Fermi-Velocity-Controlled Length Scales in Solids

One Master Formula, Many Phenomena

Graduate Lecture

July 3, 2026

Outline

1. The Master Formula
2. Temperature: The Thermal Length
3. The Gap: BCS Coherence Length
4. Kondo: The Screening Cloud
5. Disorder: Mean Free Path and Localization
6. Diffusive Generalization and Thouless Energy
7. Summary and Limits of the Framework

Opening challenge

Energy scale		Measured length
$E_{\mathrm{F}} \sim 5$ eV	$\longrightarrow$	Ångströms
$\Delta \sim 1$ meV	$\longrightarrow$	~ 200 nm
$k_{\mathrm{B}} T$ at 1 K	$\longrightarrow$	$\sim 1\ \mu$m
$k_{\mathrm{B}} T_K \sim 0.1$ meV	$\longrightarrow$	$\sim 1\ \mu$m

Question

What single conversion factor unifies Josephson junctions, superconductors, Kondo physics, and localization?

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$$\hbar v_{\mathrm{F}} \approx 0.66\ \mathrm{eV} \cdot \mathrm{nm}$$

The master formula

Low-energy physics lives on the Fermi surface; linearize:

$$\varepsilon_k \simeq \hbar v_{\mathrm{F}}(k - k_{\mathrm{F}}) \qquad \Longrightarrow \qquad \text{energy window } E \;\leftrightarrow\; \text{momentum shell } \delta k = \frac{E}{\hbar v_{\mathrm{F}}}$$

Fourier uncertainty: waves from a window δk stay phase-coherent over $\sim 1/\delta k$ only.

Master formula

$$\boxed{L_E \sim \frac{\hbar v_{\mathrm{F}}}{E}}$$

Key idea

Any mechanism with energy scale E smears states within E of the Fermi surface $\Rightarrow$ imprints real space at $L_E = \hbar v_{\mathrm{F}}/E$. The physics differs wildly; the length scale does not.

Three equivalent readings

1. **Fourier / uncertainty:** shell $\delta k = E/\hbar v_{\mathrm{F}} \Rightarrow$ structure of size $1/\delta k$.
   (Cooper pair size, Kondo cloud)
2. **Dephasing:** states split by E differ by $\delta k = E/\hbar v_{\mathrm{F}}$; relative phase $\delta k\, x \sim 1$ at $x \sim \hbar v_{\mathrm{F}}/E$.
   (Thermal Friedel decay, SNS supercurrent)
3. **Dynamical:** coherence time $\tau_E = \hbar/E$; ballistic electron covers $L_E = v_{\mathrm{F}}\tau_E = \hbar v_{\mathrm{F}}/E$.
   (Generalizes to diffusive metals: Sec. 6)

Numbers to memorize

$$\hbar v_{\mathrm{F}} \approx 0.66\ \mathrm{eV}\cdot\mathrm{nm}, \qquad k_{\mathrm{B}} \times 1\,\mathrm{K} \approx 86\ \mu\mathrm{eV}$$

Anchor case: $E = E_{\mathrm{F}}$ gives the Fermi wavelength

For a parabolic band $E_{\mathrm{F}} = \frac{1}{2}\hbar v_{\mathrm{F}} k_{\mathrm{F}}$:

$$L_{E_{\mathrm{F}}} = \frac{\hbar v_{\mathrm{F}}}{E_{\mathrm{F}}} = \frac{2}{k_{\mathrm{F}}} \sim \lambda_{\mathrm{F}} \sim \text{Å}$$

The *shortest* length in the hierarchy — where "near the Fermi surface" breaks down.

Two roles, keep separated throughout

- k_{F} sets **fast oscillation periods** (π/k_{F}: Friedel, RKKY).
- $\hbar v_{\mathrm{F}}/E$ sets **slow envelope decay lengths**.

Thermal length $L_T = \hbar v_F / 2\pi k_B T$

Set $E \sim k_B T$. More precisely: Matsubara frequencies $\omega_n = (2n+1)\pi k_B T$; longest-lived contribution from the lowest:

$$\boxed{L_T = \frac{\hbar v_F}{2\pi k_B T}} \qquad L_T(1\,\mathrm{K}) \approx 1.2\ \mu\mathrm{m}$$

Mesoscopic physics in a dilution fridge is micron-scale *for a reason*.

