Lecture 4: Abelian & Non-Abelian Fractional Chern Insulators in tMoTe2

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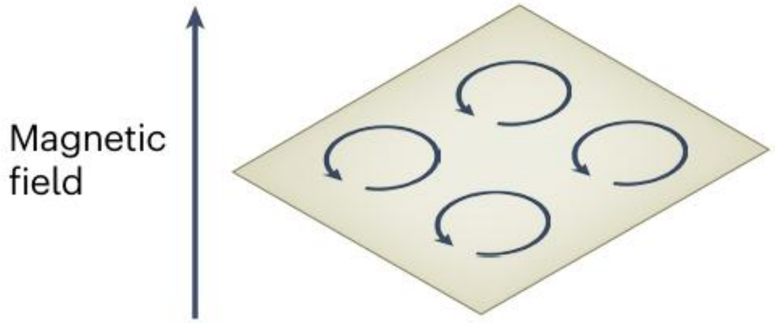
Outline

- ✓ Introduction
- ✓ Skyrmion Picture
- ✓ Generalized Landau Levels

Landau Levels

- Landau levels created by magnetic fields are flat Chern bands.



- Magnetic Bloch states:

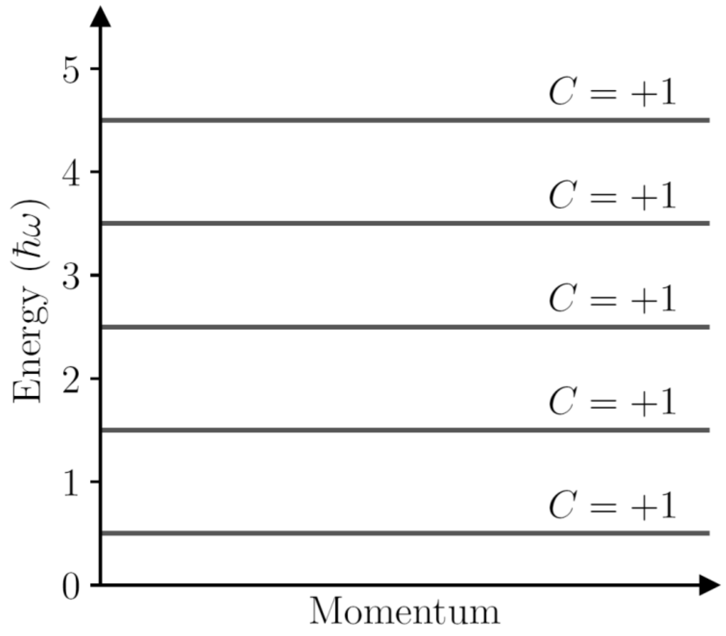
$$\Psi^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r}+\boldsymbol{a}_i) = -e^{-i\frac{1}{2\ell^2}s\boldsymbol{a}_i\times\boldsymbol{r}}e^{i\boldsymbol{k}\cdot\boldsymbol{a}_i}\Psi^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r})$$

$$|(\boldsymbol{a}_1\times\boldsymbol{a}_2)\cdot\hat{z}| = 2\pi\ell^2$$

- Uniform quantum geometry

$$\chi^{(s)}_{n,\boldsymbol{k}} = \begin{pmatrix} n+\frac{1}{2} & -\frac{is}{2} \\ \frac{is}{2} & n+\frac{1}{2} \end{pmatrix}\ell^2$$

$$\mathrm{Tr}[g_{\boldsymbol{k}}] = (2n+1)|\Omega_{\boldsymbol{k}}|$$



Wilson Loop

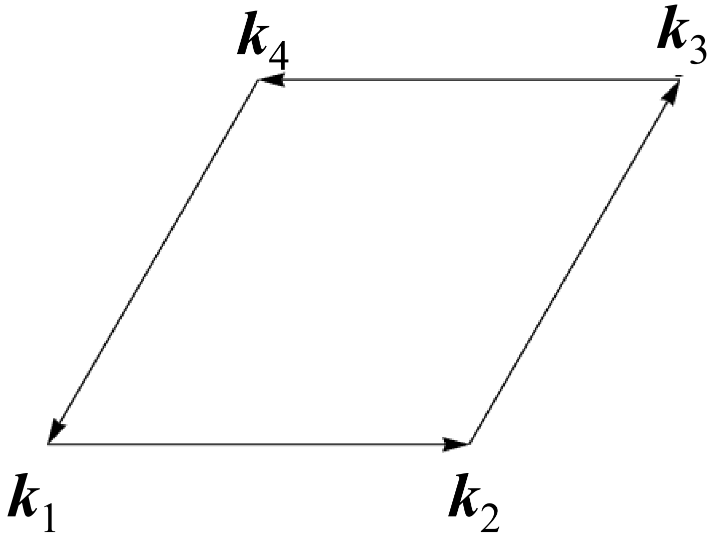
- Wilson loop:

$$\mathcal{W}(\boldsymbol{k}_1, \boldsymbol{k}_2, ..., \boldsymbol{k}_N, \boldsymbol{k}_{N+1} = \boldsymbol{k}_1) = \prod_{j=1}^{N} M_{\boldsymbol{k}_j, \boldsymbol{k}_{j+1}}$$

$$\approx \exp[i \sum_{\boldsymbol{k}_j} \mathcal{A}_\alpha(\boldsymbol{k}_j)\delta k_\alpha - \frac{1}{2}\sum_{\boldsymbol{k}_j} \hat{g}_{\alpha\beta}(\boldsymbol{k}_j)\delta k_\alpha \delta k_\beta]$$

$$\approx \exp[i \int_{\Gamma_L} \Omega_{\boldsymbol{k}} d^2\boldsymbol{k} - \frac{1}{2}\sum_{\boldsymbol{k}_j} \hat{g}_{\alpha\beta}(\boldsymbol{k}_j)\delta k_\alpha \delta k_\beta]$$

$$M_{\boldsymbol{k}_j, \boldsymbol{k}_{j+1}} = \langle u_{\boldsymbol{k}_j} | u_{\boldsymbol{k}_{j+1}} \rangle$$



Γ_L is the interior enclosed by the loop L formed by $\boldsymbol{k}_j$ for $j = 1,...,N + 1$, and $\delta\boldsymbol{k} = \boldsymbol{k}_{j+1} - \boldsymbol{k}_j$.

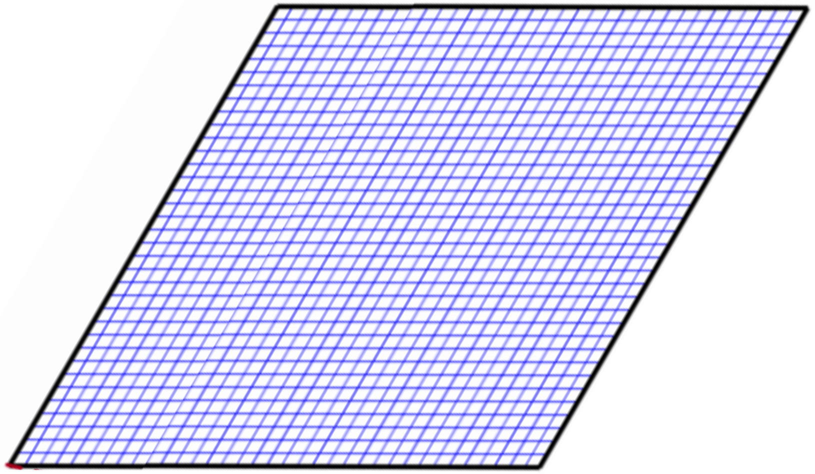
Berry Curvature

$$\mathcal{W}(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_N, \boldsymbol{k}_{N+1} = \boldsymbol{k}_1) = \prod_{j=1}^{N} M_{\boldsymbol{k}_j, \boldsymbol{k}_{j+1}}$$

$$\approx \exp[i \int_{\Gamma_L} \Omega_{\boldsymbol{k}} d^2\boldsymbol{k} - \frac{1}{2} \sum_{\boldsymbol{k}_j} \hat{g}_{\alpha\beta}(\boldsymbol{k}_j) \delta k_\alpha \delta k_\beta]$$

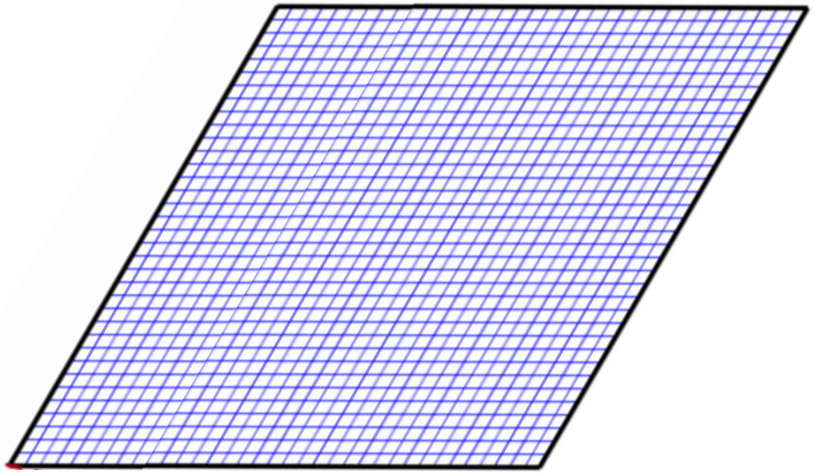
$$\Downarrow$$

$$\Omega_{\boldsymbol{k}} \approx \frac{\arg[\mathcal{W}]}{\mathcal{A}_{\Gamma_L}}$$



Quantum Metric

$$
\begin{aligned}
&\langle u_{\boldsymbol{k}-\boldsymbol{p}/2} | u_{\boldsymbol{k}+\boldsymbol{p}/2} \rangle \\
\approx & 1 + i p_\alpha \mathcal{A}_\alpha(\boldsymbol{k}) - \frac{1}{2} p_\alpha p_\beta \mathrm{Re}[\langle \partial_{k_\alpha} u_{\boldsymbol{k}} | \partial_{k_\beta} u_{\boldsymbol{k}} \rangle] \\
\approx & \exp[i p_\alpha \mathcal{A}_\alpha(\boldsymbol{k}) - \frac{1}{2} \hat{g}_{\alpha\beta}(\boldsymbol{k}) p_\alpha p_\beta],
\end{aligned}
$$



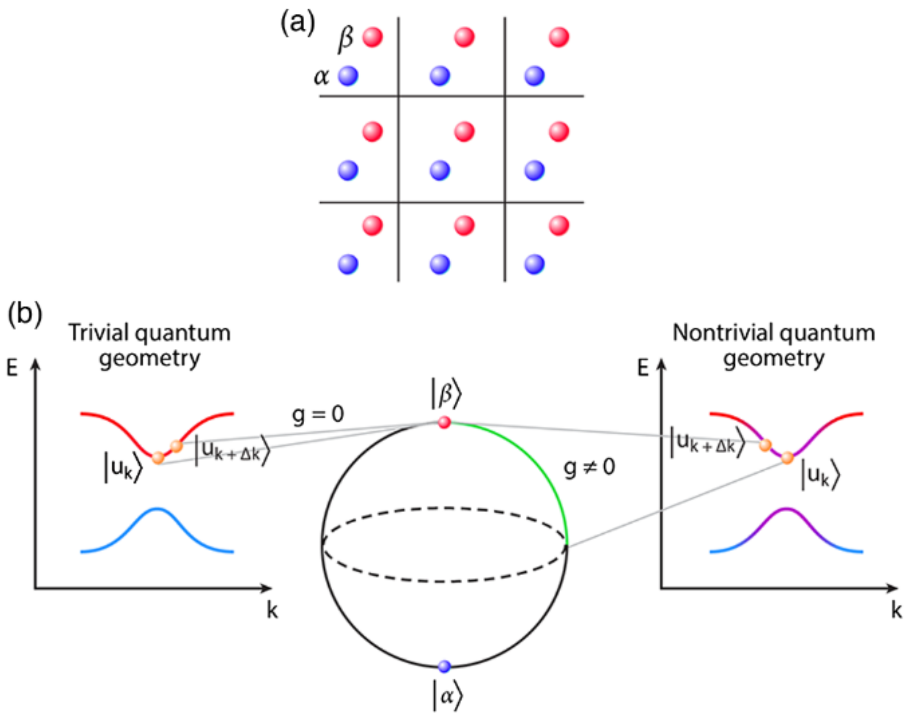
$$
\hat{g}_{\alpha\beta} p_\alpha p_\beta \approx 1 - \left| \langle u_{\boldsymbol{k}-\frac{\boldsymbol{p}}{2}} | u_{\boldsymbol{k}+\frac{\boldsymbol{p}}{2}} \rangle \right|^2
$$

Quantum Geometry

$$
\begin{aligned}
\hat{\Lambda}_{\alpha\beta}(\boldsymbol{k}) &= \langle \partial_{k_\alpha} u_{\boldsymbol{k}} | \partial_{k_\beta} u_{\boldsymbol{k}} \rangle - \mathcal{A}_\alpha(\boldsymbol{k}) \mathcal{A}_\beta(\boldsymbol{k}) \\
&= \langle \partial_{k_\alpha} u_{\boldsymbol{k}} | (\hat{I} - \hat{P}_{\boldsymbol{k}}) | \partial_{k_\beta} u_{\boldsymbol{k}} \rangle \\
&= \hat{g}_{\alpha\beta}(\boldsymbol{k}) + \frac{i}{2} \epsilon_{\alpha\beta} \Omega_{\boldsymbol{k}}
\end{aligned}
$$

- $\hat{I}$ is the identity matrix and $\hat{P}_{\boldsymbol{k}}$ is the projector $| u_{\boldsymbol{k}} \rangle \langle u_{\boldsymbol{k}} |$.
- The tensor $\widehat{\Lambda}$ is the projection of $\hat{I} - \hat{P}_{\boldsymbol{k}}$ onto the subspace spanned by $\{| \partial_{k_x} u_k \rangle, | \partial_{k_y} u_k \rangle\}$
- The projector $\hat{I} - \hat{P}_{\boldsymbol{k}}$ is semipositive definite, so is the tensor $\widehat{\Lambda}$
- $\mathrm{Tr}\widehat{\Lambda} \geq 0$ and $\det\widehat{\Lambda} \geq 0$
- $\mathrm{Tr}\widehat{\Lambda} = \mathrm{Tr}\hat{g}$ and $\det\widehat{\Lambda} = -\frac{\Omega^2}{4} + \det g$ ➔ $\mathrm{Tr}\hat{g}_{\boldsymbol{k}} \geq 0, \det\hat{g}_{\boldsymbol{k}} \geq \frac{\Omega_k^2}{4}$
- Because $\hat{g}_{\boldsymbol{k}}$ is a 2×2 matrix, $(\mathrm{Tr}\hat{g}_{\boldsymbol{k}})^2 \geq 4\det \hat{g}_{\boldsymbol{k}}$ ➔ $\mathrm{Tr}\hat{g}_{\boldsymbol{k}} \geq | \Omega_k |$ [Trace Inequality]

Quantum Geometry



- ➢ **Euclidean geometry**: determines distances between two points on a plane or on a sphere
- ➢ **Quantum geometry:** captures the phase and amplitude distances between quantum states.

- Quantum geometric tensor

$$(\chi_{\boldsymbol{k}})_{ab} = \langle \partial_{k_a} u_{\boldsymbol{k}} | I - P_{\boldsymbol{k}} | \partial_{k_b} u_{\boldsymbol{k}} \rangle$$

$$u_{\boldsymbol{k}}(\boldsymbol{r}) = e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \psi_{\boldsymbol{k}}(\boldsymbol{r})$$

$$P_{\boldsymbol{k}} = |u_{\boldsymbol{k}}\rangle\langle u_{\boldsymbol{k}}|$$

$$(\chi_{\boldsymbol{k}})_{ab} = (g_{\boldsymbol{k}})_{ab} + \frac{i}{2}\epsilon_{ab}\Omega_{\boldsymbol{k}}$$

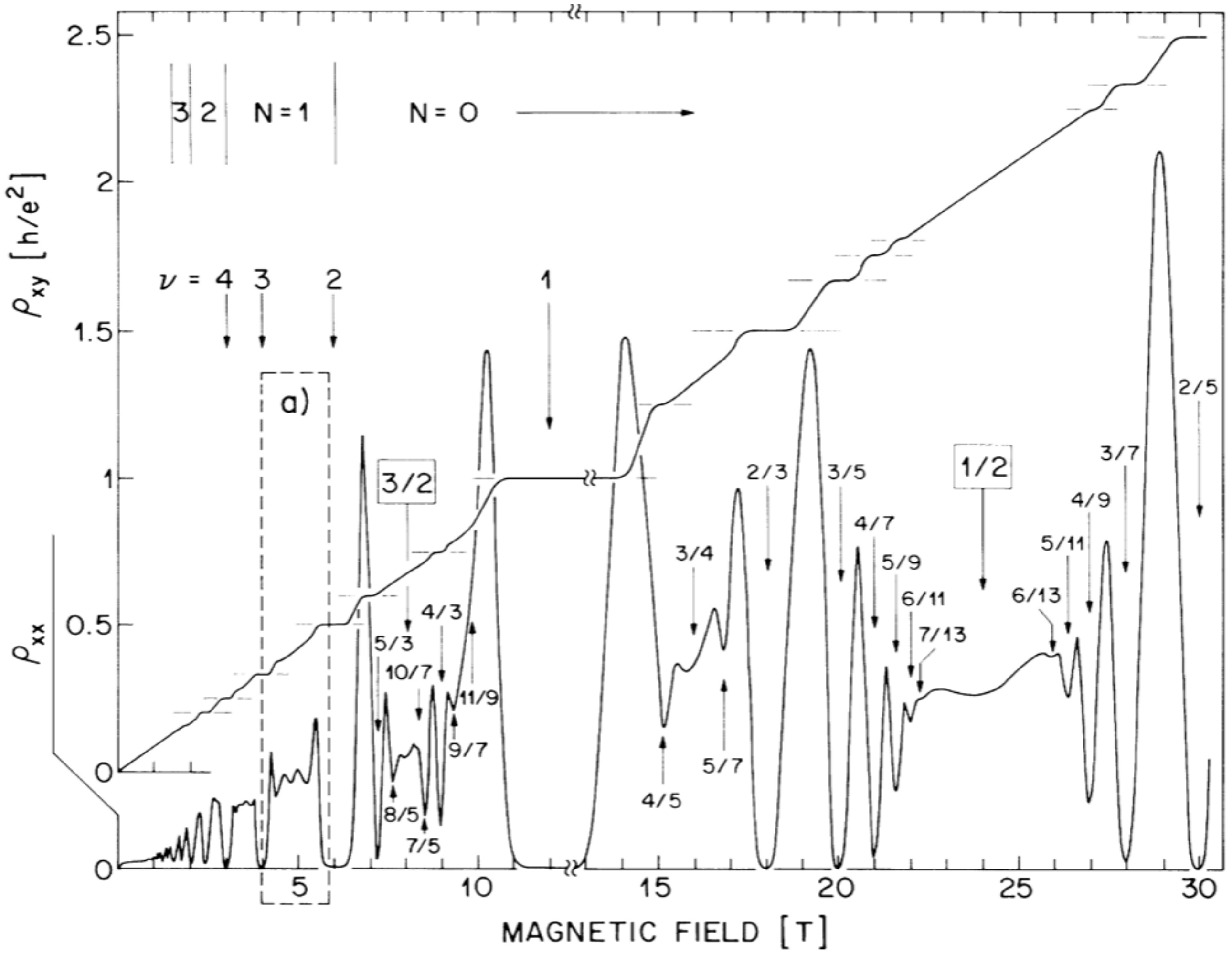
Quantum metric: amplitude distance

Berry curvature: phase distance

Chern number $\mathcal{C} = \frac{1}{2\pi}\int d\boldsymbol{k}\,\Omega_{\boldsymbol{k}}$

