The Cooper Problem

Instability of the Fermi Sea and the Origin of Pairing

Lecture Notes

July 2, 2026

Overview

The Cooper Problem (1956)

The key idea

The normal filled Fermi sea of a metal is **unstable** against forming a bound pair of electrons, provided there is *any* net attraction between them — no matter how weak.

- This bound pair — the **Cooper pair** — is the seed of BCS theory.
- Take a filled Fermi sea at $T = 0$, inert except that it blocks all states with $k < k_F$ (Pauli exclusion).
- Add **two extra electrons** just above the Fermi surface.
- Ask: can the attraction bind them into a state with energy *below* $2\epsilon_F$?

Symmetry of the pair

We look for the lowest-energy pair state.

- The interaction is strongest for pairs that can scatter into the largest number of states $\Rightarrow$ **zero total momentum**.
- The two electrons occupy states k and $-$k.
- The spatial wave function turns out to be **symmetric**.
- A symmetric spatial part requires an antisymmetric **spin-singlet** state.

The trial wave function

With zero center-of-mass momentum, the general two-particle spatial wave function is a superposition of pair states $(\mathrm{k}\uparrow, -\mathrm{k}\downarrow)$:

$$\psi(\mathrm{r}_1, \mathrm{r}_2) = \left(\sum_{\mathrm{k}} g(\mathrm{k})\, e^{i\mathrm{k}\cdot(\mathrm{r}_1 - \mathrm{r}_2)}\right) \times \text{(spin singlet)}$$

- $g(\mathrm{k})$ = amplitude for one electron in $+\mathrm{k}$, the other in $-\mathrm{k}$.
- States below the Fermi surface are already occupied, so Pauli forbids:

$$g(\mathrm{k}) = 0 \qquad \text{for } k < k_F.$$

Substituting into the Schrödinger equation

The two-particle Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right)+V(\mathrm{r}_1-\mathrm{r}_2)\right]\psi=E\,\psi$$

- Acting with the kinetic operator on $e^{i\mathrm{k}\cdot(\mathrm{r}_1-\mathrm{r}_2)}$ gives $\frac{\hbar^2k^2}{2m}$ for **each** electron.
- Define the single-particle energy $\epsilon_k=\dfrac{\hbar^2k^2}{2m}$.
- The kinetic energy of pair component k is $2\epsilon_k$.

Substituting the expansion

Inserting the expansion into the Schrödinger equation:

$$\sum_{\mathrm{k}} g(\mathrm{k})\, 2\epsilon_k\, e^{i\mathrm{k}\cdot(\mathrm{r}_1-\mathrm{r}_2)} + V(\mathrm{r}_1-\mathrm{r}_2) \sum_{\mathrm{k}} g(\mathrm{k})\, e^{i\mathrm{k}\cdot(\mathrm{r}_1-\mathrm{r}_2)} = E \sum_{\mathrm{k}} g(\mathrm{k})\, e^{i\mathrm{k}\cdot(\mathrm{r}_1-\mathrm{r}_2)}$$

- Project onto a plane wave: multiply by $e^{-i\mathrm{k}'\cdot(\mathrm{r}_1-\mathrm{r}_2)}$ and integrate over $\mathrm{r} = \mathrm{r}_1 - \mathrm{r}_2$.
- Kinetic and energy terms are diagonal; the potential gives its Fourier matrix element.

The amplitude equation

Relabeling indices yields the **equation for the amplitudes** $g(\mathrm{k})$:

$$\boxed{(2\epsilon_k - E)\, g(\mathrm{k}) = -\sum_{\mathrm{k}'} V_{\mathrm{kk}'}\, g(\mathrm{k}')}$$

with interaction matrix element

$$V_{\mathrm{kk}'} = \frac{1}{\Omega} \int V(\mathrm{r})\, e^{-i(\mathrm{k}-\mathrm{k}')\cdot \mathrm{r}}\, d^3 r,$$

where Ω is the crystal volume.

This is the Schrödinger equation rewritten in momentum space.

Cooper's model interaction

The exact $V_{\mathrm{kk}'}$ is complicated. Cooper's approximation captures the phonon-mediated attraction:

$$V_{\mathrm{kk}'} = \begin{cases} -V & \text{if } \epsilon_F < \epsilon_k,\ \epsilon_{k'} < \epsilon_F + \hbar\omega_D \\ 0 & \text{otherwise,} \end{cases} \qquad V > 0$$

- Attraction acts only in a thin shell of width $\hbar\omega_D$ (the Debye energy) above the Fermi surface.
- Constant and negative there.
- The cutoff reflects that the attraction is phonon-mediated: it operates only within a phonon energy of the Fermi surface.