Two applications, which will turn out to be *the same calculation*:

- Friedel oscillations at finite T
- Supercurrent decay in a clean SNS junction

Derivation I: thermal Friedel envelope (setup)

$T = 0$: sharp Fermi edge $\Rightarrow$ endpoint-dominated integral $\Rightarrow$ power law

$$\delta n(r) \propto \frac{\cos(2k_{\mathrm{F}} r + \delta_0)}{r^d}$$

$T > 0$: endpoint replaced by Fermi function. Linearize $\varepsilon = \hbar v_{\mathrm{F}} q$, $q = k - k_{\mathrm{F}}$:

$$\delta n(r) \;\propto\; \mathrm{Im}\left[g(k_{\mathrm{F}})\, e^{2ik_{\mathrm{F}} r} \underbrace{\int_{-\infty}^{\infty} dq\, \frac{e^{2iqr}}{e^{\beta\hbar v_{\mathrm{F}} q} + 1}}_{\text{the whole game}} \right]$$

Strategy

Close the contour in the upper half q-plane and collect the poles of the Fermi factor.

Derivation I: the Matsubara pole series

Poles where $e^{\beta\hbar v_{\mathrm{F}} q} = -1$:

$$q_n = \frac{i\,\omega_n}{\hbar v_{\mathrm{F}}}, \qquad \omega_n = (2n+1)\pi k_{\mathrm{B}} T \quad \text{— the Matsubara frequencies!}$$

Residue $= -1/(\beta\hbar v_{\mathrm{F}})$ at each pole; geometric series resums:

$$\int_{-\infty}^{\infty} \frac{dq\, e^{2iqr}}{e^{\beta\hbar v_{\mathrm{F}} q} + 1} = -\frac{2\pi i}{\beta\hbar v_{\mathrm{F}}} \sum_{n=0}^{\infty} e^{-2\omega_n r/\hbar v_{\mathrm{F}}} = -\frac{\pi i}{\beta\hbar v_{\mathrm{F}}} \frac{1}{\sinh(r/L_T)}$$

Result: universal thermal envelope

$$\delta n(r;T) = \delta n(r;0) \times \frac{x}{\sinh x}, \qquad x = \frac{r}{L_T}$$

$x \ll 1$: power law survives; $\qquad x \gg 1 : \sim 2x\, e^{-x}$ (exponential decay)

Asymptotics controlled by the pole nearest the real axis, $\omega_0 = \pi k_{\mathrm{B}} T$.

Derivation II: clean SNS junction via Andreev phases

Supercurrent carried by Andreev pairs in the normal region of length L:

$$\text{electron: } k_e = k_F + \frac{\varepsilon}{\hbar v_F}, \qquad \text{retroreflected hole: } k_h = k_F - \frac{\varepsilon}{\hbar v_F}$$

Relative phase accumulated across the junction:

$$\delta\phi(\varepsilon) = (k_e - k_h)\, L = \frac{2\varepsilon L}{\hbar v_F}$$

- $T = 0$: all energies coherent $\Rightarrow$ robust supercurrent.
- $T > 0$: thermal average of $e^{i\delta\phi(\varepsilon)}$ over window $\varepsilon \sim k_B T$ — **same integral** as the Friedel case, with $r \to L$. Same Matsubara poles, same $1/\sinh$:

$$\boxed{I_c \propto e^{-L/L_T}, \qquad L_T = \frac{\hbar v_F}{2\pi k_B T}}$$

Punchline

Friedel envelope = Josephson decay: one calculation, two costumes.

Pippard's argument, then the BCS derivation

Pippard (pre-BCS): pairing involves states within Δ of the Fermi surface $\Rightarrow$ shell $\delta k = \Delta/\hbar v_{\mathrm{F}} \Rightarrow$ pair size $\xi_0 \sim \hbar v_{\mathrm{F}}/\Delta$.

BCS: equal-time pair amplitude

$$F(\mathbf{r}) = \sum_{\mathbf{k}} u_k v_k \, e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad u_k v_k = \frac{\Delta}{2E_k}, \quad E_k = \sqrt{\xi_k^2 + \Delta^2}$$

Angular integral (3D) + linearization $\xi_k = \hbar v_{\mathrm{F}}(k - k_{\mathrm{F}})$:

$$F(r) \propto \frac{\sin k_{\mathrm{F}} r}{r} \int_{-\infty}^{\infty} d\xi \, \frac{\Delta \, \cos(\xi r/\hbar v_{\mathrm{F}})}{\sqrt{\xi^2 + \Delta^2}} = \frac{\sin k_{\mathrm{F}} r}{r} \, 2\Delta \, K_0\Big(\frac{\Delta r}{\hbar v_{\mathrm{F}}}\Big)$$

$$\xrightarrow{r \gg \hbar v_{\mathrm{F}}/\Delta} \frac{\sin k_{\mathrm{F}} r}{r} \, e^{-r/\pi\xi_0}, \qquad \boxed{\xi_0 = \frac{\hbar v_{\mathrm{F}}}{\pi \Delta}}$$

Fast k_{F} oscillations $\times$ slow $\hbar v_{\mathrm{F}}/\Delta$ envelope: the master pattern, exactly.