Quantum Weight $\mathcal{K} = \frac{1}{2\pi}\int d\boldsymbol{k}\,\mathrm{Tr}[g_{\boldsymbol{k}}]$

Integer and Fractional Quantum Hall Effects



R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, J. H. English, Phys. Rev. Lett. 59, 15 (1987)

- Integer Quantum Hall states:

$$\sigma_{xy} = \mathcal{C}\frac{e^2}{h}$$

TKNN invariant, PRL 49, 405 (1982)

- Fractional Quantum Hall states:

$$\psi(z_i) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}$$

R. B. Laughlin, PRL 50, 1395 (1983)

Fractional Quantum Hall Effects at 5/2

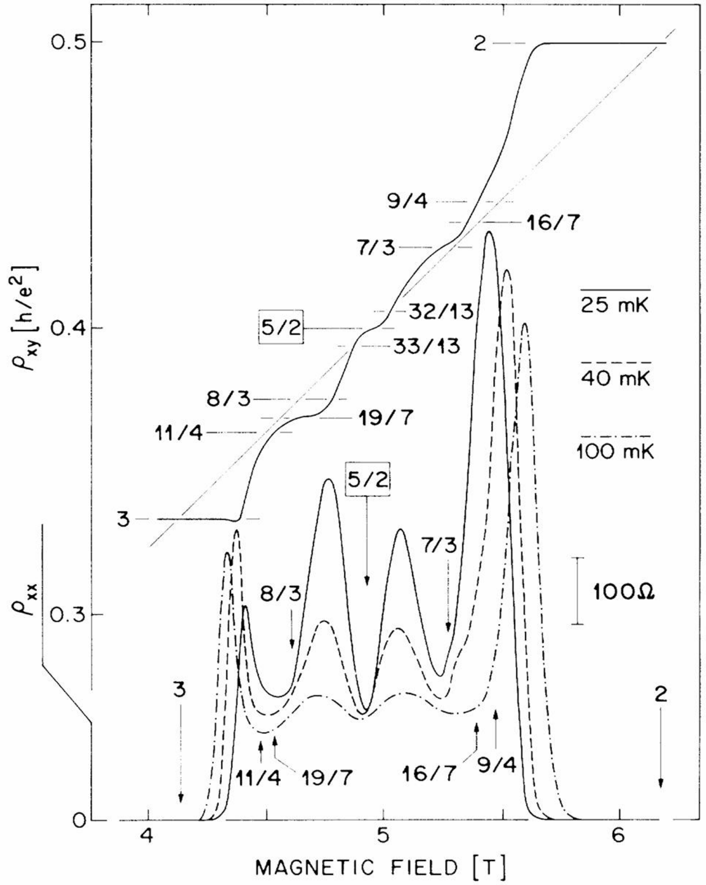


FIG. 2. Diagonal resistivity ρ_{xx} and Hall resistance ρ_{xy} [enlarged section (a) of Fig. 1] at $T=100$ to 25 mK. Filling factors ν are indicated in ρ_{xx} while quantum numbers p/q are shown in ρ_{xy}.

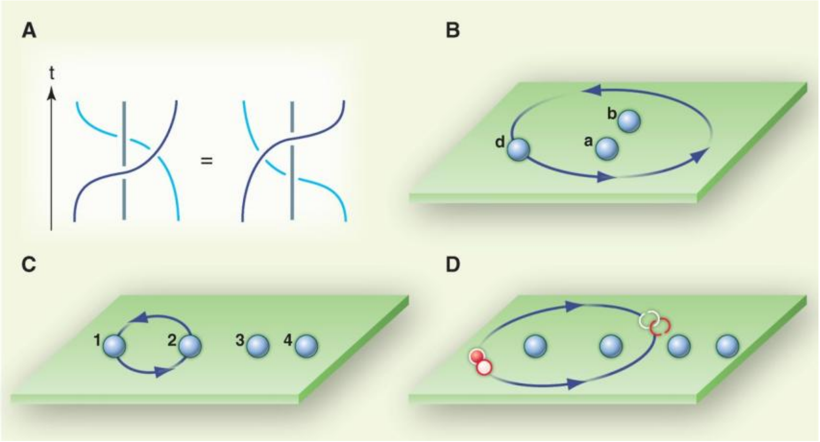
R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, J. H. English, Phys. Rev. Lett. 59, 15 (1987)

- Moore-Read Pfaffian states

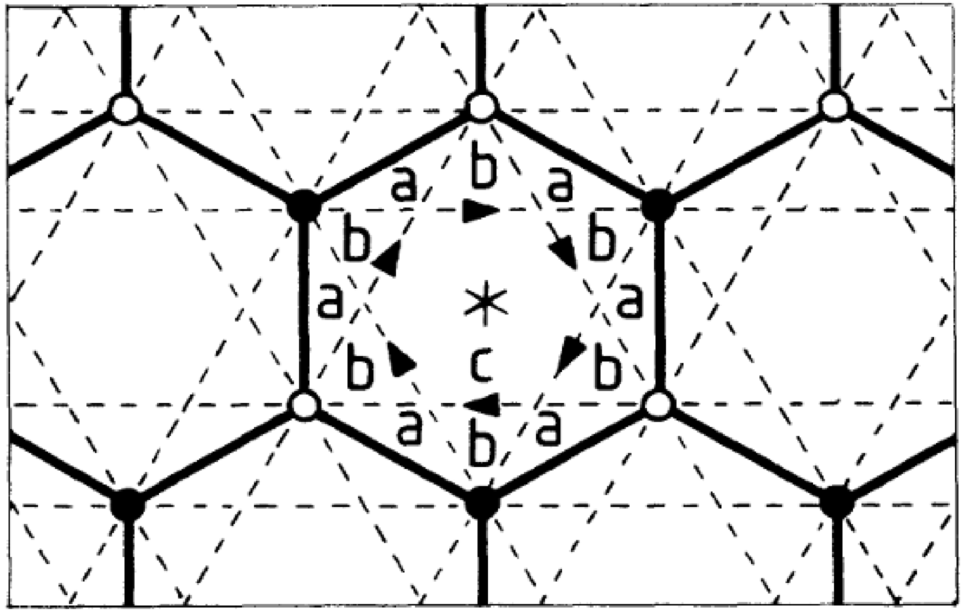
$$\Psi_{\mathrm{Pf}}(z_1,\ldots,z_N) = \mathrm{Pfaff}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^q \exp\left[-\tfrac{1}{4}\sum |z|^2\right]$$

G. Moore and N. Read, Nuclear Physics B 360, 362 (1991)

- Non-Abelian Quasiparticles

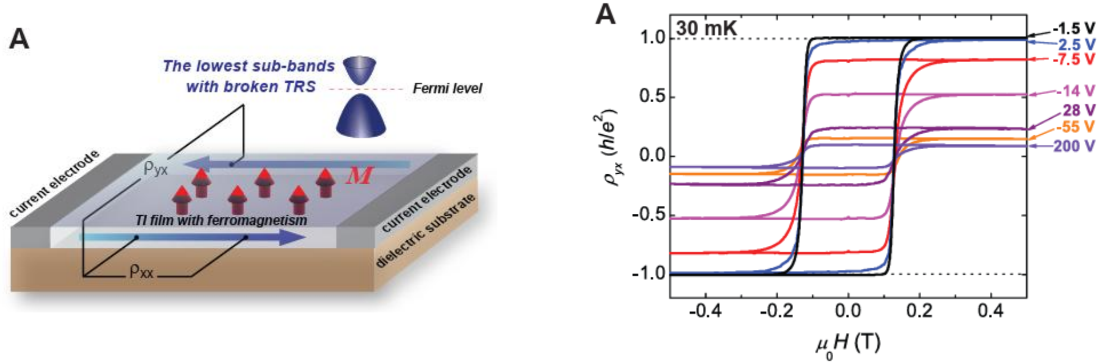


Chern Insulators

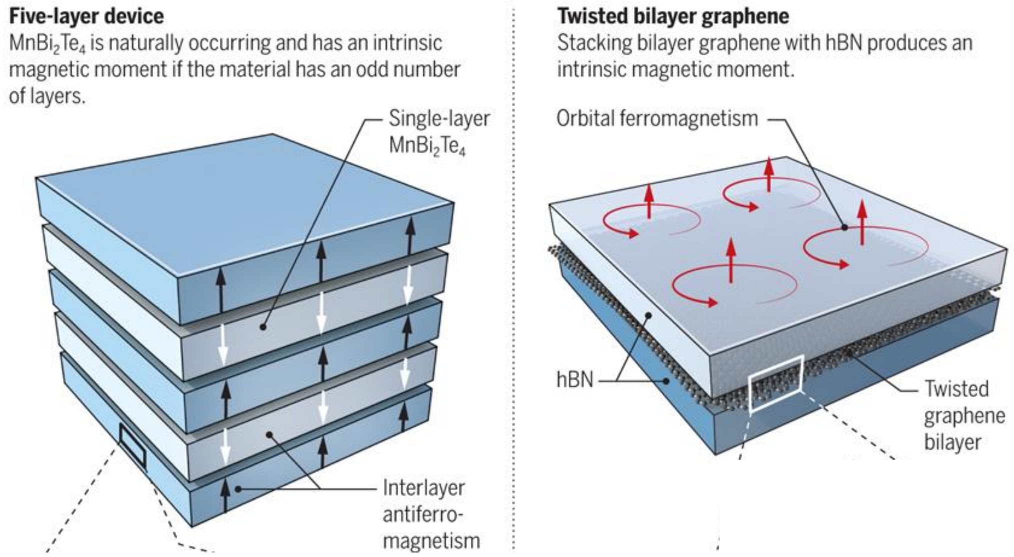


Haldane Model on honeycomb lattice

F. D. M. Haldane, PRL 61, 2015 (1988)

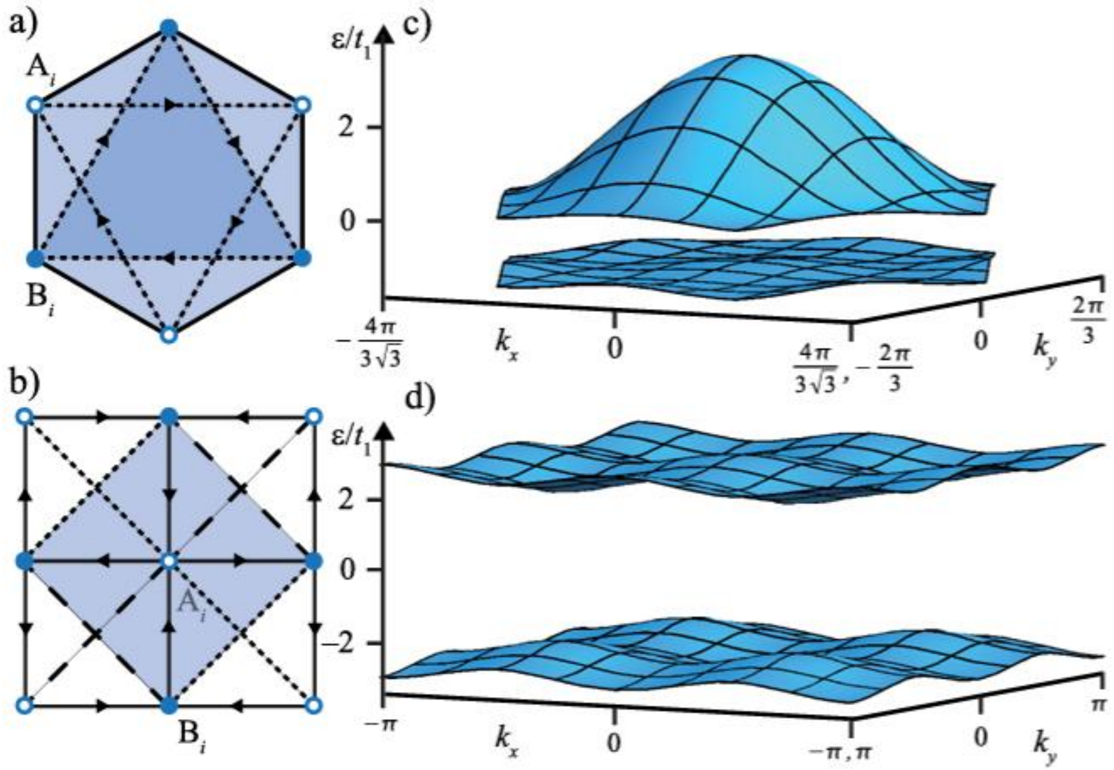


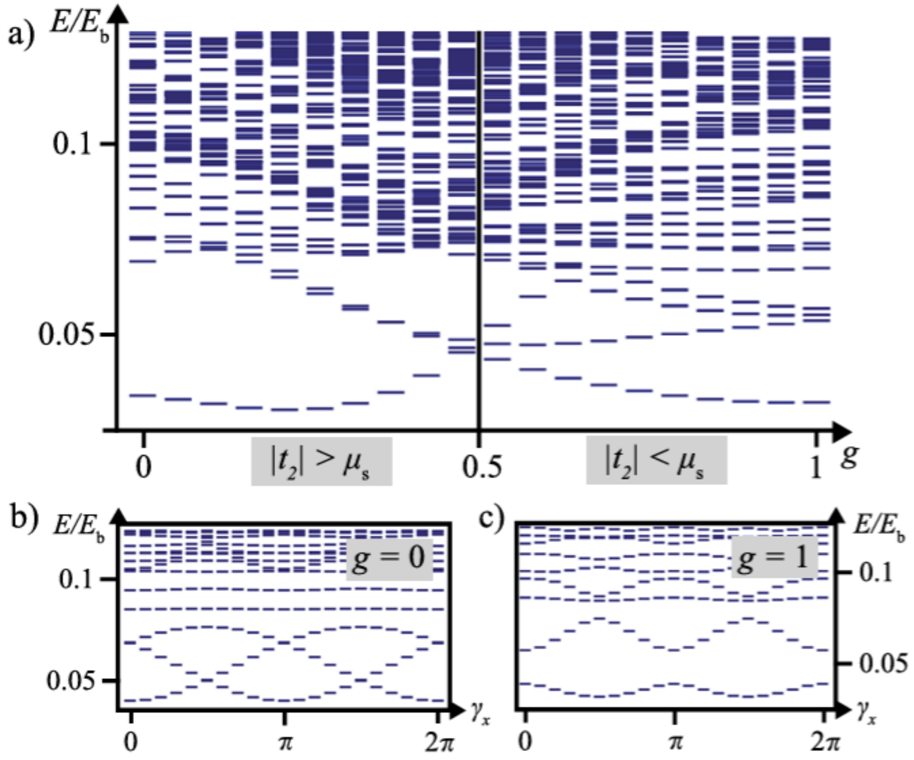
Q.-K. Xue et al., Science 340, 167 (2013)



Y. Zhang et al., Science 367, 895 (2020)
A. Young et al., Science 367, 900 (2020)

Fractional Chern Insulators





Fractional Chern insulators are lattice version of fractional quantum Hall states.

