Notes on Conformal Field Theory

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I. INTRODUCTION

Critical phenomena near phase transitions constitute a cornerstone of modern physics with profound implications across condensed matter physics, statistical mechanics, quantum field theory (QFT) and quantum gravity. Continuous transitions and related critical phenomena exhibit a remarkable property of universality, generally characterized by a set of critical exponents that describe non-analytic behavior of physical observables near critical points. For example, the liquid-gas transition shares the same universality class as the three-dimensional Ising magnetic transition. Universality offers extensive applications of the Ising model and general theories of critical phenomena, reaching fields such as pure mathematics, biology, earth science, deep learning, economics, just to name a few.

Regarding the field theory description of critical phenomena, distinct universality classes correspond to distinct renormalization group (RG) fixed points. Remarkably, at these RG fixed points, the usual Poincare symmetry (comprising Lorentz and translation symmetries) and scale invariance often enhance into a larger spacetime symmetry—conformal symmetry—first explicitly uncovered by Polyakov through the exact solution of the two-dimensional Ising transition. Conformal symmetry preserves angles but not necessarily distances, signifying invariance under non-uniform RG transformations. Beyond its elegance, conformal symmetry has proven instrumental in solving critical phenomena. In two dimensions, conformal symmetry becomes infinite-dimensional, rendering numerous theories exactly solvable and thus giving birth to conformal field theory (CFT), which significantly impacts diverse areas from condensed matter physics to quantum gravity. In higher dimensions (e.g., 3D), conformal symmetry is less restrictive, yet modern developments such as the conformal bootstrap method have demonstrated that it remains highly effective for numerically determining precise critical exponents.

What is the purpose to study the conformal symmetry and CFT? A quick answer is it is simple and useful.

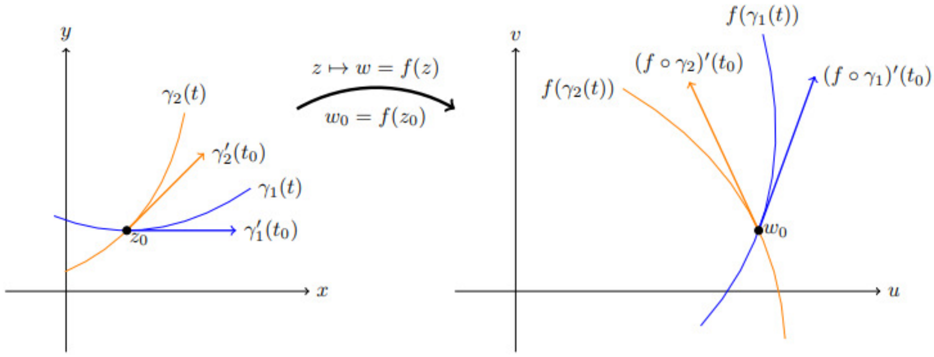

FIG. 1: A conformal map rotates and scales all tangent vectors at z_0 by the same ammount.

II. PRELIMINARY

A. Conformal Transformations

A conformal mapping, also called a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation $w = f(z)$ that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative. Conversely, any conformal mapping of a complex variable which has continuous partial derivatives is analytic. Remarks. 1) Conformality is a local phenomenon. At a different point z_1 the rotation angle and scale factor might be different. 2) Since rotations preserve the angles between vectors, a key property of conformal maps is that they preserve the angles between curves. For example, Figure 1 shows a conformal map $f(z)$ mapping two curves through z_0 to two curves through $w_0 = f(z_0)$. The tangent vectors to each of the original curves are both rotated and scaled by the same amount. The details about conformal transformation please refer to any book on *Complex Analysis*.

A fractional linear transformation is a function of the form

$$f(z) = \frac{az + b}{cz + d}, \tag{1}$$

where a, b, c, d are complex constants and $|ad - bc| \neq 0$. Note that a linear fractional transformation maps lines and circles to lines and circles.

- Scale and rotate. Let $f(z) = az$. If $a = r$ is real this scales the plane. If $a = e^{i\theta}$ it rotates the plane. If $a = re^{i\theta}$ it does both at once. f is the fractional linear transformation with coefficients $\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} = \begin{pmatrix} \mathrm{a} & 0 \\ 0 & 1 \end{pmatrix}$.

- Scale and rotate and translate. Let $f(z) = az + b$. Adding the b term introduces a translation to the previous example: $\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ 0 & 1 \end{pmatrix}$.

- Inversion. Let $f(z) = 1/z$. This is called an inversion. It turns the unit circle inside out. Note that $f(0) = \infty$ and $f(\infty) = 0$. In the figure below the circle that is outside the unit circle in the z plane is inside the unit circle in the w plane and vice-versa. Note that the arrows on the curves are reversed. f is the fractional linear transformation with coefficients $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Especially, we note that for the inversion w = 1/z. 1. Any line not through the origin is mapped to a circle through the origin. 2. Any line through the origin is mapped to a line through the origin. 3. Any circle not through the origin is mapped to a circle not through the origin. 4. Any circle through the origin is mapped to a line not through the origin.

- Let $f(z) = \frac{z-i}{z+i}$. We claim that this maps the x-axis to the unit circle and the upper half-plane to the unit disk. Similary, $f(z) = \frac{z-1}{z+1}$ maps the right half plane to the unit disk.

Besides fractional linear transformation, one could check the transformation $f(z) = e^z$ and $f(z) = Lnz$.

Example.— We want to compute

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} \,. \tag{2}$$

The standard result is $I = \pi$, but here we obtain it by mapping the real line to the unit circle.

Consider the Möbius transformation

$$w = \frac{z - i}{z + i} \,, \tag{3}$$

where $z = x + iy$. This map is an $\mathrm{SL}(2, \mathbf{R})$ transformation that conformally sends the **upper half-plane** $\mathbf{H} = \{z : \mathrm{Im}(z) > 0\}$ onto the **unit disk** $\mathbf{D} = \{w : |w| < 1\}$. The

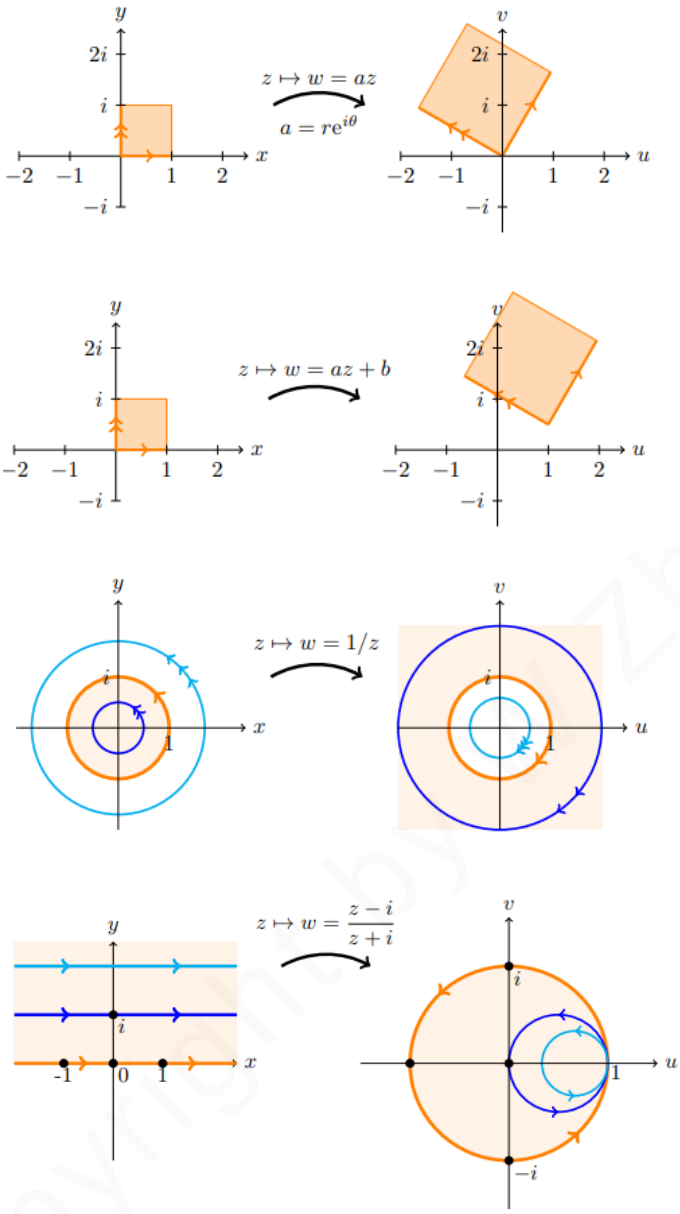

FIG. 2: A conformal map rotates and scales all tangent vectors at z_0 by the same amount.

boundary is mapped as follows: the extended real axis $\mathbf{R} \cup \{\infty\}$ goes exactly onto the unit circle $|w| = 1$. As x runs from $-\infty$ to ∞, w traverses the unit circle counterclockwise:

$$x = -\infty \;\to\; w = 1, \quad x = -1 \;\to\; w = i, \quad x = 0 \;\to\; w = -1, \quad x = 1 \;\to\; w = -i,$$

Invert the map to express z in terms of w: $z = i\,\frac{1+w}{1-w}$. Differentiate:

$$dz = i\,\frac{(1)(1-w) - (1+w)(-1)}{(1-w)^2}\,dw = \frac{2i}{(1-w)^2}\,dw\,. \tag{4}$$

Hence the integrand becomes

$$\frac{dz}{1+z^2} = \left(\frac{(1-w)^2}{-4w}\right)\left(\frac{2i}{(1-w)^2}\right) dw = \frac{-i}{2w}\, dw\,. \tag{5}$$

The original integral over the real line is now a contour integral over the unit circle:

$$I = \oint_{|w|=1} \frac{-i}{2w}\, dw\,. \tag{6}$$

The integrand has a simple pole at $w = 0$ with residue $\mathrm{Res}(1/w) = 1$. Since the contour is traversed counterclockwise,

$$\oint_{|w|=1} \frac{1}{w}\, dw = 2\pi i\,.$$

Therefore

$$I = \frac{-i}{2} \cdot 2\pi i = \pi\,,$$

as expected.

Some more highly non-trivial examples exist, for example:

$$I = \int_{-\infty}^{\infty} \frac{x^4 + 2x^2 + 3}{(1+x^2)^3} dx \tag{7}$$

III. GLOBAL CONFORMAL GROUP

A. d-dimensional conformal transformation

We denote by $g_{\mu\nu}(x)$ the metric tensor in a space-time of dimension d: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. By definition, a conformal transformation of the coordinates is an invertible mapping, which leaves the metric tensor invariant up to a scale (compared with the general case $g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x)$):

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x). \tag{8}$$

A conformal transformation $w = f(z)$ preserves local angles. Generally speaking, a conformal transformation is a coordinate transformation that is a local rescaling of the metric, which is a subgroup of the whole coordinate transformations. The set of conformal transformations manifestly forms a group, and it obviously has the Poincare group as a subgroup (in flat space), since the latter corresponds to the special case $\Lambda(x) = 1$.

To study the properties of a given group, we can start from the infinitesimal generators. The infinitesimal generators of the conformal group can be determined by considering the infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, under which

$$\begin{aligned}
ds'^2 &= g'_{\mu\nu}dx'^\mu dx'^\nu \\
&= (g_{\mu\nu} + \delta g_{\mu\nu})d(x^\mu + \epsilon^\mu(x))d(x^\nu + \epsilon^\nu(x)) \\
&\approx g_{\mu\nu}dx^\mu dx^\nu + g_{\mu\nu}(d\epsilon^\mu dx^\nu + d\epsilon^\nu dx^\mu) \equiv \Omega(x)ds^2 \\
&\to (\frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu}) = f(x)g_{\mu\nu}
\end{aligned}$$

The requirement that the transformation be conformal implies that

$$(\frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu}) = f(x)g_{\mu\nu}(x) \tag{9}$$

where $f(x) = \frac{2}{d}\partial_\mu\epsilon^\mu = \frac{2}{d}(\partial \cdot \epsilon)$ is obtained by trace on two sides ($g^{\mu\nu}(\frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu}) = f(x)g^{\mu\nu}(x)g_{\mu\nu}(x) \Rightarrow 2\partial_\mu\epsilon^\mu = df(x)$).

