The Size of a Cooper Pair in a Flat Band

Lecture Notes on Quantum Geometry and Superconductivity

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Abstract

In conventional superconductors the size of a Cooper pair – the coherence length ξ_0 – is set by how fast electrons move: $\xi_0 \sim \hbar v_F/\Delta$. In a *flat band*, electrons do not move at all ($v_F \to 0$), and this familiar estimate predicts a vanishing pair size, which cannot be correct. These notes resolve the paradox. We show that a new length scale, hidden in the *geometry* of the Bloch wavefunctions and encoded in the *quantum metric* $g_{\mu\nu}(\mathbf{k})$, takes over. The Cooper pair size becomes $\xi^2_{\text{pair}} = 2\,\langle \operatorname{tr} g\rangle_{\text{BZ}}$, exactly twice the minimal spread of a maximally localized Wannier function. We build the argument from the ground up, aiming at a first- or second-year graduate student who has seen elementary BCS theory.

Contents

1 The Question and Why It Is Subtle

A superconductor is characterized by two length scales: the penetration depth λ (how far a magnetic field leaks in) and the coherence length ξ (roughly, the physical diameter of a Cooper pair). In these notes we focus on ξ, and we ask a deceptively simple question:

How big is a Cooper pair when the electrons that form it cannot move?

The phrase "cannot move" means a **flat band**: an energy band whose dispersion $\epsilon_{\mathbf{k}}$ is independent of the crystal momentum $\mathbf{k}$ (Fig. 1). The group velocity $\mathbf{v}_{\mathbf{k}} = \hbar^{-1}\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}}$ then vanishes everywhere, so the effective mass is infinite and electrons are, in a kinetic sense, completely localized.

Flat bands are not academic curiosities. They appear in Lieb and kagome lattices, in Landau levels, and – most famously – in magic-angle twisted bilayer graphene, where superconductivity emerges from nearly flat bands. Understanding pairing in a flat band is therefore of central modern interest, and the pair *size* is one of the most basic quantities we can ask about.

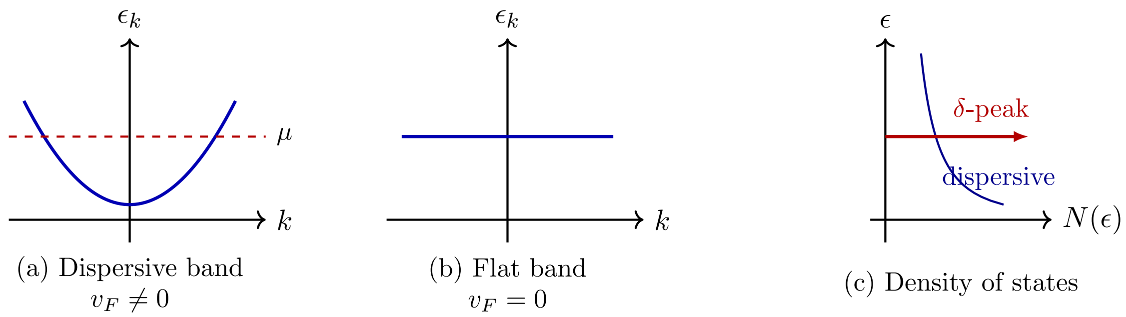


Figure 1: (a) A dispersive band has a finite Fermi velocity v_F and a smooth density of states. (b) A flat band has $\epsilon_{\mathbf{k}} = \text{const}$ so $v_F = 0$ everywhere. (c) The flat band collapses all states to a single energy, giving a divergent (δ-function) density of states. This divergence is the origin of the strong, non-perturbative pairing in flat bands.

The paradox in one line. The textbook BCS coherence length is $\xi_0 \sim \hbar v_F/\Delta$. Setting $v_F = 0$ gives $\xi_0 = 0$: a pointlike pair. But a pointlike pair of two fermions is forbidden by the Pauli principle in any single orbital, and it would also give an infinite kinetic energy. Something must replace v_F. The punchline of these notes is that the missing ingredient is the *quantum geometry* of the Bloch states.

