Fermi-Velocity-Controlled Length Scales in Solids:
One Master Formula, Many Phenomena

Graduate Lecture Notes

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Abstract

Every energy scale E in a metal — temperature, a superconducting gap, a Kondo temperature, a disorder broadening — defines a characteristic length $L_E \sim \hbar v_F/E$. These notes develop this master relation from three complementary physical pictures, then tour its realizations: the thermal length in Friedel oscillations and clean SNS Josephson junctions, the BCS coherence length, the Kondo screening cloud, and mean free path/localization lengths in disordered conductors. The diffusive generalization $L_E \to \sqrt{\hbar D/E}$ and the Thouless energy are derived, and the limits of the framework (flat bands, non-Fermi liquids, Dirac materials) are discussed. Full derivations are included.

Contents

1 The Master Formula

1.1 Setup: linearization at the Fermi surface

Low-energy physics in a metal lives on the Fermi surface, where the dispersion may be linearized:

$$\varepsilon_k \;\simeq\; \hbar v_{\mathrm{F}}\,(k-k_{\mathrm{F}}), \qquad v_{\mathrm{F}} \equiv \frac{1}{\hbar}\,\frac{\partial \varepsilon_k}{\partial k}\bigg|_{k=k_{\mathrm{F}}}. \tag{1}$$

Linearization provides a *rigid dictionary between energy and momentum*: an energy window E around the Fermi surface corresponds to a momentum shell of width

$$\delta k = \frac{E}{\hbar v_{\mathrm{F}}}. \tag{2}$$

By elementary Fourier analysis, a superposition of plane waves drawn from a window δk remains phase-coherent in real space only over a distance $\sim 1/\delta k$. Hence the master formula:

$$\boxed{L_E \;\sim\; \frac{\hbar v_{\mathrm{F}}}{E}} \tag{3}$$

Key Idea 1. *Any physical mechanism characterized by energy scale E necessarily mixes or smears electronic states within E of the Fermi surface, and therefore imprints itself on real space at the length $L_E = \hbar v_{\mathrm{F}}/E$. The physics differs wildly from case to case; the length scale does not.*

1.2 Three equivalent readings

(R1) Fourier/uncertainty reading. Momentum shell $\delta k = E/\hbar v_{\mathrm{F}} \Rightarrow$ real-space structure of size $1/\delta k = \hbar v_{\mathrm{F}}/E$. Natural for: Cooper pair size, Kondo cloud.

(R2) Dephasing reading. Two quasiparticle states differing in energy by E differ in wavevector by $\delta k = E/\hbar v_{\mathrm{F}}$. After propagating a distance x they accumulate a relative phase $\delta k\, x$; interference is washed out once $\delta k\, x \sim 1$, i.e. at $x \sim \hbar v_{\mathrm{F}}/E$. Natural for: thermal decay of Friedel oscillations, SNS supercurrent.

(R3) Dynamical reading. An energy scale E defines a coherence time $\tau_E = \hbar/E$. A ballistic electron at the Fermi surface covers, during this time,

$$L_E = v_{\mathrm{F}}\tau_E = \frac{\hbar v_{\mathrm{F}}}{E}. \tag{4}$$

This reading generalizes immediately to diffusive metals (Sec. 6).

1.3 A number to memorize

For a typical metal $v_{\rm F} \approx 10^6$ m/s, so

$$\hbar v_{\rm F} \approx 0.66\ \text{eV} \cdot \text{nm}, \qquad k_{\rm B} \times 1\,\text{K} \approx 86\ \mu\text{eV}. \tag{5}$$

With these two numbers, every length in this lecture is a back-of-envelope estimate.

1.4 The trivial anchor: the Fermi wavelength

Setting $E = E_{\rm F}$ (for a parabolic band, $E_{\rm F} = \frac{1}{2}\hbar v_{\rm F} k_{\rm F}$):

$$L_{E_{\rm F}} = \frac{\hbar v_{\rm F}}{E_{\rm F}} = \frac{2}{k_{\rm F}} \sim \lambda_{\rm F}, \tag{6}$$

a few Å in a good metal — the *shortest* length in the hierarchy, where the notion of "near the Fermi surface" itself breaks down. Throughout these notes, keep two roles cleanly separated:

$k_{\rm F}$ sets fast oscillation periods ($\pi/k_{\rm F}$ for Friedel/RKKY); $\hbar v_{\rm F}/E$ sets slow envelope decay lengths.

