Friedel Oscillations in One Dimension: The T-Matrix / Green's Function Approach

Graduate Condensed-Matter Notes

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Abstract

We give a self-contained derivation of Friedel oscillations produced by a single delta-function impurity in a one-dimensional gas of non-interacting electrons. We construct the free retarded Green's function, present its Lehmann (spectral) representation to connect $\operatorname{Im} G^R$ to the local density of states, resum the impurity scattering exactly through the Dyson equation and the T-matrix (deriving the separable form for a delta potential), and finally obtain the induced charge density. At zero temperature the density exhibits a $2k_F$ oscillation with a $1/x$ power-law envelope; at finite temperature this envelope is cut off beyond the thermal length $\xi_T = \hbar v_F/(2\pi k_B T)$ through the universal factor $(x/\xi_T)/\sinh(x/\xi_T)$.

Contents

1 Physical Setup 1

2 The Free Green's Function and Its Lehmann Representation 1
- 2.1 Lehmann representation in an eigenbasis . . . 1
- 2.2 Spectral function and density of states . . . 2
- 2.3 Free case: explicit Lehmann form . . . 2
- 2.4 Evaluating the free Green's function . . . 3

3 Dyson's Equation and the T-Matrix 3
- 3.1 Dyson equation and definition of the T-matrix . . . 3
- 3.2 Deriving the separable form for a delta impurity . . . 3
- 3.3 The diagonal correction to the Green's function . . . 4

4 Charge Density from the Green's Function 4

5 Zero-Temperature Result 5

6 Finite Temperature and the Thermal Length 5
- 6.1 Linearization about the Fermi surface . . . 5
- 6.2 The sech^2 Fourier transform . . . 6
- 6.3 Assembly: the central result . . . 6
- 6.4 Limits and interpretation . . . 6

1 Physical Setup

Consider non-interacting spinless electrons in one dimension (the spin factor of 2 can be restored at the end) with a single static impurity at the origin,

$$H = H_0 + V, \qquad H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}, \qquad V(x) = u\,\delta(x). \tag{1}$$

The impurity breaks translational invariance and back-scatters electrons. Since an incoming wave e^{ikx} and the back-scattered wave e^{-ikx} interfere, the electron density acquires a standing-wave modulation at wavevector $2k_F$. These are the Friedel oscillations. The Green's function is the natural tool because the density is built directly from it.

2 The Free Green's Function and Its Lehmann Representation

Before computing G_0^R explicitly, we record its Lehmann (spectral) representation. This makes the link between the retarded Green's function and the density of states transparent and explains *why* the density in Sections 4–6 is built from Im G^R.

2.1 Lehmann representation in an eigenbasis

For any single-particle Hamiltonian H with a complete orthonormal set of eigenstates $H\,|\alpha\rangle = \epsilon_\alpha\,|\alpha\rangle$, the retarded Green's function operator is

$$G^R(E) = \frac{1}{E - H + i0^+}. \tag{2}$$

Inserting the resolution of identity $\sum_\alpha |\alpha\rangle\,\langle\alpha| = \mathbb{1}$ and taking coordinate matrix elements $G^R(x, x'; E) = \langle x|G^R(E)|x'\rangle$ gives the **Lehmann representation**

$$\boxed{G^R(x, x'; E) = \sum_\alpha \frac{\psi_\alpha(x)\,\psi_\alpha^*(x')}{E - \epsilon_\alpha + i0^+}} \tag{3}$$

where $\psi_\alpha(x) = \langle x|\alpha\rangle$. The poles of G^R in E sit at the exact single-particle energies ϵ_α, just below the real axis; the $i0^+$ pushes them into the lower half-plane, which is precisely what makes the function *retarded* (analytic in the upper half-plane).

For a general interacting system Eq. (3) generalizes to a sum over exact many-body eigenstates,

$$G^R(x, x'; \omega) = \sum_n \left[\frac{\langle\Psi_0|\hat{\psi}(x)|\Psi_n\rangle\;\langle\Psi_n|\hat{\psi}^\dagger(x')|\Psi_0\rangle}{\hbar\omega - (E_n^{N+1} - E_0^N) + i0^+} + \frac{\langle\Psi_0|\hat{\psi}^\dagger(x')|\Psi_n\rangle\;\langle\Psi_n|\hat{\psi}(x)|\Psi_0\rangle}{\hbar\omega + (E_n^{N-1} - E_0^N) + i0^+} \right], \tag{4}$$

which reduces to Eq. (3) in the non-interacting limit.