2 Warm-up: The Cooper Pair Size in Ordinary BCS Theory

Before tackling the flat band, let us recall *why* $\xi_0 \sim \hbar v_F/\Delta$ in the conventional case, being careful about the logic so we can see exactly which step breaks down later.

2.1 The pair wavefunction

In the BCS ground state the amplitude to find a pair with momenta $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ is

$$\phi_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}\rangle = u_{\mathbf{k}} v_{\mathbf{k}} = \frac{\Delta}{2E_{\mathbf{k}}}, \qquad E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}, \tag{1}$$

with $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. The real-space relative wavefunction of the two electrons is the Fourier transform, $\phi(\mathbf{r}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\phi_{\mathbf{k}}$, and the size of the pair is its root-mean-square radius,

$$\xi_0^2 \;=\; \langle r^2 \rangle \;=\; \frac{\displaystyle\int d^d r\, r^2\, |\phi(\mathbf{r})|^2}{\displaystyle\int d^d r\, |\phi(\mathbf{r})|^2} \;=\; \frac{\displaystyle\int d^d k\, |\nabla_{\mathbf{k}}\phi_{\mathbf{k}}|^2}{\displaystyle\int d^d k\, |\phi_{\mathbf{k}}|^2}. \tag{2}$$

The last equality uses the standard fact that multiplication by $\mathbf{r}$ in real space becomes the derivative $i\nabla_{\mathbf{k}}$ in momentum space; the mean-square radius is thus a measure of how rapidly $\phi_{\mathbf{k}}$ varies across the Brillouin zone.

Intuition. A pair that is *small* in real space is *spread out* in momentum space (a broad $\phi_{\mathbf{k}}$), and vice versa. This is nothing but the uncertainty principle. The width of $\phi_{\mathbf{k}}$ in $\mathbf{k}$-space therefore controls the inverse size of the pair.

2.2 Evaluating the integral

The amplitude $\phi_{\mathbf{k}} = \Delta/2E_{\mathbf{k}}$ is sharply peaked on the Fermi surface, where $\xi_{\mathbf{k}} = 0$ and ϕ is largest. Near the surface we linearize the dispersion along its normal direction, $\xi_{\mathbf{k}} \simeq \hbar v_F\, q$, where q is the momentum measured from the Fermi surface. Then

$$\frac{d\phi}{dq} = \hbar v_F\, \frac{d\phi}{d\xi} = -\,\hbar v_F\, \frac{\Delta\,\xi}{2(\xi^2+\Delta^2)^{3/2}}. \tag{3}$$

Using the elementary integrals

$$\int_{-\infty}^{\infty} \frac{\xi^2\, d\xi}{(\xi^2+\Delta^2)^3} = \frac{\pi}{8\Delta^3}, \qquad \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2+\Delta^2} = \frac{\pi}{\Delta}, \tag{4}$$

Eq. (2) gives, after the Δ's cancel,

Conventional BCS coherence length

$$\xi_0^2 = \frac{(\hbar v_F)^2}{8\Delta^2} \qquad \Longrightarrow \qquad \boxed{\xi_0 \sim \frac{\hbar v_F}{\Delta}} \tag{5}$$

(The precise Pippard/BCS prefactor, $\xi_0 = \hbar v_F/\pi\Delta$, depends on the exact weighting used, but the scaling is robust.)

Reading the result. The size of the pair is the distance an electron travels at the Fermi velocity during the time $\hbar/\Delta$ it takes to "feel" the gap. It is entirely a *kinetic* quantity. And here is the crucial observation for what follows: **set $v_F = 0$ and the formula predicts a pointlike pair.** The derivation assumed a dispersive band; it says nothing sensible about a flat one. We need a genuinely different calculation.

3 Setting Up the Flat-Band Cooper Problem

The resolution is that a flat band is never made of a single orbital. It lives on a lattice with several orbitals per unit cell, and the Bloch eigenvector rotates among these orbitals as $\mathbf{k}$ sweeps the Brillouin zone. That internal rotation – invisible in the energy $\epsilon_{\mathbf{k}}$ – carries all the spatial information. We now make this precise.