Next, we apply derivative on $\partial^\rho\epsilon_\rho$ and have

$$\partial_\nu\partial_\mu(\partial^\rho\epsilon_\rho) = \partial_\mu\partial^\rho(\partial_\nu\epsilon_\rho + \partial_\rho\epsilon_\nu - \partial_\rho\epsilon_\nu) = \frac{2}{d}\partial_\nu\partial_\mu(\partial \cdot \epsilon) - \partial^\rho\partial_\rho\partial_\mu\epsilon_\nu \tag{10}$$

and

$$(g_{\mu\nu}\partial^\rho\partial_\rho + (d-2)\partial_\mu\partial_\nu)(\partial \cdot \epsilon) = 0 \tag{11}$$

Taking the trace, we then get

$$(d-1)\partial^\mu\partial_\mu(\partial\cdot\epsilon)=0. \tag{12}$$

These two equation imply that, 1) if $d=1$, the above equations do not impose any constraint on the function $\partial\cdot\epsilon$, and therefore any smooth transformation is conformal in one dimension. This is a trivial statement, since the notion of angle then does not exist. 2) $d=2$ is special, which is different from $d>2$. 3) in $d>2$, the condition $\partial_\mu\partial_\nu(\partial\cdot\epsilon)=0$ enforces that $\partial\cdot\epsilon$ is a linear function like $\partial\cdot\epsilon=A+B_m x_m$. If we substitute this expression into Eq. 9, we see that $\epsilon(x)$ is at most quadratic in the coordinates. We therefore write the general expression

$$\epsilon(x)_\mu=a_\mu+b_{\mu\nu}x_\nu+c_{\mu\nu\rho}x_\nu x_\rho \tag{13}$$

Thus, we have four kinds of transformations:

- translation: $\epsilon_\mu=a_\mu$, i.e. ordinary translations independent of x.
- rotaion: $\epsilon_\mu=\omega_{\mu\nu}x_\nu$
- dilatation: $\epsilon_\mu=\lambda x_\mu$
- special conformal transformation: $\epsilon_\mu=b_\mu x^2-2x_\mu b_\nu x_\nu$

Question: What is the meaning of special conformal transformation?

Or, the transformation is also written as

1. **Translations** $\quad x^\mu\to x^\mu+a^\mu$
2. **Rotations** $\quad x^\mu\to\Lambda^\mu_{\ \nu}x^\nu\ ,\quad \Lambda\in SO(d)$
3. **Dilatations** $\quad x^\mu\to\lambda\, x^\mu\ ,\quad \lambda>0$
4. **Special Conformal Transformations (SCT)** $\quad x^\mu\to x'^\mu=\dfrac{x^\mu+b^\mu x^2}{1+2b\cdot x+b^2x^2}$

In the SCT, b^μ is a constant vector and we define the factor

$$\gamma(x)\equiv 1+2b\cdot x+b^2x^2\ .$$

These four types, together with possible parity reflections, generate the whole conformal group $SO(d+1,1)$.

B. Conformal generators

Locally, the above conformal transformation can be expressed as the conformal generators: $a_\mu \partial_\mu, \omega^\mu_\nu x^\nu \partial_\mu x_\mu, \lambda x \cdot \partial, b^\mu (x^2 \partial_\mu - 2x^\mu x \cdot \partial)$, where the number of them are respectively $d, d(d-1)/2, 1, d$. Thus, the total number generators for conformal symmetry is $(d+1)(d+2)/2$.

The generators satisfy the commutation relations:

$$P_\mu = -i\partial_\mu, \tag{14}$$

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \tag{15}$$

$$D = -ix_\mu \partial_\mu, \tag{16}$$

$$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \tag{17}$$

and they satisfy

$$[D, P_\mu] = iP_\mu, \tag{18}$$

$$[D, K_\mu] = -iK_\mu, \tag{19}$$

$$[P_\rho, L_{\mu\nu}] = i(\eta_{\mu\nu} P_\nu - \eta_{\rho\nu} P_\mu), \tag{20}$$

$$[P_\mu, K_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}), \tag{21}$$

$$[L_{\mu\nu}, K_\rho] = i(\eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu), \tag{22}$$

$$[L_{\mu\nu}, D] = 0, \tag{23}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}). \tag{24}$$

Up to now, we consider the conformal transformation on the coordinates only. Here we consider how a quantum field changes under a given conformal transformation T_a: $\phi(x) \to \phi'(x') = (1 - i\epsilon_a T_a)\phi(x)$.

Some care should be taken when there is some special requirements on the quantum fields. For example, consider Lotentz group $L_{\mu\nu}\phi(0) = S_{\mu\nu}\phi(0)$. From Hausdorff formula $(e^{-A} B e^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + ...)$:

$$e^{ix^\lambda P_\lambda} L_{\mu\nu} e^{-ix^\lambda P_\lambda} = L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \tag{25}$$

Thus we have

$$P_\mu \phi(x) = -i\partial_\mu \phi(x) \tag{26}$$

$$L_{\mu\nu}\phi(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x) + S_{\mu\nu}\phi(x) \tag{27}$$

$$
\begin{aligned}
[P_\mu, D] &= (-i\partial_\mu)(-ix^\nu\partial_\nu) - (-ix^\nu\partial_\nu)(-i\partial_\mu) \\
&= (-i)^2\big(\partial_\mu(x^\nu\partial_\nu) - x^\nu\partial_\nu\partial_\mu\big) \\
&= -\big((\partial_\mu x^\nu)\partial_\nu + x^\nu\partial_\mu\partial_\nu - x^\nu\partial_\nu\partial_\mu\big) \\
&= -\delta_\mu^\nu\partial_\nu \\
&= -\partial_\mu = iP_\mu.
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
[L_{\mu\nu}, P_\rho] &= [i(x_\mu\partial_\nu - x_\nu\partial_\mu),\, -i\partial_\rho] \\
&= i(-i)[x_\mu\partial_\nu - x_\nu\partial_\mu,\, \partial_\rho] \\
&= [x_\mu\partial_\nu, \partial_\rho] - [x_\nu\partial_\mu, \partial_\rho] \\
&= -\eta_{\rho\mu}\partial_\nu + \eta_{\rho\nu}\partial_\mu \\
&= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
&= i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu).
\end{aligned}
\tag{29}
$$

We used $[x_\mu\partial_\nu, \partial_\rho] = x_\mu \underbrace{[\partial_\nu, \partial_\rho]}_{=0} - [\partial_\rho, x_\mu]\partial_\nu = -\eta_{\rho\mu}\partial_\nu$.

$$
\begin{aligned}
[L_{\mu\nu}, D] &= i(x_\mu\partial_\nu - x_\nu\partial_\mu)(-ix^\lambda\partial_\lambda) - (-ix^\lambda\partial_\lambda)i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
&= (x_\mu\partial_\nu - x_\nu\partial_\mu)(x^\lambda\partial_\lambda) - (x^\lambda\partial_\lambda)(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
&= 0.
\end{aligned}
\tag{30}
$$

C. Examples of conformal invariance

1) The simplest example of classical conformal symmetry is Maxwell's equations in the absence of sources

$$
\partial_\mu F_{\mu\nu} = 0 \tag{31}
$$

or,

$$\bigtriangledown^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\bigtriangledown^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

2) Massless scale field. Like the electromagnetic field equations above, the equation of motion for this theory is also a wave equation,

$$\bigtriangledown^2 \varphi - \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{32}$$

and is invariant under the transformation $x \to \lambda x$,$t \to \lambda t$. The name massless refers to the absence of a term $\propto m^2\varphi$ in the field equation. Such a term is often referred to as a 'mass' term, and would break the invariance under the above transformation.

3) The above field equations in the examples are all linear in the fields. One non-linear equation example is $\phi - 4$ theory:

$$\bigtriangledown^2 \phi - \frac{\partial^2 \phi}{\partial t^2} + g\phi^3 = 0 \tag{33}$$

and is invariant under the transformation $x \to \lambda x$,$t \to \lambda t, \phi \to \lambda^{-1}\phi$. The key point is that the parameter g must be dimensionless, otherwise one introduces a fixed length scale into the theory: For this theory, this is only the case in $D = 4$. Note that under these transformations the argument of the function ϕ is unchanged.

D. Primary Fields

Physical states are characterized by the eigenvalues of the dilation operator $l_0 + \bar{l}_0$ and the rotation operator $il_0 - i\bar{l}_0$. These can be written in terms of the quantities $h, \bar{h}$:

$$l_0|\psi\rangle = h|\psi\rangle, \bar{l}_0|\psi\rangle = \bar{h}|\psi\rangle \tag{34}$$

where $h, \bar{h}$ is confomal weights.

Under a local conformal map $z \to w(z)$, $\bar{z} \to w(\bar{z})$, a primary field is defined by the transformation

$$\phi'(w, \bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}}\phi(z, \bar{z}) \tag{35}$$

The above shows that a primary field of conformal dimensions $(h, \bar{h})$ transforms like the component of a covariant tensor of rank $h + \bar{h}$ having h "z" indices and $\bar{h}$ "$\bar{z}$" indices.

Please note that, this is equivalent to require the "line element" unchanged:

$$\phi(z,\bar{z})dz^h d\bar{z}^h = \phi(w,\bar{w})dw^h d\bar{w}^h = \phi(w,\bar{w})dz^h d\bar{z}^h (\frac{\partial w}{\partial z})^h (\frac{\partial \bar{w}}{\partial \bar{z}})^{\bar{h}} \tag{36}$$

If Eq. 35 holds *only* for global conformal transformations $w(z) \in SL(2,C)/Z_2$, we call this field as a quasi-primary field. Note that a primary field is always quasi-primary but the reverse is not true. Furthermore, not all fields in a CFT are primary or quasi-primary. Those fields are called secondary fields. (As we shall see, an example of a quasi- primary field that is not primary is the energy-momentum tensor.)

If the map $z \to w = z + \epsilon(z)$ is close to the identity, the variation of the field is

$$(\frac{\partial w}{\partial z})^h = 1 + h\partial_z \epsilon(z) + O(\epsilon^2) \tag{37}$$

$$\phi(z+\epsilon(z),\bar{z}) = \phi(z,\bar{z}) + \epsilon(z)\partial_z \phi(z,\bar{z}) \tag{38}$$

and

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \phi'(z,\bar{z}) - \phi(z,\bar{z}) \tag{39}$$

$$= \phi'(w-\epsilon(z),\bar{w}-\epsilon(\bar{z})) - \phi(z,\bar{z}) = \phi'(w,\bar{w}) - \epsilon(z)\partial_z\phi'(w) - \epsilon(\bar{z})\partial_{\bar{z}}\phi'(\bar{w}) - \phi(z,\bar{z}) \tag{40}$$

$$= \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) - \epsilon(z)\partial_z\phi'(w) - \epsilon(\bar{z})\partial_{\bar{z}}\phi'(\bar{w}) - \phi(z,\bar{z}) \tag{41}$$

$$\approx -[(h\partial_z\epsilon + \epsilon\partial_z) + (\bar{h}\partial_{\bar{z}}\epsilon(\bar{z})) + \epsilon(\bar{z})\partial_{\bar{z}}]\phi(z,\bar{z}) \tag{42}$$

E. Constraints on correlation functions

The objects of interest in quantum field theories are n-point correlation functions which are usually computed in a perturbative approach via either canonical quantisation or the path integral method. In this section, we will see that the exact two- and three-point functions for certain fields in a conformal field theory are already determined by the symmetries. This will allow us to derive a general formula for the OPE among quasi-primary fields.

Consider d-dimensional Euclidean space $\mathbf{R}^d$. The conformal group consists of all coordinate transformations $x \to x'$ that preserve angles.

A conformal field theory contains special operators called *primary fields* that transform in a simple way under conformal maps. In these notes we only consider *scalar* primaries (no spin indices). A scalar primary $\phi(x)$ of **scaling dimension** Δ obeys

$$\phi'(x) = \left|\frac{\partial x'}{\partial x}\right|^{\Delta/d} \phi(x').$$

Here ϕ' denotes the field in the transformed coordinates and $|\partial x'/\partial x|$ is the Jacobian determinant. The exponent Δ/d ensures that $\phi(x)(dx)^0$ has homogeneous scaling.

Example: Under a dilatation $x' = \lambda x$, the Jacobian is λ^d, so

$$\phi'(x) = \lambda^{\Delta}\,\phi(\lambda x).$$

For an SCT, $x' = \frac{x+bx^2}{\gamma(x)}$ and one finds the Jacobian $|\partial x'/\partial x| = \gamma(x)^{-d}$; therefore

$$\phi'(x) = \gamma(x)^{-\Delta}\,\phi(x').$$

The vacuum state $|0\rangle$ of a conformal field theory is invariant under the global conformal group. Consequently, correlation functions computed in the vacuum satisfy

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \langle \phi_1'(x_1)\cdots\phi_n'(x_n)\rangle.$$

This equation holds for any conformal transformation, provided we apply the transformation law (1) to each field insertion individually.

We will now exploit (4) for a single two-point function

$$G(x_1, x_2) \equiv \langle \phi_1(x_1)\phi_2(x_2)\rangle.$$

Our goal is to determine its functional form using only the symmetry constraints.

Translation and Rotation

Translation invariance means that G can only depend on the relative coordinate $x_1 - x_2$:

$$G(x_1, x_2) = F(x_1 - x_2).$$

Rotational invariance then forces F to depend only on the absolute distance

$$r \equiv |x_1 - x_2|.$$

Thus we write

$$G(x_1, x_2) = F(r).$$

Dilatation

Apply the dilatation $x \to \lambda x$ to both fields using (2) and the invariance condition (4):

$$G(x_1, x_2) = \lambda^{\Delta_1+\Delta_2}\, G(\lambda x_1, \lambda x_2).$$

In terms of $F(r)$ this becomes

$$F(r) = \lambda^{\Delta_1+\Delta_2}\, F(\lambda r).$$

Equation (8) is the functional equation of a homogeneous function. Choosing the scaling parameter $\lambda = 1/r$ we immediately find

$$F(r) = r^{-(\Delta_1+\Delta_2)}\, F(1).$$

Hence

$$\boxed{G(x_1, x_2) = \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}}},$$

where $C = F(1)$ is an undetermined constant. Dilatation alone gives a power law, but does not fix the sum of the dimensions independently ? it only ties the decay exponent to $\Delta_1 + \Delta_2$.

