The Size of a Cooper Pair in a Flat Band

Quantum Geometry and Superconductivity

Graduate Lecture

July 2, 2026

Outline

The Question and Why It Is Subtle

Two length scales characterize a superconductor:

- penetration depth λ (magnetic screening),
- coherence length ξ $\approx$ diameter of a Cooper pair.

Central question

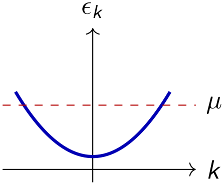
How big is a Cooper pair when the electrons that form it cannot move?

A **flat band** has $\epsilon_{\mathbf{k}} = \text{const}$, so

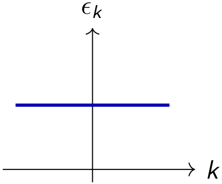
$$\mathbf{v}_{\mathbf{k}} = \hbar^{-1}\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}} = 0 \quad \forall\,\mathbf{k} \;\Rightarrow\; m^* \to \infty.$$

Realized in Lieb/kagome lattices, Landau levels, and magic-angle twisted bilayer graphene.

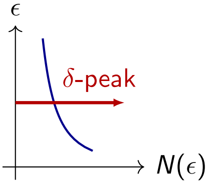
Dispersive vs. Flat Band



(a) $v_F \neq 0$



(b) $v_F = 0$



(c) DOS

The flat band's δ-function DOS is the origin of strong, non-perturbative pairing.

The Paradox in One Line

Textbook BCS coherence length:

$$\xi_0 \sim \frac{\hbar v_F}{\Delta}.$$

The problem

Set $v_F = 0 \;\Rightarrow\; \xi_0 = 0$: a *pointlike* pair.

But a pointlike pair of two fermions is:

- forbidden by Pauli in a single orbital,
- infinite in kinetic energy.

Something must replace $v_F \longrightarrow$ quantum geometry.

The BCS Pair Wavefunction

Amplitude to find a pair $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$:

$$\phi_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle = u_{\mathbf{k}} v_{\mathbf{k}} = \frac{\Delta}{2E_{\mathbf{k}}}, \qquad E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2},$$

with $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. Real-space pair: $\phi(\mathbf{r}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \phi_{\mathbf{k}}$.

Pair size = rms radius. Using $\mathbf{r} \leftrightarrow i\nabla_{\mathbf{k}}$:

Definition of the coherence length

$$\xi_0^2 = \langle r^2 \rangle = \frac{\int d^d k \, |\nabla_{\mathbf{k}} \phi_{\mathbf{k}}|^2}{\int d^d k \, |\phi_{\mathbf{k}}|^2}.$$

Uncertainty principle: small pair in real space $\Leftrightarrow$ broad $\phi_{\mathbf{k}}$ in momentum space.

Evaluating the Integral

$\phi_{\mathbf{k}}$ is peaked on the Fermi surface. Linearize $\xi_{\mathbf{k}} \simeq \hbar v_F\, q$:

$$\frac{d\phi}{dq} = \hbar v_F \, \frac{d\phi}{d\xi} = -\hbar v_F \, \frac{\Delta\, \xi}{2(\xi^2 + \Delta^2)^{3/2}}.$$

Elementary integrals:

$$\int_{-\infty}^{\infty} \frac{\xi^2 \, d\xi}{(\xi^2 + \Delta^2)^3} = \frac{\pi}{8\Delta^3}, \qquad \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + \Delta^2} = \frac{\pi}{\Delta}.$$

Substituting (the Δ's cancel):

Conventional BCS result

$$\xi_0^2 = \frac{(\hbar v_F)^2}{8\Delta^2} \quad \Longrightarrow \quad \boxed{\xi_0 \sim \frac{\hbar v_F}{\Delta}}$$

Exact Pippard prefactor $\xi_0 = \hbar v_F / \pi \Delta$; scaling is robust.

Reading the Result

$$\xi_0 \sim \frac{\hbar v_F}{\Delta} = \text{(Fermi velocity)} \times \text{(time to feel the gap } \hbar/\Delta\text{)}.$$

- The pair size is entirely a kinetic quantity.
- Set $v_F = 0 \Rightarrow$ pointlike pair.