T. Neupert et al., PRL 2011; K. Sun et al., PRL 2011; E. Tang et al., PRL 2011;
N. Regnault et al., PRX 2011; D. N. Sheng et al., NC 2011

Outline

- ✓ Introduction
- ✓ Skyrmion Picture
- ✓ Generalized Landau Levels

Layer Pseudospin Skyrmions

Moiré Hamiltonian:

$$\mathcal{H}_{\uparrow}' = U_0^{\dagger}(\boldsymbol{r})\mathcal{H}_{\uparrow}U_0(\boldsymbol{r}),$$
$$= -\frac{\hbar^2\hat{\boldsymbol{k}}^2}{2m^*}\sigma_0 + \boldsymbol{\Delta}(\boldsymbol{r})\cdot\boldsymbol{\sigma} + \Delta_0(\boldsymbol{r})\,\sigma_0,$$

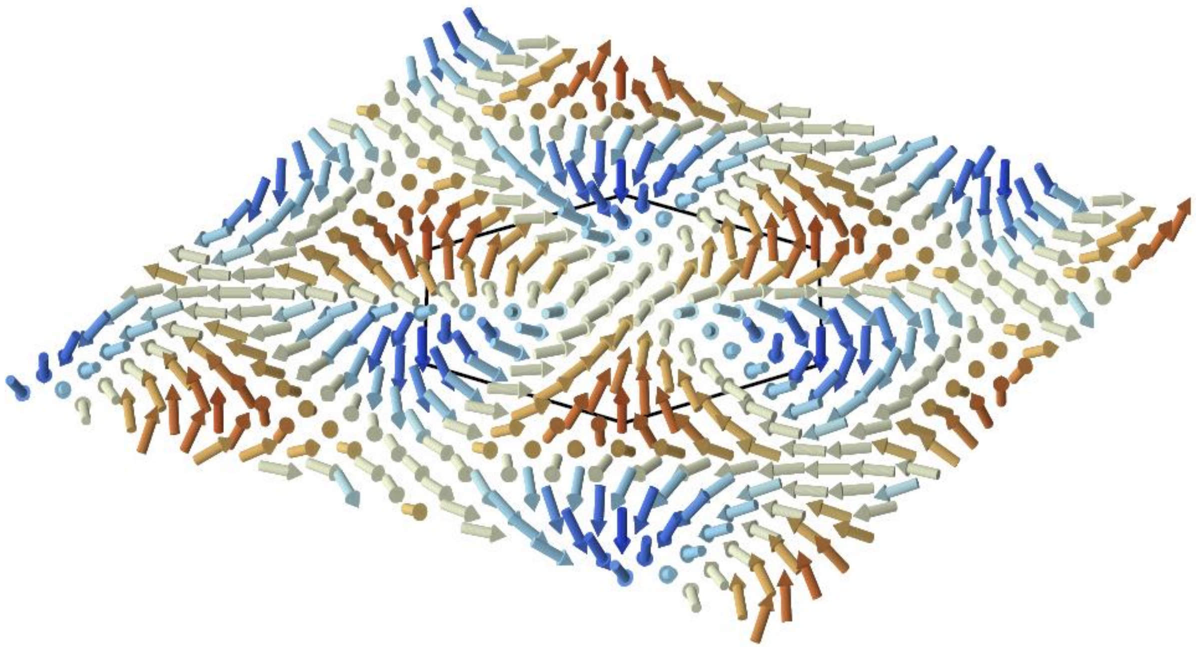
- Layer pseudospin magnetic field:

$$\boldsymbol{\Delta}(\boldsymbol{r}) = [\mathrm{Re}\,\widetilde{\Delta}_{\mathrm{T}}^{\dagger}(\boldsymbol{r}), \mathrm{Im}\,\widetilde{\Delta}_{\mathrm{T}}^{\dagger}(\boldsymbol{r}), \frac{\Delta_+(\boldsymbol{r}) - \Delta_-(\boldsymbol{r})}{2}]$$

- Layer pseudospin skyrmions:

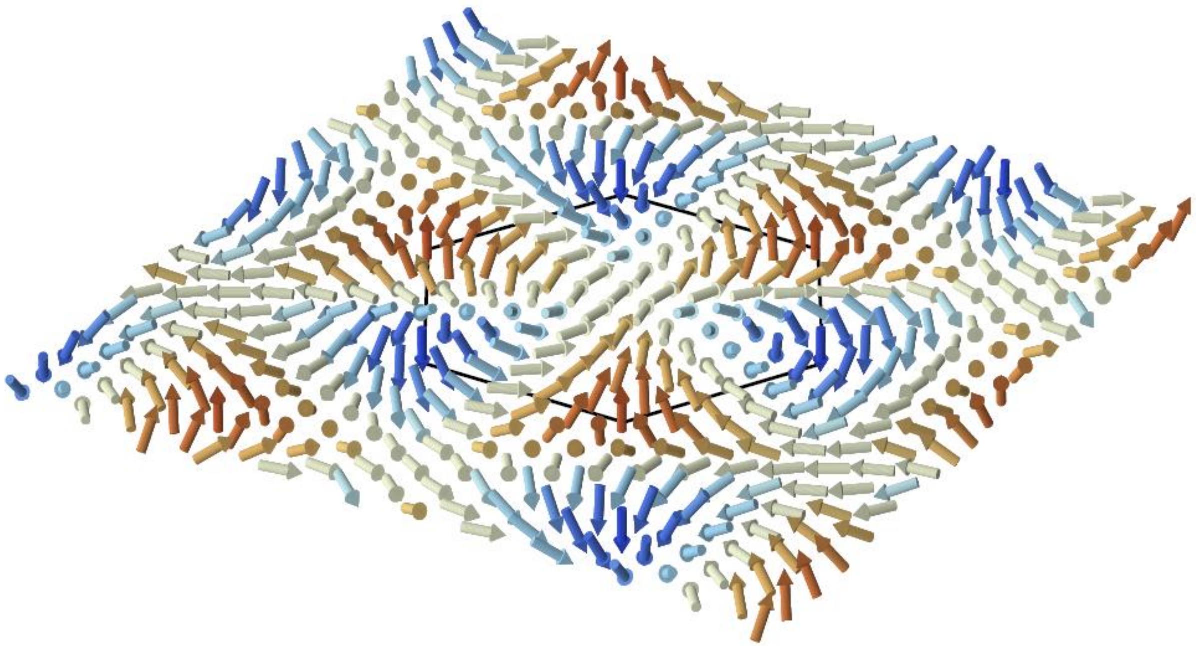
$$N_w \equiv \frac{1}{4\pi}\int_{\mathrm{MUC}} d\boldsymbol{r}\frac{\boldsymbol{\Delta}\cdot(\partial_x\boldsymbol{\Delta}\times\partial_y\boldsymbol{\Delta})}{|\boldsymbol{\Delta}|^3}$$
$$= \begin{cases} +1, & V\sin\psi > 0 \\ -1, & V\sin\psi < 0. \end{cases}$$

$$\boldsymbol{n}(\boldsymbol{r}) = \boldsymbol{\Delta}(\boldsymbol{r})/|\boldsymbol{\Delta}(\boldsymbol{r})|$$



Skyrmion Field ➔ Effective Magnetic Field

Layer pseudospin skyrmion lattice



$\boldsymbol{n}(\boldsymbol{r}) = \boldsymbol{\Delta}(\boldsymbol{r})/|\boldsymbol{\Delta}(\boldsymbol{r})|$

- Rotate the the layer pseudospin to the local frame of skyrmion field:

$$U^\dagger(\boldsymbol{r})[\boldsymbol{\Delta}(\boldsymbol{r})\cdot\boldsymbol{\sigma}]U(\boldsymbol{r}) = |\boldsymbol{\Delta}(\boldsymbol{r})|\sigma_z$$

$$\begin{aligned}
U(\boldsymbol{r}) =& (\boldsymbol{\chi}^{(+)}(\boldsymbol{r}), \boldsymbol{\chi}^{(-)}(\boldsymbol{r})),\\
\boldsymbol{\chi}^{(+)}(\boldsymbol{r}) =& e^{i\lambda_+(\boldsymbol{r})}[\cos\frac{\vartheta(\boldsymbol{r})}{2}, e^{i\phi(\boldsymbol{r})}\sin\frac{\vartheta(\boldsymbol{r})}{2}]^T,\\
\boldsymbol{\chi}^{(-)}(\boldsymbol{r}) =& e^{i\lambda_-(\boldsymbol{r})}[e^{-i\phi(\boldsymbol{r})}\sin\frac{\vartheta(\boldsymbol{r})}{2}, -\cos\frac{\vartheta(\boldsymbol{r})}{2}]^T
\end{aligned}$$

Effective Magnetic Field

- In the local frame, there is an effective non-Abelian gauge field:

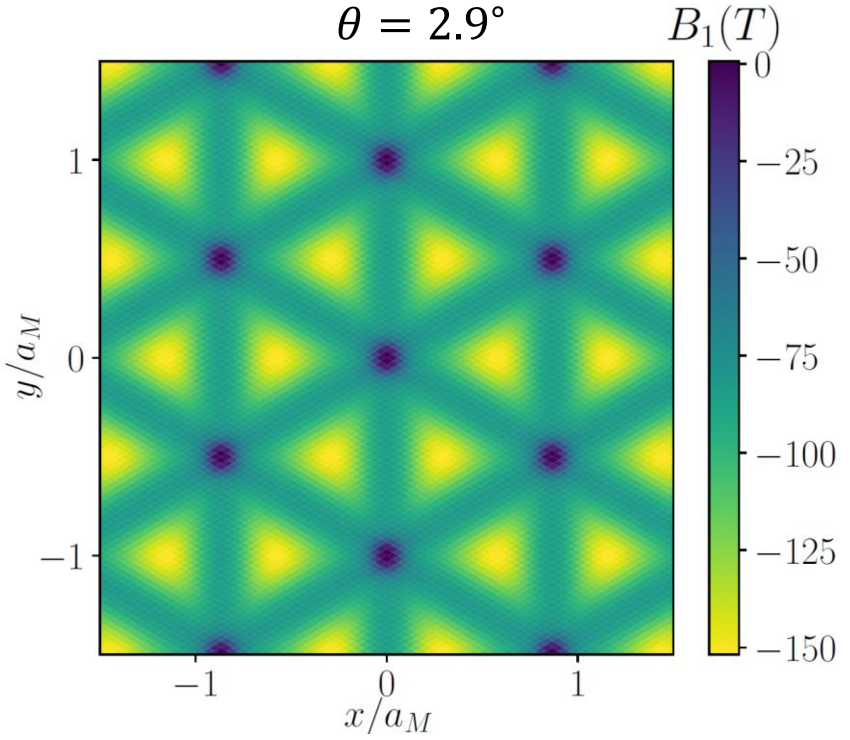
$$
\begin{aligned}
H_2 =& U^\dagger(\boldsymbol{r}) H_1 U(\boldsymbol{r}) \\
=& -\frac{\hbar^2}{2m^*}\begin{pmatrix} (\hat{\boldsymbol{k}}+\boldsymbol{A}_{11})^2 & \hat{\boldsymbol{k}}\boldsymbol{A}_{12}+\boldsymbol{A}_{12}\hat{\boldsymbol{k}} \\ \hat{\boldsymbol{k}}\boldsymbol{A}_{21}+\boldsymbol{A}_{21}\hat{\boldsymbol{k}} & (\hat{\boldsymbol{k}}+\boldsymbol{A}_{22})^2 \end{pmatrix} + |\boldsymbol{\Delta}(\boldsymbol{r})|\sigma_z + [\Delta_0(\boldsymbol{r}) - D(\boldsymbol{r})]\sigma_0,
\end{aligned}
$$

$$\boldsymbol{A} = -iU^\dagger(\boldsymbol{r})\boldsymbol{\nabla} U(\boldsymbol{r})$$

$$B_1 = \frac{\hbar}{e}\boldsymbol{\nabla}\times\boldsymbol{A}_{11} = +\frac{1}{2}\frac{\hbar}{e}\boldsymbol{n}\cdot(\partial_x\boldsymbol{n}\times\partial_y\boldsymbol{n})$$

$$\int_{\mathrm{UC}} d\boldsymbol{r}\, B_1 = -\frac{h}{e}$$

$\theta = 2.9°$



N. Morales-Durán, N. Wei, J. Shi & A. H. MacDonald, PRL (2024)
J. Shi, N. Morales-Durán, E. Khalaf & A. H. MacDonald, PRB(2024)

Landau Level Representation

Bloch WF

Dressed Landau Level WF

$$\varphi_{m,\boldsymbol{k}}(\boldsymbol{r}) = \sum_n \sum_{s=\pm} c^{(s)}_{m,n,\boldsymbol{k}} \psi^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r})$$

$$\psi^{(+)}_{n,\boldsymbol{k}}(\boldsymbol{r}) = U_0(\boldsymbol{r})\chi^{(+)}(\boldsymbol{r})\Psi^{(-)}_{n,\boldsymbol{k}}(\boldsymbol{r})$$

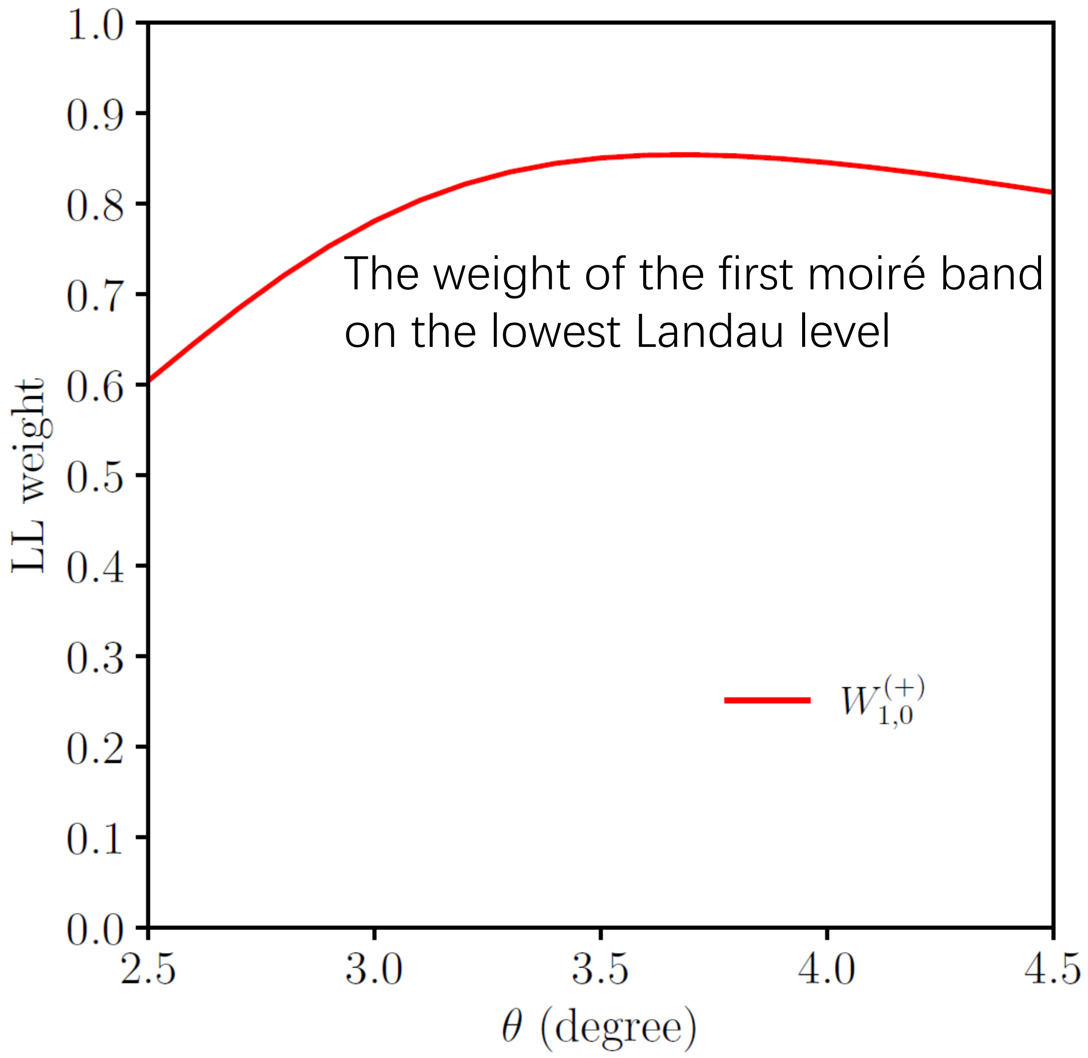
$$\psi^{(-)}_{n,\boldsymbol{k}}(\boldsymbol{r}) = U_0(\boldsymbol{r})\chi^{(-)}(\boldsymbol{r})\Psi^{(+)}_{n,\boldsymbol{k}}(\boldsymbol{r})$$

Spinor

Landau Level WF

- **Landau Level Weight:**

$$W^{(s)}_{m,n} = \frac{1}{N}\sum_{\boldsymbol{k}} \left|c^{(s)}_{m,n,\boldsymbol{k}}\right|^2$$



B. Li & F. Wu, PRB **111**, 125122 (2025)

Variational Mapping

- We generalize $\psi_{0,\boldsymbol{k}}^{(+)}(\boldsymbol{r})$ to $\Theta_{\boldsymbol{k}}(\boldsymbol{r})$ to maximize the weight

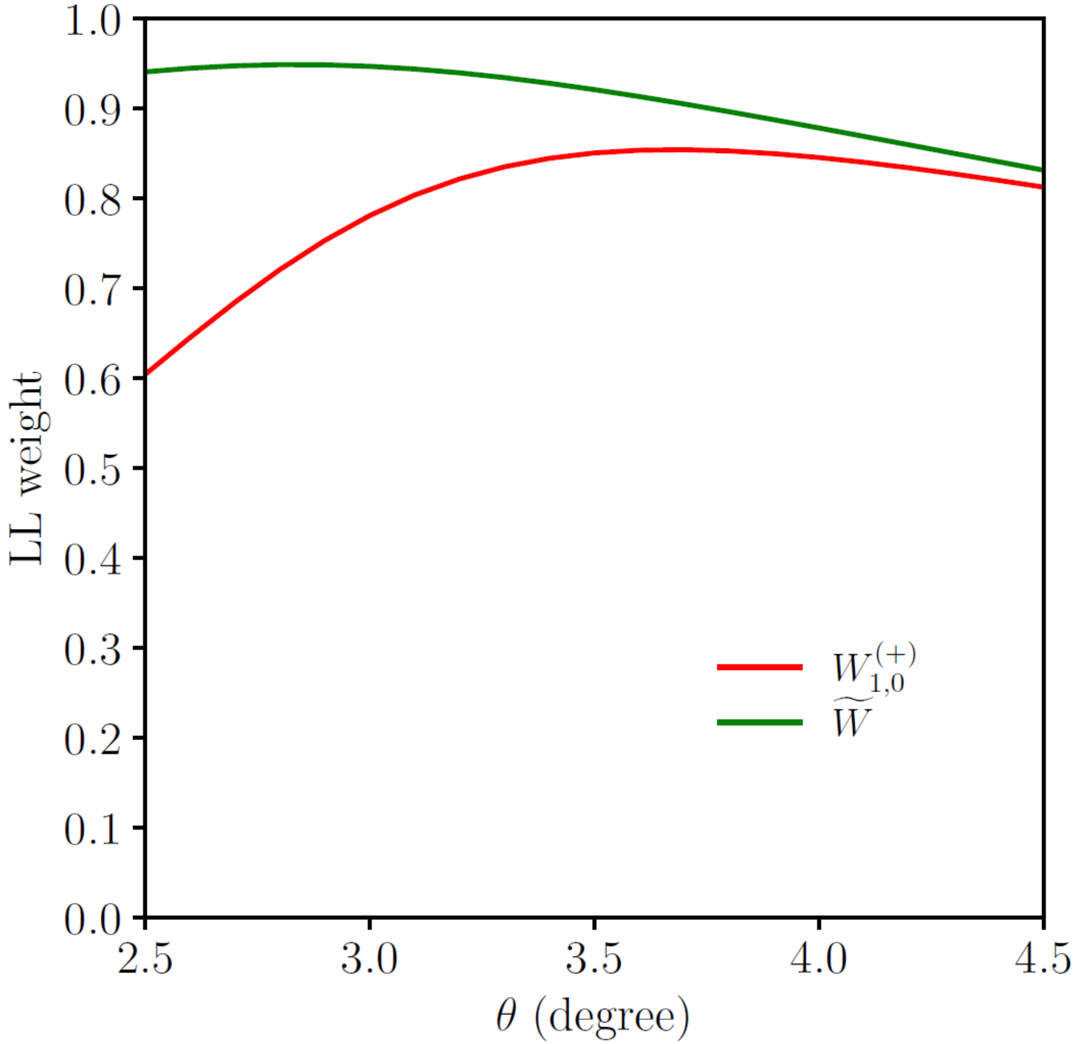
$$\psi_{0,\boldsymbol{k}}^{(+)}(\boldsymbol{r}) = U_0(\boldsymbol{r})\chi^{(+)}(\boldsymbol{r})\Psi_{0,\boldsymbol{k}}^{(-)}(\boldsymbol{r})$$

$$\Downarrow$$

$$\Theta_{\boldsymbol{k}}(\boldsymbol{r}) = \mathcal{N}_{\boldsymbol{k}} U_0(\boldsymbol{r})\mathcal{B}(\boldsymbol{r})\Psi_{0,\boldsymbol{k}}^{(-)}(\boldsymbol{r}),$$