Special Conformal Transformation

Now we further demand invariance under an SCT. Using (3) and (4):

$$\begin{aligned} G(x_1, x_2) &= \langle \varphi_1'(x_1)\varphi_2'(x_2)\rangle \\ &= \gamma(x_1)^{-\Delta_1}\gamma(x_2)^{-\Delta_2}\,\langle \varphi_1(x_1')\varphi_2(x_2')\rangle \\ &= \gamma_1^{-\Delta_1}\gamma_2^{-\Delta_2}\, G(x_1', x_2'). \end{aligned}$$

To proceed we need the distance between the transformed points. A straightforward but slightly lengthy algebraic calculation (see Appendix) yields

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{\gamma_1^{1/2}\,\gamma_2^{1/2}},$$

where we abbreviated $\gamma_i \equiv \gamma(x_i)$.

Insert the ansatz and the distance relation:

$$\frac{C}{r^{\Delta_1+\Delta_2}} = \gamma_1^{-\Delta_1}\gamma_2^{-\Delta_2}\,\frac{C}{\left(r\,\gamma_1^{-1/2}\gamma_2^{-1/2}\right)^{\Delta_1+\Delta_2}}.$$

The factor C and the common power of r cancel, leaving

$$1 = \gamma_1^{-\Delta_1 + \frac{\Delta_1+\Delta_2}{2}} \gamma_2^{-\Delta_2 + \frac{\Delta_1+\Delta_2}{2}} = \gamma_1^{\frac{\Delta_2-\Delta_1}{2}} \gamma_2^{\frac{\Delta_1-\Delta_2}{2}}.$$

The functions γ_1 and γ_2 depend on the arbitrary vector b^μ and on the positions x_1, x_2. For the identity (12) to hold for *all* possible SCT parameters, the exponents must vanish:

$$\Delta_2 - \Delta_1 = 0 \qquad \Longleftrightarrow \qquad \Delta_1 = \Delta_2 \equiv \Delta.$$

If the dimensions differ, the only consistent solution is $C = 0$; the two-point function must then vanish identically. This is a remarkable prediction: **only primary fields of the same scaling dimension can have a non-zero two-point correlator**.

Finally, with equal dimensions, the two-point correlator reduces to

$$\boxed{\langle \phi(x_1)\phi(x_2) \rangle = \frac{C}{|x_1 - x_2|^{2\Delta_\phi}}}.$$

The constant C is not fixed by conformal symmetry. It can be set to 1 (or 0 if the fields are different) by appropriately normalising the operators. The essential physical information of the power-law decay with distance is completely determined.

- The two-point function of scalar primary fields in a conformal field theory is forced to be a pure power law, with exponent 2Δ where Δ is the common scaling dimension.

- The result follows solely from the symmetries of the global conformal group; no dynamical details (Lagrangian, interactions) are needed.

- This derivation is the starting point for the conformal bootstrap: higher?point functions are similarly constrained, eventually leading to exact non?perturbative predictions.

- In $d = 2$ dimensions the conformal group is infinite-dimensional, but the global subgroup considered above still applies, giving the same form. The richer structure appears in the presence of a central charge and in the full Virasoro constraints.

- The method generalises to operators with spin: the tensor structure is also fixed by conformal symmetry, and the two-point function vanishes unless the operators have the same dimension *and* the same Lorentz representation.

Distance Transformation under SCT.— We prove the identity (11). For an SCT with constant vector b^μ, the transformation is

$$x'^\mu = \frac{x^\mu + b^\mu x^2}{\gamma(x)}\,, \qquad \gamma(x) = 1 + 2b\cdot x + b^2x^2.$$

Define $\Delta x = x_1 - x_2$. A useful intermediate step is to verify the following relation, which follows from straight algebra:

$$(x_1' - x_2')^\mu = \frac{1}{\gamma_1\gamma_2}\Big[\gamma_2 x_1^\mu - \gamma_1 x_2^\mu + b^\mu\big(\gamma_2 x_1^2 - \gamma_1 x_2^2\big)\Big].$$

Now compute the square:

$$|x_1' - x_2'|^2 = (x_1' - x_2')^\mu (x_1' - x_2')_\mu.$$

A cleaner method is to note that an SCT can be written as an inversion, followed by a translation, followed by another inversion:

$$x' = \frac{x + bx^2}{\gamma(x)} = I \circ T_b \circ I(x)\,,$$

where $I(x) = x/x^2$. Using the fact that inversion gives $|I(x_1) - I(x_2)| = \frac{|x_1 - x_2|}{|x_1||x_2|}$, one obtains after short manipulations

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{\gamma_1^{1/2}\,\gamma_2^{1/2}}.$$

The square?root factors appear because $\gamma(x) = 1 + 2b\cdot x + b^2x^2$ can be rewritten as $|x|^{-2}|x + b|x|^2|^2$ after inversion, but the end result is as claimed. For the purpose of our derivation, this identity is sufficient.

Similarly, for three-point correlator

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{f_{123}}{|z_1 - z_2|^{\Delta_1+\Delta_2-\Delta_3}|z_2 - z_3|^{\Delta_2+\Delta_3-\Delta_1}|z_3 - z_1|^{\Delta_3+\Delta_1-\Delta_2}} \tag{43}$$

The scaling dimension Δ and Operator Product Expansion (OPE) coefficients are universal numbers, and they are the most important information for a given CFT. Using these data, one can reconstruct all critical phenomena near the phase transition.

Connection with the Critical Phenomena.— CFT is very useful in the statistical physics and condensed matter physics, especially in the phase transition and critical phenomena. A key feature of critical phenomena is the universality, where diverse physical systems exhibit identical scaling behaviors around phase transition point. It is rooted in the long wavelength effective field theory governing the critical phenomena. This section elucidates how the conformal data of CFT determine measurable critical exponents, bridging abstract field theory to microscopic observables.

In Landau-Ginzburg-Wilson theory, the free energy near a critical point is governed by local fields such as order parameter ϕ and energy density ϕ^2. These map to the relevant CFT primaries:

$$\begin{aligned} \phi(\text{local order parameter}) &\longleftrightarrow \sigma \text{ (lowest symmetry-odd primary)} \\ \phi^2(\text{energy density}) &\longleftrightarrow \epsilon \text{ (lowest symmetry-even primary)}. \end{aligned} \tag{44}$$

Thus, the critical exponents relating to the local fields are solely determined by the scaling dimensions of CFT primaries. The widely used critical exponents $(\eta, \nu, \beta, \gamma, \delta)$ of Wilson-Fisher universality class relates to the scaling dimensions Δ_σ, Δ_ϵ as

$$\eta = 2\Delta_\sigma - (d-2), \tag{45}$$

$$\nu = \frac{1}{d - \Delta_\epsilon}, \tag{46}$$

$$\beta = \frac{\nu}{2}(d - 2 + \eta), \tag{47}$$

$$\gamma = \nu(2 - \eta), \tag{48}$$

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta}. \tag{49}$$

Additionally, CFT also governs physics close to or around the exact critical point. For example, in the finite-size simulation on a given size L, the subleading correction to the free energy reads:

$$F \sim L^{-d}\left(a_0 + a_1 L^{-\omega} + \cdots\right),$$

where anomalous critical exponent $\omega = \Delta_{\epsilon'} - d$ is determined by the lowest irrelevant symmetry-even primary ϵ'.

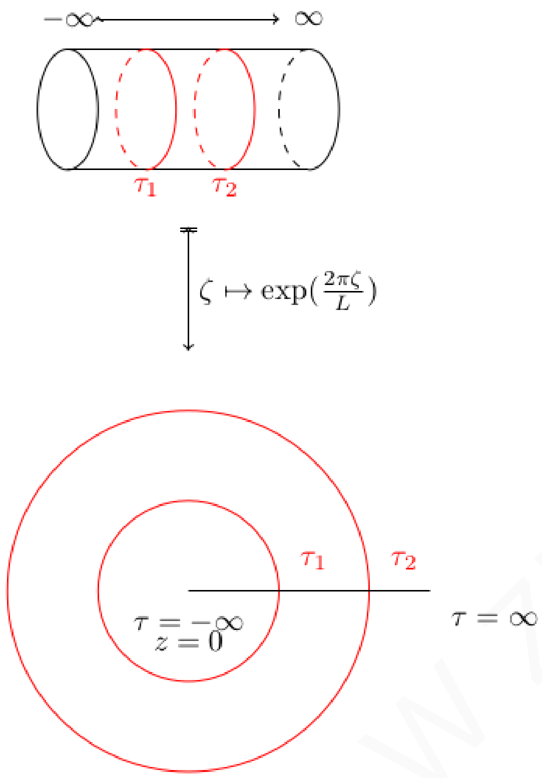

FIG. 3: Map from cylinder to plane in 2D.

F. Radial quantization

To see the consequences of conformal invariance in a two dimensional quantum field theory, we enter into some of the details of the quantization procedure (we first take conformal field theories defined on Euclidean two-dimensional flat space.). We begin with flat Euclidean "space" and "time" coordinates σ and τ (In Minkowski space, one needs to make a Wick rotation $\tau = it$). The coordinate $\sigma = \sigma + 2\pi$, is the periodic boundary condition for 1d spatial dimension (Or, it is to eliminate any infrared divergences, we compactify the space coordinate). Thus, we define the complex number $\zeta = \tau + i\sigma$.

The conformal map

$$z = \exp(\tau + i\sigma) \tag{50}$$

maps the cylinder to the complex plane. Equal time surfaces, $\tau = const$, become circles of constant radius on the z-plane, and time reversal, $\tau \to -\tau$, becomes $z \to 1/z^*$. Then the time translation $\tau- > \tau + T$ becomes dilatation on the complex plane $z- > e^T z$. So the dilatation generator on the conformal plane can be regarded as the Hamiltonian for the system, and the Hilbert space is built up on surfaces of constant radius. This procedure for defining a quantum theory on the plane is known as radial quantization (Hilbert state space defined on circles about the origin, and propagation of states in the radial direction dilation operator is the Hamiltonian; rotation operator is a spatial translation). Thus the hamiltonian on the complex plane is

$$H \leftrightarrow D \tag{51}$$

The radial quantization, in particular, is to quantize this Hamiltonian or said to quantize the generators on the complex plane.

Another way to view this relation is, under that map Eq. 109, the field changes as

$$\phi_{cyl}(\tau, \Omega) = \Lambda(r, \Omega)^{\Delta/2} \phi_{plane}(r, \Omega) \tag{52}$$

where $\Lambda = R^{-2} e^{\frac{2\tau}{R}}$ is the scale factor of this transformation and we can see it by considering the metric

$$ds^2_{plane} = dr^2 + r^2 d\Omega^2 = R^{-2} e^{\frac{2\tau}{R}} \left(d\tau^2 + R^2 d\Omega^2\right) = \Lambda(r, \Omega) \left(d\tau^2 + R^2 d\Omega^2\right) \equiv \Lambda(r, \Omega) ds^2_{cyl}. \tag{53}$$

So we have

$$\langle \phi_{cyl}(\tau) \phi_{cyl}(0) \rangle = \Lambda(r, \Omega)^{\Delta/2} \Lambda(r = 1, \Omega)^{\Delta/2} \langle \phi_{plane}(\tau) \phi_{plane}(0) \rangle \tag{54}$$

$$= R^{-2} e^{\frac{\tau\Delta}{R}} R^{-2} \frac{1}{|r-1|^{2\Delta}} \sim e^{\frac{-\tau\Delta}{R}} \tag{55}$$

This is expontial decay function. Recalling that, the inverse of correlation length is proportional the eigenenergy gap, we have

$$\Delta E = E_n - E_0 \sim \frac{1}{\xi} = \frac{\Delta}{R} \tag{56}$$

So we reach the state-operator correspondence: *On the cylinder geometry $S^{d-1} \times R$, each eigenenergy gap has one-to-one correspondence with scaling dimension of CFT operator.* [Cardy 1984, 1985].

G. Example: 2D Ising model

Let us consider an example of 2D Ising model. It was first solved by Onsager in 1944 using the transfer-matrix method. We will try to solve it using the conformal mapping.

1. Mapping from two-dimension classical Ising model to quantum transverse Ising model

Let us move to study the case in 2d. The result will be different from the case in 1d. We start with the partition function again

$$Z = \sum_{\sigma_1=\pm,\ldots,\sigma_N=\pm} \exp[-\beta H], H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j \tag{57}$$

Since we are dealing with the model in two dimension (squared mesh), it is better to introduce coordinates (p,q) for each site, where $p,q \in Z$, and denote the coordinates in x- and y-direction, respectively. We generalize our model allowing for different couplings along the x- and y-directions, and having N sites in the x-direction and M-sites in the y-direction but keeping periodic boundary conditions (p.b.c) along both directions.