2 Temperature: The Thermal Length

Set $E \sim k_{\rm B}T$. Finite temperature enters diagrammatics via Matsubara frequencies $\omega_n = (2n + 1)\pi k_{\rm B}T$; the longest-lived contribution comes from the lowest one, giving

$$\boxed{L_T = \frac{\hbar v_{\rm F}}{2\pi k_{\rm B} T}} \tag{7}$$

At $T = 1\,\text{K}$, $L_T \approx 1.2\ \mu$m: mesoscopic physics in a dilution fridge happens on the micron scale for a reason.

2.1 Application A: Friedel oscillations at finite T

2.1.1 Zero temperature: sharp edge ⇒ power law

At $T = 0$ a point impurity induces $\delta n(r) \propto \cos(2k_{\rm F}r + \delta_0)/r^d$. The power law reflects the perfectly sharp Fermi surface: the relevant integral

$$\delta n(r) \ \propto\ \text{Im} \int_0^{k_{\rm F}} dk\, g(k)\, e^{2ikr} \tag{8}$$

is endpoint-dominated at $k = k_{\rm F}$ for $k_{\rm F}r \gg 1$, and each integration by parts costs one power of $1/r$.

2.1.2 Finite temperature: the Matsubara-pole derivation

At finite T, occupation is governed by the Fermi function $f(\varepsilon) = [e^{\beta(\varepsilon-\mu)} + 1]^{-1}$, and the sharp endpoint is replaced by a smooth integral,

$$\delta n(r) \ \propto\ \text{Im} \int_0^{\infty} dk\, g(k)\, f(\varepsilon_k)\, e^{2ikr}. \tag{9}$$

Linearize around k_F: with $\varepsilon = \hbar v_\mathrm{F}(k - k_\mathrm{F})$ and $q = k - k_\mathrm{F}$,

$$\delta n(r) \;\propto\; \mathrm{Im}\left[g(k_\mathrm{F})\, e^{2ik_\mathrm{F}r} \int_{-\infty}^{\infty} dq\, \frac{e^{2iqr}}{e^{\beta\hbar v_\mathrm{F} q} + 1}\right]. \tag{10}$$

Close the contour in the upper half q-plane ($r > 0$). The Fermi factor has simple poles where $e^{\beta\hbar v_\mathrm{F} q} = -1$, i.e. at

$$q_n = \frac{i\,\omega_n}{\hbar v_\mathrm{F}}, \qquad \omega_n = (2n+1)\pi k_\mathrm{B}T, \qquad n = 0, 1, 2, \ldots, \tag{11}$$

— precisely the Matsubara frequencies. The residue at each pole is $-1/(\beta\hbar v_\mathrm{F})$, so

$$\int_{-\infty}^{\infty} \frac{dq\, e^{2iqr}}{e^{\beta\hbar v_\mathrm{F} q} + 1} = -\frac{2\pi i}{\beta\hbar v_\mathrm{F}} \sum_{n=0}^{\infty} e^{-2\omega_n r/\hbar v_\mathrm{F}} = -\frac{2\pi i}{\beta\hbar v_\mathrm{F}}\, \frac{e^{-r/L_T}}{1 - e^{-2r/L_T}} = -\frac{\pi i}{\beta\hbar v_\mathrm{F}}\, \frac{1}{\sinh(r/L_T)}, \tag{12}$$

where the geometric series over poles resummed into the sinh. The asymptotic decay is controlled by the pole nearest the real axis ($\omega_0 = \pi k_\mathrm{B}T$). Restoring the $T = 0$ normalization, the exact statement is that the zero-temperature result acquires a universal envelope:

$$\boxed{\delta n(r;T) = \delta n(r;0) \times \frac{x}{\sinh x}, \qquad x = \frac{r}{L_T} = \frac{2\pi k_\mathrm{B}T\, r}{\hbar v_\mathrm{F}}} \tag{13}$$

with limits $x/\sinh x \to 1$ ($r \ll L_T$: power law survives) and $\to 2x\, e^{-x}$ ($r \gg L_T$: exponential decay).