2.2 Spectral function and density of states

The physical content follows from the imaginary part. With the Sokhotski–Plemelj identity

$$\frac{1}{E - \epsilon_\alpha + i0^+} = \mathcal{P}\frac{1}{E - \epsilon_\alpha} - i\pi\,\delta(E - \epsilon_\alpha), \tag{5}$$

we obtain the **spectral function**

$$A(x,x';E) \equiv -\frac{1}{\pi}\,\mathrm{Im}\,G^R(x,x';E) = \sum_\alpha \psi_\alpha(x)\,\psi^*_\alpha(x')\,\delta(E-\epsilon_\alpha). \tag{6}$$

Two special cases give the quantities we need:

$$\text{Local DOS:}\quad \rho(x,E) = -\frac{1}{\pi}\,\mathrm{Im}\,G^R(x,x;E) = \sum_\alpha |\psi_\alpha(x)|^2\,\delta(E-\epsilon_\alpha), \tag{7}$$

$$\text{Total DOS:}\quad \int dx\,\rho(x,E) = \sum_\alpha \delta(E-\epsilon_\alpha) = \rho(E), \tag{8}$$

using normalization $\int dx\,|\psi_\alpha(x)|^2 = 1$. The particle density then follows by filling states with the Fermi function $f(E) = [e^{(E-\mu)/k_BT}+1]^{-1}$:

$$n(x) = \int_{-\infty}^{\infty} dE\,f(E)\,\rho(x,E) = \sum_\alpha |\psi_\alpha(x)|^2 f(\epsilon_\alpha), \tag{9}$$

which is exactly $\langle \hat{n}(x)\rangle$ for the free Fermi gas. This is the starting point of Section 4, now derived rather than asserted.

2.3 Free case: explicit Lehmann form

For H_0 the eigenstates are plane waves $\psi_k(x) = e^{ikx}/\sqrt{L}$ with $\epsilon_k = \hbar^2k^2/2m$. The Lehmann sum becomes an integral,

$$G_0^R(x,x';E) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{e^{ik(x-x')}}{E-\frac{\hbar^2k^2}{2m}+i0^+}. \tag{10}$$

Its spectral function reproduces the free local density of states

$$\rho_0(x,E) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\delta\Big(E-\tfrac{\hbar^2k^2}{2m}\Big) = \frac{1}{\pi\hbar v_E} \equiv \nu(E), \tag{11}$$

where the two roots $k = \pm k_E$ combine with the Jacobian $|d\epsilon/dk|^{-1} = m/\hbar^2 k_E$. This is x-independent, as it must be for a translationally invariant system; the oscillations arise only once the impurity breaks that symmetry.

2.4 Evaluating the free Green's function

Introduce $k_E = \sqrt{2mE}/\hbar$ so that $E = \hbar^2k_E^2/2m$, and rewrite Eq. (10) as

$$G_0^R = -\frac{2m}{\hbar^2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{e^{ik(x-x')}}{k^2-k_E^2-i0^+}. \tag{12}$$

The denominator has poles at $k = \pm(k_E + i0^+)$. For $x - x' > 0$ we close in the upper half-plane (pole at $+k_E$); for $x - x' < 0$ we close below (pole at $-k_E$). Both give a result depending only on $|x-x'|$:

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{e^{ik|x-x'|}}{k^2-k_E^2-i0^+} = \frac{i}{2k_E}\,e^{ik_E|x-x'|}. \tag{13}$$

Therefore

$$\boxed{G_0^R(x,x';E) = -\frac{im}{\hbar^2k_E}\,e^{ik_E|x-x'|} = -\frac{i}{\hbar v_E}\,e^{ik_E|x-x'|}} \tag{14}$$

with $v_E = \hbar k_E/m$. Setting $x' = x$ gives $G_0^R(x,x;E) = -i/(\hbar v_E)$, whose imaginary part reproduces $\rho_0 = 1/(\pi\hbar v_E)$, consistent with the spectral derivation above.

3 Dyson's Equation and the T-Matrix

3.1 Dyson equation and definition of the T-matrix

The full Green's function is obtained by expanding $(E - H_0 - V)^{-1}$ in powers of V, giving the Dyson equation

$$G = G_0 + G_0 V G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots . \tag{15}$$

We resum this series by defining the T**-matrix**, which packages all repeated scatterings off the impurity into an effective single-scattering vertex,

$$G = G_0 + G_0\, T\, G_0, \qquad T = V + V G_0 V + V G_0 V G_0 V + \cdots = V + V G_0 T. \tag{16}$$

3.2 Deriving the separable form for a delta impurity

For the local potential $V(x) = u\,\delta(x)$ the coordinate matrix element is

$$\langle x|V|y\rangle = V(x)\,\delta(x-y) = u\,\delta(x)\,\delta(x-y). \tag{17}$$