Let us introduce a notation for the collection of all spin values in row-n

$$\sigma^{(n)} = \{\sigma_{1,n}, \sigma_{2,n}, \ldots, \sigma_{M,n}\} \tag{58}$$

Then the total Boltzmann factor for all interactions within that row-n is

$$W_n(\{\sigma^{(n)}\}) = \exp(\beta J_x \sum_{m=1}^{M}\sigma_{m,n}\sigma_{m+1,n}) \tag{59}$$

whereas the Boltzmann factor for the interaction between rows n and $n+1$ is

$$W_{n+\frac{1}{2}}(\sigma^{(n)}, \sigma^{(n+1)}) = \exp(\beta J_y \sum_{m=1}^{M}\sigma_{m,n}\sigma_{m,n+1}) \tag{60}$$

Since the variables $\sigma_{p,q}$ have two possible values, we can represent them by a two component vector (a spinor):

$$\sigma_{p,q} = 1 = (1,0)^T, \sigma_{p,q} = -1 = (0,1)^T \tag{61}$$

W_n is a diagonal transfer matrix, we just replace $\sigma_{p,q} \to \sigma_p^z$

$$T^x = \exp[\sum_{p=1}^{M}\beta J_x \hat{\sigma}_p^z\hat{\sigma}_{p+1}^z] \tag{62}$$

(Here transfer matrix T^x should be understood by $\langle\sigma_{p,q}|T^x|\sigma_{p,q}\rangle = e^{\beta J_x\sigma_{pq}\sigma_{pq}}$.)

The transfer matrix along y-direction is

$$T^y_{\sigma_{pq},\sigma_{p,q+1}} = \begin{pmatrix} e^{\beta J_y} & e^{-\beta J_y} \\ e^{-\beta J_y} & e^{\beta J_y} \end{pmatrix}_{\sigma_{pq},\sigma_{p,q+1}} \tag{63}$$

Since a 2×2 matrix can be written in terms of Pauli matrices, we have

$$T^y = e^{\beta J_y} I + e^{-\beta J_y} \hat{\sigma}^x = e^{\beta J_y}(1 + e^{-2\beta J_y} \hat{\sigma}^x) \tag{64}$$

At this point we recall that T^y_{pq} is part of a partition function, and therefore, it would be easier to interpret what we have, if we could express it as the exponential of an operator. Since

$$e^{a\hat{\sigma}^x} = \cosh a + \sinh a\hat{\sigma}^x = \cosh a(1 + \tanh a\hat{\sigma}^x) \tag{65}$$

we can set $\tanh a = e^{-2\beta J_y}$, so we obtain

$$T^y = (\sinh a \cosh a)^{-1/2} \exp[a\hat{\sigma}^x] = (2\sinh(2J_y))^{1/2} \exp[a\hat{\sigma}^x] \tag{66}$$

where we used the relation

$$\tanh a = e^{-2J_y}, \cosh^2 a - \sinh^2 a = 1 \to \cosh^2 a = \frac{1}{1 - e^{-4J_y}} \tag{67}$$

So we have the transfer-matrix along the y-direction is

$$T^y = (\prod_{m=1}^{M} 2\sinh(2\beta J_y))^{1/2} \exp(\beta J^y \sum_{m=1}^{M} \sigma^x_m) \tag{68}$$

The total partition function is

$$Z = Tr \prod_{n=1}^{N} (T^x T^y) = Tr(T^x T^y)^N \tag{69}$$

This is clearly related to the Ising chain in transverse field with Hamiltonian

$$H = -J \sum_{m=1}^{M} \sigma^z_m \sigma^z_{m+1} - h \sum_{m=1}^{M} \sigma^x_m \tag{70}$$

$$\begin{aligned} Z &= Tre^{-\beta_Q H} \\ &\simeq Tr[\exp(\tau J \sum_n \sigma^z_m \sigma^z_{m+1}) \exp(\tau h \sum_m \sigma^x_m)]^N \end{aligned} \tag{71}$$

where $\tau = \beta_Q/N$, and when N is large enough the Trotter-Suzuki decomposition becomes exact. The so-called transverse Ising model is a canonical model in the study of quantum critical point or quantum phase transition. It is the simplest model with a quantum critical point. It is exactly solvable, as we will show below. It also connects with the conformal field theory, so it is really very important.

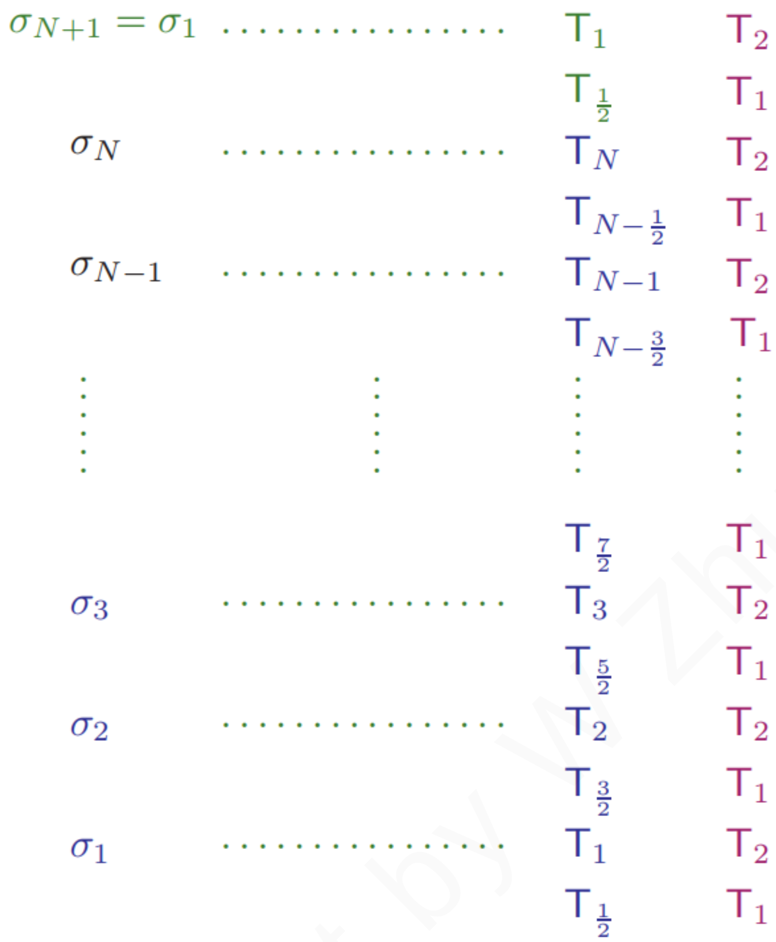

FIG. 4: Mapping from classical Ising to transverse Ising model.

2. Phase diagram

In the limit of $h \to 0$, the ground state is an Ising ferromagnet that spontaneously breaks Z_2 symmetry. The two-fold degenerate ground states are

$$|\Psi_\downarrow\rangle = |\downarrow_1\downarrow_2 \cdots\rangle, |\Psi_\uparrow\rangle = |\uparrow_1\uparrow_2 \cdots\rangle. \tag{72}$$

When $h \to \infty$, the ground state is a trivial paramagnet that preserves Ising symmetry,

$$|\Psi_x\rangle = |+_1 +_2 \cdots\rangle, + = \uparrow - \downarrow. \tag{73}$$

The above analysis matches the Monte Carlo simulations as shown in Fig. ??. The global phase diagram of the model is easy to image. The interesting problem is, where is the transition point?

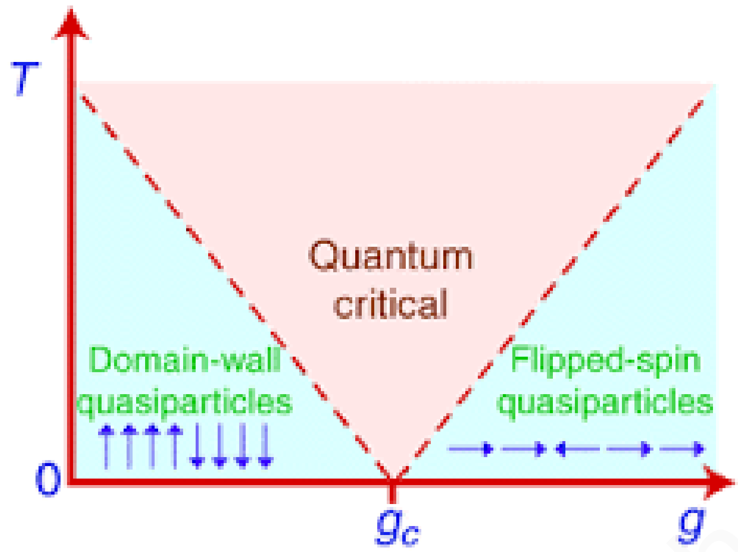

FIG. 5: Phase diagram of 1+1D transverse Ising model.

3. Duality

There is a duality transformation which defines new Pauli operators in a dual lattice

$$\tau_i^x = \sigma_i^z \sigma_{i+1}^z, \tau_i^z = \prod_{j\leq i} \sigma_i^x \tag{74}$$

then these τ_i^x and τ_i^z satisfy the same commutation and anti-commutation relations of σ_i^x and σ_i^z, i.e. $\{\tau_i^a, \tau_j^b\} = 2\delta^{ab}$. And the original Hamiltonian can be written in terms of $\tau^{x,z}$ as

$$H = -J \sum_p \tau_p^x - h \sum_p \tau_{p+1}^z \tau_p^z \tag{75}$$

where we used the condition that $\tau_{p-1}^z \tau_p^z = \prod_{j<p} \sigma_i^x \prod_{i<p+1} \sigma_i^x = \sigma_p^x$

Since these two Hamiltonian take the same algebra, they should be the same (which means energy spectra, eigenvalues are all the same). But, we notice that the parameter exchange: $J \leftrightarrow h$. This is called duality. In this case, if there is a phase transition point, it should satisfy the condition of (by setting $J = 1$)

$$h_c = \frac{1}{h_c} \Rightarrow h_c = 1 \tag{76}$$

$\diamond$ *Homework:* Please prove the commutation relation between τ_i^x and τ_i^z.

The key feature is, we got a phase transition point $h_c \neq 0$ in 2d, which is quite different from that of 1d. The global phase diagram can be mapped out easily (see Fig. 5).

4. Diagonalize quantum transverse Ising model

After the conformal mapping, 2D Ising model is mapped to the 1+1D transverse Ising model.

$$H = \sum_n \sigma_n^z \sigma_{n+1}^z + h \sum_n \sigma_n^x \tag{77}$$

The duality ensures the transition occurs at $h_c = 1$. The periodic boundary condition leads to $S^1 \times R$ geometry. Plotting the energy spectra at the transition point will give Fig. 6.

Next we solve this model with periodic boundary condition. The stategy is to use the fermionic representative [Two-dimensional Ising model as a soluble problem of many fermions, T. D. Schultz, D. C. Mattis, E. H. Lieb]. Here we make the Jordan-Wigner transformation

$$c_n = \frac{\sigma_n^x + i\sigma_n^y}{2} \prod_{m<n} \sigma_m^z, c_n^\dagger = \frac{\sigma_n^x - i\sigma_n^y}{2} \prod_{m<n} \sigma_m^z, \tag{78}$$

$$\sigma_n^+ = \prod_{m<n} (1 - 2c_m^\dagger c_m) c_n, \sigma_n^- = \prod_{m<n} (1 - 2c_m^\dagger c_m) c_n^\dagger, \sigma_n^z = 1 - 2c_n^\dagger c_n \tag{79}$$

The string $\prod_{m<n}(1 - 2c_m^\dagger c_m)$ takes values ± 1, depending on even/odd number of fermions on the left side of n. One can check that,

$$\{c_n, c_m^\dagger\} = \delta_{m,n}, \{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0 \tag{80}$$

$$[\sigma_n^+, \sigma_m^-] = \delta_{n,m}\sigma_n^z, [\sigma_n^z, \sigma_m^\pm] = \pm 2\delta_{n,m}\sigma_n^\pm \tag{81}$$

(Only the Pauli matrix with the same site index should consider the commutation relation $\{\sigma_i^a, \sigma_j^b\} = 2\delta_{ij}\delta_{ab}, [\sigma_i^+, \sigma_j^-] = \delta_{ij}\sigma_j^z$.)

Under the Jordan-Wigner transformation, the Hamiltonian becomes

$$\begin{aligned} H &= \sum_n \sigma_n^z - \sum_n \sigma_n^x \sigma_{n+1}^x \\ &= \sum_{n=1}^{N} (1 - 2c_n^\dagger c_n) - \sum_{n=1}^{N-1} [c_n^\dagger c_{n+1}^\dagger + c_n^\dagger c_{n+1} + h.c.] + (c_N^\dagger c_1^\dagger + c_N^\dagger c_1 + h.c.) e^{i\pi\mathcal{N}}, \mathcal{N} = \sum_n c_n^\dagger c_n \end{aligned} \tag{82}$$

with

$$
\begin{aligned}
\sigma_n^x \sigma_{n+1}^x &= [\prod_{m<n}(1-2c_m^\dagger c_m)](c_m + c_m^\dagger)[\prod_{k<n+1}(1-2c_k^\dagger c_k)](c_k + c_k^\dagger) \\
&= (c_n^\dagger + c_n)(1-2c_n^\dagger c_n)(c_{n+1} + c_{n+1}^\dagger) \\
&= c_n^\dagger c_{n+1} + c_n^\dagger c_{n+1}^\dagger + h.c.
\end{aligned}
\tag{83}
$$

The boundary term comes from that $\sigma_N^x \sigma_1^x = e^{i\pi \sum_{j<L} n_j} c_N^\dagger c_1 = -e^{i\pi \sum_{j\le L} n_j} c_N^\dagger c_l = -e^{i\pi \mathcal{N}} c_N^\dagger c_l$, because to the left of $c_N^\dagger$ we certainly have $n_N = 1$. This shows that boundary condition are changed by fermion parity $e^{i\pi\mathcal{N}} = (-1)^{\mathcal{N}}$ and periodic boundary condition become anti-periodic boundary condition when $\mathcal{N}$ is even. And odd $\mathcal{N}$ relates to periodic boundary condition. Therefore, the real spin problem is not exactly the same with free fermion. Next, for odd $\mathcal{N}$, we set $e^{ikN} = 1, k = \frac{2\pi n}{N}, n = -N/2+1, ..., 0, ..., N/2$, for even $\mathcal{N}$, we set $e^{ikN} = -1, k = \pm\frac{\pi(2n-1)}{N}, n = 1, ..., N/2$.