2.2 Application B: the clean SNS Josephson junction

Supercurrent through a normal metal of length L between two superconductors is carried by Andreev pairs: an electron at energy ε above E_F with

$$k_e = k_\mathrm{F} + \frac{\varepsilon}{\hbar v_\mathrm{F}}, \tag{14}$$

correlated with the retroreflected hole at

$$k_h = k_\mathrm{F} - \frac{\varepsilon}{\hbar v_\mathrm{F}}. \tag{15}$$

Traversing the normal region, the pair accumulates a relative phase

$$\delta\phi(\varepsilon) = (k_e - k_h)\, L = \frac{2\varepsilon L}{\hbar v_\mathrm{F}}. \tag{16}$$

At $T = 0$ all energies contribute coherently and the junction carries a robust supercurrent. At finite T, the current involves a thermal average of $e^{i\delta\phi(\varepsilon)}$ over the window $\varepsilon \sim k_\mathrm{B}T$. This is *the same integral* as Eq. (10) with $r \to L$: closing the contour over the Matsubara poles gives a $1/\sinh(L/L_T)$ factor, hence for a long junction

$$\boxed{I_c \propto e^{-L/L_T}, \qquad L_T = \frac{\hbar v_\mathrm{F}}{2\pi k_\mathrm{B}T}} \tag{17}$$

Key Idea 2. *The Friedel envelope and the Josephson decay are the* same calculation *— thermal dephasing of Fermi-shell interference, resolved by the Matsubara pole series — dressed in different physical clothing.*

3 The Superconducting Gap: The Coherence Length

3.1 Pippard's argument

Pairing correlations involve only electrons within Δ of the Fermi surface: momentum shell $\delta k = \Delta/\hbar v_\mathrm{F}$, hence pair size $\xi_0 \sim \hbar v_\mathrm{F}/\Delta$. This uncertainty-principle argument (Pippard, pre-BCS) fixes everything but the numerical factor.

3.2 BCS derivation of the pair wavefunction

The equal-time pair amplitude is

$$F(\mathbf{r}) = \sum_{\mathbf{k}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \, e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\mathbf{k}} u_k v_k \, e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad u_k v_k = \frac{\Delta}{2E_k}, \quad E_k = \sqrt{\xi_k^2 + \Delta^2}. \tag{18}$$

Perform the angular integral in 3D and linearize $\xi_k = \hbar v_\mathrm{F}(k - k_\mathrm{F})$:

$$F(r) \;\propto\; \frac{\sin k_\mathrm{F} r}{r} \int_{-\infty}^{\infty} d\xi \, \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}} \, \cos\!\Big(\frac{\xi r}{\hbar v_\mathrm{F}}\Big) \;=\; \frac{\sin k_\mathrm{F} r}{r} \, 2\Delta \, K_0\!\Big(\frac{\Delta r}{\hbar v_\mathrm{F}}\Big), \tag{19}$$

using the standard integral representation of the modified Bessel function K_0. Its asymptotics give

$$F(r) \;\xrightarrow{r \gg \hbar v_\mathrm{F}/\Delta}\; \frac{\sin k_\mathrm{F} r}{r} \, e^{-\Delta r/\hbar v_\mathrm{F}} \equiv \frac{\sin k_\mathrm{F} r}{r} \, e^{-r/\pi\xi_0}, \qquad \boxed{\xi_0 = \frac{\hbar v_\mathrm{F}}{\pi\Delta}} \tag{20}$$

Note the master pattern exactly: fast k_F-oscillations under a slow $\hbar v_\mathrm{F}/\Delta$ envelope.

3.3 Numbers and finite-temperature remarks

Al ($\Delta \approx 0.18$ meV, $v_\mathrm{F} \approx 2 \times 10^6$ m/s): $\xi_0 \sim 1.6$ μm. Nb ($\Delta \approx 1.5$ meV): $\xi_0 \sim 40$ nm. Bigger gap, smaller pair — obvious from Eq. (3) before any computation.

Two refinements:

1. Near T_c, the Ginzburg–Landau coherence length diverges, $\xi(T) \approx 0.74\, \xi_0 (1 - T/T_c)^{-1/2}$ (clean limit): the relevant energy scale $\Delta(T)$ collapses, so the length blows up, exactly as the master formula demands.
2. In a *proximitized normal metal* at finite T, the pair correlations decay on L_T, not ξ_0: the normal region has no gap, so temperature is the only energy scale left. Always ask: *which energy scale governs which region of the device?*

4 The Kondo Temperature: The Screening Cloud

4.1 The exponentially small scale

Below the Kondo temperature T_K, conduction electrons form a many-body singlet with an impurity spin. Yosida's variational treatment gives the singlet binding energy from the self-consistency condition

$$1 = \frac{3J\rho}{4} \int_0^D \frac{d\epsilon}{\epsilon + \Delta_b} \quad\Longrightarrow\quad \Delta_b \approx D\, e^{-4/(3J\rho)} \;\sim\; k_\mathrm{B} T_K, \tag{21}$$

the infrared logarithm coming — like the Cooper logarithm — from integrating $1/\epsilon$ down to a sharp Fermi edge.