3.1 Bloch states and band projection

Consider a tight-binding model with N_{orb} orbitals $\alpha = 1, \dots, N_{\text{orb}}$ per cell and N unit cells,

$$H_0 = \sum_{\mathbf{k}} \sum_{\alpha\beta\sigma} [H(\mathbf{k})]_{\alpha\beta}\, c^\dagger_{\mathbf{k}\alpha\sigma} c_{\mathbf{k}\beta\sigma}, \qquad H(\mathbf{k})\, |u_{n\mathbf{k}}\rangle = \epsilon_{n\mathbf{k}}\, |u_{n\mathbf{k}}\rangle\,. \tag{6}$$

Let the flat band be band $n = 0$ at energy $\epsilon_0 = 0$, with normalized eigenvector $|u_{\mathbf{k}}\rangle$ whose orbital components are $u^\alpha_{\mathbf{k}} = \langle\alpha\rangle\, u_{\mathbf{k}}$, so that $\sum_\alpha |u^\alpha_{\mathbf{k}}|^2 = 1$. The band annihilation operator is the projection of the orbital operators onto this eigenvector,

$$\gamma_{\mathbf{k}\sigma} = \sum_\alpha u^{\alpha *}_{\mathbf{k}}\, c_{\mathbf{k}\alpha\sigma}, \qquad \gamma^\dagger_{\mathbf{k}\sigma} = \sum_\alpha u^{\alpha}_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\alpha\sigma}. \tag{7}$$

Because the flat band is isolated, we keep only this band. Restricting the orbital operators to the flat-band subspace amounts to

$$c_{\mathbf{k}\alpha\sigma} \;\longrightarrow\; u^\alpha_{\mathbf{k}}\, \gamma_{\mathbf{k}\sigma}, \qquad c^\dagger_{\mathbf{k}\alpha\sigma} \;\longrightarrow\; u^{\alpha *}_{\mathbf{k}}\, \gamma^\dagger_{\mathbf{k}\sigma}. \tag{8}$$

3.2 Projecting the attractive interaction

Take the simplest attraction, an on-site Hubbard U:

$$H_{\text{int}} = -U \sum_{\mathbf{R}\alpha} n_{\mathbf{R}\alpha\uparrow} n_{\mathbf{R}\alpha\downarrow} = -\frac{U}{N} \sum_{\mathbf{k}\mathbf{k}'}$$

$\sum_\alpha c^\dagger_{\mathbf{k}+}\;,\;{}_{\alpha\uparrow} c^\dagger_{\mathbf{k}'-}\;,\;{}_{\alpha\downarrow} c_{\mathbf{k}'\alpha\downarrow} c_{\mathbf{k}\alpha\uparrow}.$(9) We retain the Cooper channel of zero center-of-mass momentum, i.e. a pair $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ scattering to $(\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow)$, which fixes $\;=\; \mathbf{k}' - \mathbf{k}$. Substituting the projection (8) and using time-reversal symmetry $u^\alpha_{-\mathbf{k}} = u^{\alpha *}_{\mathbf{k}}$ (so that the products collapse to $|u^\alpha_{\mathbf{k}}|^2$), we arrive at an effective pair-scattering vertex

Geometry-dressed pairing interaction

$$V_{\mathbf{k}\mathbf{k}'} = -\frac{U}{N} \sum_\alpha \varrho^\alpha_{\mathbf{k}}\, \varrho^\alpha_{\mathbf{k}'}, \qquad \varrho^\alpha_{\mathbf{k}} \equiv |u^\alpha_{\mathbf{k}}|^2, \quad \sum_\alpha \varrho^\alpha_{\mathbf{k}} = 1. \tag{10}$$

The bare constant $-U$ has become a $\mathbf{k}$-dependent vertex weighted by the *orbital weights* $\varrho^\alpha_{\mathbf{k}}$ – the probability that a Bloch electron at $\mathbf{k}$ sits on orbital α. This is the first place where the geometry of $|u_{\mathbf{k}}\rangle$ enters. Two electrons attract only when they occupy the *same* orbital on the *same* site; the factor $\varrho^\alpha_{\mathbf{k}}\varrho^\alpha_{\mathbf{k}'}$ measures precisely that overlap.