- Variational maximize the weight

$$\widetilde{W} = \frac{1}{N}\sum_{\boldsymbol{k}} |\tilde{c}_{\boldsymbol{k}}|^2, \qquad \tilde{c}_{\boldsymbol{k}} = \langle \varphi_{1,\boldsymbol{k}} \mid \Theta_{\boldsymbol{k}} \rangle$$



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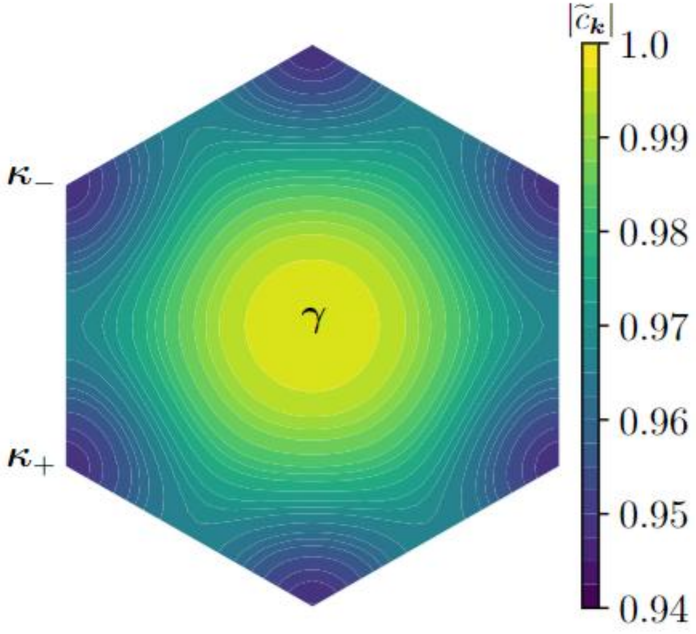
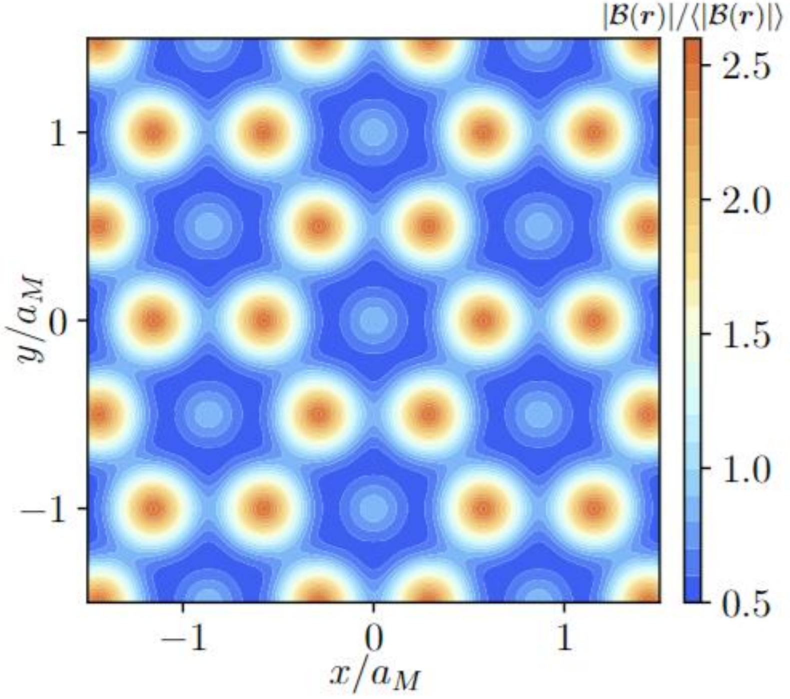
Variational Mapping

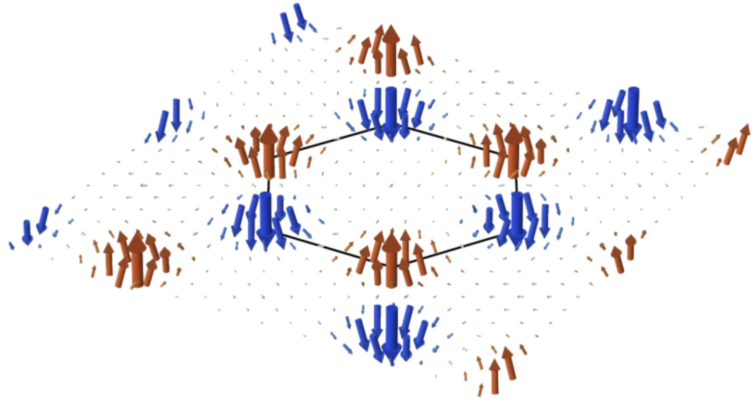
$\tilde{c}_{\boldsymbol{k}} = \langle \varphi_{1,\boldsymbol{k}} \mid \Theta_{\boldsymbol{k}} \rangle$



$|\mathcal{B}(\boldsymbol{r})|$

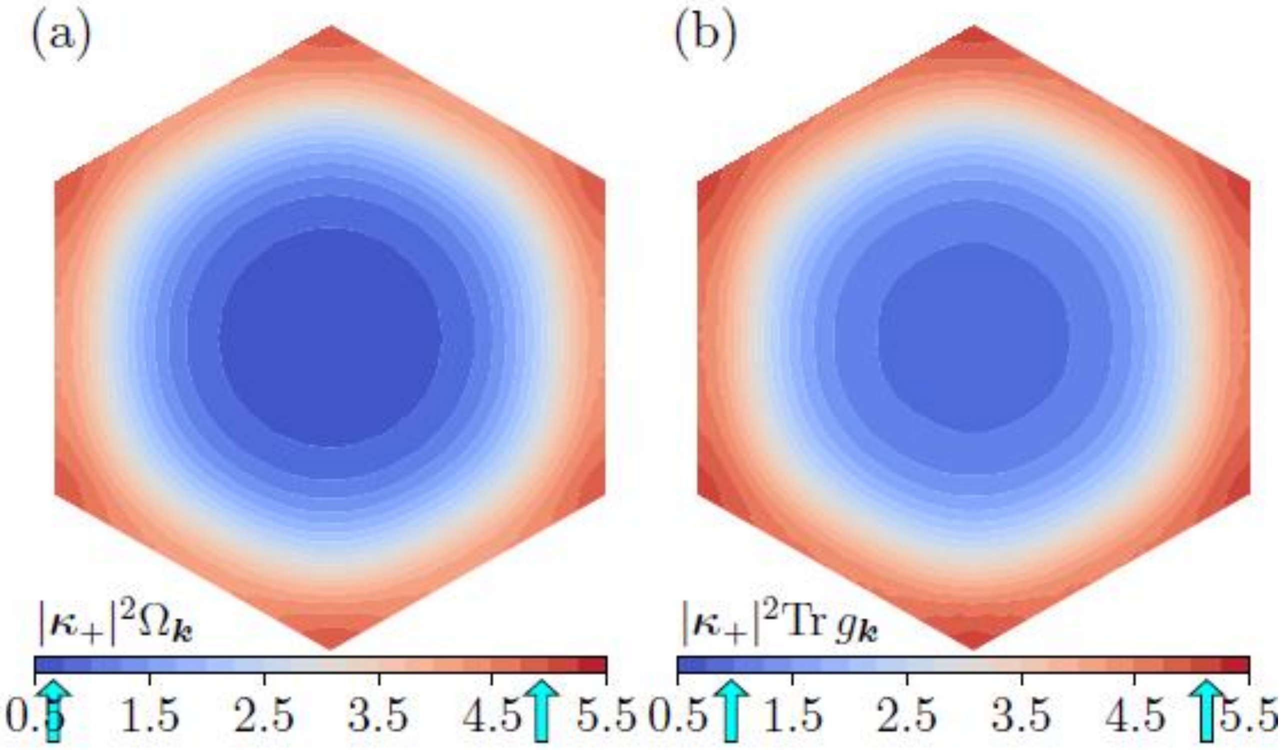


$[\mathcal{B}(\boldsymbol{r})]^\dagger \boldsymbol{\sigma} \mathcal{B}(\boldsymbol{r})$

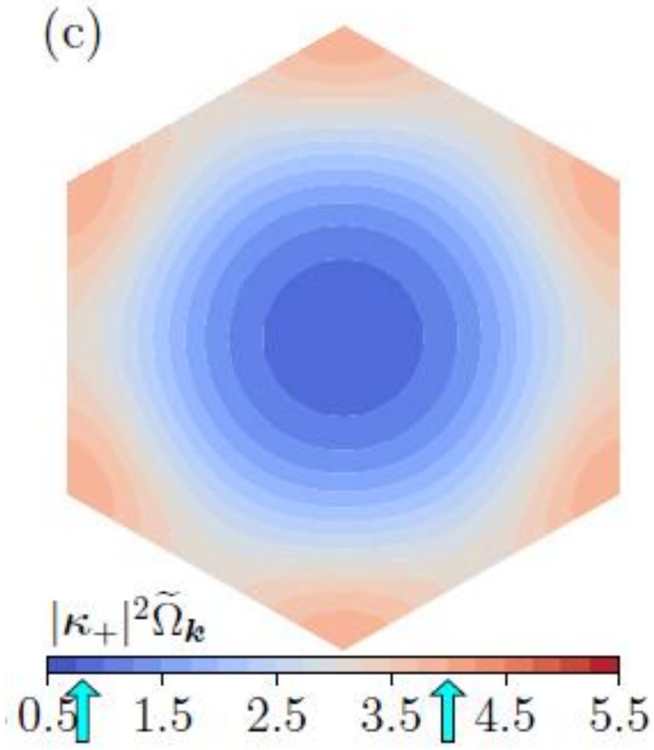


Quantum Geometry

Quantum geometry of original WF



Variational WF
(*ideal* quantum geometry)



$$\mathrm{Tr}\, \tilde{g}_{\boldsymbol{k}} = \widetilde{\Omega}_{\boldsymbol{k}}$$

Ideal vs. Nonideal Quantum Geometry

- Deviation from ideal quantum geometry

$$T = \frac{1}{2\pi}\int d^2\boldsymbol{k}\,\mathrm{Tr}\, g_{\boldsymbol{k}} - \mathcal{C}$$

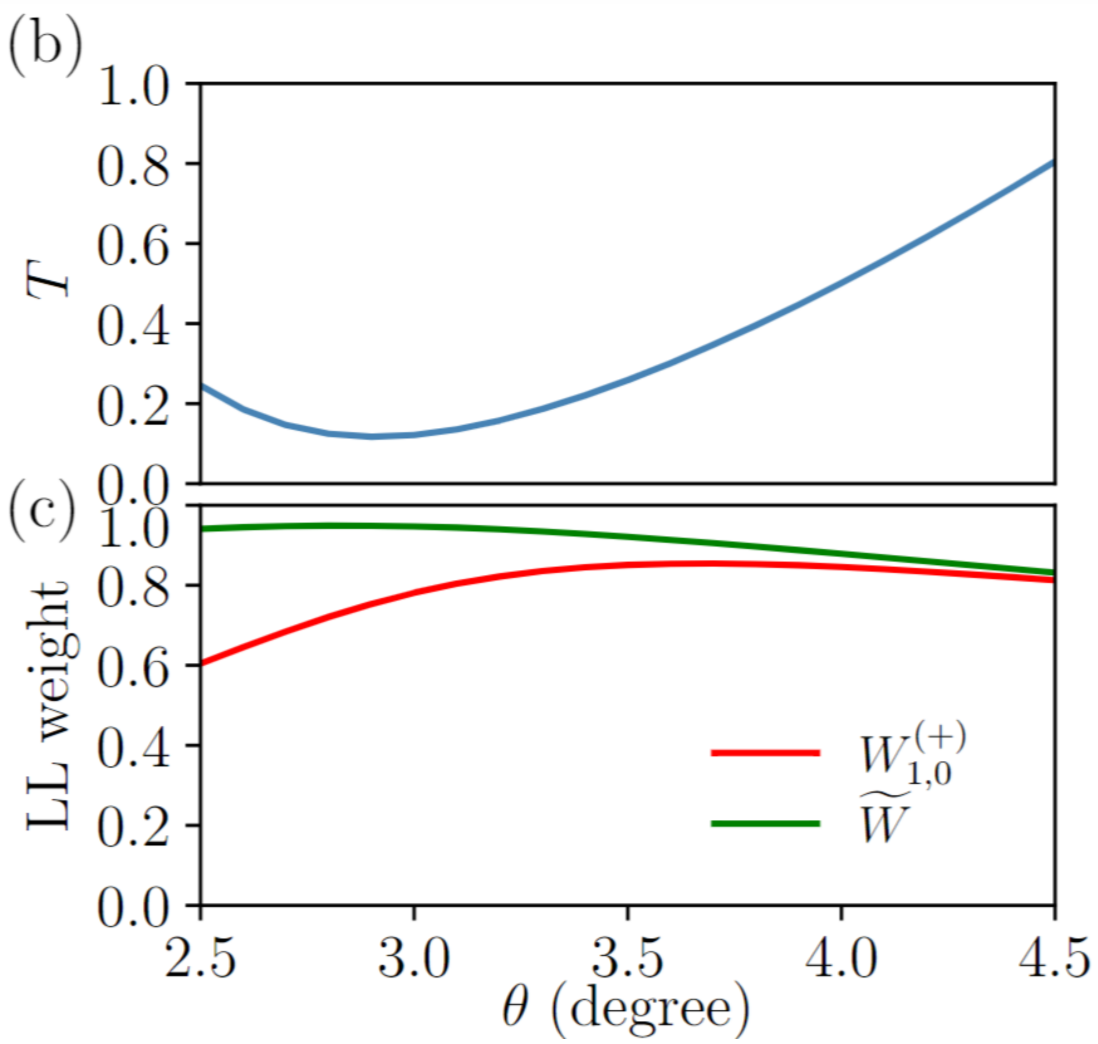
- Trace inequality: $T \geq 0$
- Ideal band $\Theta_{\boldsymbol{k}}(\boldsymbol{r}) = \mathcal{N}_{\boldsymbol{k}}\mathcal{B}(\boldsymbol{r})\Psi_{0,\boldsymbol{k}}(\boldsymbol{r})$ has $T = 0$
  [J. Wang, et al., PRL (2021)]

- In general, the quantum geometry is nonideal $T \neq 0$

Is there still a mapping between Chern bands and Landau levels?

Can physics of FQHI still be applied to understand FCI?

Quantum Geometry



- Deviation from ideal quantum geometry

$$T = \frac{1}{2\pi}\int d^2\boldsymbol{k}\,\mathrm{Tr}\, g_{\boldsymbol{k}} - \mathcal{C}$$

- Trace inequality: $T \geq 0$

- Ideal band $\Theta_{\boldsymbol{k}}(\boldsymbol{r})$ has $T = 0$
  [J. Wang, et al., PRL (2021)]

- For the moiré band in tMoTe$_2$,
  T reaches a minimum at $\theta_m \sim 2.9°$,
  where the weight $\widetilde{W}$ reaches a maximum.

