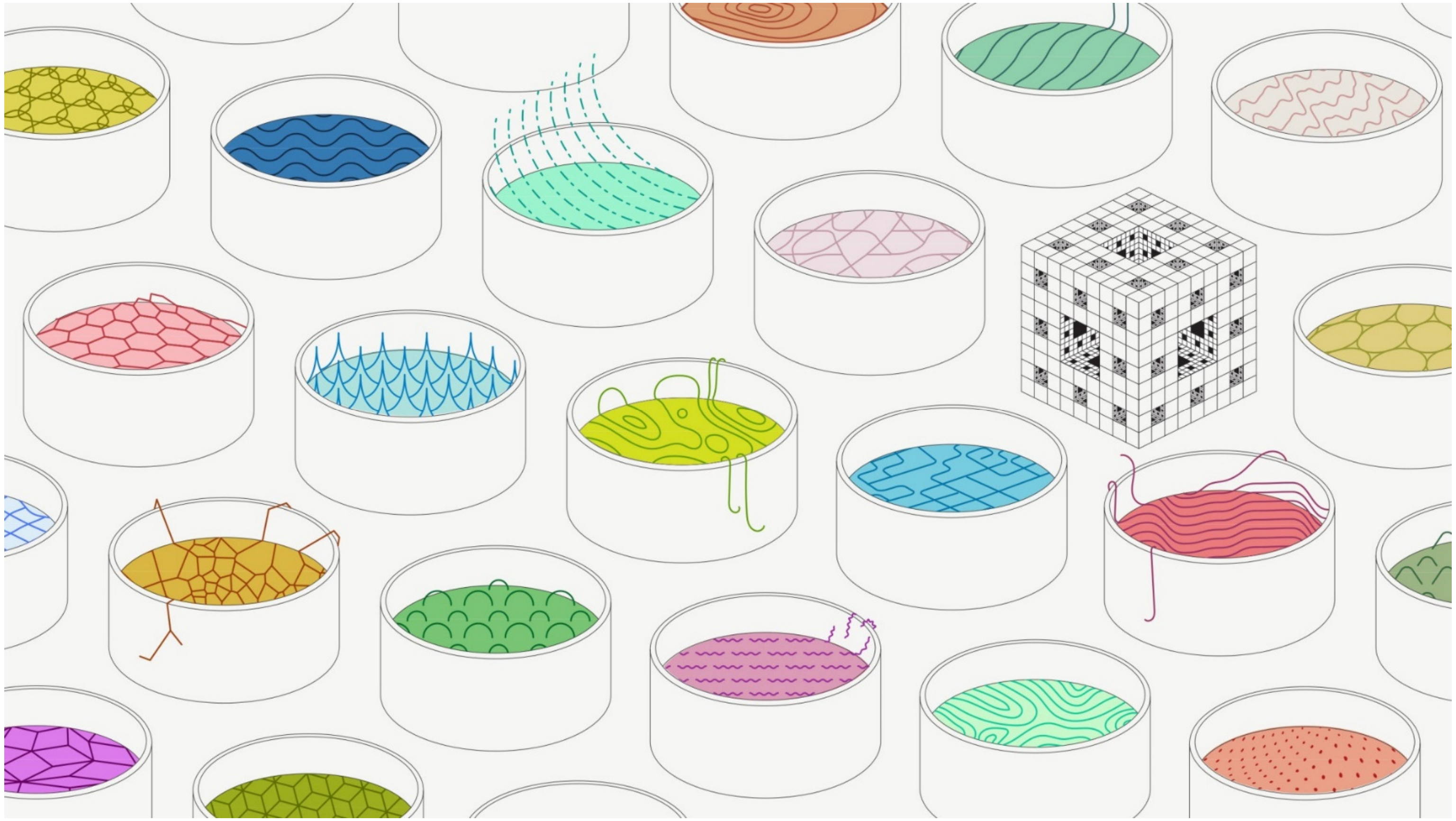
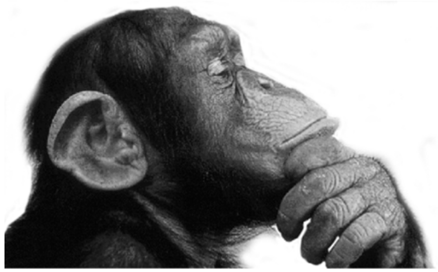
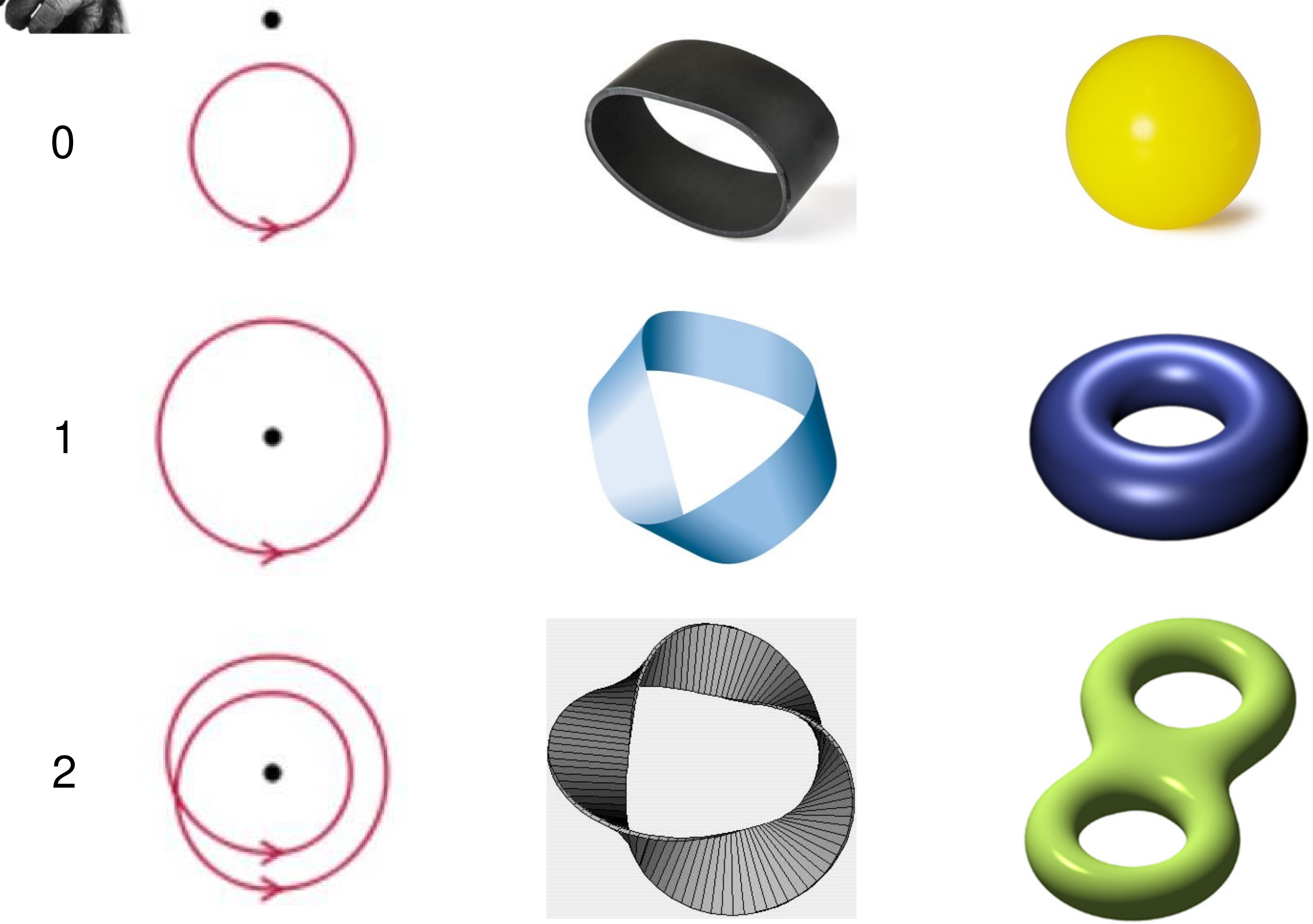