4 Solving the Gap Equation: A Linear Binding Energy

4.1 The Cooper secular equation

Following Cooper, take a single pair on top of the (empty or partially filled) band,

$$|\Psi\rangle = \sum_{\mathbf{k}} g_{\mathbf{k}}\, \gamma^\dagger_{\mathbf{k}\uparrow} \gamma^\dagger_{-\mathbf{k}\downarrow}\, |0\rangle\,. \tag{11}$$

Since the kinetic energy $\epsilon_{0\mathbf{k}} + \epsilon_{0,-\mathbf{k}} = 0$ vanishes identically, the Schrödinger equation $H\,|\Psi\rangle = E\,|\Psi\rangle$ reduces to a pure interaction problem,

$$E\, g_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} g_{\mathbf{k}'} = -\frac{U}{N} \sum_\alpha \varrho^\alpha_{\mathbf{k}} \sum_{\mathbf{k}'} \varrho^\alpha_{\mathbf{k}'} g_{\mathbf{k}'}. \tag{12}$$

4.2 Decoupling with orbital gaps

The right-hand side factorizes in $\mathbf{k}$ and $\mathbf{k}'$. Introduce one number per orbital,

$$\Delta_\alpha \equiv \frac{U}{N}\sum_{\mathbf{k}'} \varrho^{\alpha}_{\mathbf{k}'}\, g_{\mathbf{k}'} \qquad \Longrightarrow \qquad g_{\mathbf{k}} = -\frac{1}{E}\sum_{\alpha} \varrho^{\alpha}_{\mathbf{k}}\, \Delta_\alpha. \tag{13}$$

Feeding this back into the definition of Δ_α gives a finite $N_{\text{orb}} \times N_{\text{orb}}$ eigenvalue problem,

$$\Delta_\alpha = -\frac{U}{E}\sum_{\beta} M_{\alpha\beta}\Delta_\beta, \qquad M_{\alpha\beta} = \frac{1}{N}\sum_{\mathbf{k}} \varrho^{\alpha}_{\mathbf{k}}\varrho^{\beta}_{\mathbf{k}}. \tag{14}$$

So $-E/U$ is an eigenvalue of the positive matrix M, and the bound state (largest binding) corresponds to its largest eigenvalue.

4.3 The symmetric solution

If the orbitals are related by symmetry, the uniform vector $\Delta_\alpha = \Delta$ is an eigenvector. Using $\sum_\beta \varrho^{\beta}_{\mathbf{k}} = 1$,

$$\sum_{\beta} M_{\alpha\beta} = \frac{1}{N}\sum_{\mathbf{k}} \varrho^{\alpha}_{\mathbf{k}} = \frac{1}{N_{\text{orb}}}, \tag{15}$$

so the eigenvalue is $1/N_{\text{orb}}$ and $g_{\mathbf{k}} = -\Delta/E$ is *constant* across the zone. Self-consistency then fixes the binding energy:

Flat-band pair binding energy

$$\boxed{E_b = -\frac{U}{N_{\text{orb}}}} \tag{16}$$

Linear in U – not the exponentially small $\Delta \sim e^{-1/N(0)V}$ of ordinary BCS.

Why linear and not exponential? In the dispersive case the gap equation contains the logarithmically divergent integral $\int d\xi/\sqrt{\xi^2+\Delta^2}$, whose logarithm produces the exponential. In a flat band there is no ξ to integrate over: the δ-function density of states (Fig. 1c) collapses the energy denominator to a single number $-E$, and the equation becomes algebraic. Weak-coupling superconductivity is thus *strongly* enhanced in a flat band – pairing survives to arbitrarily small U with a non-exponential scale.