Generalized Laughlin States

- Bloch states of $\Theta_{\boldsymbol{k}}(\boldsymbol{r})$ enable construction of generalized Laughlin states:

$$\Phi_F = \Psi_F \prod_i U_0(\boldsymbol{r}_i)\mathcal{B}(\boldsymbol{r}_i)$$

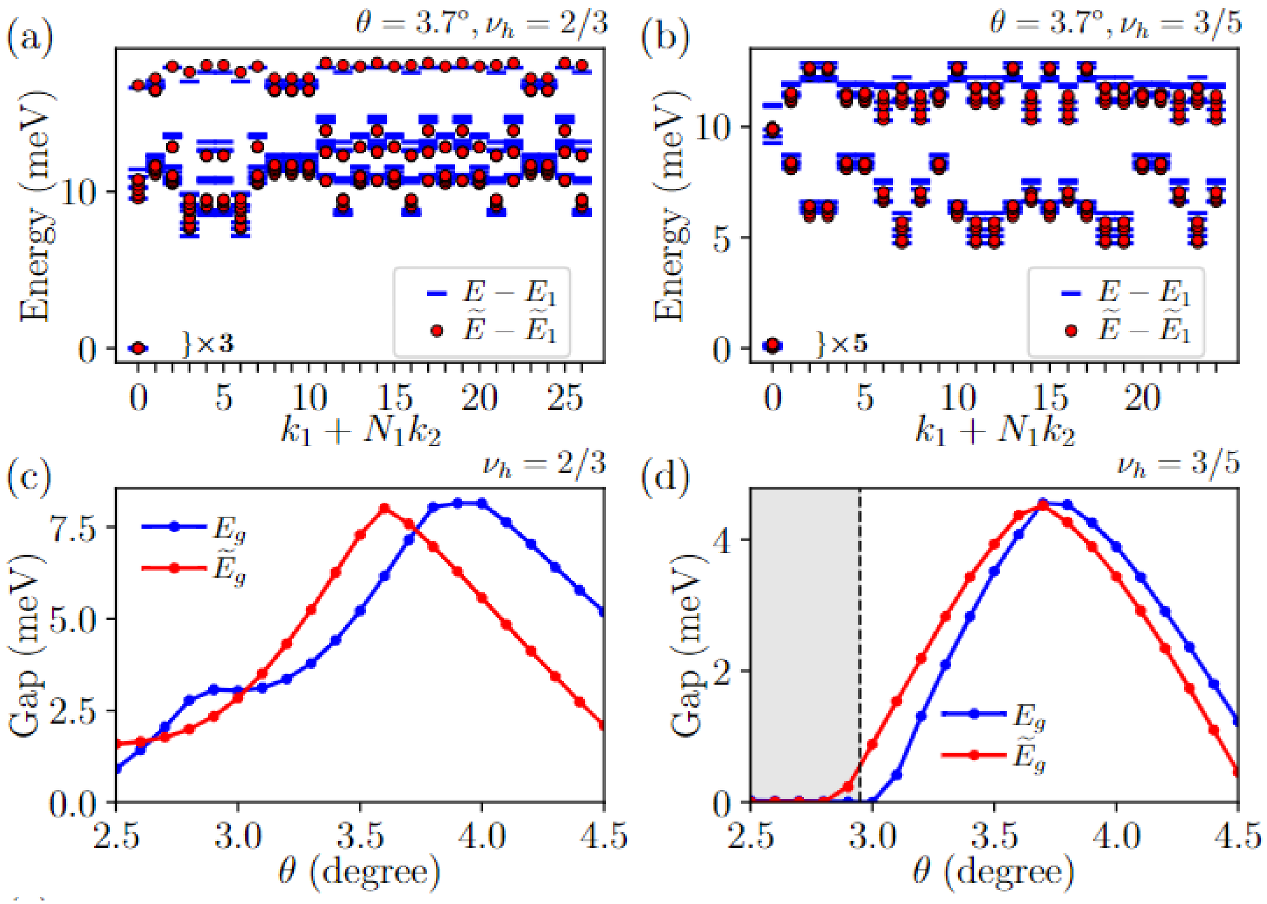
✓ Ψ_F represents the Fractional quantum Hall states in the lowest Landau level.

Fractional quantum Hall Insulators

Fractional Chern Insulators

P. J. Ledwith et al., PR Research (2020)
J. Wang, et al., PRL (2021).

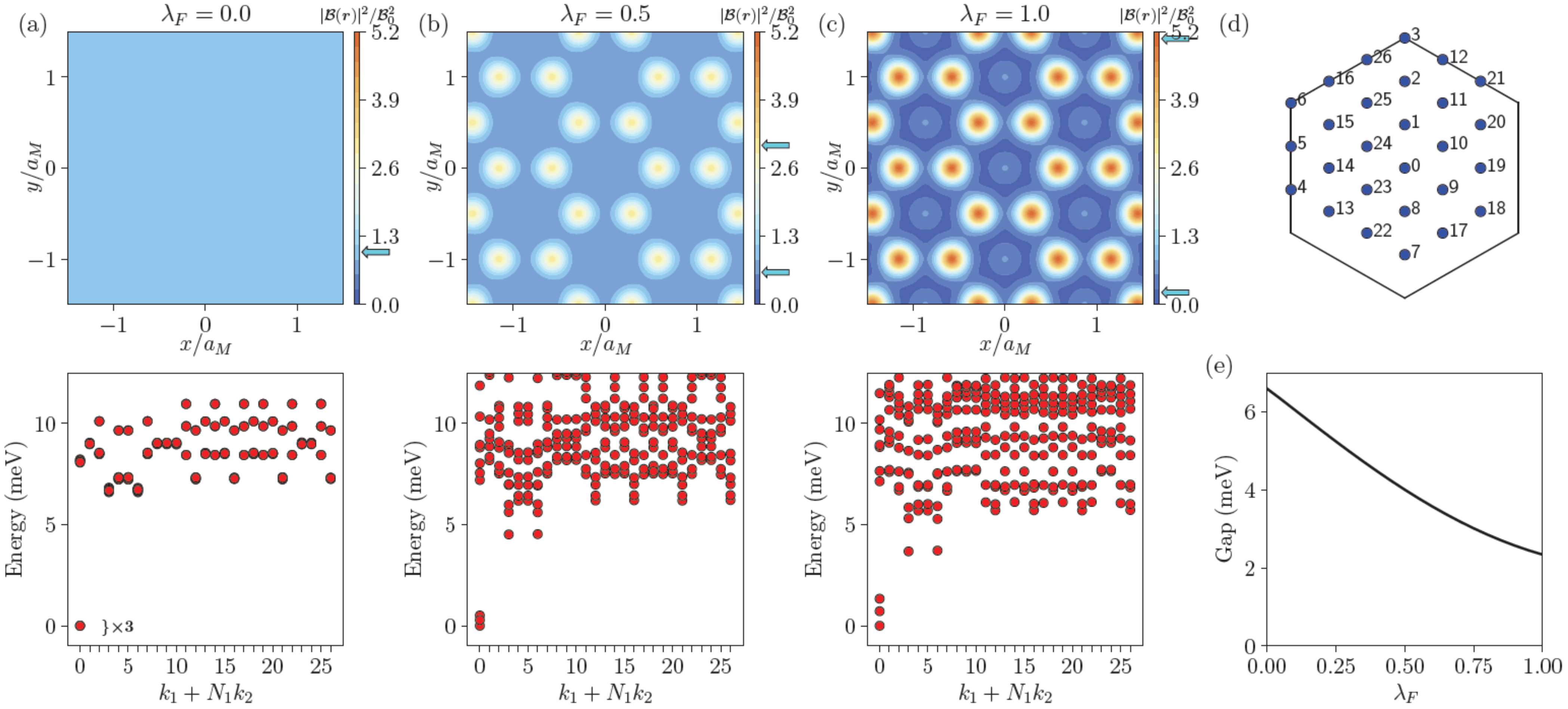
Fractional Chern Insulators: Exact-Diagonalization



Quantitative agreement between variational WF and original WF.

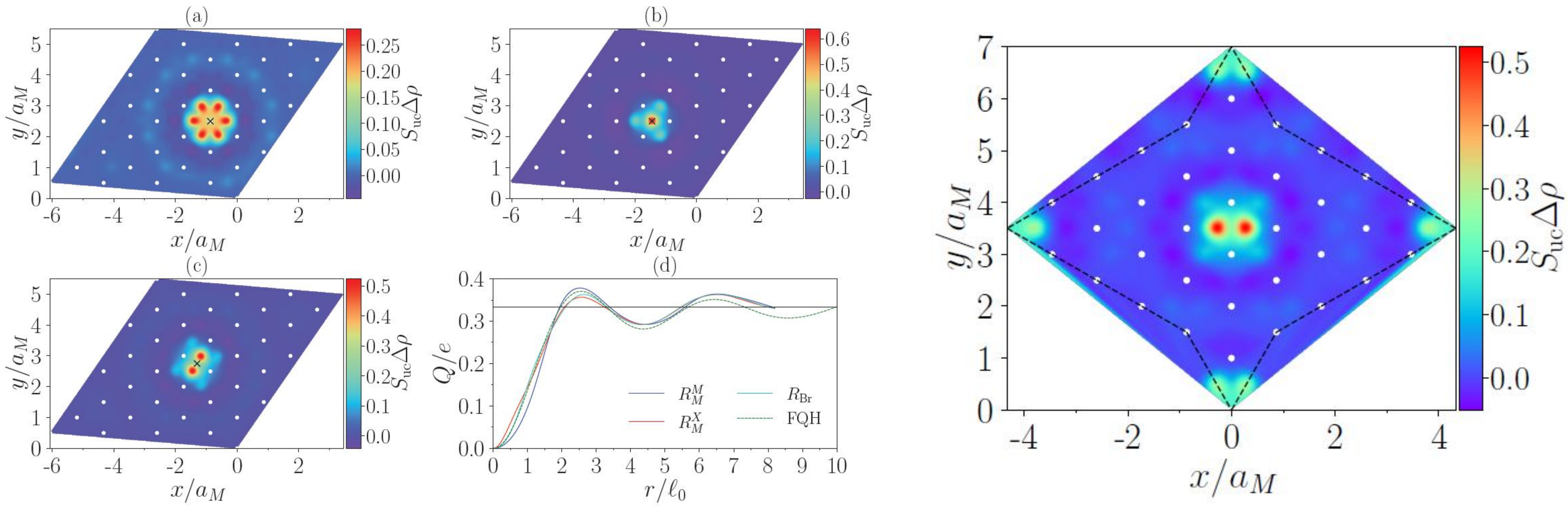
B. Li & F. Wu, PRB **111**, 125122 (2025)

Fractional Chern Insulators: Evolution from Laughlin States



$$|\mathcal{B}(\boldsymbol{r}, \lambda_F)|^2 = \mathcal{B}_0^2 \left(1 + \lambda_F \sum_{\boldsymbol{g} \neq \boldsymbol{0}} F_{\boldsymbol{g}} e^{i\boldsymbol{g}\cdot\boldsymbol{r}}\right)$$

Fractionalized Quasiparticles and Fractional Statistics



$$\Phi_L = \mathcal{N}_L' \prod_i \mathcal{B}(\boldsymbol{r}_i)\Psi_L,$$

$$\Psi_L = \mathcal{N}_L \prod_{1\le i<j\le N} (z_i - z_j)^3 e^{-\frac{1}{4\ell_B^2}\sum_i |z_i|^2}$$

$$\Phi_{L,h} = \mathcal{N}_{L,h}' \prod_i (z_i - w)\mathcal{B}(\boldsymbol{r}_i)\Psi_L$$

Z. Liu, B. Li, Y. Shi, F. Wu, Phys. Rev. B 112, 245104 (2025)

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Generalized tMoTe2 Model

generalized Moiré Hamiltonian:

$$H_+ = U_0(\boldsymbol{r}) \begin{pmatrix} H_b^0(\hat{\boldsymbol{k}}) + \Delta_b(\hat{\boldsymbol{k}}, \boldsymbol{r}) & \Delta_T(\hat{\boldsymbol{k}}, \boldsymbol{r}) \\ \Delta_T^\dagger(\hat{\boldsymbol{k}}, \boldsymbol{r}) & H_t^0(\hat{\boldsymbol{k}}) + \Delta_t(\hat{\boldsymbol{k}}, \boldsymbol{r}) \end{pmatrix} U_0^\dagger(\boldsymbol{r})$$

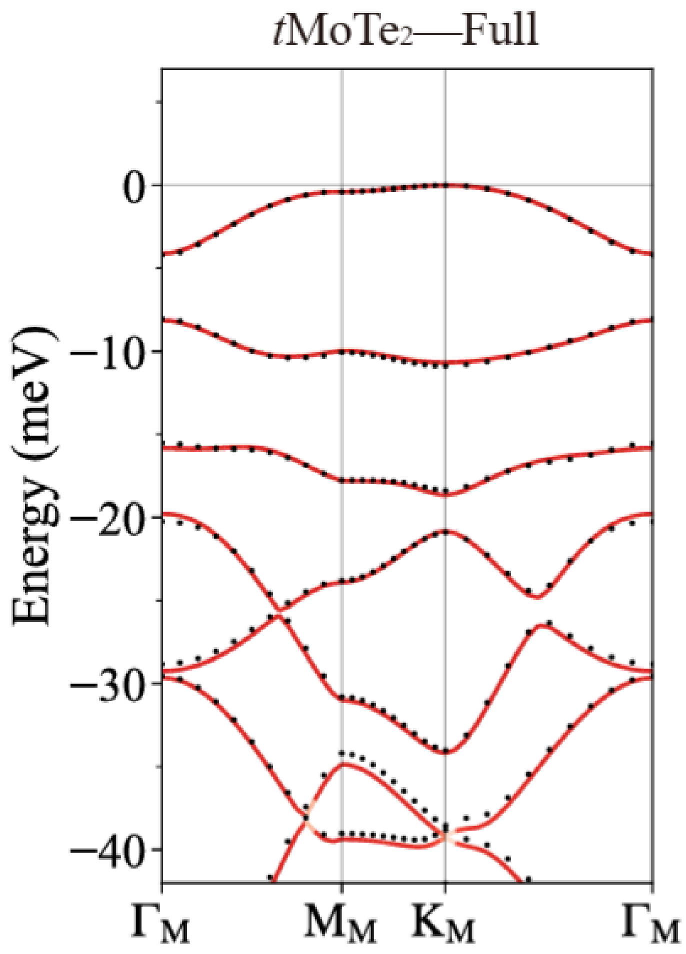
- **kinetic-energy terms**

$$H_l^0(\hat{\boldsymbol{k}}) = \sum_{M_1, M_2} \frac{\hbar^2}{2m_l^{(M_1,M_2)}} f_{\hat{\boldsymbol{k}}}^{(M_1,M_2)}$$

- **potential terms**

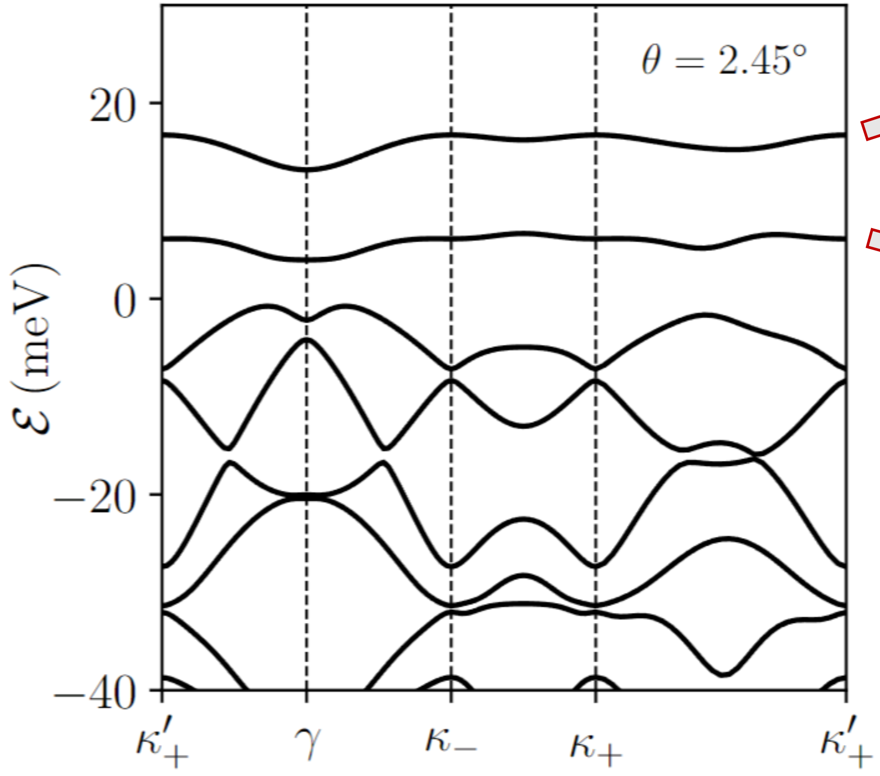
$$\Delta_l(\hat{\boldsymbol{k}}, \boldsymbol{r}) = \sum_{M_1, M_2, \boldsymbol{g}} V_{\boldsymbol{g},l}^{(M_1,M_2)} f_{\hat{\boldsymbol{k}}}^{(M_1,M_2)} e^{i\boldsymbol{g}\cdot\boldsymbol{r}} + \text{h.c.}$$

$$\Delta_T(\hat{\boldsymbol{k}}, \boldsymbol{r}) = \sum_{M_1, M_2, \boldsymbol{q}} w_{\boldsymbol{q}}^{(M_1,M_2)} [f_{\hat{\boldsymbol{k}}}^{(M_1,M_2)} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} + e^{i(\hat{R}_{2x}\boldsymbol{q})\cdot\boldsymbol{r}} f_{\hat{\boldsymbol{k}}}^{(M_2,M_1)}]$$



Y. Zhang, H. Pi, J. Liu, W. Miao, Z. Qi, N. Regnault, H. Weng, X. Dai, B. A. Bernevig, Q. Wu, J. Yu, arXiv:2411.08108

Chern Bands ⇔ Landau levels



$C = 1, K \approx 1.1,$ ~ zeroth Landau level

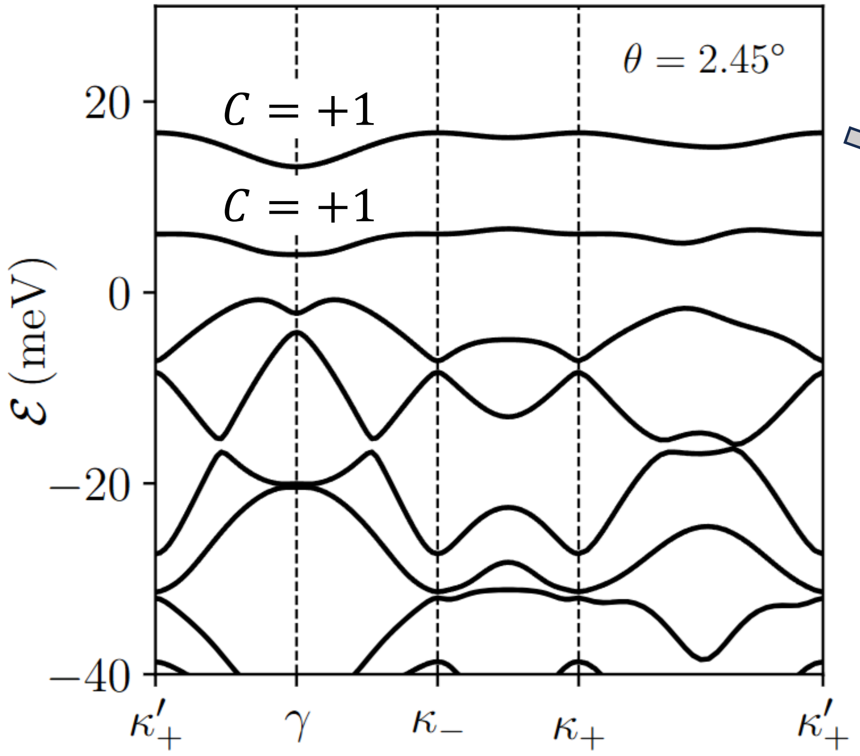
$C = 1, K \approx 3.5,$ ~ first Landau level