The delta functions force every scattering event to occur at the origin. Consider the second-order term of Eq. (16):

$$\begin{aligned}\left\langle x\middle|V G_0 V\middle|x'\right\rangle &= \int dy_1\, dy_2\ \langle x|V|y_1\rangle\ G_0(y_1, y_2; E)\ \left\langle y_2\middle|V\middle|x'\right\rangle \\ &= \int dy_1\, dy_2\ u\,\delta(x)\delta(x-y_1)\, G_0(y_1, y_2; E)\, u\,\delta(y_2)\delta(y_2 - x') \\ &= u^2\,\delta(x)\,\delta(x')\, G_0(0, 0; E). \end{aligned} \tag{18}$$

The same collapse occurs at every order: each internal integration is pinned to the origin, and each intermediate propagator contributes a factor $G_0(0, 0; E)$. The n-th order term is $u\,[u\, G_0(0, 0; E)]^{\,n-1}\,\delta(x)\delta(x')$. Hence the T-matrix is **separable**,

$$\boxed{T(x, x'; E) = \delta(x)\,\delta(x')\,t(E), \qquad t(E) = u\sum_{n=0}^{\infty}\big[u\, G_0(0, 0; E)\big]^{n} = \frac{u}{1 - u\, G_0(0, 0; E)}} \tag{19}$$

The geometric series converges to a closed form because the scattering strength enters only through the single c-number $G_0(0, 0; E)$. Using $G_0(0, 0; E) = -i/(\hbar v_E)$ from Eq. (14),

$$t(E) = \frac{u}{1 + \dfrac{iu}{\hbar v_E}}. \tag{20}$$

3.3 The diagonal correction to the Green's function

The full Green's function is then exactly

$$G^R(x, x'; E) = G_0^R(x, x'; E) + G_0^R(x, 0; E)\, t(E)\, G_0^R(0, x'; E). \tag{21}$$

The diagonal correction needed for the density, $\delta G(x, x; E) \equiv G^R(x, x; E) - G_0^R(x, x; E)$, is

$$\boxed{\delta G(x, x; E) = G_0^R(x, 0; E)\, t(E)\, G_0^R(0, x; E) = -\frac{t(E)}{(\hbar v_E)^2}\, e^{\,2ik_E|x|}} \tag{22}$$

The factor $e^{2ik_E|x|}$ is the seed of the oscillation: the electron propagates from x to the impurity and back, accumulating phase $2k_E|x|$. At the Fermi surface this becomes the $2k_F$ modulation. For the remainder we keep the leading Born approximation $t(E) \approx u$, which captures all the essential physics; the full $t(E)$ merely renormalizes the amplitude and inserts a scattering phase shift $2\delta_0$ inside the cosine.

4 Charge Density from the Green's Function

Using Eq. (9), the impurity-induced change in density is

$$\delta n(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} dE\, f(E)\ \mathrm{Im}\ \delta G(x, x; E). \tag{23}$$

With the Born expression and $\mathrm{Im}\ e^{2ik_E|x|} = \sin(2k_E|x|)$,

$$\delta n(x) = \frac{u}{\pi} \int_0^{\infty} dE\, f(E)\, \frac{\sin(2k_E|x|)}{(\hbar v_E)^2}. \tag{24}$$

Changing variables to k via $E = \hbar^2k^2/2m$ ($dE = (\hbar^2k/m)\, dk$, $(\hbar v_E)^2 = (\hbar^2k/m)^2$) yields the compact master formula

$$\boxed{\delta n(x) = \frac{um}{\pi\hbar^2} \int_0^{\infty} \frac{dk}{k}\, f(\epsilon_k)\, \sin(2k|x|)} \tag{25}$$

with $\epsilon_k = \hbar^2k^2/2m$. This single integral contains both the zero-temperature Friedel oscillation and its thermal decay.

5 Zero-Temperature Result

At $T = 0$ the Fermi function is a sharp step, $f = \theta(k_F - k)$:

$$\delta n(x) = \frac{um}{\pi\hbar^2} \int_0^{k_F} \frac{dk}{k}\, \sin(2k|x|). \tag{26}$$

The long-distance oscillation is dominated by the sharp cutoff at $k = k_F$. Integrating by parts, the boundary term gives

$$\int_0^{k_F} \frac{\sin(2k|x|)}{k}\, dk \xrightarrow{|x|\to\infty} -\frac{\cos(2k_F|x|)}{2k_F|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right), \tag{27}$$

so that

$$\boxed{\delta n(x) \simeq -\frac{u}{2\pi\hbar v_F}\, \frac{\cos(2k_F|x|)}{|x|} = -\frac{u\,\nu_F}{2}\, \frac{\cos(2k_F|x|)}{|x|}} \tag{28}$$

with $\nu_F = 1/(\pi\hbar v_F)$. This is the hallmark 1D Friedel oscillation: a $2k_F$ modulation with a $1/|x|$ power-law envelope. (In 2D the decay is $1/r^2$ and in 3D it is $1/r^3$; the slow 1D decay reflects the perfectly nested Fermi "surface" of just two points $\pm k_F$.)