Key observation

This derivation *assumed* a dispersive band. It says nothing sensible about a flat one. We need a genuinely different calculation.

The Key Idea

A flat band is **never** made of a single orbital. It lives on a lattice with several orbitals per cell, and the Bloch eigenvector *rotates among them* as **k** sweeps the BZ.

That internal rotation — invisible in $\epsilon_{\mathbf{k}}$ — carries all the spatial information.

Tight-binding model, N_{orb} orbitals, N cells:

$$H_0 = \sum_{\mathbf{k}\alpha\beta\sigma} [H(\mathbf{k})]_{\alpha\beta} c^{\dagger}_{\mathbf{k}\alpha\sigma} c_{\mathbf{k}\beta\sigma}, \quad H(\mathbf{k}) \left| u_{n\mathbf{k}} \right\rangle = \epsilon_{n\mathbf{k}} \left| u_{n\mathbf{k}} \right\rangle .$$

Flat band $n = 0$ at $\epsilon_0 = 0$; components $u^{\alpha}_{\mathbf{k}} = \langle \alpha \rangle \, u_{\mathbf{k}}$, $\sum_{\alpha} |u^{\alpha}_{\mathbf{k}}|^2 = 1$.

Band Projection

Band operators:

$$\gamma_{\mathbf{k}\sigma} = \sum_{\alpha} u_{\mathbf{k}}^{\alpha *} c_{\mathbf{k}\alpha\sigma}, \qquad \gamma_{\mathbf{k}\sigma}^{\dagger} = \sum_{\alpha} u_{\mathbf{k}}^{\alpha} c_{\mathbf{k}\alpha\sigma}^{\dagger}.$$

Since the band is isolated, project onto it:

Projection rule

$$c_{\mathbf{k}\alpha\sigma} \longrightarrow u_{\mathbf{k}}^{\alpha} \gamma_{\mathbf{k}\sigma}, \qquad c_{\mathbf{k}\alpha\sigma}^{\dagger} \longrightarrow u_{\mathbf{k}}^{\alpha *} \gamma_{\mathbf{k}\sigma}^{\dagger}.$$

All the interaction physics will be filtered through the orbital content $u_{\mathbf{k}}^{\alpha}$ of the Bloch states.

Projecting the Interaction

On-site attractive Hubbard U:

$$H_{\text{int}} = -U \sum_{\mathbf{R}\alpha} n_{\mathbf{R}\alpha\uparrow} n_{\mathbf{R}\alpha\downarrow} = -\frac{U}{N} \sum_{\mathbf{k}\mathbf{k}'}$$

Keep the zero-COM Cooper channel ($=\ \mathbf{k}' - \mathbf{k}$), project, and use TRS $u^{\alpha}_{-\mathbf{k}} = u^{\alpha *}_{\mathbf{k}}$:

Geometry-dressed pairing vertex

$$V_{\mathbf{k}\mathbf{k}'} = -\frac{U}{N} \sum_{\alpha} \varrho^{\alpha}_{\mathbf{k}}\, \varrho^{\alpha}_{\mathbf{k}'}, \qquad \varrho^{\alpha}_{\mathbf{k}} \equiv |u^{\alpha}_{\mathbf{k}}|^2, \;\; \sum_{\alpha} \varrho^{\alpha}_{\mathbf{k}} = 1.$$

Two electrons attract only when on the *same* orbital & site; $\varrho^{\alpha}_{\mathbf{k}} \varrho^{\alpha}_{\mathbf{k}'}$ measures that overlap.

The Cooper Secular Equation

Single-pair trial state:

$$|\Psi\rangle = \sum_{\mathbf{k}} g_{\mathbf{k}}\, \gamma^{\dagger}_{\mathbf{k}\uparrow}\gamma^{\dagger}_{-\mathbf{k}\downarrow}\,|0\rangle\,.$$

Kinetic energy $\epsilon_{0\mathbf{k}} + \epsilon_{0,-\mathbf{k}} = 0$ vanishes, so $H\,|\Psi\rangle = E\,|\Psi\rangle$ becomes pure interaction:

$$E\,g_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\,g_{\mathbf{k}'} = -\frac{U}{N}\sum_{\alpha}\varrho^{\alpha}_{\mathbf{k}}\sum_{\mathbf{k}'}\varrho^{\alpha}_{\mathbf{k}'}\,g_{\mathbf{k}'}\,.$$

The RHS factorizes in $\mathbf{k}$ and $\mathbf{k}'$ — exploit this next.