Topology in solid state physics, a historical overview



Topology

the property of an object that is invariant under continuous deformation



topological invariant: winding number twist genus (hole)

- Gauss-Bonnet theorem (for a 2D closed surface)

$$\int_M d^2a\, G(\vec{r}) = 2\pi\chi, \quad \chi = 2(1-g)$$

Euler characteristic 歐拉特徵數
(a topological invariant)

虧格 $g=0$

$g=1$

$g=2$

- 2D surface with boundary

$$\int_M da\, G + \int_{\partial M} ds\, k_g = 2\pi\,\chi(M,\partial M)$$

- Can be generalized to higher (even) dimension.

Marder, Phys Today, Feb 2007

2016 Nobel prize

Michael Kosterlitz, David Thouless and Duncan Haldane.

"For theoretical discoveries of topological phase transitions and topological phases of matter."

拓樸相變和拓樸物質的理論研究

"There are two types of work that win Nobel Prizes. One involves specific achievements, … the other is for a set of new ideas that gradually become more important in the way we think. The 2016 prize belongs to the latter group."

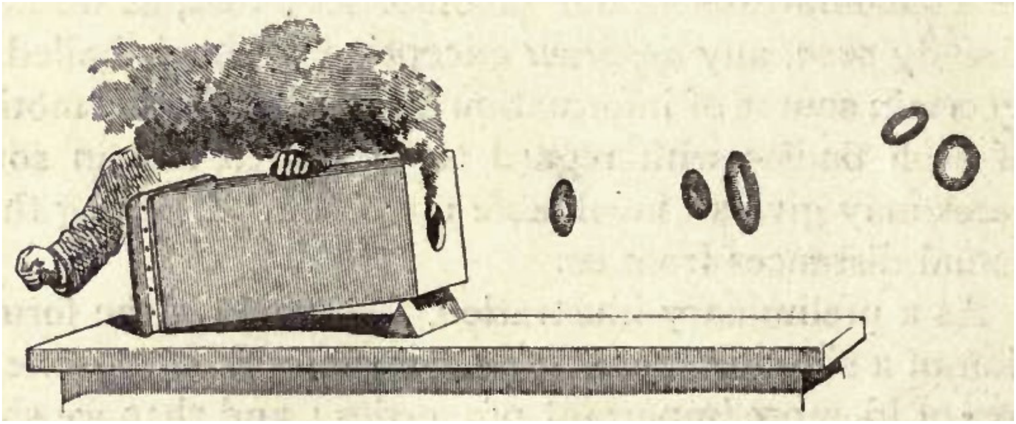
J. Chalker

APS news (Apr, 2016)
This month in physics

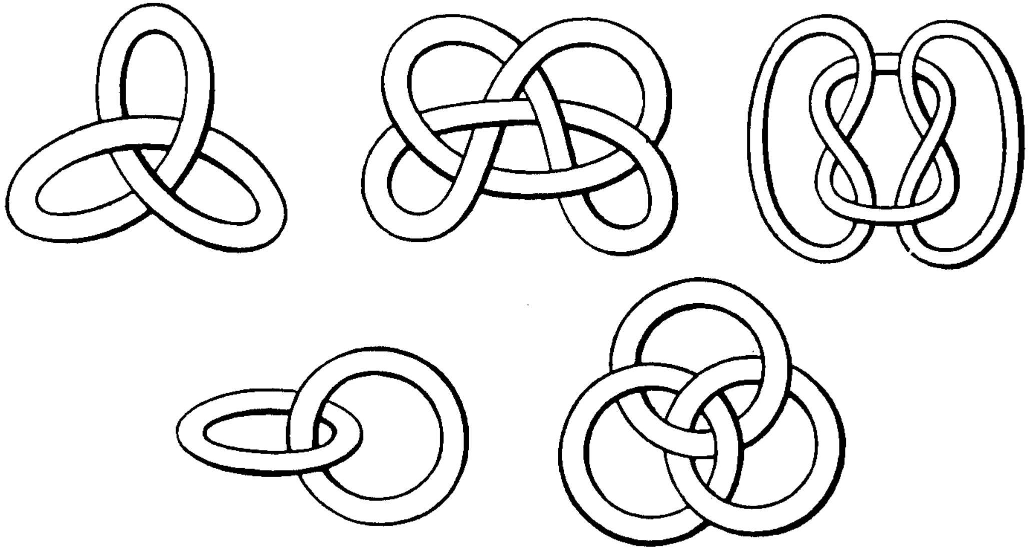
Precursor

Hydrodynamics theory of vortex (Helmholtz)

Smoke ring experiment (P.G. Tait): the smoke rings were remarkably stable, traveling across the room without dissipating.

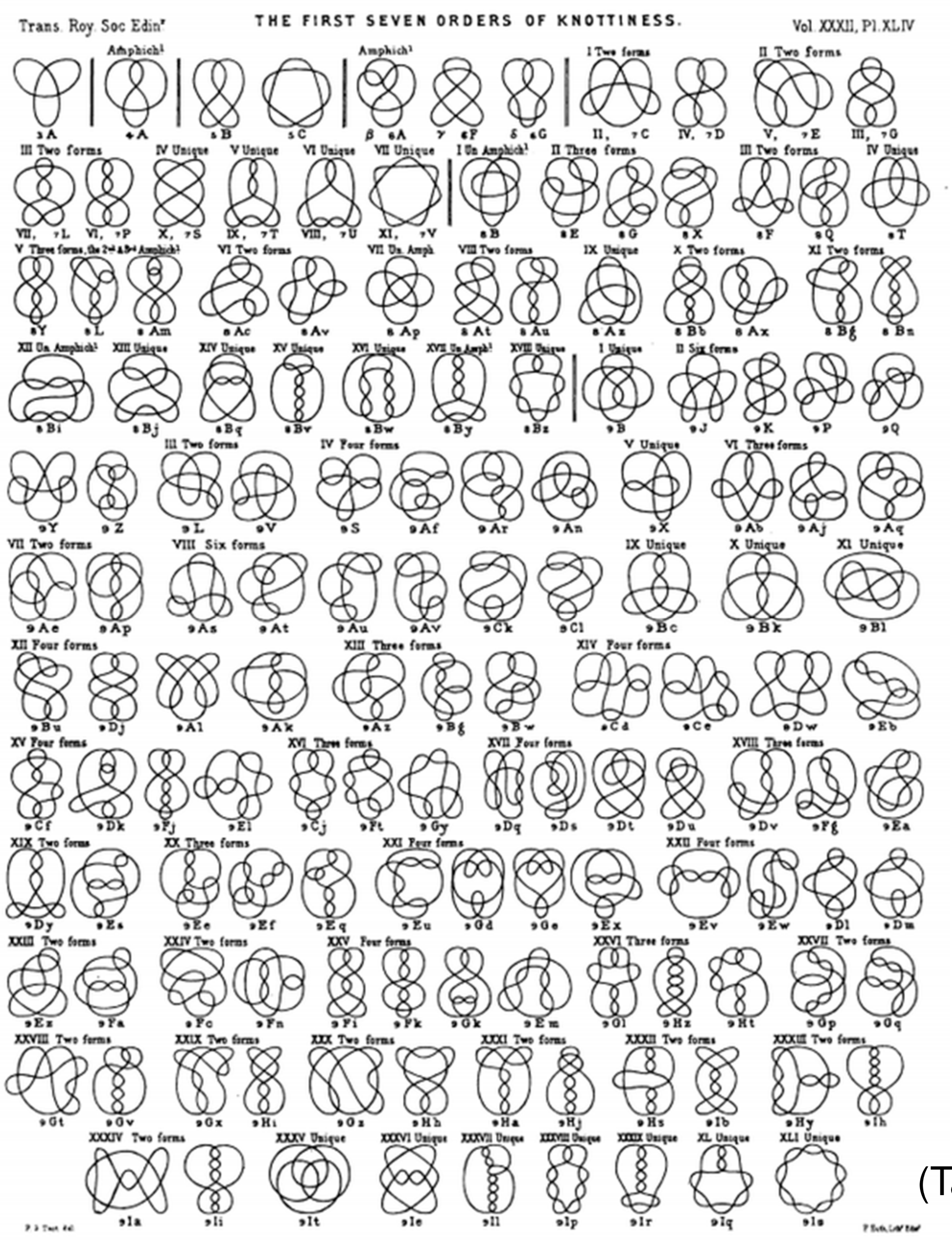


What if atoms are knotted vortices of the ether?
(Lord Kelvin, 1867)