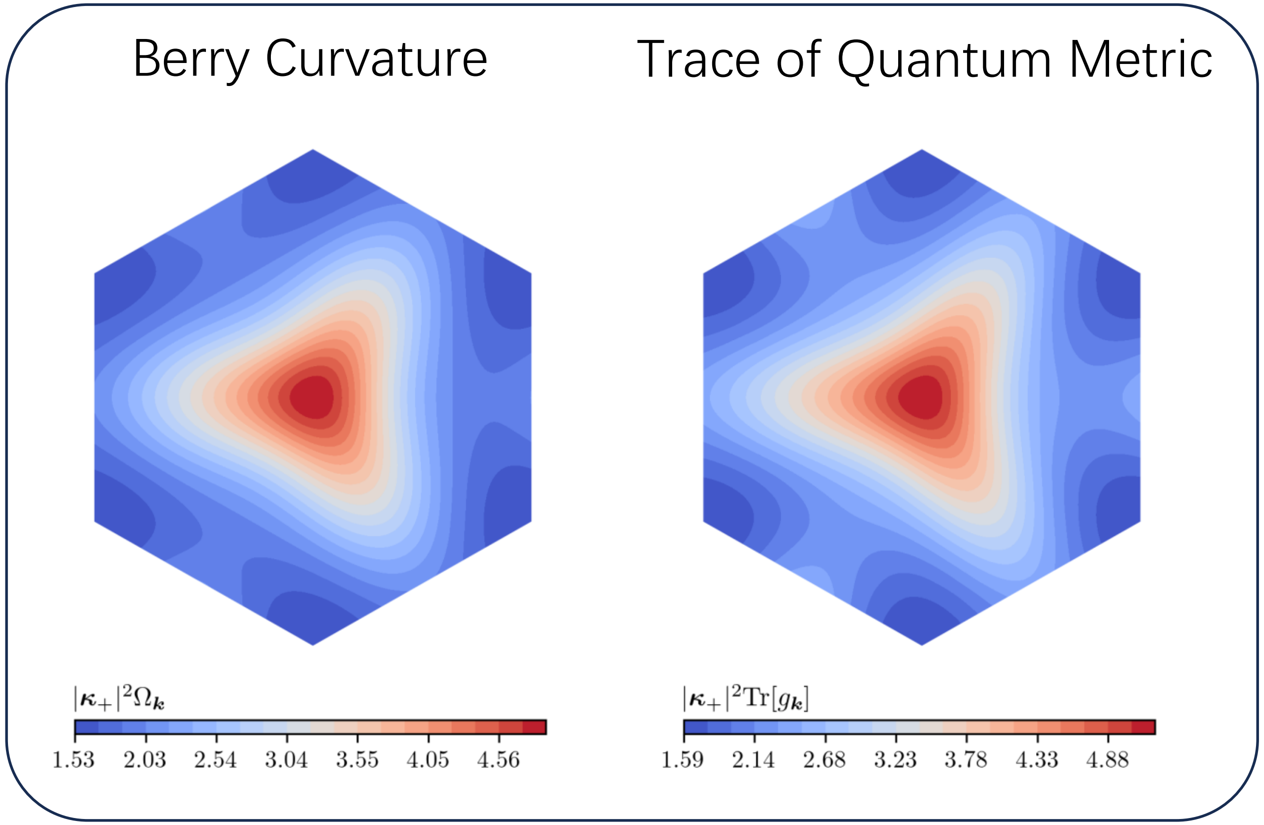
Chern number: $C = \frac{1}{2\pi}\int d\boldsymbol{k}\,\Omega_{\boldsymbol{k}}$

Quantum weight: $K = \frac{1}{2\pi}\int d\boldsymbol{k}\,\mathrm{Tr}\,g_{\boldsymbol{k}}$

Nearly "Ideal" band in $tMoTe_2$

Topological Moiré Bands





- Nearly flat band, bandwidth < 5 meV
- Nearly Ideal Quantum Geometry, Tr $g_{\boldsymbol{k}} \approx \Omega_{\boldsymbol{k}}$

Ideal Quantum Geometry ➔ Fractional Chern Insulators

- A Chern band and the zeroth Landau level can have an *exact* correspondence when the quantum geometry is *ideal* [Jie Wang et al., PRL 127, 246403 (2021)]

$$\mathrm{Tr}\, g_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}} \quad \Rightarrow \quad \Theta_{\boldsymbol{k}}(\boldsymbol{r}) = \mathcal{N}_{\boldsymbol{k}}\, \mathcal{B}(\boldsymbol{r}) \Psi_{0,\boldsymbol{k}}(\boldsymbol{r})$$

- A Chern band of $\Theta_{\boldsymbol{k}}(\boldsymbol{r})$ can host FCI states [P. J. Ledwith et al., PRR(2020)]

$$\Phi_F = \Psi_F \prod_i \mathcal{B}(\boldsymbol{r}_i)$$

Ψ_F represents the Fractional quantum Hall states in the lowest Landau level.

Ideal vs. Nonideal Quantum Geometry

- Deviation from ideal quantum geometry

$$T = \frac{1}{2\pi}\int d^2\boldsymbol{k}\,\mathrm{Tr}\, g_{\boldsymbol{k}} - |\mathcal{C}|$$

- Trace inequality: $T \geq 0$
- Ideal band $\Theta_{\boldsymbol{k}}(\boldsymbol{r}) = \mathcal{N}_{\boldsymbol{k}}\mathcal{B}(\boldsymbol{r})\Psi_{0,\boldsymbol{k}}(\boldsymbol{r})$ has $T = 0$
  [J. Wang, et al., PRL (2021)]

- In general, the quantum geometry is nonideal $T \neq 0$

Is there still a mapping between Chern bands and Landau levels?

Can physics of FQHI still be applied to understand FCI?

Generalized Landau Level

- Generalized nLL wave function [Zhao Liu… Jie Wang, PRX 15, 031019 (2025)]

$$e^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r}) = \mathcal{B}^{(s)}(\boldsymbol{r})\Psi^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r})$$

$$\Downarrow$$

$$\Theta^{(s)}_{n,\mathbf{k}}(\mathbf{r}) = \begin{cases} \mathcal{N}_{0,\mathbf{k}}\mathcal{B}^{(s)}(\mathbf{r})\Psi^{(s)}_{n,\mathbf{k}}(\mathbf{r}) & n = 0 \\ \mathcal{N}_{n,\mathbf{k}}\left[\mathcal{B}^{(s)}(\mathbf{r})\Psi^{(s)}_{n,\mathbf{k}}(\mathbf{r}) - \sum_{m=0}^{n-1}\left\langle \Theta^{(s)}_{m,\mathbf{k}} \mid e^{(s)}_{n,\mathbf{k}}\right\rangle \Theta^{(s)}_{m,\mathbf{k}}(\mathbf{r})\right] & n \geq 1, \end{cases}$$

$$\Downarrow$$

Integrated form of trace condition $\mathcal{K} = (2n+1)|\mathcal{C}| = 2n+1$

Generalized Landau Level

- Generalized nLL wave function [Zhao Liu… Jie Wang, PRX 2025]

$$e^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r}) = \mathcal{B}^{(s)}(\boldsymbol{r})\Psi^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r}) \Longrightarrow \Theta^{(s)}_{n,\mathbf{k}}(\mathbf{r}) = \begin{cases} \mathcal{N}_{0,\mathbf{k}}\mathcal{B}^{(s)}(\mathbf{r})\Psi^{(s)}_{n,\mathbf{k}}(\mathbf{r}) & n=0 \\ \mathcal{N}_{n,\mathbf{k}}\left[\mathcal{B}^{(s)}(\mathbf{r})\Psi^{(s)}_{n,\mathbf{k}}(\mathbf{r}) - \sum_{m=0}^{n-1}\left\langle \Theta^{(s)}_{m,\mathbf{k}} \mid e^{(s)}_{n,\mathbf{k}}\right\rangle \Theta^{(s)}_{m,\mathbf{k}}(\mathbf{r})\right] & n \geq 1, \end{cases}$$

- Generalized LL decomposition

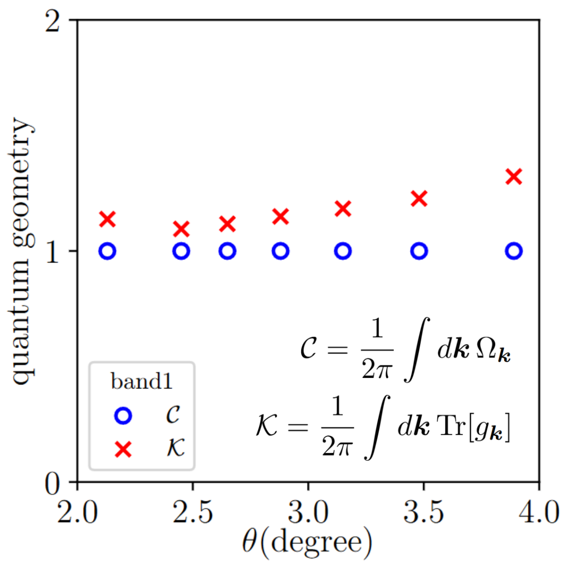
$$\psi_{+,m,\boldsymbol{k}}(\boldsymbol{r}) = \sum_n \sum_{s=\pm} c^{(s)}_{m,n,\boldsymbol{k}}\Theta^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r})$$

- Variational mapping approach of maximizing $W^{(s_0)}_{m,n_0}$

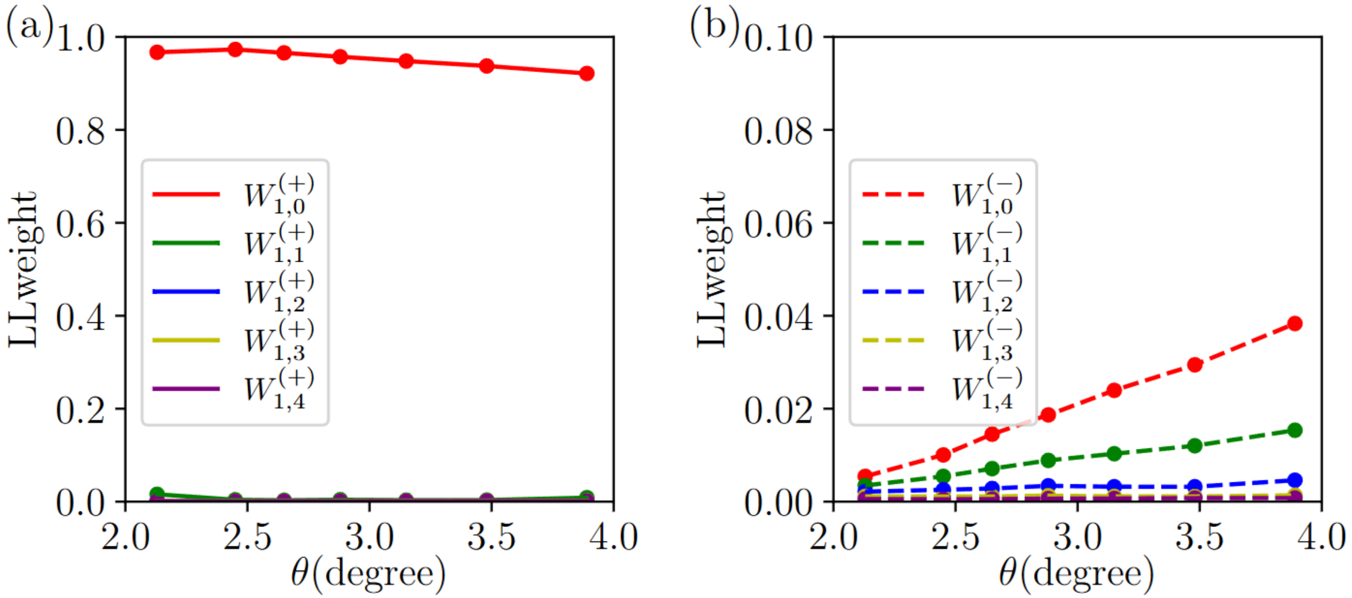
$$W^{(s)}_{m,n} = \frac{1}{N}\sum_{\mathbf{k}}\left|\left\langle \Theta^{(s)}_{n,\mathbf{k}} \mid \psi_{+,m,\mathbf{k}}\right\rangle\right|^2 \qquad \mathcal{B}^{(s_0)}_i(\boldsymbol{r}) \rightarrow \mathcal{B}^{(s_0)}_i(\boldsymbol{r}) + \zeta\frac{\delta W^{(s_0)}_{m,n_0}}{\delta\overline{\mathcal{B}_i}^{(s_0)}(\boldsymbol{r})}$$

1^{st} Moiré Band ~ Generalized 0LL

- Quantum Geometry



- Generalized Landau Level weights

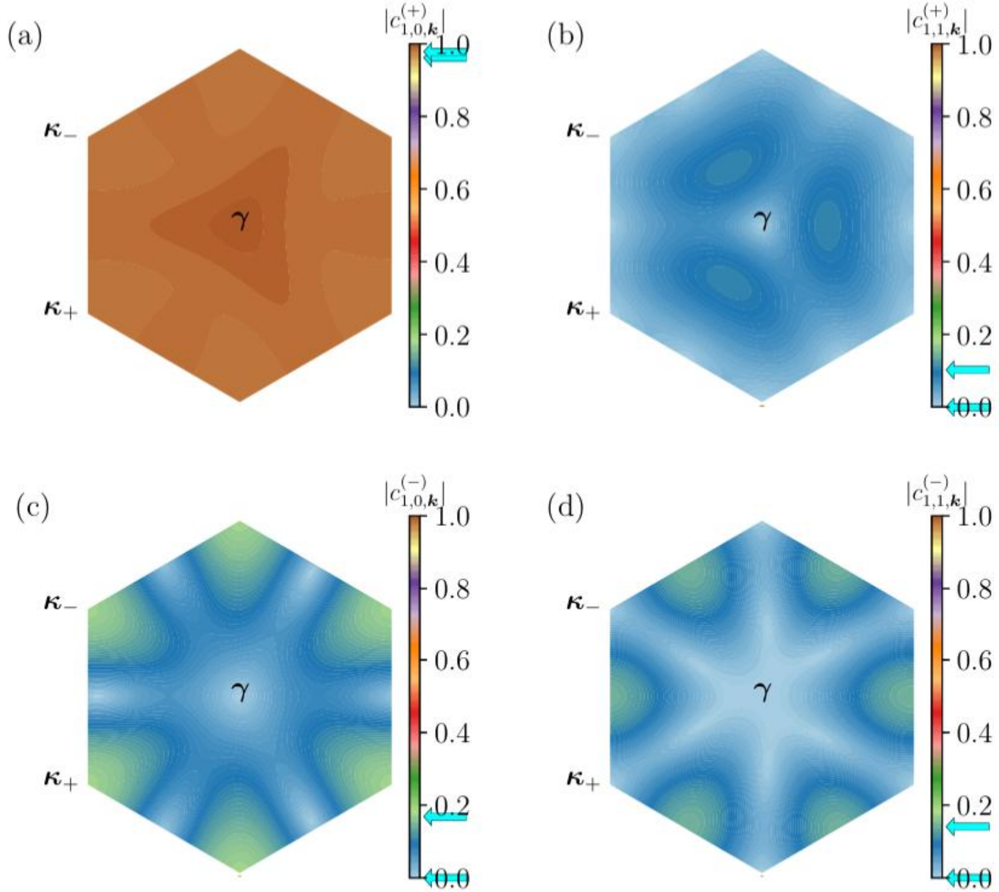


- The 1^{st} moiré band has nearly ideal quantum geometry, Tr $g_{\boldsymbol{k}} \approx \Omega_{\boldsymbol{k}}$ ➔ $\mathcal{K} \approx \mathcal{C}$
- It is dominated by the generalized 0th Landau level.