Coherence length: numbers and refinements

	Δ	ξ_0
Al	0.18 meV	$\sim 1.6\ \mu$m
Nb	1.5 meV	~ 40 nm

Bigger gap $\Rightarrow$ smaller pair: obvious from $L_E = \hbar v_{\mathrm{F}}/E$ before any computation.

Two finite-T remarks

1. Near T_c: $\xi(T) \approx 0.74\,\xi_0(1 - T/T_c)^{-1/2}$ (clean GL) — $\Delta(T)$ collapses, length diverges, as the master formula demands.
2. Proximitized *normal* metal: decay length is L_T, not ξ_0 — no gap in N, so $k_{\mathrm{B}}T$ is the only scale left.

Skill to build: which energy scale governs which region of the device?

Kondo cloud I: the exponentially small energy scale

Yosida singlet variational state; self-consistency for the binding energy:

$$1 = \frac{3J\rho}{4}\int_0^D \frac{d\epsilon}{\epsilon + \Delta_b} \qquad \Longrightarrow \qquad \Delta_b \approx D\, e^{-4/(3J\rho)} \sim k_{\mathrm{B}}\, T_K$$

- Infrared logarithm from integrating $1/\epsilon$ down to a *sharp Fermi edge* — same mechanism as the Cooper log.
- Bound orbital built from amplitudes $\alpha_k \propto (\epsilon_k + \Delta_b)^{-1}$: a shell of width $k_{\mathrm{B}}\, T_K$ above E_{F}.

Kondo cloud II: real-space profile

Fourier transform with linearized dispersion (s-wave):

$$\phi(r) \propto \frac{1}{r}\,\mathrm{Im}\int_0^D d\epsilon\,\frac{e^{i(k_\mathrm{F}+\epsilon/\hbar v_\mathrm{F})\,r}}{\epsilon + k_\mathrm{B}\,T_K}$$

Pole at distance $k_\mathrm{B}\,T_K$ from the edge $\Rightarrow$ envelope scale

$$\boxed{\xi_K = \frac{\hbar v_\mathrm{F}}{k_\mathrm{B}\,T_K}} \qquad |\phi(r)|^2 \sim \begin{cases} \dfrac{\sin^2 k_\mathrm{F} r}{r^2}\,\ln^2\dfrac{\xi_K}{r}, & r \ll \xi_K \\[2ex] \dfrac{\cos^2 k_\mathrm{F} r}{r^2}\,\dfrac{\xi_K^2}{r^2}, & r \gg \xi_K \end{cases}$$

The shock

T_K exponentially small $\Rightarrow$ ξ_K exponentially *large*: $T_K \sim 1$ K gives $\xi_K \sim$ microns — bigger than the device! Eluded detection for 50 years; observed in a quantum-dot interferometer (Borzenets *et al.*, Nature 2020), confirming the $\hbar v_\mathrm{F}/k_\mathrm{B}\,T_K$ scaling.

Disorder: $E = \hbar/\tau$ gives the mean free path

Elastic scattering time $\tau \Rightarrow$ level broadening $\hbar/\tau$:

$$L_{\hbar/\tau} = \frac{\hbar v_{\mathrm{F}}}{\hbar/\tau} = v_{\mathrm{F}}\tau = \ell$$

Strict 1D derivation sketch. Transfer matrices ($\det T_n = 1$) + Furstenberg $\Rightarrow$ positive Lyapunov exponent $\gamma = 1/\xi_{\mathrm{loc}}$; weak disorder:

$$\xi_{\mathrm{loc}} = \frac{8t^2 \sin^2(ka)}{\langle \epsilon_n^2 \rangle}$$

Restore $\hbar v(E) = 2ta\sin(ka)$ and use the 1D Born rate $\hbar/\tau = 2\pi\langle\epsilon^2\rangle\rho_{1\mathrm{D}}a$, $\rho_{1\mathrm{D}} = 1/\pi\hbar v$:

$$\boxed{\xi_{\mathrm{loc}} \simeq 4\ell = 4v_{\mathrm{F}}\tau}$$

Same master formula, with the disorder-induced energy scale.

Localization: dimensionality dressing (and honesty)

$$\xi_{\mathrm{loc}} \sim \ell \ \text{(1D)}, \qquad \xi_{\mathrm{loc}} \sim N_{\mathrm{ch}}\,\ell \ \text{(quasi-1D)}, \qquad \xi_{\mathrm{loc}} \sim \ell\, e^{\pi k_{\mathrm{F}} \ell/2} \ \text{(2D)}$$

Honesty clause

The master formula supplies the *brick* ℓ. The 2D exponential and the channel-number factor come from interference physics (weak localization, scaling theory) — genuinely beyond dimensional analysis.