In terms of momentum space $c_j = \frac{1}{\sqrt{N}}\sum_k e^{ikj} c_k$, the Hamiltonian becomes

$$
\begin{aligned}
H &= -\sum_k [2\cos(k) c_k^\dagger c_k + (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] + \sum_k (2c_k^\dagger c_k - 1) \\
&= \sum_k [(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] \\
&= \sum_{k>0} [(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] + \sum_{k<0} ... \\
&= \sum_{k>0} [2(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (2i\sin(k) c_k^\dagger c_{-k}^\dagger - 2i\sin(k) c_{-k} c_k)] \\
&= \sum_{k>0} (c_k^\dagger, c_{-k}) \begin{pmatrix} 2(1-\cos(k)) & -2i\sin(k) \\ 2i\sin(k) & 2(1-\cos(k)) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}
\end{aligned}
\tag{84}
$$

where we used $\sum_k 2\cos(k) c_k^\dagger c_k = \sum_k \cos(k)(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger)$, and $\sum_k (2c_k^\dagger c_k - 1) = \sum_k (c_k^\dagger c_k - c_{-k} c_{-k}^\dagger)$.

The diagonalization is akin to Bogovliubov transformation, and all eigenvalues can be calculated:

$$
\Lambda(k) = \pm 2\sqrt{(\cos(k)-1)^2 + \sin^2(k)} = \pm 2\sin\frac{k}{2}
\tag{85}
$$

So we obtain that, for odd N, we have $e^{ikN} = 1, k = \frac{2\pi n}{N}, n = -N/2, ..., 0, ..., N/2,$

$$H = \sum_{n=0}^{N-1} \Lambda^-(n)(\eta_n^\dagger \eta_n - \frac{1}{2}) + const. \tag{86}$$

$$\Lambda^-(n) = [(1-\cos\frac{2\pi n}{N})^2 + (\sin\frac{2\pi n}{N})^2]^{1/2} = 2\sin\frac{2\pi n}{2N} \tag{87}$$

where Bogoliubov particle as $\begin{pmatrix} c_q \\ c_{-q}^\dagger \end{pmatrix} = \begin{pmatrix} u_q & -iv_q \\ -iv_q & u_q \end{pmatrix} \begin{pmatrix} \eta_q \\ \eta_{-q}^\dagger \end{pmatrix}$. (We have used that $H = \sum_{k>0} \Lambda(k)(\eta_k^\dagger \eta_k + \eta_{-k}\eta_{-k}^\dagger) = \sum_k \Lambda(k)(\eta_k^\dagger \eta_k - 1/2)$) For even N, we know the boundary condition is $e^{ikN} = -1, k = \pm\frac{\pi(2n-1)}{N}, n = 1, ..., N/2.$

$$H = \sum_{n=1}^{N/2} \Lambda^+(n)(\eta_n^\dagger \eta_n - \frac{1}{2}) + const. \tag{88}$$

$$\Lambda^+(n) = [(1-\cos\frac{\pi(2n-1)}{N})^2 + (\sin\frac{\pi(2n-1)}{N})^2]^{1/2} = 2\sin\frac{\pi(2n-1)}{2N} \tag{89}$$

The expression for H in above allows to immediately conclude that the ground state of the Hamiltonian must be the Bogoliubov vacuum state $|0\rangle$ which annihilates the $\vec{\eta}_k|0\rangle = 0$ for all k. Thus, the ground state energy is

$$E_0^+ = -\frac{1}{2}\sum_{n=1}^{N/2} \Lambda^+(n) + const. = -csc\frac{\pi}{2N} + const \approx -\frac{2N}{\pi} - \frac{\pi}{12N} + ... \tag{90}$$

$$E_0^- = -\frac{1}{2}\sum_{n=1}^{N/2-1} \Lambda^-(n) + const. = -cot\frac{\pi}{2N} + const \approx -\frac{2N}{\pi} + \frac{\pi}{6N} + ... = E_0^+ + \frac{\pi}{4N} \tag{91}$$

The lowest excited energy in even sector is

$$E_1^+ = \Lambda^+(1) + \Lambda^+(N/2) + E_0^+ = 4\sin\frac{\pi}{2N} + E_0^+ \approx \frac{2\pi}{N} + E_0^+ \tag{92}$$

Finally, according to the CFT, we have

$$H = \frac{1}{2\pi}\int_0^N dy[T'(w) + \bar{T}'(\bar{w})] = \frac{2\pi}{N}(L_0 + \bar{L}_0) - \frac{\pi c}{6N} + K \tag{93}$$

where K is a constant. Then we use the state $|\Delta\rangle$, $L_0|\Delta+n\rangle = (\Delta+n)|\Delta+n\rangle, \bar{L}_0|\bar{\Delta}+n\rangle = (\bar{\Delta}+n)|\bar{\Delta}+n\rangle$. We got the eigenvalues as

$$E = \frac{2\pi}{N}(\Delta + \bar{\Delta} + n + \bar{n}) - \frac{\pi c}{6N} + K, \tag{94}$$

$$E_0 = -\frac{\pi c}{6N} + K \tag{95}$$

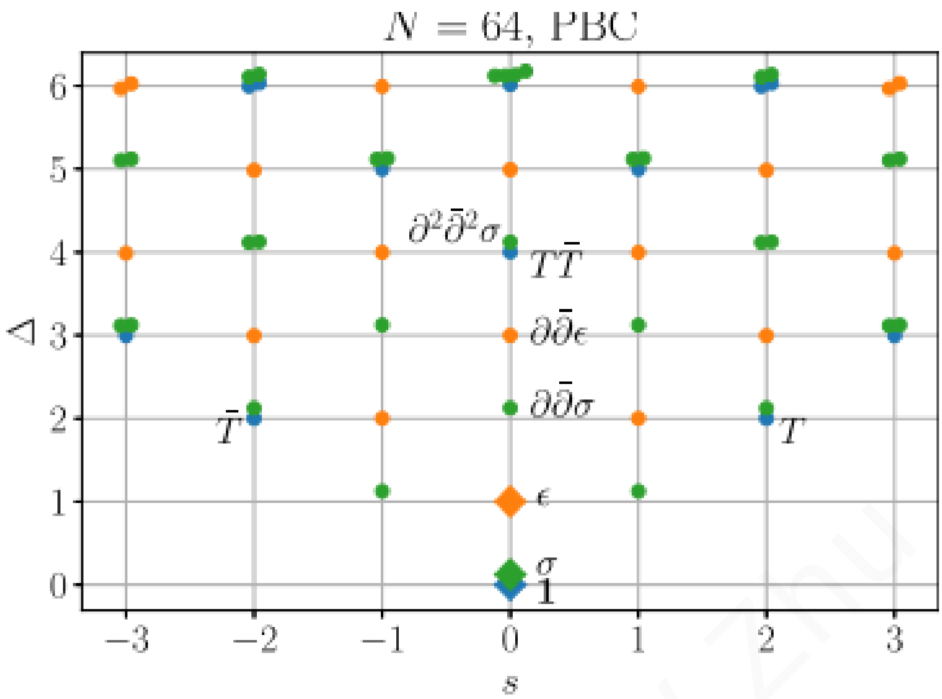

FIG. 6: Operator spectra of 1+1 transverse Ising model at the transition point.

Compare E_0^+ ($1/N$ term) with CFT result, we have $c = 1/2$. And we used $csc(x) \approx \frac{1}{x} + \frac{x}{6} + .., cot(x) \approx \frac{1}{x} - \frac{x}{3} + ..$

Comparing E_1^+ with CFT, we have

$$\Delta_\epsilon = \bar{\Delta}_\epsilon = 1/2 \tag{96}$$

Comparing E_0^- with CFT

$$\Delta_\sigma = \bar{\Delta}_\sigma = 1/16 \tag{97}$$

In conclusion, the state-operator correspondence works in 2D Ising transition.

Once we have all energies, we can go further calculate the partition function. And all thermal quantities can be derived by the partition function. Here we omit these part of discussion and show some results directly. Around T_c, the magnetization is [The Spontaneous Magnetization of a Two-Dimensional Ising Model, C. N. YANG, Phys. Rev. 85, 808 (1951)]

$$m(B = 0, T) \approx \begin{cases} [8\sqrt{2}\frac{J}{k_B T_c}(1 - \frac{T}{T_c})]^{1/8}, T \leq T_c \\ 0, T > T_c \end{cases} \tag{98}$$

IV. CONFORMAL TRANSFORMATION IN TWO DIMENSION

CFT in 2D has very different properties. 2D CFT is Virasoro algebra, which is the central extension of Witt algebra.

A. Holomorphic function

In two dimension, Eq. 9 reduces to

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1 \tag{99}$$

This is famous Cauchy-Riemann condition, which is sufficient conditions for holomorphic functions. (A complex function satisfying the Cauchy-Riemann equations is a holomorphic function in some open set.) Two dimensional conformal transformations thus coincide with the analytic coordinate transformations $z- > f(z) = z + \epsilon(z)$ and $\bar{z}- > f(\bar{z}) = \bar{z} + \epsilon(\bar{z})$.

That is, in 2D (on complex plane), all analytic or holomorphic transformations coincide with the conformal transformations. note that, there are infinite number of such kinds of holomorphic transformations in 2D, which is important for 2D (compared to $d > 2$ dimension).

B. Witt algebra

To calculate the commutation relations of the generators of the conformal algebra, i.e. infinitesimal transformations of the form $f(z)$, we take for basis

$$z \to z' = z + \epsilon(z), \quad \bar{z} \to \bar{z}' = \bar{z} + \epsilon(\bar{z}) \tag{100}$$

where take a Laurent expansion

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}, \epsilon(\bar{z}) = \sum_{n=-\infty}^{\infty} c_n \bar{z}^{n+1} \tag{101}$$

and the c_n are assumed to be infinitesmal.

To get the infiniteesimal generators for 2D CFT, let us compute the action on functions

$$e^{c_n\ell_n + \bar{c}_n\bar{\ell}_n}\psi(z,\bar{z}) = \psi(z - \epsilon(z), \bar{z} - \bar{\epsilon}(\bar{z})) \tag{102}$$

which is equivalent to

$$(1 + c_n \ell_n + \bar{c}_n \bar{\ell}_n)\psi(z, \bar{z}) = (1 - \epsilon(z)\partial_z - \bar{\epsilon}(\bar{z})\partial_{\bar{z}})\psi(z, \bar{z}) \tag{103}$$

So the generators of the conformal algebra, or the corresponding infinitesimal generators are $\ell_n = -z^{n+1}\partial_z$ and $\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$. They satisfy the algebras

$$[\ell_m, \ell_n] = z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) = (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z = (m-n)\ell_{m+n}, \tag{104}$$

$$[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}, \tag{105}$$

$$[\ell_n, \bar{\ell}_m] = 0 \tag{106}$$

The first and second equations are copies of the Witt algebra. In the quantum case, the above algebras will be corrected to include an extra term proportional to a central charge. Since two independent algebras naturally arise, it is frequently useful to regard z and $\bar{z}$ as independent coordinates.

Remark. 1) Treating z and $\bar{z}$ as independent variables yields two copies of the Witt Algebra $A \oplus \bar{A}$. 2) Imposing the 'physical condition' leaves us with the subalgebra of $A \oplus \bar{A}$ generated by $\ell_n + \bar{\ell}_n$ and $i(\ell_n - \bar{\ell}_n)$ for all n. 3) For a Quantum Theory, we need a central extension of the Witt Algebra, which is the so-called Virasoro Algebra.

C. Global conformal transformations

The above algebra is defined locally, which may be ill-defined on some singular points.

Holomorphic conformal transformations are generated by vector fields

$$v(z) = \sum_n a_n \ell_n = \sum_n a_n z^{n+1}\partial_z \tag{107}$$

Non-singularity of $v(z)$ as $z \to 0$ allows $n \geq -1$. Similarly, non-singularity at $z \to \infty$ requires $n \leq 1$. Thus, we see that only the conformal transformations generated by a ℓ_n for $n = 0, \pm 1$ are globally defined.

In two dimensions the global conformal group is defined to be the group of conformal transformations that are well-defined and invertible on the Riemann sphere $C \cup \infty$. How does this finite subalgebra correspond to the momentum, rotation, etc. generators?