6 Finite Temperature and the Thermal Length

Write the oscillatory integral of Eq. (25) as

$$\delta n(x,T) = \operatorname{Im}\int_0^\infty dk\, g(k)\, f(\epsilon_k)\, e^{2ikx}, \qquad g(k) = \frac{um}{\pi\hbar^2 k}. \tag{29}$$

Integrating by parts moves the oscillation onto f. Since $g(k)$ is smooth while df/dk is sharply peaked at the Fermi surface over a window k_BT, the $2k_F$ oscillation comes entirely from the derivative term:

$$\delta n(x,T) \simeq -\operatorname{Im}\left\{\frac{g(k_F)}{2ix}\int_0^\infty dk\,\frac{df}{dk}\,e^{2ikx}\right\}. \tag{30}$$

6.1 Linearization about the Fermi surface

Set $k = k_F + q$ and linearize $\epsilon_k - \mu \approx \hbar v_F q$. Using $f(\epsilon) = \frac{1}{2}[1 - \tanh\frac{\epsilon-\mu}{2k_BT}]$,

$$\frac{df}{dk} = \hbar v_F f'(\epsilon_k) = -\frac{\hbar v_F}{4k_BT}\,\operatorname{sech}^2\!\left(\frac{\hbar v_F q}{2k_BT}\right). \tag{31}$$

Because this is sharply peaked we extend the q-integration over the whole real axis and substitute $s = \hbar v_F q/(2k_BT)$:

$$\int_0^\infty dk\,\frac{df}{dk}\,e^{2ikx} = -\frac{e^{2ik_Fx}}{2}\int_{-\infty}^\infty ds\,\operatorname{sech}^2(s)\,e^{i\alpha s}, \qquad \alpha = \frac{4k_BT\,x}{\hbar v_F}. \tag{32}$$

6.2 The sech^2 Fourier transform

The required transform is

$$\int_{-\infty}^\infty \operatorname{sech}^2(s)\,e^{i\alpha s}\,ds = \frac{\pi\alpha}{\sinh(\pi\alpha/2)}. \tag{33}$$

Defining the **thermal length**

$$\boxed{\xi_T = \frac{\hbar v_F}{2\pi k_BT}} \tag{34}$$

we have $\pi\alpha/2 = x/\xi_T$, and Eq. (33) becomes e^{2ik_Fx} times a real, even envelope.

6.3 Assembly: the central result

Collecting the factors in Eq. (30), with $g(k_F) = u/(\pi\hbar v_F k_F)$, the smooth prefactors combine exactly as in the $T = 0$ case and we obtain

$$\boxed{\delta n(x,T) = -\frac{u}{2\pi\hbar v_F}\,\frac{\cos(2k_Fx)}{x}\cdot\frac{x/\xi_T}{\sinh(x/\xi_T)}} \tag{35}$$

6.4 Limits and interpretation

Equation (35) interpolates between all regimes:

- **Zero temperature.** As $T \to 0$, $\xi_T \to \infty$ and $(x/\xi_T)/\sinh(x/\xi_T) \to 1$, recovering Eq. (28).

- **Short distances $x \ll \xi_T$.** The envelope ≈ 1; oscillations keep their $1/x$ form and thermal effects are invisible.

- **Long distances $x \gg \xi_T$.** With $\sinh(x/\xi_T) \approx \frac{1}{2} e^{x/\xi_T}$,

$$\delta n(x, T) \simeq -\frac{u}{\pi \hbar v_F \, \xi_T} \cos(2k_F x) \, e^{-x/\xi_T}, \tag{36}$$

  an exponential cutoff with decay length ξ_T.

Physically, at temperature T the Fermi surface is smeared over a momentum window $\delta k \sim k_B T/(\hbar v_F)$. Electrons in this window contribute to the $2k_F$ standing wave with slightly different wavevectors and dephase over $x \sim 1/\delta k \sim \xi_T$. Beyond the thermal length the interference washes out and the oscillations decay exponentially. The same universal factor $(x/\xi_T)/\sinh(x/\xi_T)$ governs the finite-T RKKY interaction and the 1D polarization function, reflecting the generic thermal dephasing of $2k_F$ processes in one dimension.