The Cooper Problem

Lecture Notes

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Abstract

In 1956 Leon Cooper showed that the normal (filled) Fermi sea of a metal is unstable against the formation of a bound pair of electrons, provided there is *any* net attractive interaction between them, no matter how weak. This bound pair—the "Cooper pair"—is the seed of the BCS theory of superconductivity. Below is the full derivation.

Contents

1 Physical setup

Consider a filled Fermi sea at $T = 0$, completely inert except that it blocks all states with $k < k_F$ through the Pauli exclusion principle. We now add **two extra electrons** just above the Fermi surface and ask whether the attractive interaction between them can bind them into a state whose energy lies *below* $2\epsilon_F$.

We look for the lowest-energy pair state. General arguments (and the fact that the interaction is strongest for pairs that can scatter into the largest number of states) show that the pair should have **zero total momentum**, so the two electrons occupy states $\mathbf{k}$ and $-\mathbf{k}$. Furthermore the spatial wave function will turn out to be symmetric, which requires an antisymmetric **spin-singlet** state.

2 The trial wave function

With zero center-of-mass momentum, the most general two-particle spatial wave function is a superposition of pair states $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \left(\sum_{\mathbf{k}} g(\mathbf{k})\, e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \right) \times (\text{spin singlet}), \tag{1}$$

where $g(\mathbf{k})$ is the amplitude for finding one electron in $+\mathbf{k}$ and the other in $-\mathbf{k}$. Because the states below the Fermi surface are already occupied, the Pauli principle forbids the added electrons from using them:

$$g(\mathbf{k}) = 0 \qquad \text{for } k < k_F. \tag{2}$$

3 Substituting into the Schrödinger equation

The two-particle Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m} \left(\nabla_1^2 + \nabla_2^2 \right) + V(\mathbf{r}_1 - \mathbf{r}_2) \right] \psi = E\, \psi. \tag{3}$$

Acting with the kinetic operator on the plane wave $e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}$ gives a factor $\frac{\hbar^2k^2}{2m}$ for **each** electron. Defining the single-particle energy

$$\epsilon_k = \frac{\hbar^2 k^2}{2m}, \tag{4}$$

the kinetic energy of the pair component $\mathbf{k}$ is $2\epsilon_k$. Substituting the expansion:

$$\sum_{\mathbf{k}} g(\mathbf{k})\, 2\epsilon_k\, e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} + V(\mathbf{r}_1-\mathbf{r}_2) \sum_{\mathbf{k}} g(\mathbf{k})\, e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} = E \sum_{\mathbf{k}} g(\mathbf{k})\, e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}. \tag{5}$$

Now project onto a particular plane wave by multiplying by $e^{-i\mathbf{k}'\cdot(\mathbf{r}_1-\mathbf{r}_2)}$ and integrating over the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The kinetic and energy terms are diagonal, while the potential produces its Fourier matrix element. Relabeling indices, we obtain the **equation for the amplitudes** $g(\mathbf{k})$:

$$\boxed{(2\epsilon_k - E)\, g(\mathbf{k}) = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\, g(\mathbf{k}')} \tag{6}$$

where the interaction matrix element is

$$V_{\mathbf{k}\mathbf{k}'} = \frac{1}{\Omega} \int V(\mathbf{r})\, e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\, d^3r, \tag{7}$$

with Ω the crystal volume. This is an integral equation for $g(\mathbf{k})$; it is exactly the Schrödinger equation rewritten in momentum space.

4 Cooper's model interaction

The exact $V_{\mathbf{k}\mathbf{k}'}$ is complicated, so Cooper introduced a simple approximation that captures the essential physics of the phonon-mediated attraction. The attraction acts only within a thin shell of width $\hbar\omega_D$ (the Debye energy) above the Fermi surface, and is taken to be a negative constant there:

$$V_{\mathbf{k}\mathbf{k}'} = \begin{cases} -V & \text{if } \epsilon_F < \epsilon_k,\ \epsilon_{k'} < \epsilon_F + \hbar\omega_D \\ 0 & \text{otherwise,} \end{cases} \qquad V > 0. \tag{8}$$

The cutoff $\hbar\omega_D$ reflects the fact that the attraction is caused by lattice (phonon) exchange and only operates for electrons within a phonon energy of the Fermi surface.