It is clear that $\ell_{-1} = \partial_z = \frac{1}{2}(\partial_0 - i\partial_1)$ and $\bar{\ell}_{-1} = \frac{1}{2}(\partial_0 + i\partial_1)$ generate translations on the complex plane. Similarly, ℓ_1 and $\bar{\ell}_1$ generate special conformal translations (to see it, make the transformation of $w- > z^{-1}$, then l_1 is the translational symmetry on w). To understand ℓ_0, we can consider complex polar coordinates $z = r\exp^{i\theta}$. In terms of these variables, $\ell_0 = r\partial_r - i\partial_\theta$, $\bar{\ell}_0 = r\partial_r + i\partial_\theta$. Then the useful linear combinations are easily seen to be $\ell_0 + \bar{\ell}_0 = r\partial_r$, $i(\ell_0 - \ell_0) = \partial_\theta$. Therefore, for definition, we identify ℓ_{-1} and $\bar{\ell}_{-1}$ is the translational operator, $i(\ell_0 - \bar{\ell}_0)$ as rotation operator and $\ell_0 + \bar{\ell}_0$ as dilation operator.

- translation: $a = 1, b, c = 0, d = 1$, $l_{-1} + \bar{l}_{-1}, i(l_{-1} - \bar{l}_{-1})$
- rotation: $a = e^{i\theta/2}, b = 0, c = e^{-i\theta/2}, d = 0$, i$(l_0 - \bar{l}_0)$
- dialtation: $a = \lambda, b = 0, c = \lambda^{-1}, d = 0$, $l_0 + \bar{l}_0$
- special conformal transformation: $a = 1, b = 0, c, d = 1$, $l_1 + \bar{l}_1, i(l_1 - \bar{l}_1)$

Together, these operators generate transformations of the form $z \to \frac{az+b}{cz+d}, ad - bc = 1$ ($\in SL(2,C)/Z_2 = SO(3,1)$ on Riemann sphere $C \cup \infty$), which is precisely the conformal group (or Mobius group) $PSL(2,C)$ (The quotient by Z_2 is due to the fact that $z \to \frac{az+b}{cz+d}$ is unaffected by taking all of a, b, c, d to minus themselves; For this transformation to be invertible, it requires $|ad - bc| \neq 0$, then it to get $ad - bc = 1$).

The distinction encountered here between global and local conformal groups is unique to two dimensions (in higher dimensions there exists only a global conformal group).

D. Virasoro algebra from central extension

Here we discuss the central extension of Witt algebra. By introducing the central extension of Witt algebra (or said in group level, a central extension of the group PSL(2, C)). Why we need to consider such a group extension? Remaind that the transformation we considered is projective, in physics, this means that the state $|\psi\rangle$ is indistinguishable from any nonzero scalar multiple $c|\psi\rangle$. In general, the projective representation cannot be lifted to a true representation (ordinary linear representation), but the central extension allows you to write a true representation of a different group (in the case of the original group is a Lie group, in our case, the group is SL(2, C)) which is a Lie group). In our case, the Witt

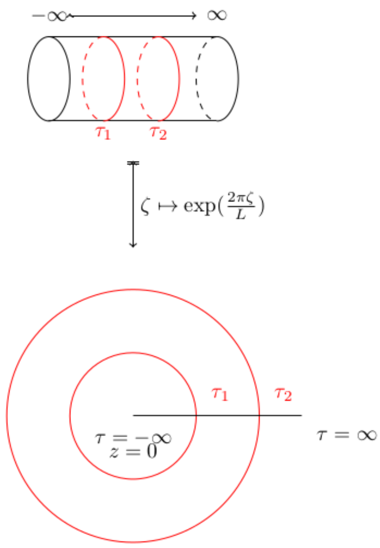

FIG. 7: Map from cylinder to plane.

algebra becomes the Virasoro algebra (We leave the derivation of this relation for future)

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{108}$$

where $L_n(n \in Z)$ denotes the elements of the central extension of the Witt algebra, and c is central charge. This is the most important results for 2D CFT. Crucially, the study of 2D CFT is actually equivalent to analysis of representation of this infinite Virasoro algebra.

E. Radial quantization

To probe more carefully the consequences of conformal invariance in a two dimensional quantum field theory, we enter into some of the details of the quantization procedure (we will focus our studies on conformal field theories defined on Euclidean two-dimensional flat space.). We begin with flat Euclidean "space" and "time" coordinates σ_1 and σ_0 (In Minkowski space, one needs to make a Wick rotation $\sigma_0 = it$). The coordinate $\sigma_1 = \sigma_1 + 2\pi$, is the periodic boundary condition for 1d spatial dimension (Or, it is to eliminate any infrared divergences, we compactify the space coordinate). Thus, we define the complex number $\zeta = \sigma_0 + i\sigma_1$.

The conformal map

$$z = exp(\sigma_0 + i\sigma_1) \tag{109}$$

maps the cylinder to the complex plane. Equal time surfaces, $\sigma_0 = const$, become circles of constant radius on the z-plane, and time reversal, $\sigma_0 \to -\sigma_0$, becomes $z \to 1/z^*$. Then the time translation $\sigma_0 - > \sigma_0 + T$ becomes dilatation on the complex plane $z- > e^T z$. So the dilatation generator on the conformal plane can be regarded as the Hamiltonian for the system, and the Hilbert space is built up on surfaces of constant radius. This procedure for defining a quantum theory on the plane is known as radial quantization (Hilbert state space defined on circles about the origin, and propagation of states in the radial direction dilation operator is the Hamiltonian; rotation operator is a spatial translation). Thus the hamiltonian on the complex plane is

$$H = (\ell_0 + \bar{\ell}_0). \tag{110}$$

The radial quantization, in particular, is to quantize this Hamiltonian or said to quantize the generators on the complex plane. And the quantized hamiltonian becomes

$$H = (L_0 + \bar{L}_0). \tag{111}$$

F. State-operator correspondence

We make the assumptions: i) There is a vacuum state $|0\rangle$ from which we can construct the Hilbert space of states in terms of creation operators (positive frequency modes); ii) As $\tau \to \pm\infty$, that is, as $(z, \bar{z}) \to \infty, 0$, the Hilbert space of states looks like the Hilbert space of states for a theory of free fields.

We can define the *operator-state correspondence*

$$|\phi_{in}\rangle = \lim_{z\to 0} \phi(z, \bar{z})|0\rangle, \langle\phi_{out}| = \lim_{z\to 0} \langle 0|\phi^\dagger(z, \bar{z}) \tag{112}$$

Let us clarify here, the hermitian conugation on Euclidean time $\tau \to -\tau$, leads to $z \to \frac{1}{z^*}$. For the primary fields,

$$\phi^\dagger(z, z) = z^{-2\bar{h}} \bar{z}^{-2h} \phi(\frac{1}{\bar{z}}, \frac{1}{z}). \tag{113}$$

Then we can compute

$$\langle\phi_{out}|\phi_{in}\rangle = \lim_{z\to 0} \lim_{w\to 0} \langle 0|\phi^\dagger(w, \bar{w})\phi(z, \bar{z})|0\rangle \tag{114}$$

$$= \lim_{z,w} w^{-2\bar{h}} \bar{w}^{-2h} \langle 0|\phi(\frac{1}{\bar{w}}, \frac{1}{w})\phi(z, \bar{z})|0\rangle = \lim_{z,w} w^{-2\bar{h}} \bar{w}^{-2h} \langle\phi(\frac{1}{\bar{w}}, \frac{1}{w})\phi(z, \bar{z})\rangle \tag{115}$$

$$= \lim_{z,w\to 0} \frac{C}{w^{2\bar{h}} \bar{w}^{2h} (\frac{1}{\bar{w}} - z)^{2h} (\frac{1}{w} - \bar{z})^{2\bar{h}}} = C \tag{116}$$

where we have used before.

We now consider a field $\phi(z,\bar{z})$ with conformal dimensions $(h,\bar{h})$ for which we can perform a Laurent expansion around $z_0 = \bar{z}_0 = 0$, thus we have the Mode expansion for quasi-primary fields

$$\phi(z,\bar{z}) = \sum_{m,n} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n} \tag{117}$$

$$\phi_{m,n} = \oint \frac{dz}{2\pi i} z^{m+h-1} \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+\bar{h}-1}\phi(z,\bar{z}) \tag{118}$$

where the $\phi_{m,n}$ are operators, and $\phi^{\dagger}_{m,n} = \phi_{-m,-n}$ (where the additional factors of h in the exponents can be explained by the radial conformal map). This is from:

$$\phi^{\dagger}(z,\bar{z}) = z^{-2\bar{h}}\bar{z}^{-2h}\phi(\frac{1}{\bar{z}},\frac{1}{z}) \tag{119}$$

$$\phi^{\dagger}(z,\bar{z}) = \sum_{m,n} \bar{z}^{-m-h}z^{-n-h}\phi^{\dagger}_{m,n} \tag{120}$$

$$z^{-2\bar{h}}\bar{z}^{-2h}\phi(\frac{1}{\bar{z}},\frac{1}{z}) = z^{-2\bar{h}}\bar{z}^{-2h}\sum_{m,n}(\frac{1}{\bar{z}})^{-m-h}(\frac{1}{z})^{-n-\bar{h}}\phi_{m,n} = \sum_{m,n} \bar{z}^{-m-h}z^{-n-\bar{h}}\phi_{-m,-n} \tag{121}$$

Additionally, with the help of the condition $\lim_{z->0}\phi(z,\bar{z}) = finite$ gives the condition $\phi_{m,n}|0\rangle = 0$ for $m > -h$ and $n > -\bar{h}$.

G. Energy-momentum tensor

We introduce the energy-momentum tensor from several ways. under an arbitrary transformation of the coordinates $x^\mu - > x^\mu + \epsilon^\mu$, the action or Lagrangian changes as follows:

$$
\begin{aligned}
& S = \int d^d x \mathcal{L}[\phi, \partial_\mu \phi] \to \int d^d x \mathcal{L}[\phi(x+\epsilon), \partial_\mu \phi(x+\epsilon)] \\
& \phi(x) \to \phi(x+\epsilon) = \phi(x) + \epsilon^\rho \partial_\rho \phi(x) = \phi(x) + \delta\phi(x) \\
& \partial_\mu \phi(x) \to \partial_\mu \phi(x+\epsilon) = \partial_\mu \phi(x) + \partial_\mu(\epsilon^\rho \partial_\rho \phi(x)) = \partial_\mu \phi(x) + \delta \partial_\mu \phi(x) \\
& \mathcal{L}[\phi(x+\epsilon), \partial_\mu \phi(x+\epsilon)] = \mathcal{L}[\phi(x), \partial_\mu \phi(x)] + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \\
& \ = \mathcal{L}[\phi(x), \partial_\mu \phi(x)] + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu [\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}] \delta\phi + \partial_\mu [\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\phi)] \\
& \ = \mathcal{L}[\phi(x), \partial_\mu \phi(x)] + \partial_\mu [\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\phi)] \\
& \mathcal{L}[\phi(x+\epsilon), \partial_\mu \phi(x+\epsilon)] = \mathcal{L}[\phi(x), \partial_\mu \phi(x)] + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu \\
& \ \Rightarrow \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu = \partial_\mu [\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\phi)] \Rightarrow \partial_\mu [\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\rho \phi - \mathcal{L} g^{\mu\rho}] \epsilon_\rho = \partial_\mu T^{\mu\rho} \epsilon_\rho = 0
\end{aligned}
\tag{122}
$$

where in the last line we used $\delta\phi = \epsilon^\rho \partial_\rho \phi = \epsilon^\rho \delta_\rho^\beta \partial_\beta \phi = \epsilon^\rho g_{\rho\alpha} g^{\alpha\beta} \partial_\beta \phi = \epsilon_\rho \partial^\rho \phi$ and the energy-momentum tensor is defined by

$$
T^{\mu\rho} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\rho \phi - \mathcal{L} g^{\mu\rho} \tag{123}
$$

Next we discuss some properties of $T^{\mu\nu}$. First, let us recall Noether's theorem which states that for every continuous symmetry in a field theory, there is a current j^μ which is conserved, i.e. $\partial_\mu j^\mu = 0$. Since we are interested in theories with a conformal symmetry $x^\mu \to x^\mu + \epsilon^\mu(x)$, we have a conserved current which can be written as

$$
j^\mu = T^{\mu\nu} \epsilon_\nu \tag{124}
$$

where symmetric tensor $T_{\mu\nu}$ is energy momentum tensor. For a special case of conformal transformation $\epsilon^\mu = 1$, we have

$$
0 = \partial_\mu j^\mu = \partial_\mu (T^{\mu\nu} \epsilon_\mu) = (\partial_\mu T^{\mu\nu}) \epsilon_\mu, \Rightarrow \partial^\mu T_{\mu\nu} = 0 \tag{125}
$$

which can be also obtained immediately from Eq. 122.

Second, under an arbitrary transformation of the coordinates $x^\mu - > x^\mu + \epsilon^\mu$ as follows:

$$
\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu = \frac{1}{2} \int d^d x T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{126}
$$

where T is the energy-momentum tensor, assumed to be symmetric (In Lagrangian theories, it is the response of the Lagrangian to the variation of space-time). According to the definition of Eq. 9, we have

$$\delta S = \frac{1}{d}\int d^d x T^{\mu\nu} g_{\mu\nu}\partial_\rho \epsilon^\rho = \frac{1}{d}\int d^d x T^\mu_\mu \partial_\rho \epsilon^\rho = 0 \tag{127}$$

The tracelessness of the energy-momentum tensor $T^\mu_\mu = 0$ then implies the invariance of the action under conformal transformations.