4.2 Size of the cloud

The bound orbital is built from amplitudes $\alpha_k \propto (\epsilon_k + \Delta_b)^{-1}$, i.e. a shell of width $k_B T_K$ above the Fermi surface. Fourier transforming with linearized dispersion (s-wave channel),

$$\phi(r) \propto \frac{1}{r} \operatorname{Im} \int_0^D d\epsilon \, \frac{e^{i(k_F + \epsilon/\hbar v_F)\, r}}{\epsilon + k_B T_K}, \tag{22}$$

which yields fast k_F oscillations under an envelope crossing over at

$$\boxed{\xi_K = \frac{\hbar v_F}{k_B T_K}} \qquad |\phi(r)|^2 \sim \begin{cases} \dfrac{\sin^2 k_F r}{r^2} \ln^2(\xi_K/r), & r \ll \xi_K, \\ \dfrac{\cos^2 k_F r}{r^2} \dfrac{\xi_K^2}{r^2}, & r \gg \xi_K. \end{cases} \tag{23}$$

4.3 Why this shocks students

T_K is exponentially small in the exchange coupling, so ξ_K is exponentially *large*: for $T_K \sim 1$ K, $\xi_K \sim$ several microns — larger than many devices containing the impurity. This is why the Kondo cloud eluded direct detection for half a century; its observation in a quantum-dot interferometer (Borzenets *et al.*, *Nature* 2020) confirmed the $\hbar v_F / k_B T_K$ scaling directly.

5 Disorder: Mean Free Path and Localization

5.1 The building block

Set $E = \hbar/\tau$, the disorder broadening of quasiparticle levels (τ the elastic scattering time). The master formula returns

$$L_{\hbar/\tau} = \frac{\hbar v_F}{\hbar/\tau} = v_F \tau = \ell, \tag{24}$$

the elastic mean free path.

5.2 Strict 1D: transfer matrices

The tight-binding equation $-t(\psi_{n+1} + \psi_{n-1}) + \epsilon_n \psi_n = E\psi_n$ is an $SL(2,\mathbb{R})$ recursion; Furstenberg's theorem guarantees a positive Lyapunov exponent $\gamma(E) = 1/\xi_{\text{loc}}$, so all 1D states are exponentially localized. Weak-disorder perturbation theory gives

$$\xi_{\text{loc}} = \frac{8\, t^2 \sin^2(ka)}{\langle \epsilon_n^2 \rangle}. \tag{25}$$

To expose the universal structure, restore the clean-band group velocity $\hbar v(E) = 2ta \sin(ka)$ and use the 1D Born scattering rate $\hbar/\tau = 2\pi \langle \epsilon^2 \rangle \rho_{1D} a$ with $\rho_{1D} = 1/(\pi \hbar v)$:

$$\boxed{\xi_{\text{loc}} \simeq 4\,\ell = 4 v_F \tau} \tag{26}$$

— again $\hbar v_F / E^*$ with $E^* = \hbar/\tau$.

5.3 Dimensionality dressing

$$\xi_{\text{loc}} \sim \ell \;\; \text{(1D)}, \qquad \xi_{\text{loc}} \sim N_{\text{ch}}\, \ell \;\; \text{(quasi-1D)}, \qquad \xi_{\text{loc}} \sim \ell\, e^{\pi k_F \ell/2} \;\; \text{(2D)}. \tag{27}$$

Honesty clause: the master formula supplies the *brick* ℓ, but the 2D exponential and the channel-number enhancement come from interference physics (weak localization, scaling theory) genuinely beyond dimensional analysis. *The framework tells you the bricks; not always the architecture.*

6 The Diffusive Generalization and the Thouless Energy

The dynamical reading (R3) — distance traveled during the coherence time $\hbar/E$ — extends everything to dirty metals. When $L_E \gg \ell$, propagation over $\tau_E = \hbar/E$ is diffusive with $D = v_\mathrm{F}\ell/d = v_\mathrm{F}^2\tau/d$, so $L = v_\mathrm{F}\tau_E \to L = \sqrt{D\tau_E}$:

$$\boxed{L_E^{\text{ballistic}} = \frac{\hbar v_\mathrm{F}}{E} \qquad \longrightarrow \qquad L_E^{\text{diffusive}} = \sqrt{\frac{\hbar D}{E}}} \tag{28}$$