Solving for $g(\mathrm{k})$

With this interaction, the right-hand side becomes independent of k:

$$(2\epsilon_k - E)\, g(\mathrm{k}) = V \sum_{\mathrm{k}'} g(\mathrm{k}') \equiv VC, \qquad C \equiv \sum_{\mathrm{k}'} g(\mathrm{k}').$$

Hence the amplitude has the revealing form

$$\boxed{g(\mathrm{k}) = \frac{VC}{2\epsilon_k - E}}$$

- $g(\mathrm{k})$ is largest at the Fermi surface, falling off as $1/\epsilon_k$.
- The pair is built predominantly from states in the attractive shell.

The self-consistency (eigenvalue) condition

Sum both sides over all k in the shell. Since $\sum_{\mathrm{k}} g(\mathrm{k}) = C$, the constant C cancels:

$$1 = V \sum_{\mathrm{k}} \frac{1}{2\epsilon_k - E}.$$

Convert to an energy integral using the density of states **per spin** $N(0)$ (nearly constant across the thin shell):

$$1 = VN(0) \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} \frac{d\epsilon}{2\epsilon - E}.$$

Evaluating the integral

Carrying out the integral:

$$1 = \frac{N(0)V}{2} \ln\left(\frac{2\epsilon_F + 2\hbar\omega_D - E}{2\epsilon_F - E}\right).$$

Introduce the binding energy Δ

$$E = 2\epsilon_F - \Delta, \qquad \Delta > 0 \text{ means a bound state.}$$

Then $2\epsilon_F - E = \Delta$ and $2\epsilon_F + 2\hbar\omega_D - E = 2\hbar\omega_D + \Delta$.

The bound-state energy

Substituting gives

$$\frac{2}{N(0)V} = \ln\left(\frac{2\hbar\omega_D + \Delta}{\Delta}\right).$$

In the **weak-coupling limit** $N(0)V \ll 1$, the left side is large, forcing $\Delta \ll \hbar\omega_D$. Drop Δ next to $2\hbar\omega_D$:

$$\frac{2}{N(0)V} \approx \ln\left(\frac{2\hbar\omega_D}{\Delta}\right) \quad \Longrightarrow \quad \boxed{\Delta = 2\hbar\omega_D\, e^{-2/N(0)V}}$$

So the pair energy is

$$E = 2\epsilon_F - 2\hbar\omega_D\, e^{-2/N(0)V}.$$

Significance I: A bound state always forms

- For *any* attraction $V > 0$, however weak, $\Delta > 0$: the electrons bind below $2\epsilon_F$.
- Sharply different from the 3D two-body problem in free space, which needs a *minimum* interaction strength to bind.
- The difference is entirely due to the Fermi sea: the cutoff at k_F effectively reduces the problem to **one dimension** near the Fermi surface, where any attraction binds.
- The restriction $g(k) = 0$ for $k < k_F$ makes the density of states finite and nonzero right at the relevant energy.

Significance II: Non-perturbative and unstable

Non-perturbative

$\Delta \propto e^{-2/N(0)V}$ has an essential singularity at $V = 0$. Every derivative in V vanishes there — Δ can never be reached by any finite order of perturbation theory. This is why superconductivity eluded explanation for so long.

Instability of the normal state

Since the Fermi sea can always lower its energy by pairing, it is *unstable*. The system reorganizes into a new ground state of many overlapping Cooper pairs — the BCS superconducting state.

Connection to full BCS theory

The full BCS gap equation yields

$$\Delta_{\mathrm{BCS}} = 2\hbar\omega_D \, e^{-1/N(0)V},$$

- Same characteristic exponential, but exponent $1/N(0)V$ instead of $2/N(0)V$.
- Reason: in BCS *all* electrons near the Fermi surface participate self-consistently, not just a single added pair.

A note on conventions

Here $N(0)$ is the density of states *per spin*. Using the total (both-spin) DOS removes the factor of 2 in the exponent. Textbooks (Tinkham, de Gennes, Fetter–Walecka) differ — always check the definition.

Summary

- Amplitude equation: $(2\epsilon_k - E)\, g(\mathrm{k}) = -\sum_{\mathrm{k}'} V_{\mathrm{kk}'}\, g(\mathrm{k}')$
- Amplitude: $g(\mathrm{k}) = \dfrac{VC}{2\epsilon_k - E}$
- Eigenvalue condition: $1 = VN(0) \displaystyle\int \frac{d\epsilon}{2\epsilon - E}$
- **Result:** $\Delta = 2\hbar\omega_D\, e^{-2/N(0)V}$
- An arbitrarily weak attraction binds a pair $\Rightarrow$ the Fermi sea is unstable $\Rightarrow$ superconductivity.

Thank you!