“The Victorian theory of everything”

Vortex atoms:
Periodic table
of elements

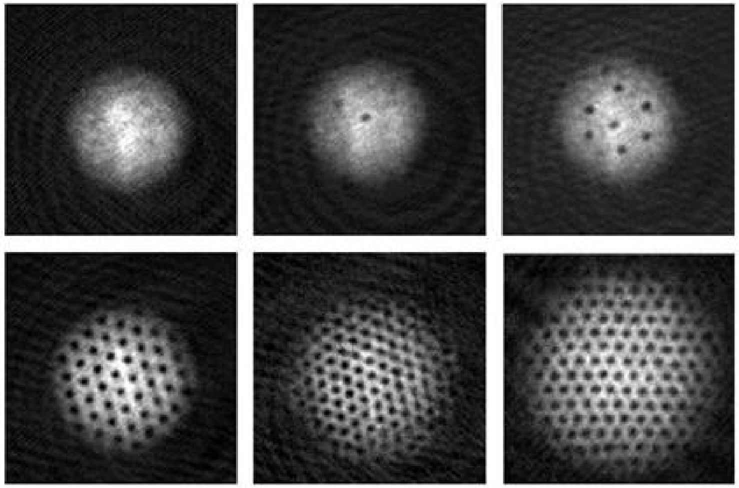


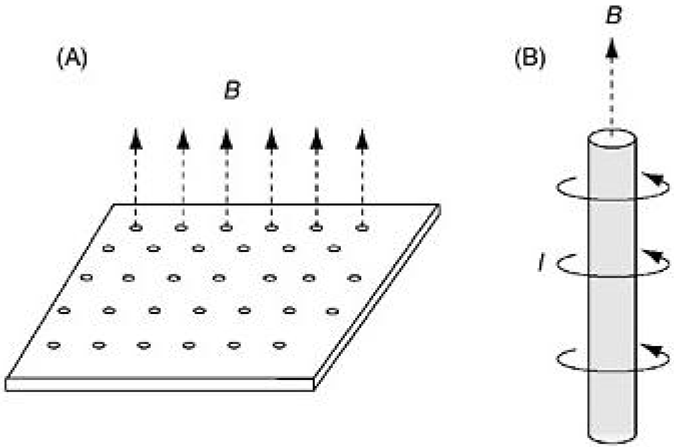
(Tait, 1885)

Unfortunately,
there is no ether!
(Michelson-Morley
experiment, 1887)

Before 1970

- Quantum magnetic monopole (Dirac, 1931)
- Quantum vortex in superfluid (Onsager, 1947)
- Quantum vortex in superconductor (Abrikosov, 1957)
- …



