

Anderson Localization in One Dimension

Transfer Matrices, Lyapunov Exponents, and the Localization Length

Graduate Lecture Notes

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Abstract

We introduce Anderson localization through the exactly tractable one-dimensional tight-binding model. We derive the transfer-matrix formulation, identify the localization length with the inverse Lyapunov exponent of a product of random 2×2 matrices, and compute it analytically in the weak-disorder limit. Two illustrative examples—the clean chain and weak box disorder—bracket the physics. Programmatically generated figures show disorder on the lattice, the contrast between extended and localized states, and the energy dependence of the localization length.

1 The physical idea

In 1958 Anderson asked what happens to an electron when the periodic crystal potential is spoiled by randomness. In a perfect crystal the eigenstates are Bloch waves, extended over the whole sample; an injected electron diffuses away forever. Anderson showed that sufficiently strong disorder can completely suppress diffusion: the eigenstates become *exponentially localized*, $\psi(x) \sim e^{-|x-x_0|/\xi}$, and the material becomes an insulator even where band theory predicts a metal. The mechanism is quantum interference of multiply back-scattered partial waves. In one dimension this interference is so effective that *any* disorder localizes *every* eigenstate, which makes 1D the cleanest possible setting.

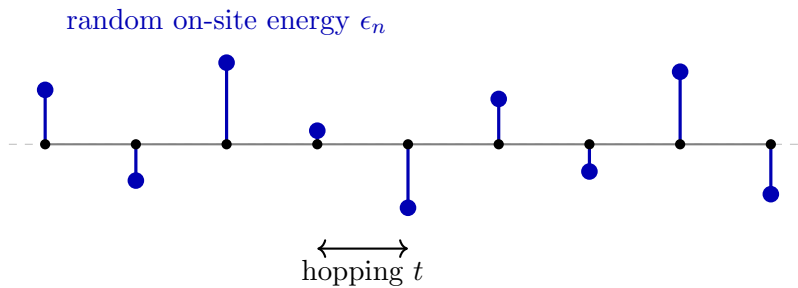


Figure 1: The 1D Anderson model: one orbital per site, nearest-neighbor hopping t , and i.i.d. random on-site energies ϵ_n .

2 The tight-binding Hamiltonian

With one orbital per site,

$$H = \sum_n \epsilon_n |n\rangle \langle n| + t \sum_n (|n\rangle \langle n+1| + |n+1\rangle \langle n|),$$

where the ϵ_n are i.i.d. with $\langle \epsilon_n \rangle = 0$ and $\langle \epsilon_n \epsilon_m \rangle = \sigma^2 \delta_{nm}$. For a box distribution on $[-W/2, W/2]$ one has $\sigma^2 = W^2/12$. Writing $|\psi\rangle = \sum_n \psi_n |n\rangle$, the eigenvalue equation $H|\psi\rangle = E|\psi\rangle$ becomes the three-term recurrence

$$\epsilon_n \psi_n + t(\psi_{n+1} + \psi_{n-1}) = E \psi_n. \quad (1)$$

3 The transfer matrix

Equation (1) is local: two consecutive amplitudes determine the rest. Solving for ψ_{n+1} and packaging neighbors into a vector gives

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = T_n \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}, \quad T_n = \begin{pmatrix} \frac{E - \epsilon_n}{t} & -1 \\ 1 & 0 \end{pmatrix}.$$

Iterating, $(\psi_{N+1}, \psi_N)^\top = \mathcal{M}_N (\psi_1, \psi_0)^\top$ with $\mathcal{M}_N = T_N \cdots T_1$. Two facts are central. First, $\det T_n = 1$, so $T_n \in \text{SL}(2, \mathbb{R})$ and $\det \mathcal{M}_N = 1$ (flux conservation, constant Wronskian). Second, the T_n are i.i.d. random matrices, so $\mathcal{M}_N$ is a product of random matrices, and all the physics lies in its growth.

4 Lyapunov exponent and localization length

Because $\det \mathcal{M}_N = 1$, the singular values are $\lambda \geq 1$ and $1/\lambda$. By the Furstenberg and Oseledec theorems the growth rate exists and is non-random (self-averaging),

$$\gamma(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \|\mathcal{M}_N\|.$$

A normalizable eigenstate decays both ways from its center, $|\psi_n| \sim e^{-\gamma|n-n_0|}$, so

$$\xi(E) = \frac{1}{\gamma(E)}. \quad (2)$$

In 1D, $\gamma > 0$ for any nonzero disorder: localization is generic.

5 Example 1: the clean chain

For $\epsilon_n \equiv 0$, $\mathcal{M}_N = T_0^N$ with eigenvalues obeying $\lambda + 1/\lambda = E/t$. Inside the band $|E| \leq 2|t|$, set $E = 2t \cos k$: then $\lambda = e^{\pm ik}$ lies on the unit circle, $\gamma = 0$, $\xi = \infty$, and the states are extended Bloch waves $\psi_n \sim e^{ikn}$. Outside the band the eigenvalues are real evanescent solutions.