A warning we must confront. The pairing amplitude $g_{\mathbf{k}} = \text{const}$ is completely uniform in $\mathbf{k}$. Recall from Eq. (2) that a constant $\phi_{\mathbf{k}}$ would give $\nabla_{\mathbf{k}}\phi = 0$ and hence a pointlike pair. If $g_{\mathbf{k}}$ carried all the information, we would be back to the paradox! The escape, as we now show, is that the pair wavefunction is *not* $g_{\mathbf{k}}$ alone – it also carries the Bloch spinors $|u_{\mathbf{k}}\rangle$, and *their* variation across the zone provides the size.

5 Quantum Geometry: The Missing Length Scale

We pause the superconductivity calculation to introduce the geometric object we need: the *quantum metric*. This is the single most important concept in these notes.

5.1 States, not just energies

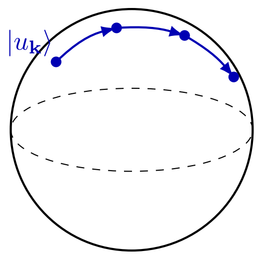
Band theory usually focuses on the eigenvalues $\epsilon_{n\mathbf{k}}$. But the eigenvectors $|u_{\mathbf{k}}\rangle$ also carry physics. As $\mathbf{k}$ moves, the state $|u_{\mathbf{k}}\rangle$ traces a curve in the Hilbert space of orbitals (Fig. 2). We can ask: *how much does the state itself change* between $\mathbf{k}$ and $\mathbf{k} + d\mathbf{k}$? The natural (gauge-invariant) measure of that change is the quantum distance,

$$ds^2 = 1 - |\,\langle u_{\mathbf{k}}\rangle\, u_{\mathbf{k}+d\mathbf{k}}|^2 = g_{\mu\nu}(\mathbf{k})\, dk^\mu dk^\nu, \tag{17}$$

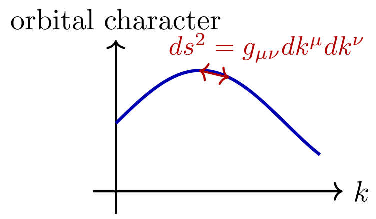
which defines the **quantum metric**

$$g_{\mu\nu}(\mathbf{k}) = \mathrm{Re}\Big[\,\langle \partial_\mu u_{\mathbf{k}}\rangle\, \partial_\nu u_{\mathbf{k}} - \langle \partial_\mu u_{\mathbf{k}}\rangle\, u_{\mathbf{k}} \langle u_{\mathbf{k}}\rangle\, \partial_\nu u_{\mathbf{k}}\Big]. \tag{18}$$

The subtraction removes the piece that is pure phase (the Berry connection); what remains is the physical, gauge-invariant spread of the state in orbital space.



(a) State traces a path as $\mathbf{k}$ varies



(b) Rate of change $\rightarrow$ metric

Figure 2: The quantum metric measures how the Bloch *state* $|u_{\mathbf{k}}\rangle$ (its orbital content), not its energy, changes with $\mathbf{k}$. (a) As $\mathbf{k}$ sweeps the Brillouin zone the normalized state moves along a path in Hilbert space (schematically, on a generalized Bloch sphere). (b) The infinitesimal quantum distance between neighboring states defines $g_{\mu\nu}(\mathbf{k})$. A flat band can have a large metric even though its energy is perfectly constant.

5.2 The quantum geometric tensor

The metric is the real, symmetric part of a larger object, the **quantum geometric tensor**,

$$Q_{\mu\nu}(\mathbf{k}) = \langle \partial_\mu u_{\mathbf{k}}|\, \big(1 - P_{\mathbf{k}}\big)\, |\partial_\nu u_{\mathbf{k}}\rangle = \underbrace{g_{\mu\nu}(\mathbf{k})}_{\text{metric}} - \frac{i}{2} \underbrace{\Omega_{\mu\nu}(\mathbf{k})}_{\text{Berry curvature}}, \tag{19}$$

where $P_{\mathbf{k}} = |u_{\mathbf{k}}\rangle\langle u_{\mathbf{k}}|$ projects onto the occupied state. The imaginary part is the familiar Berry curvature $\Omega_{\mu\nu}$; the real part is the metric. Both are gauge invariant. The positivity of Q implies a fundamental inequality in 2D,

$$\mathrm{tr}\, g(\mathbf{k}) \;\geq\; |\Omega(\mathbf{k})|, \tag{20}$$

which we will use to derive a topological lower bound on the pair size.