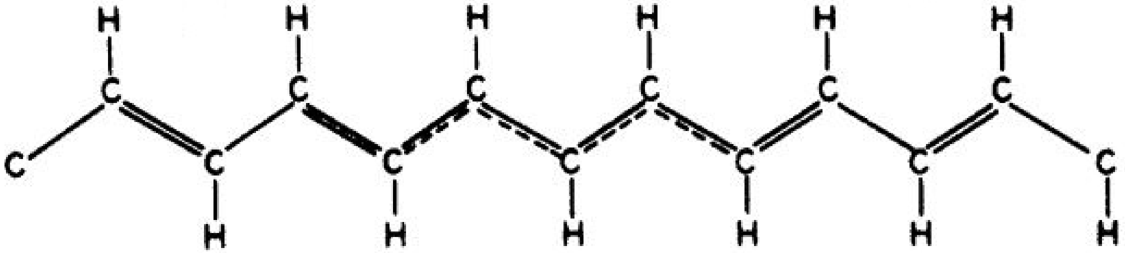


1970's

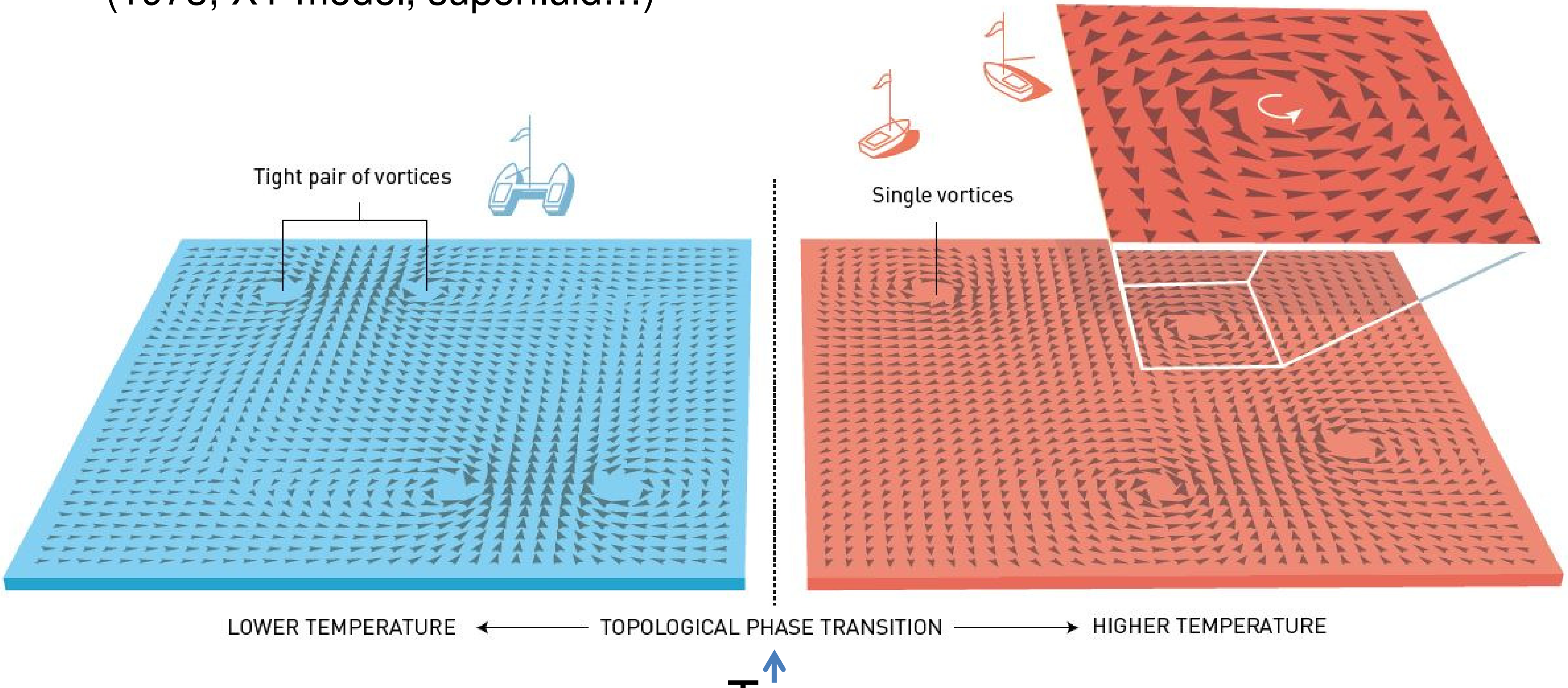
- Kosterlitz-Thouless transition (Kosterlitz and Thouless, 1973)
- Non-Abelian magnetic monopole ('t Hooft, Polyakov 1974)
- Instanton (Belavin et al, 1975) 瞬子
- Gauge theory and fiber bundle (Wu and Yang, 1975) 纖維束
- Defect in (liquid-)crystal (point, line, disclination…) 向錯

See Chainkin and Lubensky

- Soliton with charge ½ (Jackiw and Rebbi, 1976) 固子
- Soliton in SSH model (Su, Schrieffer and Heeger 1979)
- …



Kosterlitz-Thouless transition in 2D
(1973, XY model, superfluid…)



T_{KT}

- Vortex-antivortex binding phase

(quasi-long-range order)

No order parameter

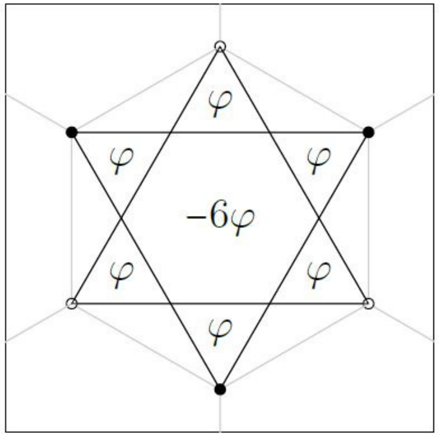
- vortex unbinding phase

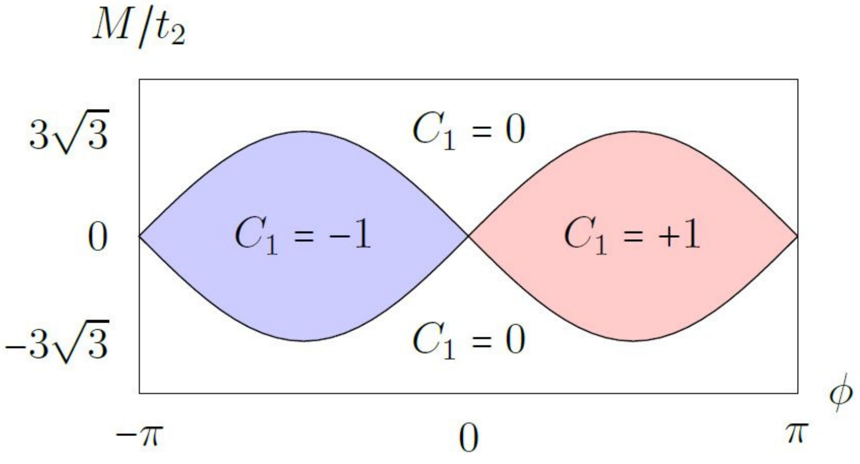
(no order)

Note: O(3) in 2D, superconductor in 2D: No order, no phase transition
(see p.9, p.16 40 years of KT transition)

1980’s

- Discovery of IQHE (1980, von Klitzing) and FQHE (Tsui et al, 1982)
- Fractional statistics in 2D: anyon (Leinaas, 1977; Wilczek, 1982) 任意子
- Energy gap in spin-1 chain (Haldane, 1983)
- TKNN theory of IQHE (Thouless et al, 1984)
- Theory of geometric phase (Berry, 1984)
- Anyon with fractional charge in FQHS (Arovas, Schrieffer, and Wilczek, 1985)
- Haldane graphene model (Haldane, 1989)
- Topological field theory (Witten, 1988)
- …