The framework tells you the bricks, not always the architecture.

Dirty metals: $L_E \to \sqrt{\hbar D/E}$

Dynamical reading (R3): distance traveled during $\tau_E = \hbar/E$. When $L_E \gg \ell$, motion is diffusive, $D = v_{\mathrm{F}}\ell/d$:

$$L = v_{\mathrm{F}}\tau_E \;\longrightarrow\; L = \sqrt{D\tau_E} \qquad \Longrightarrow \qquad \boxed{\frac{\hbar v_{\mathrm{F}}}{E} \;\longrightarrow\; \sqrt{\frac{\hbar D}{E}}}$$

Every ballistic length has a dirty partner:

$$\xi_0 = \frac{\hbar v_{\mathrm{F}}}{\pi\Delta} \;\to\; \sqrt{\frac{\hbar D}{\Delta}} \sim \sqrt{\xi_0 \ell}, \qquad L_T \;\to\; \sqrt{\frac{\hbar D}{2\pi k_{\mathrm{B}} T}}$$

v_{F} hasn't disappeared — it hides inside D.

Unified statement (both regimes)

The length associated with E is the distance a Fermi-surface electron propagates during the coherence time $\hbar/E$.

The Thouless energy: the inverse dictionary

Given a sample of size L, the associated energy:

$$\eth = \frac{\hbar v_{\mathrm{F}}}{L} \ \text{(ballistic)}, \qquad \eth = \frac{\hbar D}{L^2} \ \text{(diffusive)}$$

Application: short vs. long Josephson junction

$$\Delta \lessgtr \eth \quad \Longleftrightarrow \quad \xi \gtrless L$$

Classifying the device = comparing two entries of the dictionary.

Goal for students: translate *freely in both directions* between energies and lengths.

Summary table: the hierarchy

Energy E	**Length $\hbar v_{\mathrm{F}}/E$**	**Physics**	**Size**
E_{F}	$\lambda_{\mathrm{F}} \sim 1/k_{\mathrm{F}}$	Friedel/RKKY period	Å
$\hbar/\tau$	$\ell = v_{\mathrm{F}}\tau$	mean free path	nm–μm
Δ	$\xi_0 = \hbar v_{\mathrm{F}}/\pi\Delta$	Cooper pair size	10 nm–μm
$2\pi k_{\mathrm{B}} T$	L_T	Friedel/SNS decay	μm @ 1 K
$k_{\mathrm{B}} T_K$	ξ_K	Kondo cloud	μm
$\eth$	L (by def.)	size $\to$ energy	—

Top to bottom: energies large $\to$ small, lengths short $\to$ long.

Why quasiparticles work at all

The nesting $\lambda_{\mathrm{F}} \ll \ell \ll \xi_0, L_T, \xi_K$ is what makes the quasiparticle description self-consistent.

Where the framework fails

The construction needs a sharp Fermi surface and finite v_{F}. Three frontiers break it:

- **Flat bands** (magic-angle TBG): $v_{\mathrm{F}} \to 0$ by design. All lengths collapse to the lattice scale; interactions win; correlated insulators and superconductivity emerge. Flat-band engineering = deliberate demolition of this lecture.
- **Non-Fermi liquids**: no quasiparticles; ω/T scaling. The Planckian *time* $\hbar/k_{\mathrm{B}}T$ survives, but there is no clean v_{F} to convert it into a length.
- **Dirac materials**: no Fermi surface at neutrality, yet the velocity survives as an emergent "speed of light." Master formula still works with E measured from the Dirac point.

Problems

1. Derive the $x/\sinh x$ thermal envelope of 1D Friedel oscillations from the finite-T Lindhard function; identify the controlling Matsubara pole.
2. Clean SNS junction: compute $\delta\phi(\varepsilon)$ and show $I_c \propto e^{-L/L_T}$. Redo for a diffusive junction using $\eth = \hbar D/L^2$.
3. Estimate ξ_K for $T_K = 0.5$ K, $v_F = 1.5 \times 10^6$ m/s; compare with a quantum-dot device. What does "cloud bigger than the device" mean physically?
4. (Open-ended) Quadratic dispersion near E_F: how does the energy–length dictionary change? Which table entries are modified?

Takeaway

One number,

$$\hbar v_{\mathrm{F}} \approx 0.66\ \mathrm{eV}\cdot\mathrm{nm},$$

converts every energy scale in a metal into a length:

$$L_E \sim \frac{\hbar v_{\mathrm{F}}}{E} \quad \text{(ballistic)}, \qquad L_E \sim \sqrt{\frac{\hbar D}{E}} \quad \text{(diffusive)}.$$

Josephson junctions, superconductors, Kondo screening, and localization: one subject, not a list of phenomena.