Friedel Oscillations in One Dimension

The T-Matrix / Green's Function Approach

Graduate Condensed-Matter Notes

July 3, 2026

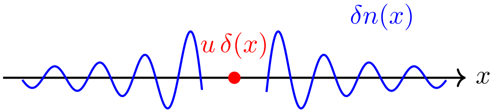
Outline

Physical Setup

Non-interacting spinless electrons in 1D (restore spin factor of 2 at the end), with a single static impurity at the origin:

$$H = H_0 + V, \quad H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}, \quad V(x) = u\,\delta(x).$$

- Impurity **back-scatters** electrons.
- e^{ikx} and e^{-ikx} **interfere**.
- Density modulates at $2k_F \Rightarrow$ **Friedel oscillations**.
- Green's function builds the density directly.



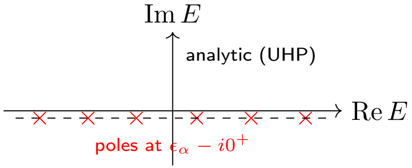
Standing-wave density modulation around the impurity.

Why start with the Lehmann representation?

Before computing G_0^R explicitly, we record its **Lehmann (spectral)** representation — it links the retarded Green's function to the **density of states** and explains why the density is built from $\mathrm{Im}\, G^R$.

Plan

Lehmann $\to$ spectral function $\to$ density $\to$ separable T-matrix $\to$ oscillations.



Retarded G^R: poles in the lower half-plane.

Lehmann Representation in an Eigenbasis

For any single-particle H with complete orthonormal eigenstates $H\,|\alpha\rangle = \epsilon_\alpha\,|\alpha\rangle$:

$$G^R(E) = \frac{1}{E - H + i0^+}.$$

Insert $\sum_\alpha |\alpha\rangle\,\langle\alpha| = \mathbb{1}$ and take coordinate matrix elements $G^R(x, x'; E) = \langle x|G^R(E)|x'\rangle$:

Lehmann representation

$$G^R(x, x'; E) = \sum_\alpha \frac{\psi_\alpha(x)\,\psi_\alpha^*(x')}{E - \epsilon_\alpha + i0^+}, \qquad \psi_\alpha(x) = \langle x|\alpha\rangle\,.$$

Poles sit at the exact energies ϵ_α, just below the real axis; the $i0^+$ makes the function **retarded** (analytic in the UHP).

Generalization to Interacting Systems

For a general interacting system the representation becomes a sum over exact many-body eigenstates:

$$G^R(x,x';\omega) = \sum_n \Bigg[\frac{\langle \Psi_0 | \hat{\psi}(x) | \Psi_n \rangle \; \langle \Psi_n | \hat{\psi}^\dagger(x') | \Psi_0 \rangle}{\hbar\omega - (E_n^{N+1} - E_0^N) + i0^+} + \frac{\langle \Psi_0 | \hat{\psi}^\dagger(x') | \Psi_n \rangle \; \langle \Psi_n | \hat{\psi}(x) | \Psi_0 \rangle}{\hbar\omega + (E_n^{N-1} - E_0^N) + i0^+} \Bigg].$$

This reduces to the single-particle form in the non-interacting limit.

Spectral Function and Density of States

Take the imaginary part via Sokhotski–Plemelj:

$$\frac{1}{E-\epsilon_\alpha+i0^+} = \mathcal{P}\frac{1}{E-\epsilon_\alpha} - i\pi\,\delta(E-\epsilon_\alpha).$$

Spectral function

$$A(x,x';E) \equiv -\frac{1}{\pi}\,\mathrm{Im}\,G^R(x,x';E) = \sum_\alpha \psi_\alpha(x)\,\psi_\alpha^*(x')\,\delta(E-\epsilon_\alpha).$$

Local DOS: $\rho(x,E) = -\frac{1}{\pi}\,\mathrm{Im}\,G^R(x,x;E) = \sum_\alpha |\psi_\alpha(x)|^2\,\delta(E-\epsilon_\alpha),$

Total DOS: $\displaystyle\int dx\,\rho(x,E) = \sum_\alpha \delta(E-\epsilon_\alpha) = \rho(E).$

From DOS to Particle Density

Fill states with the Fermi function $f(E) = \left[e^{(E-\mu)/k_B T} + 1\right]^{-1}$:

Density

$$n(x) = \int_{-\infty}^{\infty} dE\, f(E)\, \rho(x, E) = \sum_{\alpha} |\psi_\alpha(x)|^2 f(\epsilon_\alpha).$$

This is exactly $\langle \hat{n}(x) \rangle$ for the free Fermi gas — now *derived* rather than asserted. It is the starting point for the density calculation later.