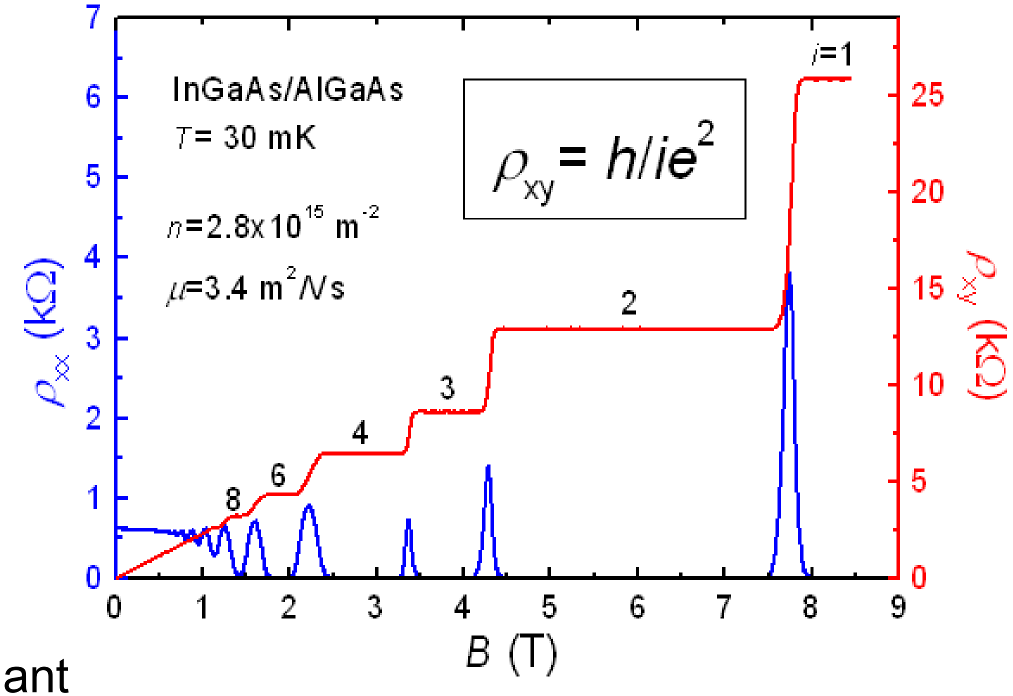


Fruchart and Carpentier, 2013

Hall conductivity as a topological invariant

Thouless et al (1982)

- The Hall conductivity



- Integer topological invariant in *momentum space*

$$\sigma_H = \frac{e^2}{h} C, \quad C \in Z \quad \left(\sigma_H = \frac{1}{\rho_{xy}} \right)$$

$$\boxed{C = \frac{1}{2\pi} \int_{BZ} d^2k \, \Omega_z(\vec{k}) \quad \in Z}$$

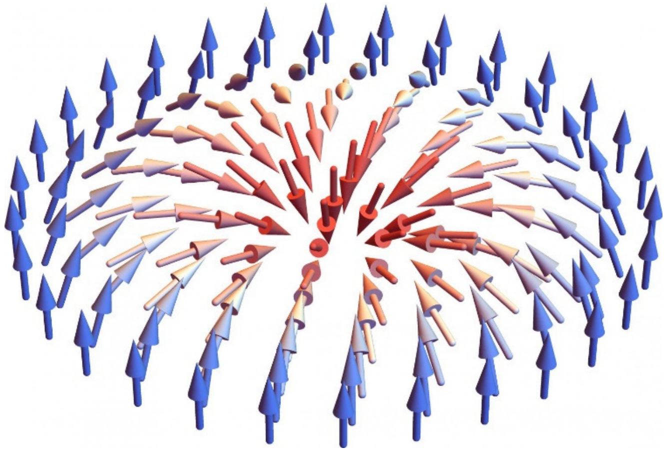
Cf: Gauss-Bonnet theorem

$$\chi = \frac{1}{2\pi} \int_M d^2a \, G(\vec{r}) \quad \in Z$$

Euler characteristic

1990's

- Ground state degeneracy in FQHS (Wen and Niu, 1990)
- Moore-Read state (spin-polarized p-wave pairing) in FQHS (Moore and Read, 1991)
- Skyrmion in QHS (Skyrme, 1962; Sondhi et al, 1993) 斯格明子
- Modern theory of polarization (Resta, King-Smith and Vanderbilt, 1993)
- Berry curvature in semiclassical electron dynamics (Chang and Niu, 1995)
- Fault-tolerant quantum computation by anyons (Kitaev, 1997)
- …