Decoupling with Orbital Gaps

Define one number per orbital:

$$\Delta_\alpha \equiv \frac{U}{N}\sum_{\mathbf{k}'} \varrho^\alpha_{\mathbf{k}'} g_{\mathbf{k}'} \implies g_{\mathbf{k}} = -\frac{1}{E}\sum_\alpha \varrho^\alpha_{\mathbf{k}} \Delta_\alpha.$$

Substitute back $\rightarrow$ finite $N_{\mathrm{orb}} \times N_{\mathrm{orb}}$ eigenproblem:

$$\Delta_\alpha = -\frac{U}{E}\sum_\beta M_{\alpha\beta}\Delta_\beta, \qquad M_{\alpha\beta} = \frac{1}{N}\sum_{\mathbf{k}} \varrho^\alpha_{\mathbf{k}} \varrho^\beta_{\mathbf{k}}.$$

$-E/U =$ eigenvalue of positive matrix M; ground pair $\rightarrow$ largest eigenvalue.

The Symmetric Solution

For symmetry-related orbitals, try $\Delta_\alpha = \Delta$. Using $\sum_\beta \varrho_{\mathbf{k}}^\beta = 1$:

$$\sum_\beta M_{\alpha\beta} = \frac{1}{N}\sum_{\mathbf{k}} \varrho_{\mathbf{k}}^\alpha = \frac{1}{N_{\mathrm{orb}}}.$$

Then $g_{\mathbf{k}} = -\Delta/E$ is constant across the BZ, and:

Flat-band binding energy

$$\boxed{E_b = -\frac{U}{N_{\mathrm{orb}}}} \qquad \textit{linear} \text{ in } U.$$

Contrast: dispersive $\Delta \sim e^{-1/N(0)V}$ (exponential). The δ-DOS removes the $\int d\xi$ log divergence $\Rightarrow$ algebraic equation.

A Warning We Must Confront

Danger

The pairing amplitude $g_{\mathbf{k}} = \text{const}$ is uniform in $\mathbf{k}$. A constant $\phi_{\mathbf{k}}$ gives $\nabla_{\mathbf{k}}\phi = 0 \Rightarrow$ pointlike pair. Are we back to the paradox?

No. The pair wavefunction is not $g_{\mathbf{k}}$ alone — it also carries the Bloch spinors:

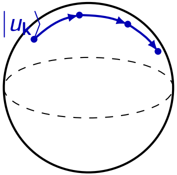
$$|\tilde{\Phi}(\mathbf{k})\rangle = g_{\mathbf{k}} \, |u_{\mathbf{k}}\rangle \otimes |u_{-\mathbf{k}}\rangle \, .$$

The variation of $|u_{\mathbf{k}}\rangle$ across the zone provides the size. Time to introduce the quantum metric.

States, Not Just Energies

As $\mathbf{k}$ moves, $|u_{\mathbf{k}}\rangle$ traces a curve in orbital Hilbert space. Ask: how much does the *state* change from $\mathbf{k}$ to $\mathbf{k} + d\mathbf{k}$?
Quantum distance:

$$ds^2 = 1 - |\langle u_{\mathbf{k}} \rangle\, u_{\mathbf{k}+d\mathbf{k}}|^2 = g_{\mu\nu}(\mathbf{k})\, dk^{\mu} dk^{\nu}.$$



Quantum metric

$$g_{\mu\nu} = \mathrm{Re}\left[\, \langle \partial_\mu u \rangle\, \partial_\nu u - \langle \partial_\mu u \rangle\, u\, \langle u \rangle\, \partial_\nu u \right]$$

Subtraction removes the pure-phase (Berry connection) piece.