Free Case: Explicit Lehmann Form

Plane waves $\psi_k(x) = e^{ikx}/\sqrt{L}$, $\epsilon_k = \hbar^2 k^2/2m$. The Lehmann sum becomes an integral:

$$G_0^R(x, x'; E) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{E - \frac{\hbar^2 k^2}{2m} + i0^+}.$$

Its spectral function reproduces the free local DOS:

$$\rho_0(x, E) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \delta\Big(E - \tfrac{\hbar^2 k^2}{2m}\Big) = \frac{1}{\pi \hbar v_E} \equiv \nu(E),$$

where the two roots $k = \pm k_E$ combine with the Jacobian $|d\epsilon/dk|^{-1} = m/\hbar^2 k_E$.

Note: ρ_0 is x-independent — translational invariance. Oscillations appear only once the impurity breaks it.

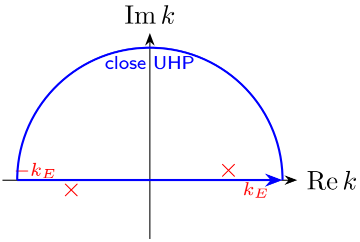
Evaluating the Free Green's Function

With $k_E = \sqrt{2mE}/\hbar$:

$$G_0^R = -\frac{2m}{\hbar^2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{e^{ik(x-x')}}{k^2 - k_E^2 - i0^+}.$$

Poles at $k = \pm(k_E + i0^+)$. Close in the upper/lower half-plane for $x - x' \gtrless 0$:

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{e^{ik|x-x'|}}{k^2 - k_E^2 - i0^+} = \frac{i}{2k_E}\,e^{ik_E|x-x'|}.$$



$x - x' > 0$: pick up the $+k_E$ pole.

Free retarded Green's function

$$G_0^R(x, x'; E) = -\frac{im}{\hbar^2 k_E}\,e^{ik_E|x-x'|} = -\frac{i}{\hbar v_E}\,e^{ik_E|x-x'|}, \quad v_E = \frac{\hbar k_E}{m}.$$

$x' = x$: $G_0^R = -i/(\hbar v_E) \Rightarrow \rho_0 = 1/(\pi\hbar v_E)$. ✓

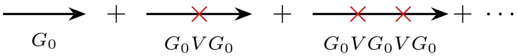
Dyson Equation and the T-Matrix

Expand $(E - H_0 - V)^{-1}$ in powers of V:

$$G = G_0 + G_0 V G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots .$$

T-matrix

$$G = G_0 + G_0\,T\,G_0, \qquad T = V + V G_0 V + V G_0 V G_0 V + \cdots = V + V G_0 T.$$



Each cross ($\times$) is a scattering event off the impurity at $x = 0$.

Separable Form for a Delta Impurity (I)

Local potential $V(x) = u\,\delta(x)$ has matrix element

$$\langle x|V|y\rangle = V(x)\,\delta(x-y) = u\,\delta(x)\,\delta(x-y).$$

The delta functions force **every scattering event to the origin**. Second-order term:

$$\begin{aligned}\langle x|VG_0V|x'\rangle &= \int dy_1\,dy_2\;\langle x|V|y_1\rangle\;G_0(y_1,y_2;E)\;\langle y_2|V|x'\rangle \\ &= \int dy_1\,dy_2\;u\,\delta(x)\delta(x-y_1)\,G_0(y_1,y_2;E)\,u\,\delta(y_2)\delta(y_2-x') \\ &= u^2\,\delta(x)\,\delta(x')\,G_0(0,0;E).\end{aligned}$$

Separable Form for a Delta Impurity (II)

The same collapse occurs at every order: each internal integration is pinned to the origin; each intermediate propagator gives $G_0(0,0;E)$. The n-th order term is $u\,[u\,G_0(0,0;E)]^{n-1}\delta(x)\delta(x')$.