6 The Flat-Band Pair Size

We now return to superconductivity, armed with the metric.

6.1 The pair wavefunction carries the Bloch states

The physical pair amplitude, resolved on orbitals, is

$$\tilde{\Phi}^{\alpha\beta}(\mathbf{k}) = \langle 0|\, c_{\mathbf{k}\alpha\uparrow} c_{-\mathbf{k}\beta\downarrow} \,|\Psi\rangle = g_{\mathbf{k}}\, u^{\alpha}_{\mathbf{k}} u^{\beta *}_{\mathbf{k}}, \tag{21}$$

that is, as a state in the two-orbital space,

$$|\tilde{\Phi}(\mathbf{k})\rangle = g_{\mathbf{k}}\, |u_{\mathbf{k}}\rangle \otimes |u_{-\mathbf{k}}\rangle\,. \tag{22}$$

Even though the envelope $g_{\mathbf{k}} = g$ is constant, the state (22) depends on $\mathbf{k}$ through the Bloch spinors. *This* is the dependence that Eq. (2) will differentiate.

6.2 Mean-square size

The two electrons sit at $\mathbf{k}$ and $-\mathbf{k}$, so the relative momentum is $\mathbf{k}$ itself and the relative-coordinate operator is $i\nabla_{\mathbf{k}}$. The mean-square pair size is

$$\xi^2_{\text{pair}} = \langle r^2 \rangle = \frac{\sum_{\mathbf{k}} \langle \nabla_{\mathbf{k}}\tilde{\Phi}|\,(1-P)\,|\nabla_{\mathbf{k}}\tilde{\Phi}\rangle}{\sum_{\mathbf{k}} \langle \tilde{\Phi}\rangle\, \tilde{\Phi}}, \tag{23}$$

where the projector $(1-P)$ removes the center-of-mass part of the position (the Berry-connection piece), which is a gauge artifact and not part of the *internal* pair structure.

Denominator. Since $|u_{\pm\mathbf{k}}\rangle$ are normalized, $\langle\tilde{\Phi}\rangle\, \tilde{\Phi} = |g|^2$ and the denominator is $N|g|^2$.

Numerator. With $g_{\mathbf{k}}$ constant, the gradient acts only on the Bloch factors,

$$\nabla_{\mathbf{k}}\, |\tilde{\Phi}\rangle = g\big[\, |\partial u_{\mathbf{k}}\rangle \otimes |u_{-\mathbf{k}}\rangle - |u_{\mathbf{k}}\rangle \otimes |\partial u\rangle_{-\mathbf{k}} \big]. \tag{24}$$

Taking the norm, summing over Cartesian directions μ, and using the decomposition

$$\langle \partial_\mu u_{\mathbf{k}}\rangle\, \partial_\mu u_{\mathbf{k}} = \underbrace{g_{\mu\mu}(\mathbf{k})}_{\text{metric}} + \underbrace{|\,\langle u_{\mathbf{k}}\rangle\, \partial_\mu u_{\mathbf{k}}|^2}\, \text{Berry connection}^2, \tag{25}$$

the projector $(1-P)$ discards the connection terms (they build the center-of-mass position) and leaves only the metric of both partners. With time-reversal symmetry $g_{\mu\nu}(-\mathbf{k}) = g_{\mu\nu}(\mathbf{k})$, the numerator becomes $2|g|^2 \sum_{\mathbf{k}} \operatorname{tr} g(\mathbf{k})$. Dividing:

Flat-band Cooper pair size

$$\boxed{\xi^2_{\text{pair}} = \frac{2}{V_{\text{BZ}}} \int_{\text{BZ}} d^d k\ \operatorname{tr} g(\mathbf{k}) \equiv 2\, \langle \operatorname{tr} g\rangle_{\text{BZ}}} \tag{26}$$

The pair size is set entirely by the Brillouin-zone average of the quantum metric – and is *independent of* U.