The Quantum Geometric Tensor

The metric is the real part of a larger object:

$$Q_{\mu\nu}(\mathbf{k}) = \langle\partial_\mu u_{\mathbf{k}}|\,(1 - P_{\mathbf{k}})\,|\partial_\nu u_{\mathbf{k}}\rangle = \underbrace{g_{\mu\nu}}_{\text{metric}} - \frac{i}{2}\underbrace{\Omega_{\mu\nu}}_{\text{Berry curv.}},$$

with $P_{\mathbf{k}} = |u_{\mathbf{k}}\rangle\langle u_{\mathbf{k}}|$.

Fundamental inequality (2D) — from positivity of Q

$$\operatorname{tr} g(\mathbf{k}) \;\geq\; |\Omega(\mathbf{k})|.$$

Both g and Ω are gauge invariant. The inequality will give a topological lower bound on the pair size.

Pair Wavefunction Carries the Bloch States

Orbital-resolved pair amplitude:

$$\tilde{\Phi}^{\alpha\beta}(\mathbf{k}) = \langle 0|\, c_{\mathbf{k}\alpha\uparrow} c_{-\mathbf{k}\beta\downarrow} \,|\Psi\rangle = g_{\mathbf{k}}\, u_{\mathbf{k}}^{\alpha} u_{\mathbf{k}}^{\beta *},$$

i.e.

$$|\tilde{\Phi}(\mathbf{k})\rangle = g_{\mathbf{k}}\, |u_{\mathbf{k}}\rangle \otimes |u_{-\mathbf{k}}\rangle\,.$$

Mean-square size ($\mathbf{r} \leftrightarrow i\nabla_{\mathbf{k}}$):

$$\xi_{\mathrm{pair}}^2 = \frac{\sum_{\mathbf{k}} \langle \nabla_{\mathbf{k}}\tilde{\Phi}|\,(1-P)\,|\nabla_{\mathbf{k}}\tilde{\Phi}\rangle}{\sum_{\mathbf{k}} \langle \tilde{\Phi}\rangle\, \tilde{\Phi}}.$$

$(1-P)$ removes the center-of-mass (Berry connection) part.

Evaluating Numerator and Denominator

Denominator: $|u_{\pm\mathbf{k}}\rangle$ normalized $\Rightarrow \langle\tilde{\Phi}\rangle\,\tilde{\Phi} = |g|^2$, sum $= N|g|^2$.

Numerator: $g_{\mathbf{k}}$ constant, so

$$\nabla_{\mathbf{k}}\,|\tilde{\Phi}\rangle = g\big[\,|\partial u_{\mathbf{k}}\rangle \otimes |u_{-\mathbf{k}}\rangle - |u_{\mathbf{k}}\rangle \otimes |\partial u\rangle_{-\mathbf{k}}\,\big].$$

Decompose each norm:

$$\langle\partial_\mu u\rangle\,\partial_\mu u = \underbrace{g_{\mu\mu}(\mathbf{k})}_{\text{metric}} + \underbrace{|\,\langle u\rangle\,\partial_\mu u|^2}_{\text{connection}^2}.$$

$(1-P)$ discards the connection; TRS gives $g_{\mu\nu}(-\mathbf{k}) = g_{\mu\nu}(\mathbf{k})$. Numerator $= 2|g|^2\sum_{\mathbf{k}}\operatorname{tr} g(\mathbf{k})$.

The Central Result

Flat-band Cooper pair size

$$\boxed{\xi_{\mathrm{pair}}^2 = \frac{2}{V_{\mathrm{BZ}}} \int_{\mathrm{BZ}} d^d k \; \mathrm{tr}\, g(\mathbf{k}) \equiv 2 \, \langle \mathrm{tr}\, g \rangle_{\mathrm{BZ}}}$$

- v_F has been replaced by a purely geometric quantity.
- Independent of U.
- The pair is as large as the region over which the Bloch state changes its orbital character.

Topological Lower Bound

Combine $\xi_{\mathrm{pair}}^2 = 2\,\langle \operatorname{tr} g\rangle$ with $\operatorname{tr} g \geq |\Omega|$ and $\int_{\mathrm{BZ}} \Omega\, d^2k = 2\pi C$:

$$\int_{\mathrm{BZ}} \operatorname{tr} g\, d^2k \geq \int_{\mathrm{BZ}} |\Omega|\, d^2k \geq 2\pi |C|.$$

With $V_{\mathrm{BZ}} = (2\pi)^2/A_{\mathrm{cell}}$:

Topological floor

$$\boxed{\xi_{\mathrm{pair}}^2 \gtrsim \frac{|C|\, a^2}{\pi}}$$

A nontrivial Chern number ($C \neq 0$) forbids an arbitrarily small pair — topology sets a minimum size $\sim a\sqrt{|C|}$.