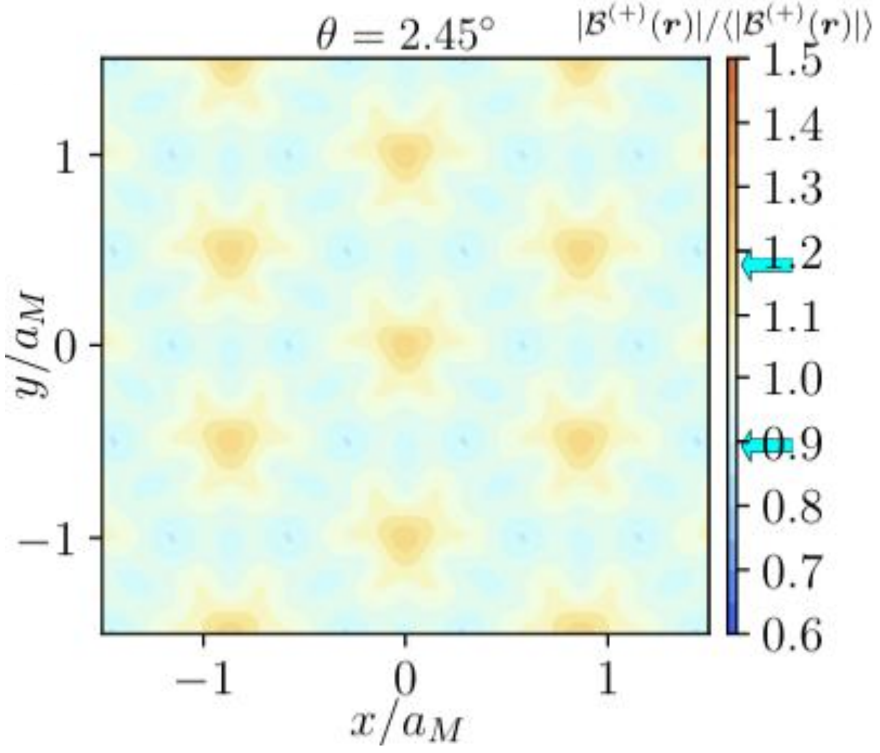
B. Li, Y. Ouyang, F. Wu, Phys. Rev. B 113, 195129 (2026) [Editors' Suggestion]

Generalized LL Decomposition

Decomposition Coefficients

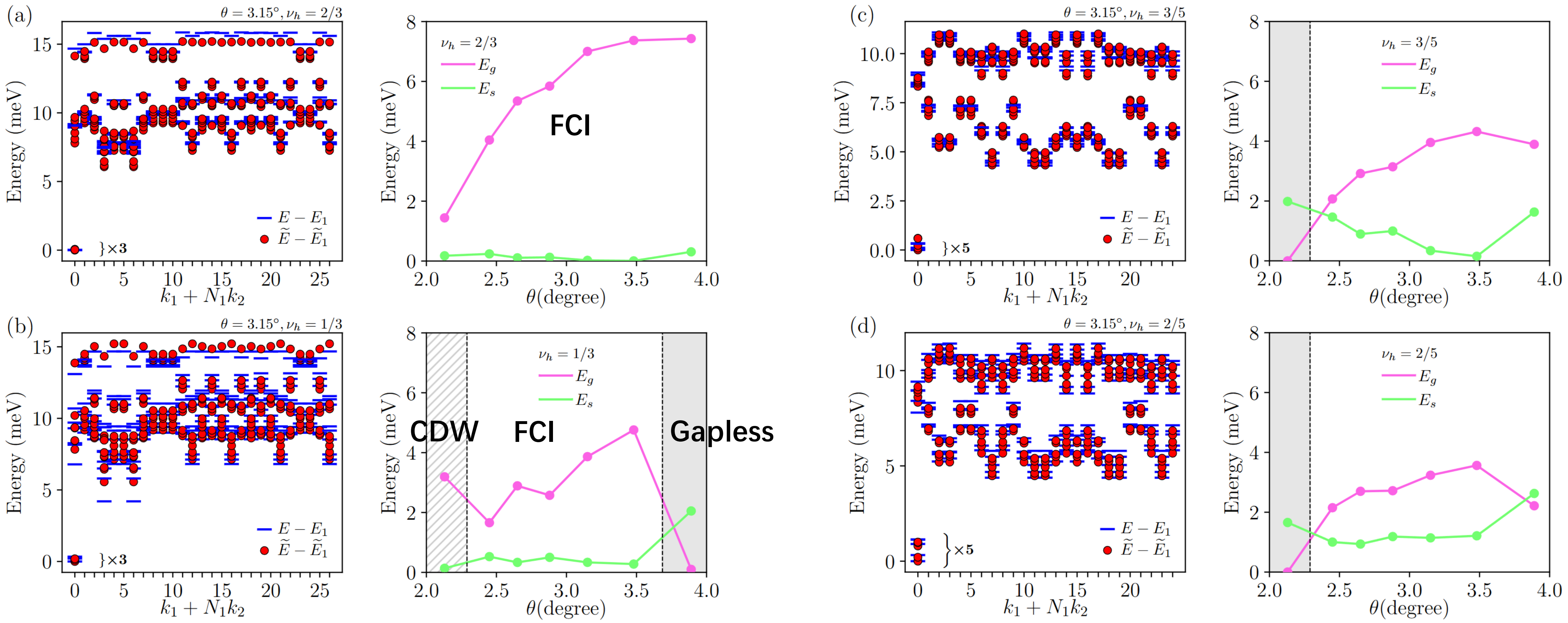


Modulation Function $\mathcal{B}(\boldsymbol{r})$

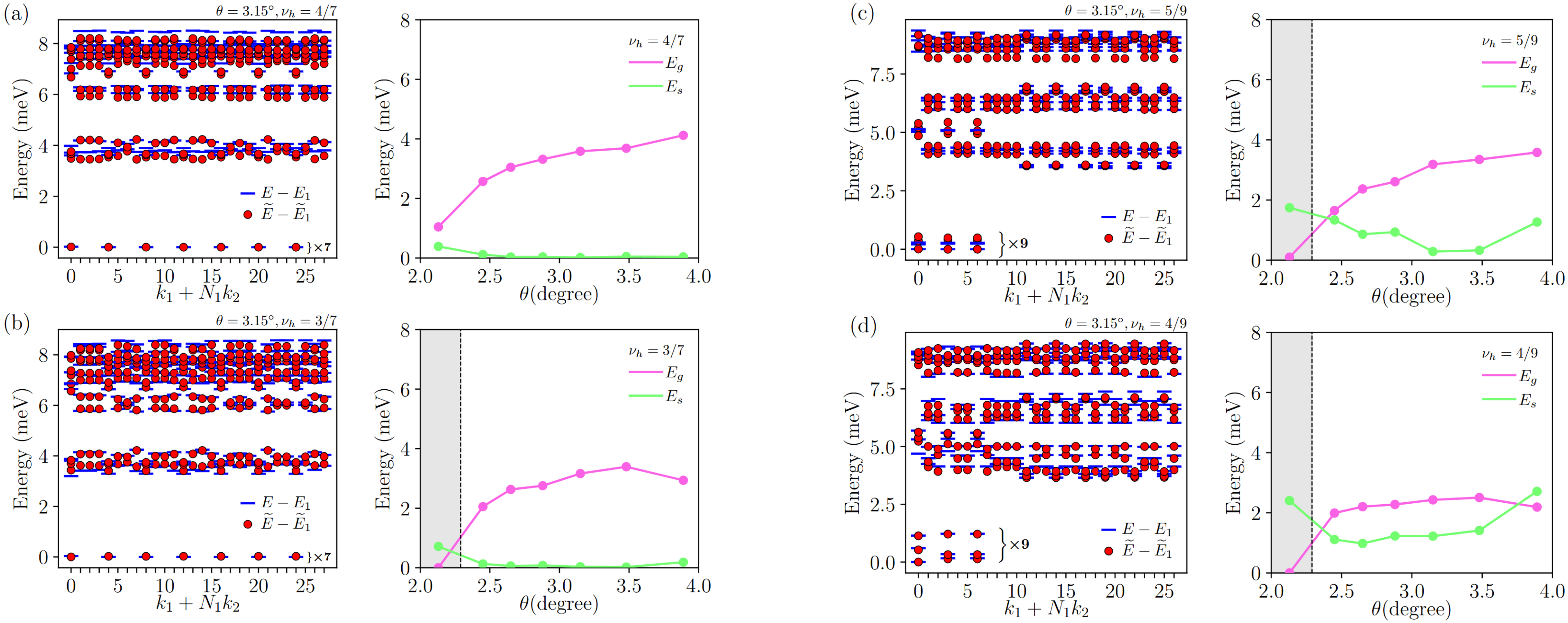


$$\psi_{+,m,\boldsymbol{k}}(\boldsymbol{r}) = \sum_n \sum_{s=\pm} c^{(s)}_{m,n,\boldsymbol{k}} \Theta^{(s)}_{n,\boldsymbol{k}}(\boldsymbol{r})$$

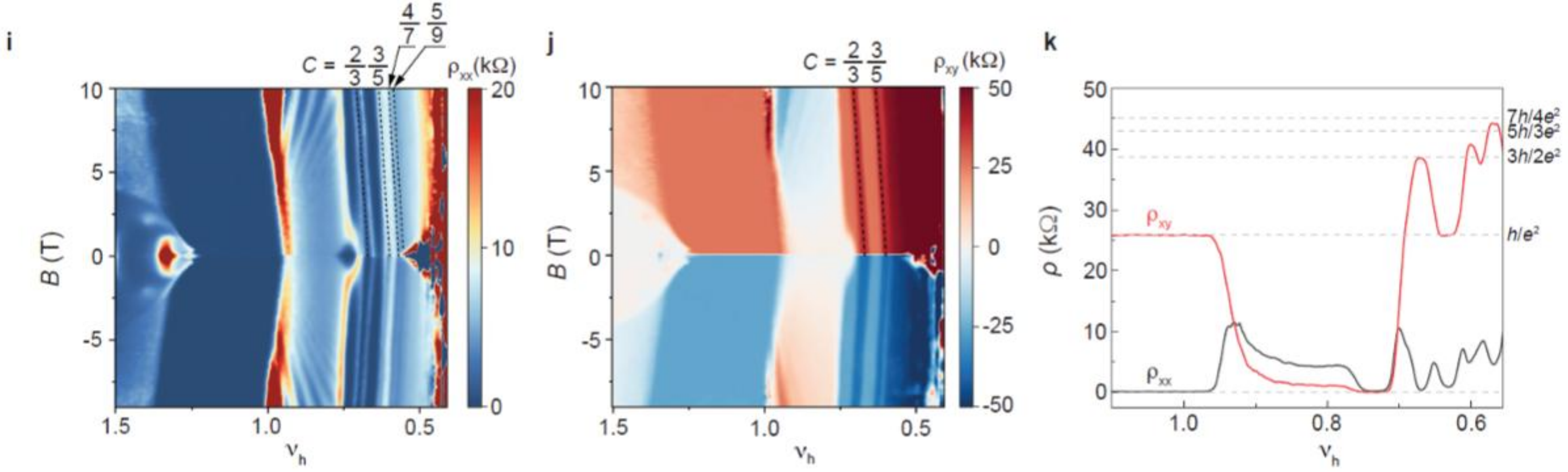
Abelian Fractionalized States in the 1st Moiré Band



Abelian Fractionalized States in the 1st Moiré Band



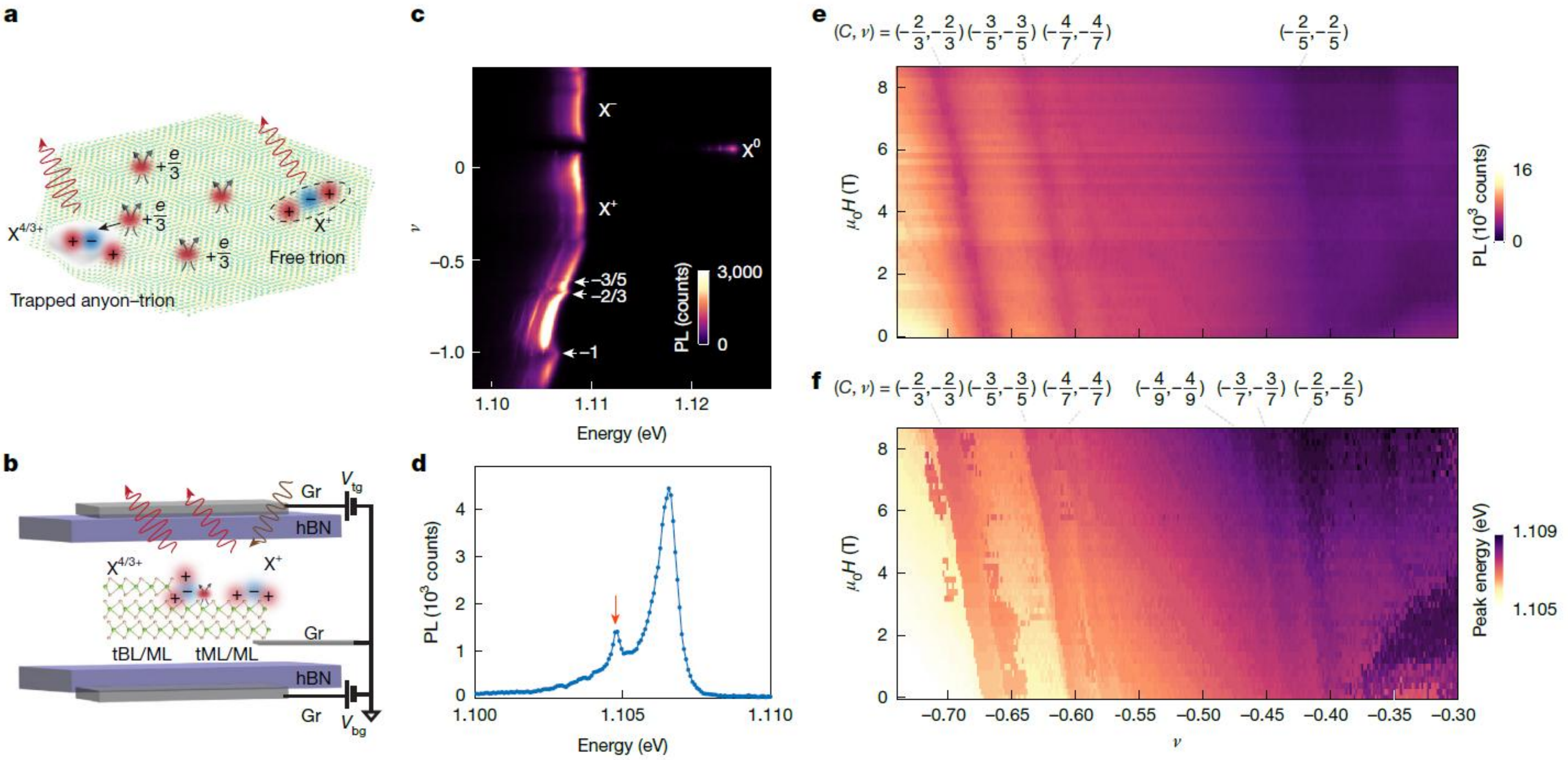
Transport Measurement



- FQAH effects at 2/3, 3/5, 4/7 fillings down to zero B field.
- FCI state at 5/9 stabilized by a finite B field.
- Twist angle ~ 3.83°

F. Xu, …, T. Li, arXiv:2504.06972

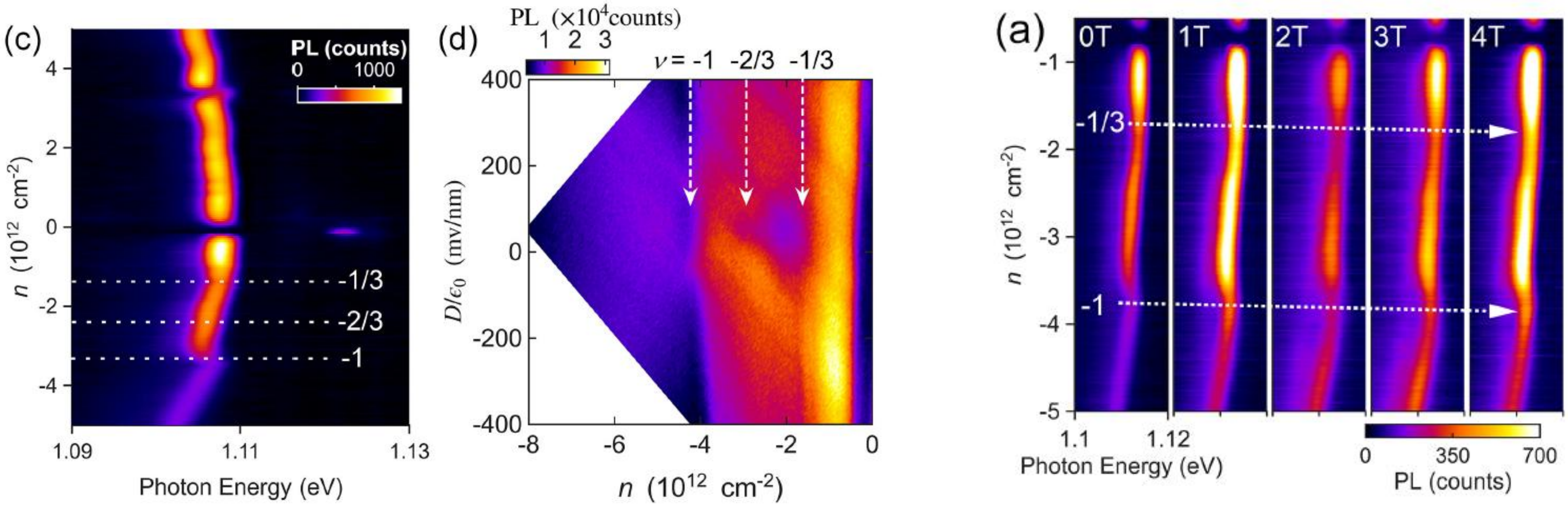
Optical Signatures



- Optical signatures of FCIs at Jain sequences: 2/3, 3/5, 4/7, 4/9, 3/7, 2/5
- Twist angle ~ 3.3°