Separable T-matrix

$$T(x,x';E) = \delta(x)\,\delta(x')\,t(E), \qquad t(E) = u\sum_{n=0}^{\infty}\Big[u\,G_0(0,0;E)\Big]^n = \frac{u}{1-u\,G_0(0,0;E)}.$$

The geometric series converges because the strength enters only through the single c-number $G_0(0,0;E) = -i/(\hbar v_E)$:

$$t(E) = \frac{u}{1+\dfrac{iu}{\hbar v_E}}.$$

Diagonal Correction to the Green's Function

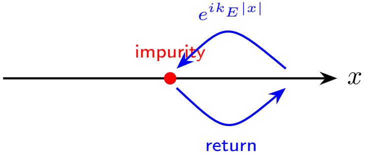
The full Green's function is exactly

$$G^R(x,x';E) = G_0^R(x,x';E) + G_0^R(x,0;E)\,t(E)\,G_0^R(0,x';E).$$

The diagonal correction $\delta G(x,x;E) \equiv G^R(x,x;E) - G_0^R(x,x;E)$:

$$\delta G(x,x;E) = G_0^R(x,0;E)\,t(E)\,G_0^R(0,x;E) = -\frac{t(E)}{(\hbar v_E)^2}\,e^{2ik_E|x|}.$$

- $e^{2ik_E|x|}$ is the **seed of the oscillation**: propagate $x \to 0 \to x$, phase $2k_E|x|$. At the Fermi surface $\to 2k_F$.
- Born approximation $t(E) \approx u$; the full $t(E)$ only renormalizes amplitude and inserts a phase shift $2\delta_0$.



Round trip $\Rightarrow$ phase $2k_E|x|$.

Charge Density from the Green's Function

The impurity-induced change in density:

$$\delta n(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dE\, f(E)\ \mathrm{Im}\ \delta G(x,x;E).$$

With the Born expression and $\mathrm{Im}\ e^{2ik_E|x|} = \sin(2k_E|x|)$:

$$\delta n(x) = \frac{u}{\pi}\int_{0}^{\infty} dE\, f(E)\, \frac{\sin(2k_E|x|)}{(\hbar v_E)^2}.$$

Change variables $E = \hbar^2 k^2/2m$:

Master formula

$$\delta n(x) = \frac{um}{\pi\hbar^2}\int_{0}^{\infty} \frac{dk}{k}\, f(\epsilon_k)\, \sin(2k|x|).$$

This single integral contains both the $T = 0$ oscillation and its thermal decay.

Zero-Temperature Result

At $T = 0$, $f = \theta(k_F - k)$:

$$\delta n(x) = \frac{um}{\pi\hbar^2}\int_0^{k_F}\frac{dk}{k}\,\sin(2k|x|).$$

Dominated by the sharp cutoff at $k = k_F$; integration by parts gives

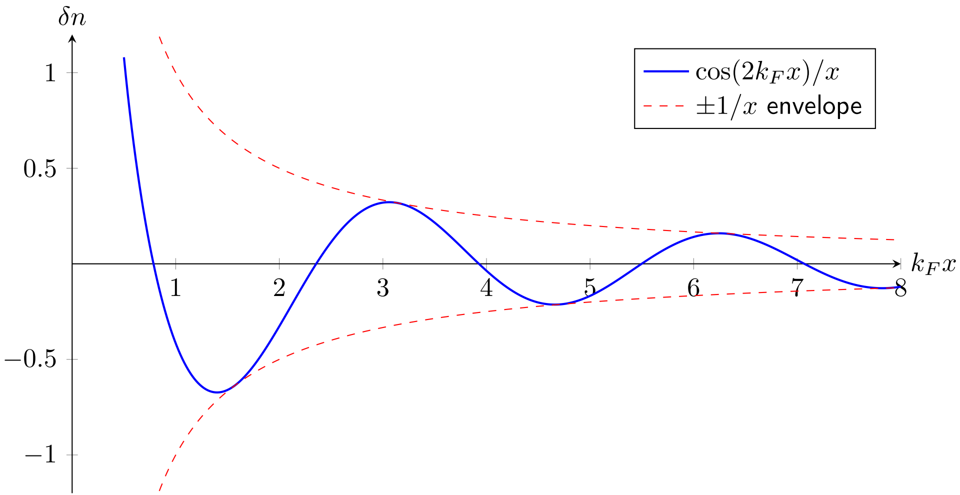
$$\int_0^{k_F}\frac{\sin(2k|x|)}{k}\,dk \xrightarrow{|x|\to\infty} -\frac{\cos(2k_F|x|)}{2k_F|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