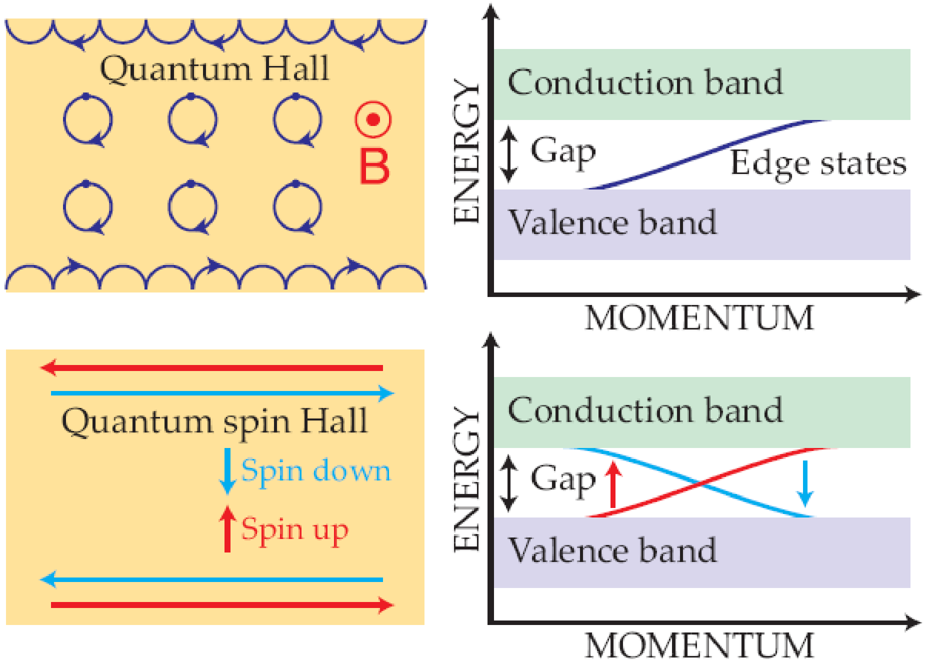
2000's

- Quantum spin Hall effect (Kane and Mele, 2005)
- Theory of topological insulator (Fu and Kane, 2006)

馬約拉納
費米子

- Majorana fermion in hybrid TI-SC (Majorana, 1937; Fu and Kane 2008)
- Topological photonics (Haldane and Raghu; Wang et al, 2008),

  topological phononics (Prodan and Prodan, 2009)
- Classification of TI/TSC (Schnyder et al, Kitaev, 2009)
- Magnetic skyrmion in MnSi (Mühlbauer et al, 2009)
- …



The garden of *electronic* topological phases

- **Symmetry-protected topological (SPT) phase**

	Fermion	**Boson**
Non-interacting	• Integer quantum Hall effect • Topological insulator • … ✓	• The topology in systems with photon, phonon, or magnon …
Interacting	• Chern insulator •	• Haldane spin chain • …

- **Topological-order phase**

Strongly Interacting

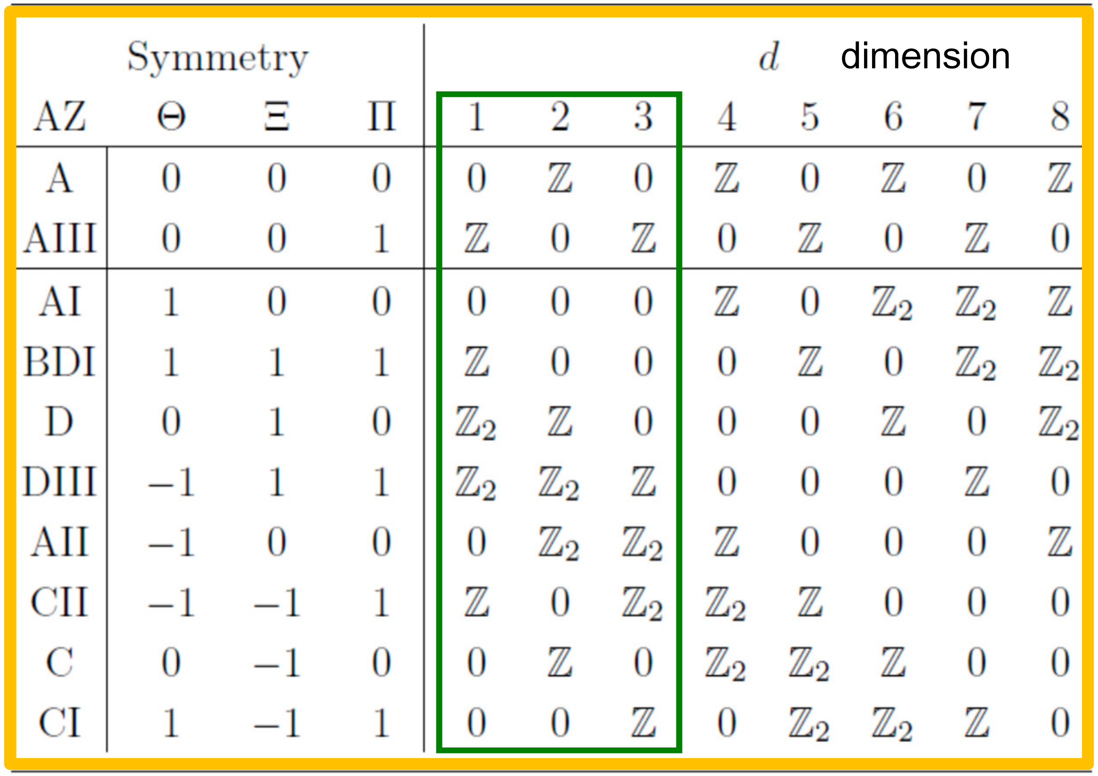
- Fractional quantum Hall effect
- Toric code
- …

(***degenerate GND state, fractional QP, long-range entanglement***)

Non-interacting fermions

The periodic table of topological phases!