Every ballistic length has a dirty-limit partner:

$$\xi_0 = \frac{\hbar v_\mathrm{F}}{\pi\Delta} \ \longrightarrow\ \xi_{\text{dirty}} \sim \sqrt{\frac{\hbar D}{\Delta}} \sim \sqrt{\xi_0\,\ell}, \tag{29}$$

$$L_T = \frac{\hbar v_\mathrm{F}}{2\pi k_\mathrm{B}T} \ \longrightarrow\ L_T^{\text{diff}} = \sqrt{\frac{\hbar D}{2\pi k_\mathrm{B}T}}. \tag{30}$$

Note v_F has not disappeared — it hides inside D. The unified, regime-independent statement:

The length associated with energy scale E is the distance a Fermi-surface electron propagates during the coherence time $\hbar/E$.

The Thouless energy: the inverse dictionary. Given a sample of size L, the associated energy is

$$\eth = \frac{\hbar v_\mathrm{F}}{L} \ \text{(ballistic)}, \qquad \eth = \frac{\hbar D}{L^2} \ \text{(diffusive)}. \tag{31}$$

The short- versus long-junction classification of a Josephson device is simply: which is smaller, Δ or $\eth$? — equivalently, which length is longer, ξ or L? Students should translate freely in both directions between energies and lengths.

7 Summary Table

Energy scale E	**Length $\hbar v_\mathrm{F}/E$**	**Physics**	**Typical size**
E_F	$\lambda_\mathrm{F} \sim 1/k_\mathrm{F}$	Fermi wavelength; Friedel/RKKY period	Å
$\hbar/\tau$	$\ell = v_\mathrm{F}\tau$	mean free path; seed of ξ_{loc}	nm–μm
Δ	$\xi_0 = \hbar v_\mathrm{F}/\pi\Delta$	Cooper pair size	10 nm–μm
$2\pi k_\mathrm{B}T$	L_T	Friedel envelope; SNS decay	μm at 1 K
$k_\mathrm{B}T_K$	ξ_K	Kondo screening cloud	μm
$\eth = \hbar v_\mathrm{F}/L$	L (by definition)	inverse dictionary: size → energy	—

Reading top to bottom is reading the hierarchy of energies from largest to smallest, and correspondingly the lengths from shortest to longest. The nested structure $\lambda_\mathrm{F} \ll \ell \ll \xi_0, L_T, \xi_K$ is what makes the quasiparticle description self-consistent in the first place.

8 Where the Framework Fails

The construction rests on a sharp Fermi surface and finite v_F. Three modern frontiers break it:

Flat bands. In magic-angle twisted bilayer graphene, $v_F \to 0$ by design. Every length in the table collapses toward the lattice scale, kinetic energy loses to interactions, and correlated insulators and superconductivity emerge — flat-band engineering is the deliberate demolition of this lecture.

Non-Fermi liquids. Near quantum critical points, well-defined quasiparticles may not exist; ω/T scaling replaces the dispersion. The coherence *time* $\hbar/k_B T$ survives (the "Planckian" time of strange-metal transport), but there is no clean v_F to convert it into a length.

Dirac materials. At the neutrality point of graphene or a topological semimetal there is no Fermi surface, yet the velocity survives as a fundamental constant of the low-energy theory (an emergent "speed of light"). The master formula still works, with v_F fixed and E measured from the Dirac point.

Suggested Problems

1. Derive the $x/\sinh x$ thermal envelope of Friedel oscillations in 1D from the finite-T Lindhard function, identifying the Matsubara pole controlling the asymptotic decay.
2. For a clean SNS junction, compute the Andreev phase $\delta\phi(\varepsilon)$ and show that thermal averaging reproduces $I_c \propto e^{-L/L_T}$. Redo the estimate for a diffusive junction using $\eth = \hbar D/L^2$.
3. Estimate ξ_K for $T_K = 0.5$ K and $v_F = 1.5 \times 10^6$ m/s; compare with the size of a typical quantum-dot device. What does "the cloud is bigger than the device" mean physically?
4. (Open-ended) Suppose the dispersion near E_F were quadratic in $k - k_F$ rather than linear. How does the energy–length dictionary change? Which lengths in the table are modified?