1D Friedel oscillation

$$\delta n(x) \simeq -\frac{u}{2\pi\hbar v_F}\frac{\cos(2k_F|x|)}{|x|} = -\frac{u\,\nu_F}{2}\frac{\cos(2k_F|x|)}{|x|}, \qquad \nu_F = \frac{1}{\pi\hbar v_F}.$$

(2D: $1/r^2$, 3D: $1/r^3$ — slow 1D decay = perfectly nested Fermi points $\pm k_F$.)

Visualization: The $T = 0$ Friedel Oscillation



A $2k_F$ oscillation decaying with a slow $1/x$ power-law envelope.

Finite Temperature: Setup

Write the master integral as

$$\delta n(x,T) = \mathrm{Im} \int_0^\infty dk\, g(k)\, f(\epsilon_k)\, e^{2ikx}, \qquad g(k) = \frac{um}{\pi\hbar^2 k}.$$

Integrate by parts. Since $g(k)$ is smooth while df/dk is sharply peaked over a window $k_B T$, the $2k_F$ oscillation comes entirely from the derivative term:

$$\delta n(x,T) \simeq -\,\mathrm{Im} \left\{ \frac{g(k_F)}{2ix} \int_0^\infty dk\, \frac{df}{dk}\, e^{2ikx} \right\}.$$

Linearization About the Fermi Surface

Set $k = k_F + q$, linearize $\epsilon_k - \mu \approx \hbar v_F q$. With $f(\epsilon) = \frac{1}{2}[1 - \tanh \frac{\epsilon-\mu}{2k_BT}]$:

$$\frac{df}{dk} = \hbar v_F f'(\epsilon_k) = -\frac{\hbar v_F}{4k_BT}\,\operatorname{sech}^2\!\left(\frac{\hbar v_F q}{2k_BT}\right).$$

Sharply peaked $\Rightarrow$ extend q over the whole real axis; substitute $s = \hbar v_F q/(2k_BT)$:

$$\int_0^\infty dk\,\frac{df}{dk}\,e^{2ikx} = -\frac{e^{2ik_Fx}}{2}\int_{-\infty}^\infty ds\,\operatorname{sech}^2(s)\,e^{i\alpha s}, \qquad \alpha = \frac{4k_BT\,x}{\hbar v_F}.$$

The sech^2 Fourier Transform & Thermal Length

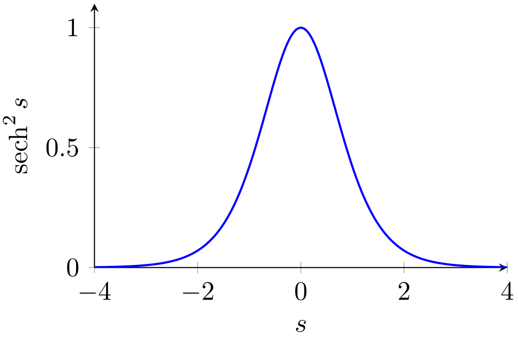
The required transform:

$$\int_{-\infty}^{\infty} \operatorname{sech}^2(s)\, e^{i\alpha s}\, ds = \frac{\pi\alpha}{\sinh(\pi\alpha/2)}.$$

Thermal length

$$\xi_T = \frac{\hbar v_F}{2\pi k_B T}.$$

Then $\pi\alpha/2 = x/\xi_T$: transform $= e^{2ik_F x}$ times a real even envelope.



