, \quad \mathcal{M}_N = T_N T_{N-1} \cdots T_1.$$

Two structural facts

Area preservation

$\det T_n = 1$ for every $n \Rightarrow T_n \in \text{SL}(2, \mathbb{R})$ and $\det \mathcal{M}_N = 1$. This is flux conservation (constant Wronskian).

Random matrix product

The ϵ_n are i.i.d. $\Rightarrow$ the T_n are i.i.d. random matrices, and $\mathcal{M}_N$ is a *product of random matrices*. All the localization physics is in how $\mathcal{M}_N$ grows with N .

Because $\det \mathcal{M}_N = 1$, its singular values are reciprocal, $\lambda \geq 1$ and $1/\lambda$. A generic vector grows like $\lambda \sim \|\mathcal{M}_N\|$.

Lyapunov exponent and localization length

Furstenberg (norm grows exponentially, a.s.) + **Oseledec** (rate is non-random, self-averaging):

$$\gamma(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \|\mathcal{M}_N\|.$$

A generic solution behaves as $|\psi_n| \sim e^{\gamma n}$; a normalizable eigenstate must decay both ways from its center:

$$|\psi_n| \sim e^{-\gamma |n-n_0|} \quad \Rightarrow \quad \boxed{\xi(E) = \frac{1}{\gamma(E)}}$$

Key 1D fact

$\gamma > 0$ for *any* nonzero disorder: localization in 1D is generic, not a threshold phenomenon.

Example 1 — the clean chain ($\epsilon_n \equiv 0$)

Every transfer matrix is identical:

$$T_0 = \begin{pmatrix} E/t & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{M}_N = T_0^N.$$

Eigenvalues obey $\lambda + 1/\lambda = E/t$.

- **In band** $|E| \leq 2|t|$: set $E = 2t \cos k$, then $\lambda = e^{\pm ik}$ lie on the unit circle $\Rightarrow \gamma = 0$, $\xi = \infty$. Extended Bloch waves $\psi_n \sim e^{ikn}$.
- **In gap** $|E| > 2|t|$: λ real, $|\lambda| \neq 1$ — evanescent, not eigenstates of the infinite chain.

The transfer matrix reproduces ordinary band theory in the clean limit.

Extended vs. localized: the two regimes

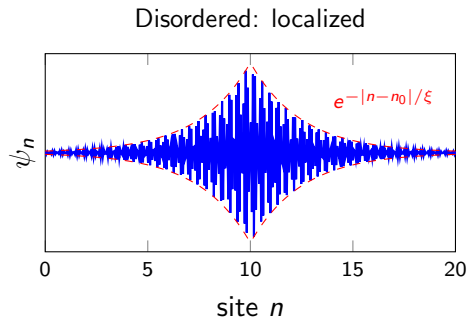
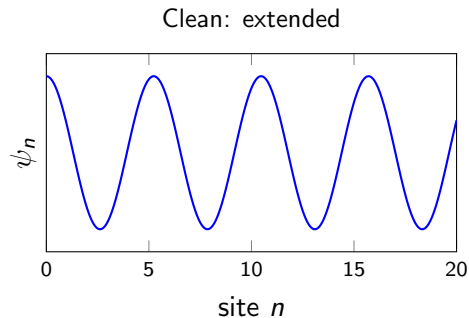


Figure 2: Left: extended Bloch wave (clean). Right: exponentially localized state with envelope $e^{-|n-n_0|/\xi}$ (disordered).

Example 2 — weak disorder: Prüfer variables

Set $t = 1$, $E = 2 \cos k$. Amplitude–phase parametrization:

$$\psi_n = r_n \sin \theta_n, \quad \psi_{n-1} = r_n \sin(\theta_n - k).$$

Clean limit: $\theta_n = kn + \text{const}$, $r_n = \text{const}$.

Insert into the recurrence and use $2 \cos k \sin \theta_n - \sin(\theta_n - k) = \sin(\theta_n + k)$. Writing $\theta_{n+1} = \theta_n + k + \delta_n$, expand order by order in ϵ_n :

$$\delta_n = \frac{\sin^2 \theta_n}{\sin k} \epsilon_n + \frac{\cos \theta_n \sin^3 \theta_n}{\sin^2 k} \epsilon_n^2 + \dots$$

$$\ln \frac{r_{n+1}}{r_n} = -\cot \theta_n \delta_n + \frac{1}{2 \sin^2 \theta_n} \delta_n^2 + \dots$$

Example 2 — disorder averaging

Lyapunov exponent = averaged growth rate, $\gamma = \langle \ln(r_{n+1}/r_n) \rangle$.