6 Example 2: weak disorder and the Thouless formula

Set $t = 1$, $E = 2 \cos k$, and use Prüfer variables $\psi_n = r_n \sin \theta_n$, $\psi_{n-1} = r_n \sin(\theta_n - k)$. Writing $\theta_{n+1} = \theta_n + k + \delta_n$ and expanding the recurrence,

$$\delta_n = \frac{\sin^2 \theta_n}{\sin k} \epsilon_n + \frac{\cos \theta_n \sin^3 \theta_n}{\sin^2 k} \epsilon_n^2 + \cdots, \quad \ln \frac{r_{n+1}}{r_n} = -\cot \theta_n \delta_n + \frac{\delta_n^2}{2 \sin^2 \theta_n} + \cdots$$

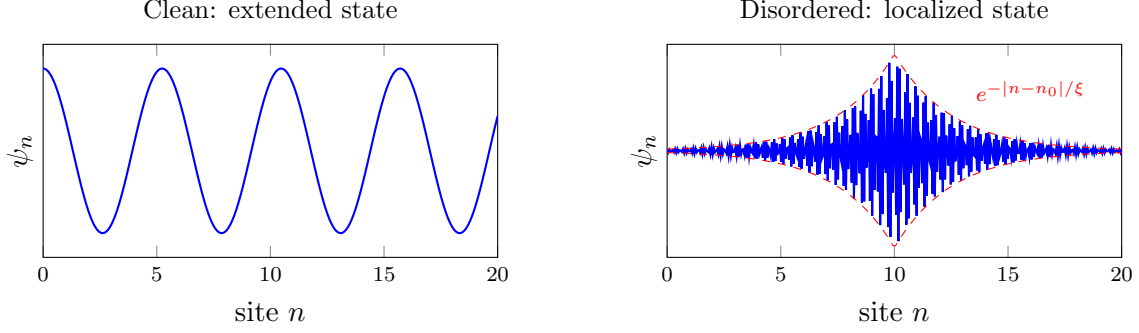


Figure 2: Extended Bloch wave (clean chain) versus an exponentially localized eigenstate (disordered chain) with its envelope $e^{-|n-n_0|/\xi}$.

The Lyapunov exponent is $\gamma = \langle \ln(r_{n+1}/r_n) \rangle$. Under the random-phase approximation (θ_n uniform and independent of ϵ_n), with $\langle \sin^2 \theta \rangle = \frac{1}{2}$ and $\langle \sin^2 \theta \cos^2 \theta \rangle = \frac{1}{8}$, the first-order term vanishes and the second-order terms combine to

$$\gamma = \left(\frac{1}{4 \sin^2 k} - \frac{1}{8 \sin^2 k} \right) \sigma^2 = \frac{\sigma^2}{8 \sin^2 k}.$$

Restoring t ,

$$\gamma(E) = \frac{\langle \epsilon_n^2 \rangle}{8 t^2 \sin^2 k}, \quad \xi(E) = \frac{8 t^2 \sin^2 k}{\langle \epsilon_n^2 \rangle} = \frac{2(4t^2 - E^2)}{\sigma^2}. \quad (3)$$

At the band center ($E = 0$, box disorder $\sigma^2 = W^2/12$), $\xi(0) = 8t^2/\sigma^2 = 96t^2/W^2$. Thus $\xi \propto 1/W^2$ (always finite), it is *maximal at the band center* and shrinks toward the band edges, where perturbation theory fails (true edge scaling $\gamma \sim \sigma^{2/3}$), with an additional anomaly exactly at $E = 0$.

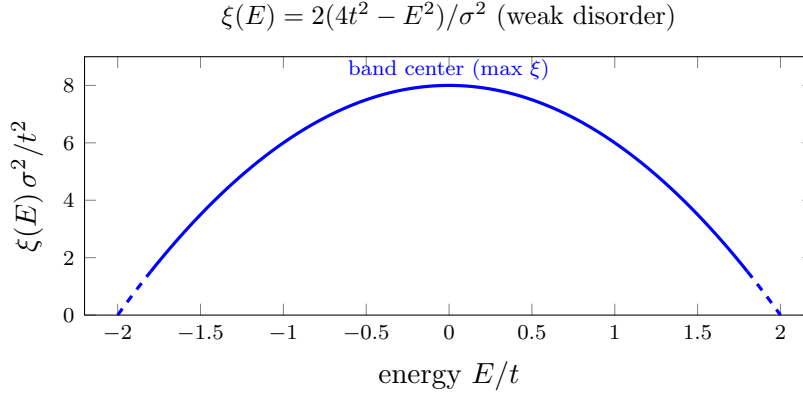


Figure 3: Localization length across the band. Dashed segments indicate where perturbation theory breaks down near the edges.

7 Numerics and the bigger picture

Numerically one multiplies the random T_n with periodic QR re-orthogonalization, $\mathcal{M} \rightarrow QR$, keeping Q and accumulating $\sum \ln R_{11}$ to extract γ stably. The 1D result is the extreme case of the scaling theory of localization: all states localize for arbitrarily weak disorder in $d = 1, 2$, while

$d = 3$ hosts a metal–insulator transition with a mobility edge E_c . The same transfer-matrix idea, promoted to $2M \times 2M$ matrices on quasi-1D bars, provides the sharpest numerical probe of that transition.