Peaked at the Fermi surface, width $\sim k_B T$.

Central Result

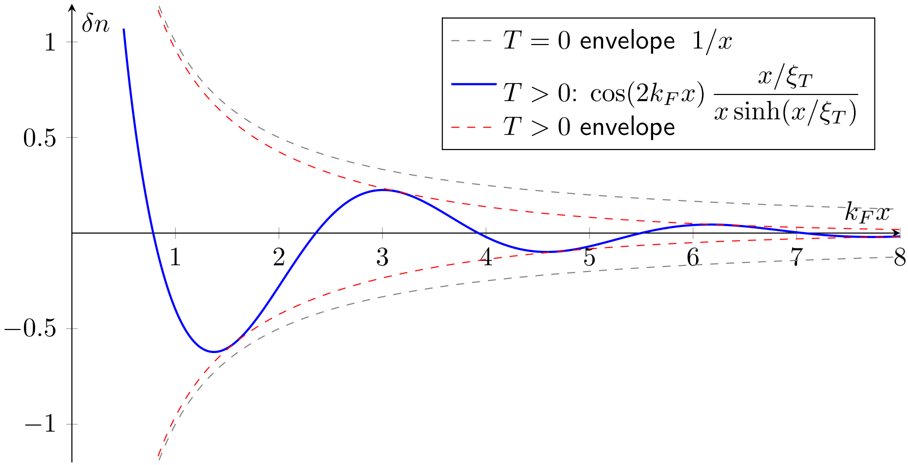
Collecting factors with $g(k_F) = u/(\pi\hbar v_F k_F)$, the smooth prefactors combine exactly as at $T = 0$:

Finite-temperature Friedel oscillation

$$\delta n(x,T) = -\frac{u}{2\pi\hbar v_F}\,\frac{\cos(2k_F x)}{x}\cdot\frac{x/\xi_T}{\sinh(x/\xi_T)}.$$

The universal factor $\dfrac{x/\xi_T}{\sinh(x/\xi_T)}$ carries all the temperature dependence.

Visualization: Thermal Cutoff of the Oscillations



For $x \gg \xi_T$ the power law crosses over to an exponential decay e^{-x/ξ_T} (here $k_F\xi_T = 2$).

Limits and Interpretation

- **Zero temperature:** $\xi_T \to \infty$, $(x/\xi_T)/\sinh(x/\xi_T) \to 1$ — recovers the $T = 0$ result.
- **Short distances** $x \ll \xi_T$**:** envelope ≈ 1; $1/x$ oscillations, thermal effects invisible.
- **Long distances** $x \gg \xi_T$**:** with $\sinh(x/\xi_T) \approx \frac{1}{2}e^{x/\xi_T}$,

$$\delta n(x,T) \simeq -\frac{u}{\pi\hbar v_F\, \xi_T}\cos(2k_F x)\, e^{-x/\xi_T}.$$

Physics: the Fermi surface is smeared over $\delta k \sim k_B T/(\hbar v_F)$; electrons dephase over $x \sim \xi_T$. Beyond it the $2k_F$ interference washes out. The same factor governs finite-T RKKY and the 1D polarization function.

Summary

1. **Lehmann representation** $\Rightarrow \operatorname{Im} G^R$ is the local DOS.
2. **Free Green's function:** $G_0^R(x, x'; E) = -\dfrac{i}{\hbar v_E} e^{ik_E|x-x'|}$.
3. **Separable T-matrix:** $t(E) = \dfrac{u}{1 - uG_0(0, 0; E)}$, delta impurity resummed exactly.
4. $T = 0$**:** $\delta n \sim -\dfrac{u\nu_F}{2} \dfrac{\cos 2k_F|x|}{|x|}$.
5. **Finite T:** multiply by $\dfrac{x/\xi_T}{\sinh(x/\xi_T)}$, with $\xi_T = \dfrac{\hbar v_F}{2\pi k_B T}$.