Random-phase approximation

θ_n depends only on $\epsilon_1, \dots, \epsilon_{n-1} \Rightarrow$ independent of ϵ_n , and (away from band center/edges) uniform on $[0, 2\pi)$.

Use $\langle \epsilon_n \rangle = 0$, $\langle \epsilon_n^2 \rangle = \sigma^2$, $\langle \sin^2 \theta \rangle = \frac{1}{2}$, $\langle \sin^2 \theta \cos^2 \theta \rangle = \frac{1}{8}$. The first-order term averages to zero. The two second-order pieces give

$$\underbrace{\left\langle \frac{1}{2 \sin^2 \theta} (a_1 \epsilon)^2 \right\rangle}_{\frac{\sigma^2}{4 \sin^2 k}} + \underbrace{\left\langle -\cot \theta a_2 \epsilon^2 \right\rangle}_{-\frac{\sigma^2}{8 \sin^2 k}} = \frac{\sigma^2}{8 \sin^2 k},$$

where $a_1 = \frac{\sin^2 \theta}{\sin k}$, $a_2 = \frac{\cos \theta \sin^3 \theta}{\sin^2 k}$.

Example 2 — the Thouless result

Restoring the hopping t :

$$\gamma(E) = \frac{\langle \epsilon_n^2 \rangle}{8 t^2 \sin^2 k}, \quad E = 2t \cos k$$

$$\xi(E) = \frac{8 t^2 \sin^2 k}{\langle \epsilon_n^2 \rangle} = \frac{2(4t^2 - E^2)}{\sigma^2}.$$

At band center ($E = 0$, box disorder $\sigma^2 = W^2/12$):

$$\xi(0) = \frac{8t^2}{\sigma^2} = \frac{96 t^2}{W^2}.$$

- $\xi \propto 1/W^2$: weak disorder $\Rightarrow$ large *but finite* ξ — all states localized.
- ξ is *largest at band center, smallest toward the band edges* (states in the tails are strongly localized).
- Perturbation theory fails right at the edges (true edge scaling $\gamma \sim \sigma^{2/3}$) and shows an anomaly exactly at $E = 0$.

Localization length across the band

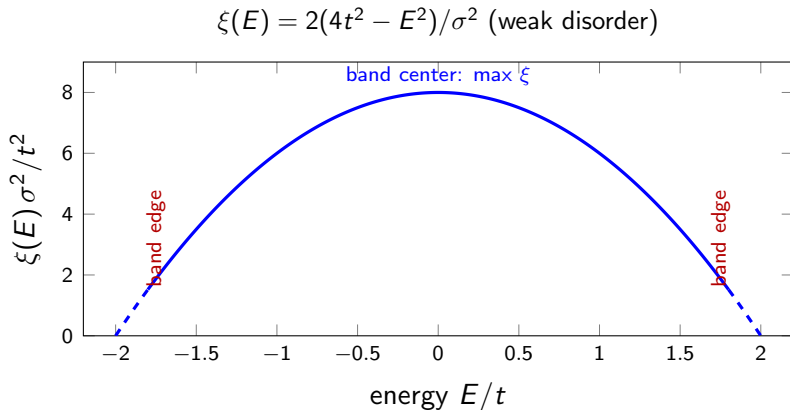


Figure 3: Localization length is maximal at the band center and shrinks toward the edges. Dashed: perturbation theory breaks down at the edges.

Computing γ numerically (transfer matrices)

- 1 Draw random $\{\epsilon_n\}$, form the T_n , multiply.
- 2 $\|\mathcal{M}_N\|$ overflows and the expanding direction swamps the contracting one.
- 3 **Cure: periodic QR re-orthogonalization.** Write the running product as $\mathcal{M} \rightarrow QR$, keep Q as the new state, accumulate $\sum \ln R_{11}$:

$$\gamma = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\text{steps}} \ln R_{11}.$$

- 4 Self-averaging: one very long chain (or a few realizations) suffices.

Numerics match $\gamma = \sigma^2 / (8t^2 \sin^2 k)$ at weak disorder, and reveal the $E = 0$ and band-edge anomalies at stronger disorder.

The bigger picture

- **Scaling theory** (Abrahams–Anderson–Licciardello–Ramakrishnan, 1979): all states localized for arbitrarily weak disorder in $d = 1, 2$.
- In $d = 3$: a genuine *metal–insulator transition* at critical disorder, with a *mobility edge* E_c separating localized band tails from extended states.
- Transfer matrices generalize to quasi-1D bars: $2M \times 2M$ matrices, a full *Lyapunov spectrum*, and finite-size scaling of the smallest exponent — the sharpest numerical probe of the 3D transition.

Takeaway

The humble 2×2 random-matrix product is the conceptual seed of the entire machinery used to locate mobility edges in real disordered systems.