(Kitaev, AIP Conf Proc 2009)



Symmetry				d dimension							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	−1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

IQHE, QAHE

π-flux state

polyacetylene

Kitaev chain, chiral p-wave **(spinless)**

helical p-wave **(spinful),** **He3**

2D/3D TI

d+id, d-id SC

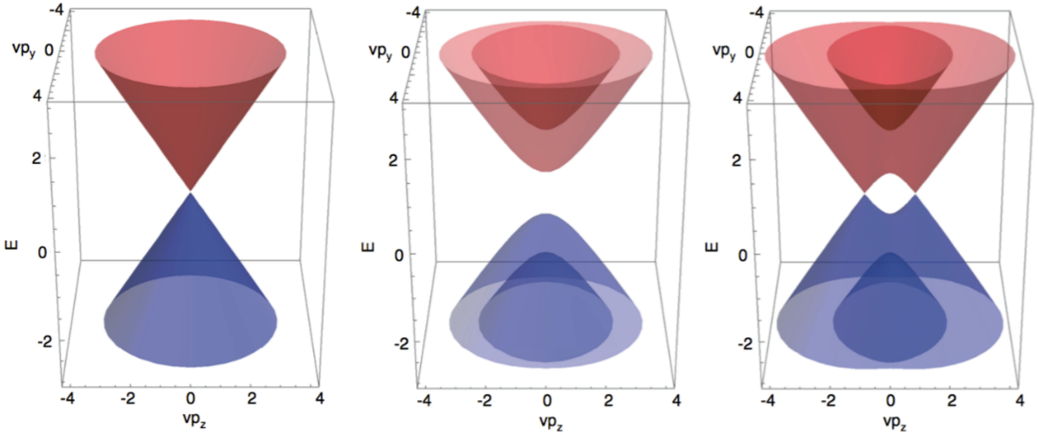
d_{xy}, d_{x2-y2} singlet SC

- Altland and Zirnbauer
- Ryu, Schnyder, Furusaki, and Ludwig

"***Topology as an organizing principle for thinking about matter***." C. Kane

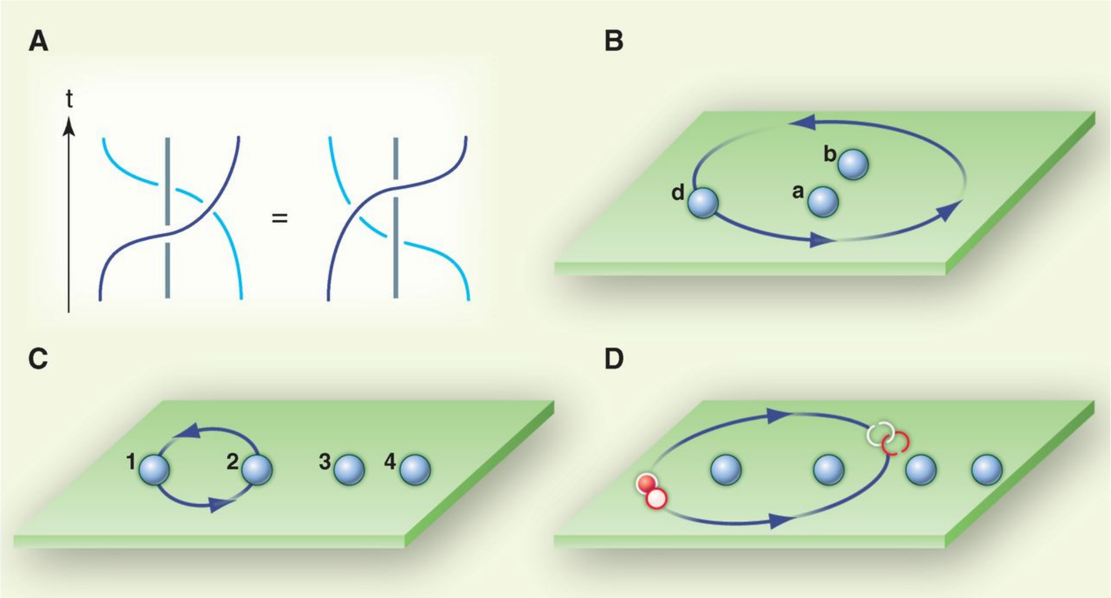
2010’s

- Weyl semi-metal (Wan et al, Burkov and Balent, 2011)
- TCI, higher-order TI, symmetry indicator (Fu, Bernevig 2011 …)
- Topological mechanics (Kane and Lubensky, 2014)
- Classification of the topology of interacting systems (… Wen…)
- …



2020 and beyond

- Quantum device using majorana fermion
- Topological quantum computation
- …



A bit of the action

In the race to build a quantum computer, companies are pursuing many types of quantum bits, or qubits, each with its own strengths and weaknesses.

	Superconducting loops	Trapped ions	Silicon quantum dots	Topological qubits	Diamond vacancies
	A resistance-free current oscillates back and forth around a circuit loop. An injected microwave signal excites the current into superposition states.	Electrically charged atoms, or ions, have quantum energies that depend on the location of electrons. Tuned lasers cool and trap the ions, and put them in superposition states.	These "artificial atoms" are made by adding an electron to a small piece of pure silicon. Microwaves control the electron's quantum state.	Quasiparticles can be seen in the behavior of electrons channeled through semiconductor structures. Their braided paths can encode quantum information.	A nitrogen atom and a vacancy add an electron to a diamond lattice. Its quantum spin state, along with those of nearby carbon nuclei, can be controlled with light.
Longevity (seconds)	0.00005	>1000	0.03	N/A	10
Logic success rate	99.4%	99.9%	~99%	N/A	99.2%
Number entangled	9	14	2	N/A	6
Company support	Google, IBM, Quantum Circuits	ionQ	Intel	Microsoft, Bell Labs	Quantum Diamond Technologies
⊕ **Pros**	Fast working. Build on existing semiconductor industry.	Very stable. Highest achieved gate fidelities.	Stable. Build on existing semiconductor industry.	Greatly reduce errors.	Can operate at room temperature.
⊖ **Cons**	Collapse easily and must be kept cold.	Slow operation. Many lasers are needed.	Only a few entangled. Must be kept cold.	Existence not yet confirmed.	Difficult to entangle.

Science, 354, 1090 (2016)