Third, the energy-momentum tensor $T_{\mu\nu}$ generates local coordinate transformations. We get the conserved charge

$$P_\mu = \int d^d x T_{\mu\mu} \tag{128}$$

which are the momentum operators that are translationally symmetric.

Next, we need to make some detailed calculation to get the energy-momentum tensor in complex plane.

$$z = x_1 + ix_2, \bar{z} = x_1 - ix_2 \tag{129}$$

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \tag{130}$$

We can then compute the energy-momentum tensor

$$T_{zz} = \frac{\partial x_1}{\partial z}\frac{\partial x_1}{\partial z}T_{11} + \frac{\partial x_1}{\partial z}\frac{\partial x_2}{\partial z}T_{12} + \frac{\partial x_2}{\partial z}\frac{\partial x_1}{\partial z}T_{21} + \frac{\partial x_2}{\partial z}\frac{\partial x_2}{\partial z}T_{22} = \frac{1}{4}[T_{11} - iT_{12} - iT_{21} - T_{22}] \tag{131}$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}[T_{11} + iT_{12} + iT_{21} - T_{22}] \tag{132}$$

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{\partial x_1}{\partial z}\frac{\partial x_1}{\partial \bar{z}}T_{11} + \frac{\partial x_1}{\partial z}\frac{\partial x_2}{\partial \bar{z}}T_{12} + \frac{\partial x_2}{\partial z}\frac{\partial x_1}{\partial \bar{z}}T_{21} + \frac{\partial x_2}{\partial z}\frac{\partial x_2}{\partial \bar{z}}T_{22} = \frac{1}{4}[T_{11} + T_{22}] \tag{133}$$

On the Euclidean space $g_{\mu\nu} = \delta_{\mu\nu}$, or $ds^2 = dx^2 + dy^2 = dzd\bar{z}$, giving

$$g_{zz} = g_{\bar{z}\bar{z}} = g_{\mu\nu}\frac{\partial x_\mu}{\partial z}\frac{\partial x_\nu}{\partial z} = 0 \tag{134}$$

$$g_{z\bar{z}} = g_{\bar{z}z} = g_{\mu\nu}\frac{\partial x_\mu}{\partial z}\frac{\partial x_\nu}{\partial \bar{z}} = \frac{1}{2} \tag{135}$$

The conservation law $g^{\alpha\mu}\partial_\alpha T_{\mu\nu} = 0$ gives:

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0 \tag{136}$$

$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0 \tag{137}$$

In 2d case, the traceless equation reads

$$T_{z\bar{z}} = T_{\bar{z}z} = 0 \tag{138}$$

which implies that the off-diagonal component vanishes in complex coordinates. Then the conservation equation $\partial_\nu T_{\mu\nu} = 0$ becomes

$$\partial_z T_{\bar{z}\bar{z}} = 0, \partial_{\bar{z}} T_{zz} = 0 \tag{139}$$

which implies that the non-vanishing components T_{zz} are holomorphic (anti-holomorphic), respectively. This motivates the introduction of the notation $T(z) = T_{zz}$, which is chiral (only depends on z but not on $\bar{z}$).

The energy-momentum tensor is the most important quantity in CFT, because it enables one to derive simple differential equations for correlation function of primary fields, and this is often called energy-momentum tensor method or stress-energy tensor method. Consider the current $j^\mu = T^{\mu\nu}\epsilon_\nu$, the corresponding conserved charge

$$Q = \frac{1}{2\pi i}\oint (dzT(z)\epsilon(z) + z- > \bar{z}) \tag{140}$$

And the infinitesimal symmetry variation in any field A should be $\delta_\epsilon A = -\epsilon[Q, A]$.

This allows us to determinate the infinitesimal transformation of a field ($\delta_\epsilon\phi(x) = \phi'(x) - \phi(x) = -\epsilon[Q, \phi(x)]$) (Eq. **??**)

$$\delta_\epsilon\phi(w,\bar{w}) = -\epsilon[Q, \phi(w,\bar{w})] = -\frac{1}{2\pi i}\oint (dz\epsilon(z)\mathcal{R}T(z)\phi(w,\bar{w}) + z- > \bar{z}) \tag{141}$$

Compared with Eq. 42,

$$\delta_\epsilon\phi(w,\bar{w}) = -\epsilon[Q, \phi(w,\bar{w})] = -\frac{1}{2\pi i}\oint (dz\epsilon(z)\mathcal{R}T(z)\phi(w,\bar{w}) + z- > \bar{z}) \tag{142}$$

$$= -h(\partial_w\epsilon(w))\phi(w,\bar{w}) - \epsilon(w)\partial_w\phi(w,\bar{w}) \tag{143}$$

With the help of the relation

$$\oint \frac{dz}{2\pi i}\frac{\epsilon(z)\phi(w,\bar{w})}{(z-w)^2} = \partial_w(\epsilon(w))\phi(w,\bar{w}) \tag{144}$$

$$\oint \frac{dz}{2\pi i}\frac{\epsilon(z)\partial_w\phi(w,\bar{w})}{(z-w)} = \epsilon(w)\partial_w\phi(w,\bar{w}) \tag{145}$$

Thus we get

$$RT(z)\phi(w,\bar{w}) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + ... \tag{146}$$

Here the method we used is so-called opertor product expansion(OPE). OPE is a tool that is used repetitively: it means that when two opertors come close to each other, it is possible to replace their product by a sum of local operators: $A(x)B(y) \sim \sum_n C_n(x-y)O_n$. The OPE is the idea that two local operators inserted at nearby points can be approximated by a string of operators at one of these points. Rather than using our original definitions, we could have defined a primary field as one whose OPE with the stress-energy tensor takes the form Eq. 146.

Furthermore, one can prove that, the OPE of $T(z)$ with $\partial\phi(w)$ as (applying derivation of ∂_w on Eq. 146)

$$T(z)\partial_w\phi(w) = \frac{2h}{(z-w)^3}\phi(w,\bar{w}) + \frac{h+1}{(z-w)^2}\partial_w\phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w^2\phi(w,\bar{w}) + ... \tag{147}$$

Also one can get that

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + ... \tag{148}$$

Here, we have to mention that, energy-momentum tensor is not primary field (for non-vanishing central charges). Instead, $T(z)$ is a chiral quasi-primary field with conformal weight $h=2$.

At last, under the conformal transformation $z->f(z)$,

$$T'(z) = (\frac{\partial f(z)}{\partial z})^2 T(f(z)) + \frac{c}{12}S(f(z),z) \tag{149}$$

with the Schwarzian derivative defined as

$$S(w,z) = \frac{1}{(\partial_z w)^2}[(\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2] \tag{150}$$

We will not prove Schwarzian derivative in detail but verify it on the level of infinitesimal conformal transformations $f(z) = z + \epsilon(z)$. In order to do so, we first use the OPE of the energy-momentum tensor to find

$$\delta_\epsilon T(z) = \frac{1}{2\pi i}\oint_{C(z)} dw\epsilon(w)T(w)T(z) \tag{151}$$

$$= \frac{1}{2\pi i}\oint_{C(z)} dw\epsilon(w)[\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + ...] \tag{152}$$

$$= \frac{c}{12}\partial_z^3\epsilon(z) + 2T(z)\partial_z\epsilon(z) + \epsilon(z)\partial_z T(z) \tag{153}$$

Let us compare this expression with Schwarzian derivative. For infiniteesimal transformations $f(z) = z + \epsilon(z)$, the leading order contribution to the Schwarzian derivative reads

$$\delta_\epsilon T(z) = T'(z) - T(z) = (1 + \partial_z \epsilon(z))^2 (T(z) + \epsilon(z)\partial_z T(z)) + \frac{c}{12}\partial_z^3 \epsilon(z) - T(z) \tag{154}$$

$$= \frac{c}{12}\partial_z^3 \epsilon(z) + 2T(z)\partial_z \epsilon(z) + \epsilon(z)\partial_z T(z) \tag{155}$$

which is the same. We have thus verified Schwarzian derivative at the level of infinitesimal conformal transformations.

H. Operator product expansions

We have seen the property of OPE in energy-momentum tensor in CFT. OPE describes the behavior of radially ordered quantum fields as the coincide, which is a useful tool. It means that when two opertors come close to each other, it is possible to replace their product by a sum of local operators: $A(x)B(y) \sim \sum_n C_n(x-y)O_n$. The OPE is the idea that two local operators inserted at nearby points can be approximated by a string of operators at one of these points.

For example, for $A(z,\bar{z})$ and $B(z,\bar{z})$

$$\mathcal{R}A(z)B(w) = \sum_{n=-\infty}^{N} \sum_{m=-\infty}^{M} \frac{\{AB\}_{n,m}(w,w)}{(z-w)^n(\bar{z}-\bar{w})^m} \tag{156}$$

In this case, we get a finite number of singular terms. These singular terms play a special role:

$$\overline{A(z)B}(w) = \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{\{AB\}_{n,m}(w,w)}{(z-w)^n(\bar{z}-\bar{w})^m} \tag{157}$$

, we need to separate them. In this taking, the OPE of $A(z)$ with $B(w)$ is expressed as

$$A(z)B(w) = \overline{A(z)B}(w) + :A(z)B(w): \tag{158}$$

where $:A(z)B(w):$ stands for the complete sequence of regular terms whose explicit forms can be extracted from the Taylor expansion of $A(z)$ around w.

$$:A(z)B(w) := \sum_{k\geq 0} \frac{(z-w)^k}{k!}(\partial^k AB)(w) \tag{159}$$

The degree zero term is known as the normal order product: $:AB:(w) = \{AB\}_{0,0}(w,w) = \oint_w dz \oint_w dz \frac{\mathcal{R}A(z)B(w)}{(w-z)(\bar{w}-\bar{z})}$. Or we have

$$:AB:(w) = \lim_{z\to w}[A(z)B(w) - \overline{A(z)B}(w)] \Rightarrow :AB:(w) = \{AB\}_{0,0}(w,w) \tag{160}$$

I. Example: Free Boson

Let us consider the free boson,

$$S \equiv \int d^2x\mathcal{L} = \frac{1}{2}\int d^2x\partial^\mu\phi\partial_\mu\phi = \frac{1}{2}\int d^x d^2y\phi(x)A(x,y)\phi(y) \tag{161}$$

where $A(x,y) = \delta(x-y)\partial_\mu\partial^\mu$.

The two-point correlation, Feynman propagator, is given by ($G(x,y) = A^{-1}(x,y)$): $\partial^2 G(x,y) = -\delta(x-y)$. To solve it, there are many ways. Here we used a simplest one by integrating the equation within a disk of radius r centered around y: $1 = 2\pi\int_0^r d\rho\rho[-\frac{\partial}{\rho\partial\rho}(\rho G'(\rho))] = -2\pi r G'(r)$, thus $G(r) = -\frac{1}{2\pi}\ln r$. ($\nabla^2 u = \partial_{rr}u + r^{-1}\partial_r u + r^{-2}\partial_{\theta\theta}u$)

Or we can get the Green's function from the variation on action:

$$S = \frac{1}{2}\int d^2x g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}\int d^2z g^{z\bar{z}}\partial_z\phi\partial_{\bar{z}}\phi + g^{\bar{z}z}\partial_{\bar{z}}\phi\partial_z\phi = \int d^2z\partial^{\bar{z}}\phi\partial_{\bar{z}}\phi + \partial^z\phi\partial_z\phi \tag{162}$$

$$\delta S = 2\int d^2z[\partial_z\delta\phi\partial_{\bar{z}}\phi + \partial_z\phi\partial_{\bar{z}}\delta\phi] = 4\int d^z\delta\phi\partial_z\partial_{\bar{z}}\phi = 0 \tag{163}$$

$$\Rightarrow \partial_z\partial_{\bar{z}}\phi = 0 \tag{164}$$

Again, this equation of motion shows z and $\bar{z}$ are independent, therefore chiral and anti-chiral are decoupled. Using the equation of motion, we can define the green's function as

$$\partial_z\partial_{\bar{z}}G(z,w) = -\delta^2(z-w) \Rightarrow G(z,w) = -\frac{1}{2\pi}\ln|z-w|^2 \tag{165}$$

where we used the relation $\partial_z\frac{1}{\bar{z}} = 2\pi\delta^2(z)$, or we can solve Laplacian equation directly as above: $\nabla^2\ln r = 2\pi\delta(\mathbf{r})$.

Thus, we have

$$\langle\phi(x)\phi(y)\rangle = -\frac{1}{2\pi}\ln|x-y|, \quad \langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\frac{1}{4\pi}[\ln(z-w) + \ln(\bar{z}-\bar{w})]. \tag{166}$$

The holomorphic and antiholomorphic components can be separated by taking the derivatives

$$\langle\partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w})\rangle = -\frac{1}{4\pi}\frac{1}{(z-w)^2}. \tag{167}$$

This OPE reflects the bosonic character of the field: exchanging the two factors does not affect the correlator. Importantly, it shows the operator $\partial_z\phi(z,\bar{z})$ has conformal weight $h = 1$.