W. Li, …, X. Xu, Nature 651, 48 (2026)

Optical Signature of FCI at 1/3 filling

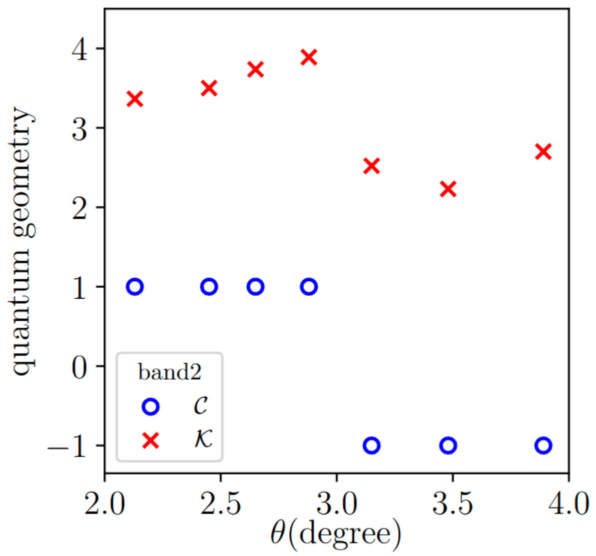


Twist angle ~ 3.5°

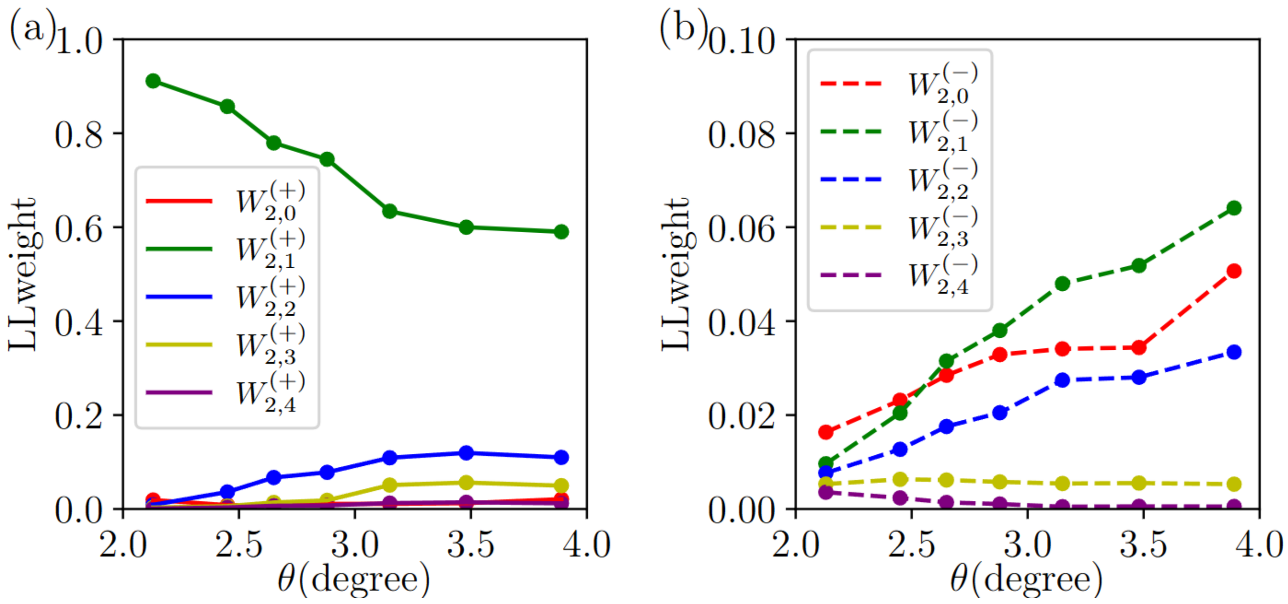
H. Pan, … , W. Gao, Phys. Rev. Lett. 136, 056601 (2026)

2nd Moiré Band ~ Generalized 1LL

- Quantum Geometry



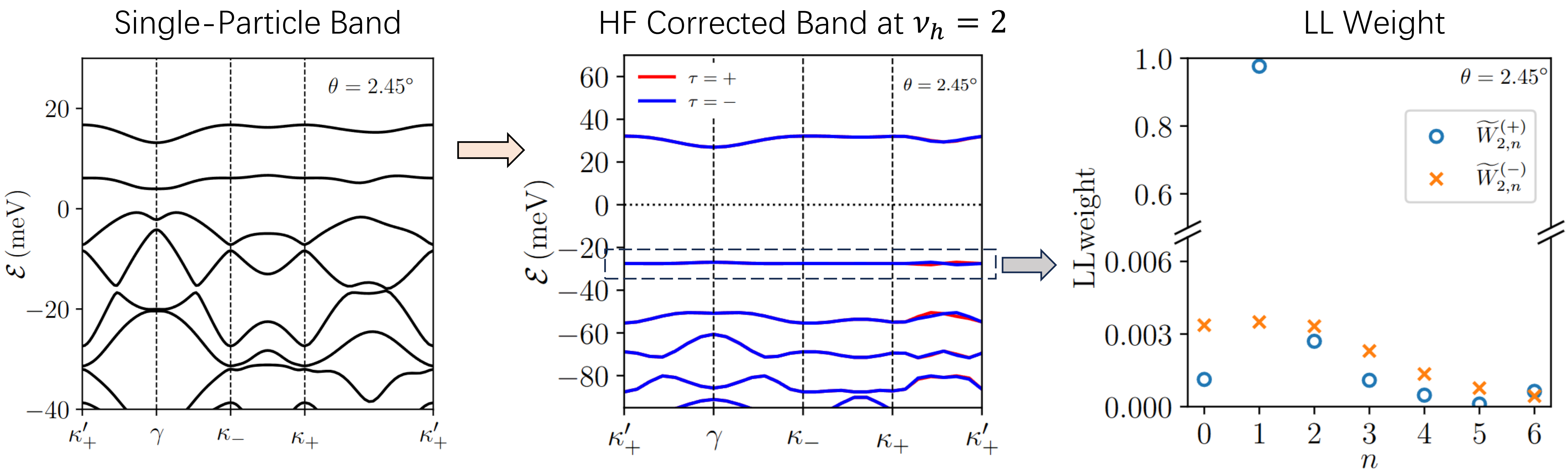
- Generalized Landau Level weights



- For the second moiré band, $\mathcal{K} \approx 3\,\mathcal{C}$ near 2° twist angle.
- It is dominated by the generalized 1st Landau level.

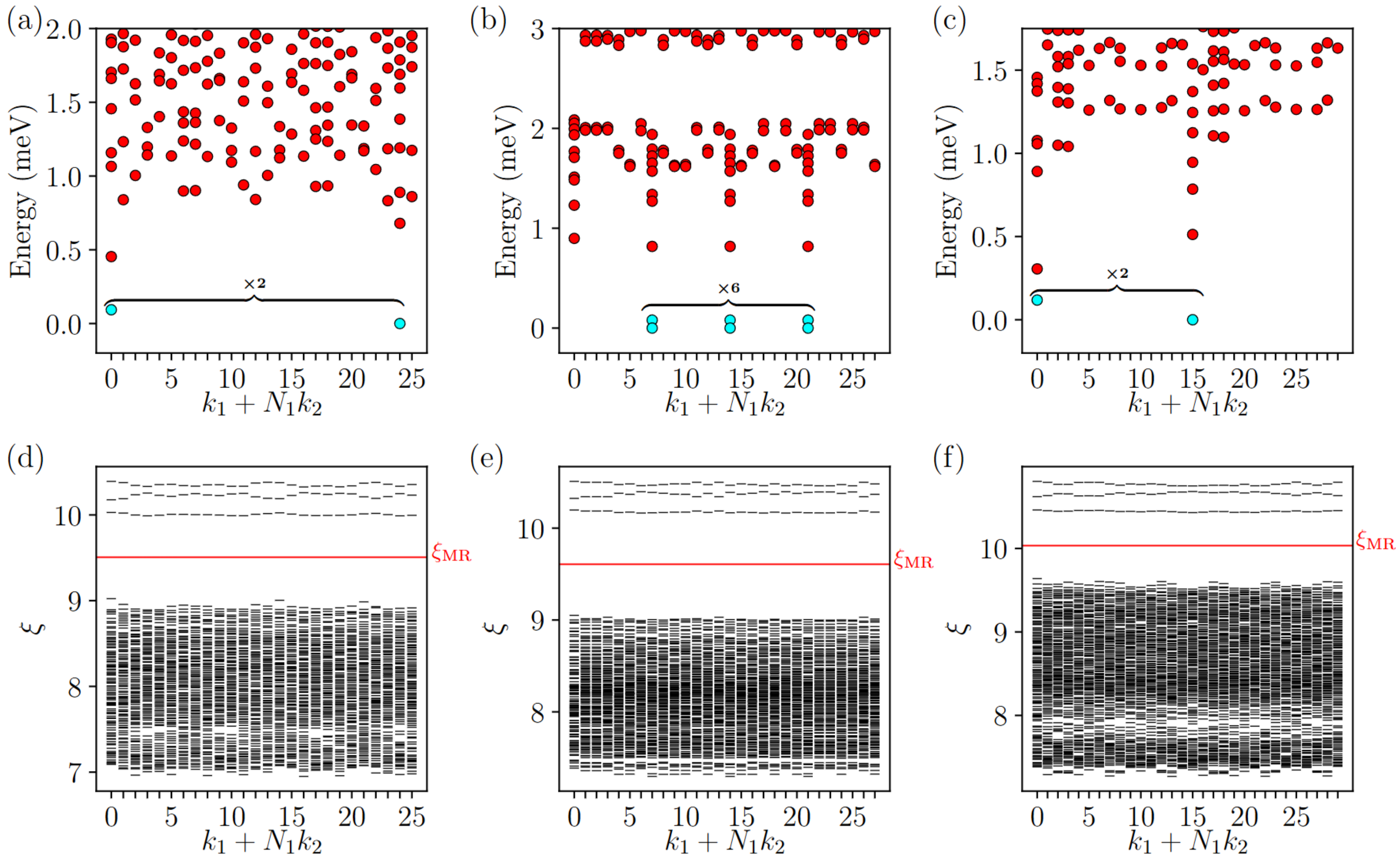
B. Li, Y. Ouyang, F. Wu, Phys. Rev. B 113, 195129 (2026) [Editors' Suggestion]

2^{nd} Moiré Band (corrected by Hartree-Fock) ~ Generalized 1LL



- After the HF correction at $\nu_h = 2$, the 2^{nd} band becomes flatter and more close to the generalized 1^{st} LL.

Non-Abelian Fractionalized States at $\nu_h = 5/2$

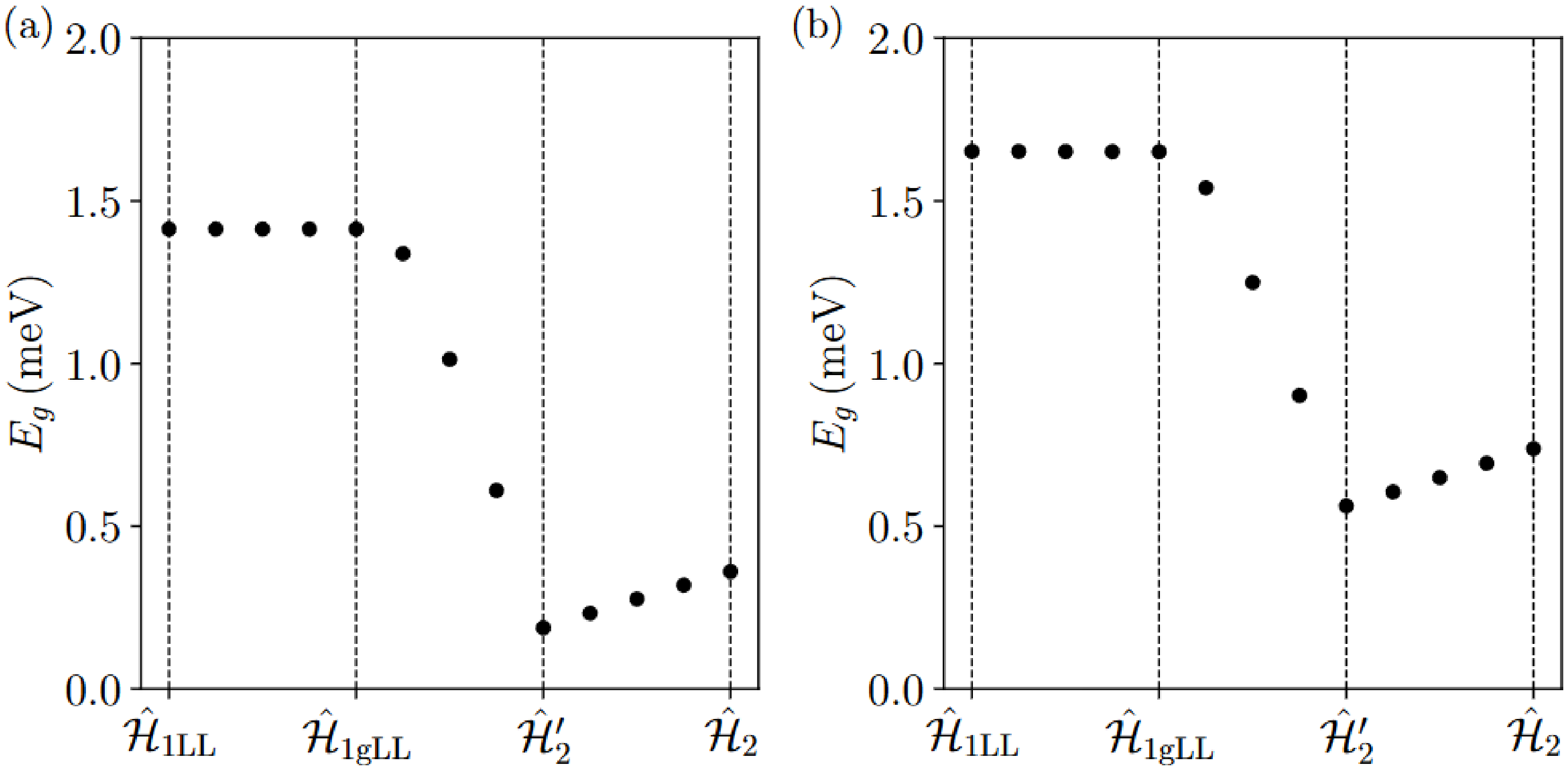


Exact Diagonalization Spectrum

Particle Entanglement Spectrum

See also C.-E. Ahn et al., PRB 110, L161109 (2024); A. P. Reddy et al., PRL 133, 166503 (2024); C. Xu et al., PRL 134, 066601 (2025); C. Wang et al., PRL 134, 076503 (2025); F. Chen et al., Nat. Comm. 16, 2115 (2025)

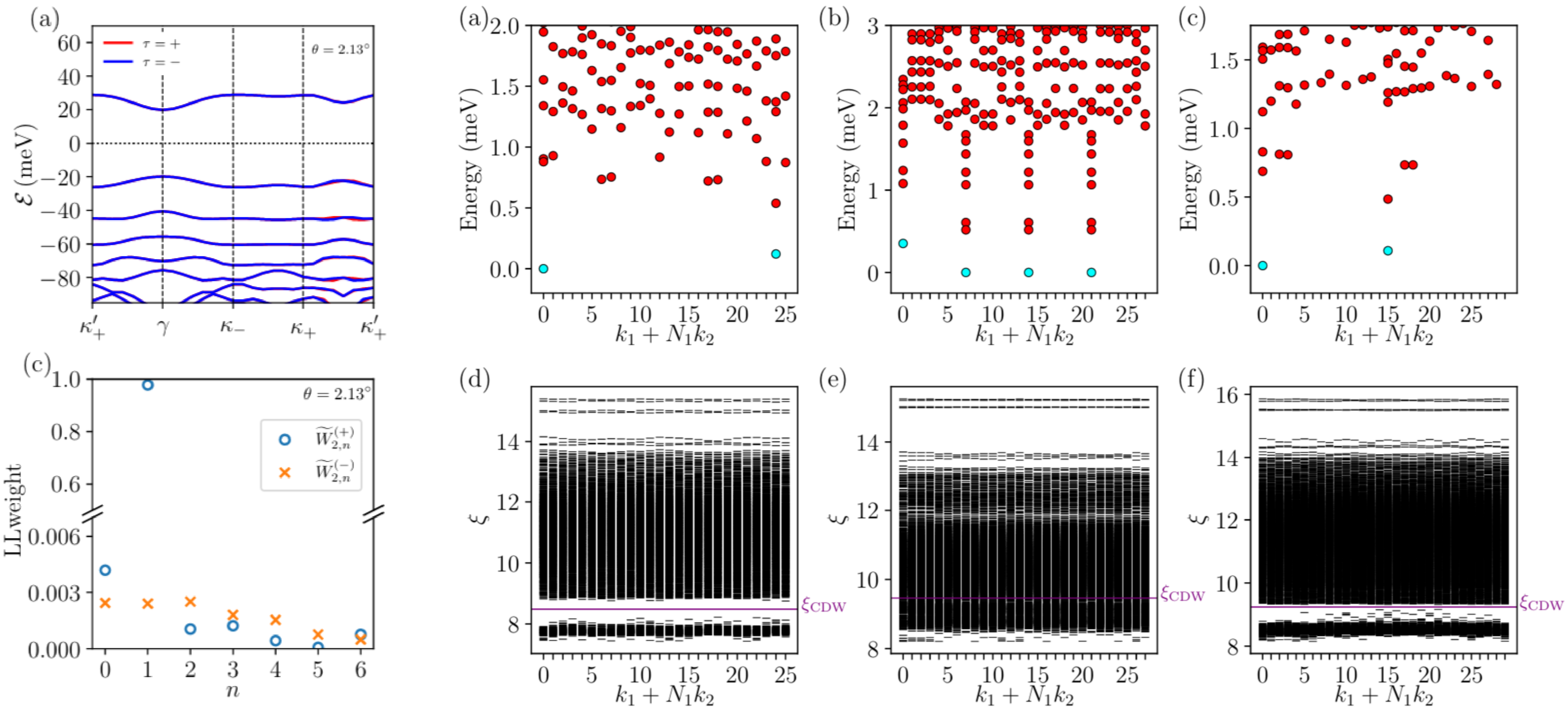
Non-Abelian Fractionalized States at $\nu_h = 5/2$



Adiabatic evolution between the Moore-Read state in the 1st LL and that in tMoTe2

B. Li, Y. Ouyang, F. Wu, Phys. Rev. B 113, 195129 (2026) [Editors' Suggestion]

Competitor: Charge Density Wave State



The Moore-Read state is fragile and sensitive to details.

Summary

- The 1st moiré band in tMoTe2 closely resembles the generalized zeroth LL
  ➔ Abelian FCI in the Jain sequences $\nu_h = \frac{p}{2p+1}$ and $\frac{p+1}{2p+1}$

- The 2nd moiré band in tMoTe2 closely resembles the generalized 1st LL near $\theta = 2.45$
  ➔ non-Abelian FCI at $\nu_h = \frac{5}{2}$

- Quantum Geometry + Topology + Interaction ➔ Exotic Physics

Thank You!