This is the central result. The velocity v_F of Eq. (5) has been replaced by a purely geometric quantity. The pair is not pointlike: it is as large as the region over which the Bloch state changes its orbital character.

6.3 Topological lower bound

Combining Eq. (26) with the inequality (20) and the quantization $\int_{\text{BZ}} \Omega\, d^2k = 2\pi C$ (with C the Chern number),

$$\int_{\text{BZ}} \operatorname{tr} g\, d^2k \geq \int_{\text{BZ}} |\Omega|\, d^2k \geq 2\pi |C|. \tag{27}$$

Using $V_{\text{BZ}} = (2\pi)^2/A_{\text{cell}}$ gives

Topological floor on the pair size

$$\xi_{\text{pair}}^2 \gtrsim \frac{|C|\,a^2}{\pi}. \tag{28}$$

A topologically nontrivial flat band ($C \neq 0$) *cannot* host an arbitrarily small pair: topology enforces a minimum size of order the lattice constant times $\sqrt{|C|}$. This is a beautiful example of a physical observable being bounded by a topological invariant.

7 The General Case: Narrow Bands and the Interpolation

Real materials are neither perfectly flat nor infinitely wide. What happens in between? Repeat the pair-size calculation (23) but now with the full BCS envelope $\phi_{\mathbf{k}} = \Delta/2E_{\mathbf{k}}$ attached to the Bloch spinors,

$$|\Phi(\mathbf{k})\rangle = \phi_{\mathbf{k}}\, |u_{\mathbf{k}}\rangle \otimes |u_{-\mathbf{k}}\rangle\,. \tag{29}$$

The gradient now hits both the envelope and the spinors:

$$\nabla_{\mathbf{k}}\,|\Phi\rangle = (\nabla_{\mathbf{k}}\phi_{\mathbf{k}})\,|u_{\mathbf{k}}\rangle\otimes|u_{-\mathbf{k}}\rangle + \phi_{\mathbf{k}}\,\nabla_{\mathbf{k}}\big(|u_{\mathbf{k}}\rangle\otimes|u_{-\mathbf{k}}\rangle\big). \tag{30}$$

The cross term is proportional to $\operatorname{Re}\langle u\rangle\,\partial u = \frac{1}{2}\partial(\langle u\rangle\,u) = 0$ and thus *vanishes identically*. The two contributions add in quadrature:

Universal interpolation formula

$$\boxed{\xi_{\text{pair}}^2 = \underbrace{\frac{\hbar^2\,\langle v_F^2\rangle}{8\Delta^2}}_{\text{conventional}} + 2\,\underbrace{\langle \operatorname{tr} g\rangle_{\text{FS}}}_{\text{geometric}}} \tag{31}$$

The first term is the kinetic BCS contribution of Eq. (5); the second is the geometric contribution of Eq. (26) (here averaged with the pairing weight, concentrated near the Fermi surface). This single formula contains both limits, illustrated in Fig. 3:

- **Wide band** (v_F large): the kinetic term dominates, $\xi_{\text{pair}} \sim \hbar v_F/\Delta$ – textbook BCS.
- **Flat band** ($v_F \to 0$): the kinetic term dies, the pairing weight spreads over the whole zone, $\langle\cdot\rangle_{\text{FS}} \to \langle\cdot\rangle_{\text{BZ}}$, and we recover $\xi_{\text{pair}}^2 = 2\,\langle \operatorname{tr} g\rangle_{\text{BZ}}$.

8 Why It Is the Wannier Spread: A Unifying Statement

There is a deep and satisfying way to understand the geometric length. The same metric that appears above also governs how localized one can make the *Wannier functions* of the band.