Narrow Bands: The Interpolation

Restore the full envelope $\phi_{\mathbf{k}} = \Delta/2E_{\mathbf{k}}$:

$$|\Phi(\mathbf{k})\rangle = \phi_{\mathbf{k}} |u_{\mathbf{k}}\rangle \otimes |u_{-\mathbf{k}}\rangle .$$

Gradient hits both factors:

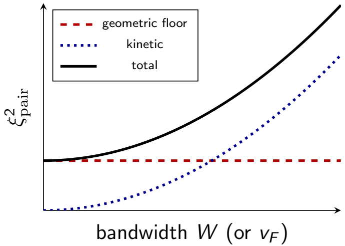
$$\nabla_{\mathbf{k}} |\Phi\rangle = (\nabla_{\mathbf{k}}\phi_{\mathbf{k}}) |u\rangle \otimes |u\rangle + \phi_{\mathbf{k}} \nabla_{\mathbf{k}}(|u\rangle \otimes |u\rangle).$$

Cross term $\propto \mathrm{Re} \langle u\rangle \partial u = \frac{1}{2}\partial \langle u\rangle u = 0$ vanishes. Terms add in quadrature:

Universal interpolation formula

$$\boxed{\xi^2_{\mathrm{pair}} = \underbrace{\frac{\hbar^2 \left\langle v_F^2 \right\rangle}{8\Delta^2}}_{\text{kinetic}} + \underbrace{2 \left\langle \mathrm{tr}\, g \right\rangle_{\mathrm{FS}}}_{\text{geometric}}}$$

The Two Regimes



Wide band (v_F large):

$$\xi_{\mathrm{pair}} \sim \frac{\hbar v_F}{\Delta} \quad \text{(BCS)}.$$

Flat band ($v_F \to 0$): kinetic term dies, $\langle\cdot\rangle_{\mathrm{FS}} \to \langle\cdot\rangle_{\mathrm{BZ}}$,

$$\xi^2_{\mathrm{pair}} = 2\,\langle \operatorname{tr} g\rangle_{\mathrm{BZ}}\,.$$

Size saturates at the geometric floor — never collapses.

It Is the Wannier Spread

Marzari–Vanderbilt spread of the band's Wannier functions:

$$\Omega_{\rm spread} = \Omega_I + \widetilde{\Omega}, \quad \Omega_I = \frac{1}{V_{\rm BZ}} \int_{\rm BZ} d^d k \; \mathrm{tr}\, g(\mathbf{k}), \quad \widetilde{\Omega} \geq 0.$$

Ω_I = gauge-invariant floor on localization (cannot be reduced by any gauge choice). Comparing:

Unifying identity

$$\boxed{\xi^2_{\rm pair}\Big|_{\rm geom} = 2\, \langle \mathrm{tr}\, g \rangle_{\rm BZ} = 2\, \Omega_I}$$

The pair is exactly as big as the optimally localized Wannier orbital from which it is built.

Summary

1. **Dispersive band:** $\xi_0 \sim \hbar v_F/\Delta$ (kinetic), would vanish for a flat band.
2. **Flat band:** $\xi_{\rm pair}^2 = 2\,\langle \mathrm{tr}\, g\rangle_{\rm BZ}$ (geometric), set by how the Bloch state changes across the zone.
3. **Topology:** bounded below, $\xi_{\rm pair}^2 \gtrsim |C|a^2/\pi$.
4. **Wannier:** equals twice the minimal Wannier spread, $2\Omega_I$.

Bigger picture

The same $\langle \mathrm{tr}\, g\rangle$ controls the superfluid weight $D_s \propto \langle \mathrm{tr}\, g\rangle$ (Peotta–Törmä). Trivial geometry $\Rightarrow$ immobile pairs, no supercurrent. Quantum geometry makes flat-band superconductivity possible.

Thank you

Questions?