5 Solving for $g(\mathbf{k})$

With this interaction, the right-hand side of the amplitude equation becomes independent of $\mathbf{k}$, since the sum over $\mathbf{k}'$ just runs over the shell:

$$(2\epsilon_k - E)\, g(\mathbf{k}) = V \sum_{\mathbf{k}'} g(\mathbf{k}') \equiv VC, \tag{9}$$

where we have defined the constant

$$C = \sum_{\mathbf{k}'} g(\mathbf{k}'). \tag{10}$$

Therefore the amplitude has the simple, revealing form

$$\boxed{g(\mathbf{k}) = \frac{VC}{2\epsilon_k - E}} \tag{11}$$

Notice that $g(\mathbf{k})$ is largest just at the Fermi surface and falls off as $1/\epsilon_k$, so the pair is built predominantly from states in the attractive shell.

6 The self-consistency (eigenvalue) condition

To find E, sum both sides over all $\mathbf{k}$ in the shell. Since $\sum_{\mathbf{k}} g(\mathbf{k}) = C$, the constant C cancels and we obtain the eigenvalue condition:

$$1 = V \sum_{\mathbf{k}} \frac{1}{2\epsilon_k - E}. \tag{12}$$

Converting the sum into an energy integral using the density of states **per spin** at the Fermi level, $N(0)$ (which is nearly constant across the thin shell):

$$1 = VN(0) \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} \frac{d\epsilon}{2\epsilon - E}. \tag{13}$$

Carrying out the integral:

$$1 = \frac{N(0)V}{2} \ln\left(\frac{2\epsilon_F + 2\hbar\omega_D - E}{2\epsilon_F - E}\right). \tag{14}$$

7 The bound-state energy

Introduce the **binding energy** Δ measured relative to the unbound pair energy $2\epsilon_F$:

$$E = 2\epsilon_F - \Delta, \qquad \Delta > 0 \text{ means a bound state.} \tag{15}$$

Then $2\epsilon_F - E = \Delta$ and $2\epsilon_F + 2\hbar\omega_D - E = 2\hbar\omega_D + \Delta$, giving

$$\frac{2}{N(0)V} = \ln\left(\frac{2\hbar\omega_D + \Delta}{\Delta}\right). \tag{16}$$

In the **weak-coupling limit** $N(0)V \ll 1$, the left side is large, forcing $\Delta \ll \hbar\omega_D$. We may then drop Δ next to $2\hbar\omega_D$ in the numerator:

$$\frac{2}{N(0)V} \approx \ln\left(\frac{2\hbar\omega_D}{\Delta}\right) \quad \Longrightarrow \quad \boxed{\Delta = 2\hbar\omega_D\, e^{-2/N(0)V}} \tag{17}$$

so the energy of the pair is

$$E = 2\epsilon_F - 2\hbar\omega_D\, e^{-2/N(0)V}. \tag{18}$$

8 Physical significance

Three features of this result are profound.

A bound state always forms. For *any* attraction $V > 0$, no matter how weak, $\Delta > 0$. The two electrons bind below $2\epsilon_F$. This is dramatically different from the two-body problem in free space in three dimensions, where a minimum interaction strength is required to form a bound state. The difference comes entirely from the presence of the Fermi sea: the sharp cutoff of allowed states at k_F effectively reduces the problem to one dimension near the Fermi surface, where an arbitrarily weak attraction always binds. It is the restriction $g(k) = 0$ for $k < k_F$ that makes the density of states nonzero and finite right at the relevant energy.

The result is non-perturbative. The binding energy $\Delta \propto e^{-2/N(0)V}$ has an essential singularity at $V = 0$. Every derivative in V vanishes there, so Δ can never be obtained by any finite order of perturbation theory. This is why superconductivity eluded explanation for so long.

Instability of the normal state. Since the normal Fermi sea can always lower its energy by forming such pairs, it is *unstable*. The system must reorganize into a new ground state built from many overlapping Cooper pairs—which is precisely the BCS superconducting state. In full BCS theory the analogous gap equation yields $\Delta_{\mathrm{BCS}} = 2\hbar\omega_D\, e^{-1/N(0)V}$, with the same characteristic exponential but a factor of $1/N(0)V$ rather than $2/N(0)V$, because in BCS all electrons near the Fermi surface participate self-consistently rather than just a single added pair.

A note on conventions. Here $N(0)$ is the density of states at the Fermi level *for one spin direction*. If instead one uses the total density of states including both spins, the factor of 2 in the exponent disappears; different textbooks (Tinkham, de Gennes, Fetter–Walecka) differ on this point, so it is worth checking the definition when comparing formulas.