Wannier functions $|w_{\mathbf{R}}\rangle$ are the real-space orbitals built by superposing Bloch states across the zone. Marzari and Vanderbilt showed that their total spread splits into a gauge-invariant piece and a gauge-dependent piece,

$$\Omega_{\text{spread}} = \Omega_I + \widetilde{\Omega}, \qquad \Omega_I = \frac{1}{V_{\text{BZ}}}\int_{\text{BZ}} d^d k\ \operatorname{tr} g(\mathbf{k}), \quad \widetilde{\Omega} \geq 0. \tag{32}$$

The gauge-invariant part Ω_I is *exactly* the band-averaged quantum metric. One minimizes $\widetilde{\Omega}$ over gauge choices to obtain the *maximally localized* Wannier function – but no choice can reduce Ω_I. It is an irreducible floor on localization. Comparing with Eq. (26),

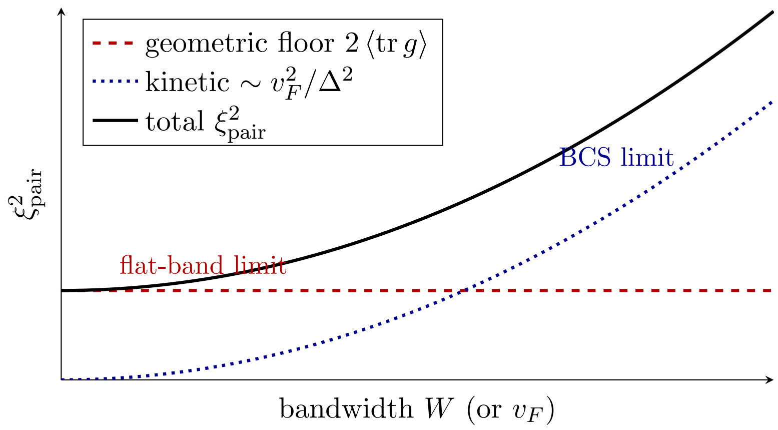


Figure 3: The Cooper pair size across regimes, Eq. (31). For a wide band the kinetic (dotted) term dominates and one recovers conventional BCS. As the band flattens ($v_F \to 0$) the kinetic term vanishes but the pair size does *not* collapse: it saturates at the geometric floor (dashed) set by the quantum metric. The full result (solid) smoothly interpolates.

The unifying identity

$$\boxed{\xi^2_{\mathrm{pair}}\Big|_{\mathrm{geom}} = 2\,\langle \mathrm{tr}\, g\rangle_{\mathrm{BZ}} = 2\,\Omega_I} \tag{33}$$

In words: **the flat-band Cooper pair is exactly as big as the optimally localized Wannier function from which it is built.** A pair cannot be smaller than its constituent orbitals, and those orbitals cannot be smaller than $\sqrt{\Omega_I}$. This is why the size is finite even when electrons cannot move: the finite spread of the Wannier orbital sets the floor.

9 Summary and Physical Picture

The story in three sentences. (1) In a dispersive band the pair size is kinetic, $\xi_0 \sim \hbar v_F/\Delta$, and would vanish in a flat band. (2) In a flat band the size is instead geometric, $\xi^2_{\mathrm{pair}} = 2\,\langle \mathrm{tr}\, g\rangle_{\mathrm{BZ}}$, set by how the Bloch state's orbital character varies across the zone. (3) This equals twice the minimal Wannier spread and is bounded below by the Chern number, $\xi^2_{\mathrm{pair}} \gtrsim |C|a^2/\pi$.

The lesson transcends the pair size. The very same quantity $\langle \mathrm{tr}\, g\rangle$ also controls the *superfluid weight* $D_s \propto \langle \mathrm{tr}\, g\rangle$ that governs whether a flat-band superconductor can carry a supercurrent at all (Peotta–Törmä). A flat band with trivial quantum geometry ($g = 0$) would have pointlike, immobile pairs and no supercurrent; a flat band with nontrivial geometry has finite-size, mobile pairs and genuine superconductivity. Quantum geometry, invisible in the energy spectrum, is what makes flat-band superconductivity possible.

Caveats on prefactors. The exact numerical factors (2 in $\xi^2 = 2\,\langle \mathrm{tr}\, g\rangle$, or π in the conventional formula) depend on conventions – whether one weights by $|\phi|^2$ or $|\nabla\phi|^2$, and whether one measures the relative or per-particle spread. The robust, convention-independent physics is: (i) the binding energy is linear in U, (ii) the pair size is pinned to the quantum metric, and (iii) it is bounded below by topology.