The energy-momentum tensor associated with the free massless boson is

$$T_{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\phi)}\partial_{\nu}\phi - g_{\mu\nu}\mathcal{L} = \partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\partial^{\rho}\phi\partial_{\rho}\phi \tag{168}$$

where we have used $\mathcal{L} = \frac{1}{2}g_{\rho\alpha}\partial^{\alpha}\phi\partial^{\rho}\phi$, $\frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)} = \frac{1}{2}g_{\rho\alpha}\partial^{\alpha}\phi\delta_{\rho\mu} + \frac{1}{2}g_{\rho\alpha}\delta_{\alpha\mu}\partial^{\rho}\phi = \partial_{\mu}\phi$.

In complex coordinates, we can compute $T(z) = T_{zz} = \frac{1}{4}(T_{11} - 2iT_{12} - T_{22}) = \partial_z\phi\partial_z\phi$. Like all composite fields, the energy-momentum tensor has to be normal ordered, in order to ensure the vanishing of its vacuum expectation value. More explicitly, the exact meaning of the above expression is

$$T(z) = -2\pi : \partial_z\phi\partial_z\phi := -2\pi \lim_{w->z}[\partial_z\phi(z)\partial_w\phi(w) - \langle\partial_z\phi(z)\partial_w\phi(w)\rangle] \tag{169}$$

The product of $T(z)$ on field $\partial\phi(w)$ is

$$\begin{aligned} T(z)\partial_w\phi(w) &= -2\pi : \partial_z\phi(z)\partial_z\phi(z) : \partial_w\phi(w) \\ &= -2\pi : \partial_z\phi(z)\partial_z\phi(z) : \partial_w\phi(w) - 2\pi : \partial_z\phi(z)\partial_z\phi(z) : \partial_w\phi(w) \\ &= \frac{\partial_z\phi(z)}{(z-w)^2} = \lim_{z\to w}\frac{\partial_w\phi(w) + (z-w)\partial_w\phi(w)}{(z-w)^2} \\ &\sim \lim_{z->w}\frac{\partial_w\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)} + ... \end{aligned} \tag{170}$$

Which tells us that $\partial_z\phi(z,\bar{z})$ is a chiral primary field with conformal weight $h = 1$ (see Eq. 146).

Wick's theorem also allows us to calculate the OPE of the energy-momentum tensor with itself:

$$\begin{aligned} T(z)T(w) &= 4\pi : \partial_z\phi(z)\partial_z\phi(z) :: \partial_w\phi(w)\partial_w\phi(w) : \\ &= 4\pi[2 : \partial_z\phi(z)\partial_z\phi(z) :: \partial_w\phi(w)\partial_w\phi(w) : +4 : \partial_z\phi(z)\partial_z\phi(z)\partial_w\phi(w)\partial_w\phi(w) :] \\ &= \frac{1/2}{(z-w)^4} - 4\pi\frac{: \partial_z\phi\partial_z\phi :}{(z-w)^2} \\ &= \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + ... \end{aligned} \tag{171}$$

where central charge $c = 1$ (see Eq. 148). Note that $T(z)$ is not primary because of the central charge term. It is 'almost' a chiral quasi-primary field of conformal weight $h = 2$. (check definition of Eq. 35)

J. Conformal anomaly and central extensions

It is convenient to define a Laurent expansion of the stress-energy tensor,

$$T(z) = \sum_n z^{-n-2} L_n, \bar{T}(\bar{z}) = \sum_n \bar{z}^{-n-2} \bar{L}_n \tag{172}$$

$$L_n = \oint \frac{dz}{2\pi i} T(z) z^{n+1}, \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{T}(\bar{z}) \bar{z}^{n+1} \tag{173}$$

which gives us the commutation relations

$$[L_n, L_m] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{n+1} w^{m+1} \mathcal{R}(T(z)T(w)) \tag{174}$$

$$= \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{n+1} w^{m+1} [\frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}] \tag{175}$$

$$= \oint_0 \frac{dw}{2\pi i} w^{m+1} [\frac{c}{12}(n+1)n(n-1) w^{n-2} w^{m+1} + 2(n+1) w^n w^{m+1} T(w) + w^{n+1} w^{m+1} \partial T(w)] \tag{176}$$

$$= \frac{c}{12}(n^3 - n)\delta_{m+n,0} + 2(m+1) L_{m+n} + \oint \frac{dw}{2\pi i} \{\partial(w^{m+n+2} T(w)) - (m+n+2) w^{m+n+1} T(w)\} \tag{177}$$

$$= \frac{c}{12}(n^3 - n)\delta_{m+n,0} + 2(m+1) L_{m+n} - (m+n+2) L_{m+n} \tag{178}$$

$$= \frac{c}{12}(n^3 - n)\delta_{m+n,0} + (n-m) L_{m+n} \tag{179}$$

Similarly, we have

$$[\bar{L}_n, \bar{L}_m] = \frac{c}{12}(n^3 - n)\delta_{m+n,0} + (n-m)\bar{L}_{n+m} \tag{180}$$

$$[L_n, \bar{L}_m] = 0 \tag{181}$$

These relations definies the Virasoro Algebra (originally discovered in the context of string theory), which is the central extension of Witt algebra (Eq. 104).

Additionally, we can see that, $T(z)$ is an 'almost' quasi-primary field with $h = 2$. Under a finite global conformal transformation takes the form $z \to w(z) = \frac{az+b}{cz+d}$,

- $w = z + c$: Schwarzian derivative Eq. 149 $\{\lambda z; z\} = 0$;
- $w = \lambda z$: $\{\lambda z; z\} = 0$;
- $w = -\frac{1}{z}$: $\{\frac{1}{z}; z\} = 0$;

As a result, we see that $\{w(z); z\} = 0$ for any PSL(2,C) transformations. This gives

$$T'(w) = (\frac{\partial w}{\partial z})^{-2} T(z) \tag{182}$$

Again, it shows $h = 2$ as we obtained before.

Here we see $T(z)$ is related to PSL(2,C), which is a projective of SL(2,C). This means the operator L_n those are defined by $T(z)$ should have some 'internal' degree of freedom.

K. Central charge

We will further elucidate the physical meaning of central charge from several aspects. Crucially, one should ask: Why we need to consider such a group extension? Remaind that the transformation we considered is projective, in physics, this means that the state $|\psi\rangle$ is indistinguishable from any nonzero scalar multiple $c|\psi\rangle$. In general, the projective representation cannot be lifted to a true representation (ordinary linear representation), but the central extension allows you to write a true representation of a different group (in the case of the original group is a Lie group, in our case, the group is SL(2, C)) which is a Lie group). In our case, the Witt algebra becomes the Virasoro algebra. The appearance of the central charge c, also known as the conformal anomaly.

A formal way

Let us now come back to the Witt algebra of infinitesimal conformal transformations. As it turns out, this algebra admits a so-called central extension. Without providing a mathematically rigorous definition, we state that the central extension $\tilde{g} = g \bigoplus C$ of a Lie algebra g by C is characterised by the commutation relations

$$[\tilde{x}, \tilde{y}]_{\tilde{g}} = [x, y]_g + cp(x, y) \tag{183}$$

$$[\tilde{x}, c]_{\tilde{g}} = 0, [c, c]_{\tilde{g}} = 0 \tag{184}$$

where $p : g \times g \to C$ is bilinear. Central extensions of algebras are closely related to projective representations which are common to Quantum Mechanics. In the following, we are going to allow for such additional structure.

More concretely, let us denote the elements of the central extension of the Witt algebra by L_n with $n \in Z$ and write their commutation relations as

$$[L_m, L_n] = cp(m,n) + (m-n)L_{n+m} \tag{185}$$

The precise form of $p(m,n)$ is determined in the following way:

1. it is clear that $p(m,n) = -p(n,m)$ in order for $p(m,n)$ to be compatible with the anti-symmetry of the Lie bracket.

2. one can always arrange for $p(1,-1) = 0$ and $p(n,0) = 0$ by a redefinition

$$\hat{L}_n = L_n + \frac{cp(n,0)}{n}, \quad n \neq 0 \tag{186}$$

$$\hat{L}_0 = L_0 + \frac{cp(1,-1)}{2} \tag{187}$$

   Indeed, for the modified generators we see that the $p(n,m)$ vanishes

$$[\hat{L}_n, \hat{L}_0] = nL_n + cp(n,0) = n\hat{L}_n \tag{188}$$

$$[\hat{L}_1, \hat{L}_{-1}] = 2L_0 + cp(1,-1) = n\hat{L}_0 \tag{189}$$

3. Next, to compute the following particular Jacobi identity:

$$0 = [[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] \tag{190}$$

$$0 = (m-n)cp(m+n,0) + np(n,m) - mcp(m,n) \tag{191}$$

$$0 = (m+n)cp(n,m) \tag{192}$$

   from which we infer that in the case $n \neq -m$, we have $p(n,m) = 0$. Therefore, the only non-vanishing central extensions are $p(n,-n)$ for $|n| \geq 2$.

4. to compute Jacobi identity:

$$0 = [[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] \tag{193}$$

$$0 = (-2n+1)cp(1,-1) + (n+1)cp(n-1,-n+1) + (n-2)cp(-n,n) \tag{194}$$

$$\Rightarrow p(n,-n) = \frac{n+1}{n-2}p(n-1,-n+1) = \frac{1}{2}\begin{pmatrix} n+1 \\ 3 \end{pmatrix} = \frac{1}{12}(n+1)n(n-1) \tag{195}$$

   where we have normalised $p(2,-2) = 1/2$. This normalisation is chosen such that the constant c has a particular value for the standard example of the free boson.

In summary, we have $p(m,n) = \delta_{m,-n}(m^3 - m)$.

Rigorous definition

In a rigorous mathematical definition, we can compute the second cohomology group H^2 of the Witt algebra. We recall exactness of the sequence means that the kernel of every map in the sequence equals the image of the previous map

$$1 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 1 \tag{196}$$

The conclusion is $G_3 \cong G_2/G_1$. (Think about examples: $S^1 = E^1/Z$, $T^2 = E^2/(z \bigoplus z)$) The extension is called central if G_1 is abelian and its image is in the center of E.

A trivial example is:

$$1 \to A \xrightarrow{f} A \times G \xrightarrow{g} G \to 1 \tag{197}$$

where $A \times G$ denotes the product group and where $A \to G$ is given by $a \to (a, 1)$. This extension is central.

In our case, the universal covering group of the Lorentz group $SO(1,3)$ is (isomorphic to) a central extension of $SO(1,3)$ by the group $\{+1, -1\}$. In fact, there is the exact sequence of Lie groups

$$1 \to \{+1, -1\} \xrightarrow{f} SL(2, C) \xrightarrow{g} SO(1,3) \to 1 \tag{198}$$

where g is the 2-to-1 covering. (think about example $1 \to \{+1, -1\} \xrightarrow{f} SU(2) \xrightarrow{g} SO(3) \to 1$)

That is, there is a short exact sequence of Lie algebra homomorphisms

$$1 \to a \xrightarrow{f} h \xrightarrow{g} g \to 1 \tag{199}$$

is called a central extension of g by a, if $[a, h] = 0$. Here we identify a with the corresponding subalgebra of h. For such a central extension the abelian Lie algebra a is realized as an ideal in h and the homomorphism: $h \to g$ serves to identify g with h/a.

Following classification of central extensions of Lie algebras, the cohomology group $H^2(g, a)$ is in one-to-one correspondence with the set of equivalence classes of central extensions of g by a, i.e. non-trivial central extensions are classified by the cohomology group

$$H^2(g, a) = \frac{Z^2(g, a)}{B^2(g, a)} \tag{200}$$

Here, $\Theta \in Z^2(g,a)$ is so-called cocycle:

$$\Theta : g \times g \to a \tag{201}$$

$$\Theta(X,Y) = -\Theta(Y,X) \tag{202}$$

$$\Theta(X,[Y,Z]) + \Theta(Z,[X,Y]) + \Theta(Y,[Z,X]) = 0 \tag{203}$$

$\Theta \in B^2(g,a)$ is exact cocycles (boundary), if

$$\mu : g \to a \tag{204}$$

$$\Theta(X,Y) = \mu([X,Y]) \tag{205}$$

Given a representative of an equivalence class $[\Theta] \in H^2(g,a)$, we see that Θ defines a Lie bracket for h by the formula:

$$[L \bigoplus \Theta, L' \bigoplus \Theta] = [L,L'] + \Theta([L,L']) \tag{206}$$

Finally, we come back to our Virasoro algebra. Here we can prove $H^2(W,C) \cong C \cong <\Theta>$. The Lie algebra is L_n, and we make a linear map: $\omega : W \times W \to C$:

$$\omega(L_n, L_m) = \delta_{n+m} \frac{n(n^2-1)}{12} \tag{207}$$

defines a nontrivial central extension of W by C and up to equivalence this is the only nontrivial extension of W by C.

In order to do this we prove

$$1.\omega \in Z^2(W,C) \tag{208}$$

$$2.\omega not \in B^2(W,C) \tag{209}$$

$$3.\Theta \in Z^2(W,C) \to \Theta \sim \lambda\omega \tag{210}$$

For step 1, it is straightforward to check by definition above.

For step 2, we have for all n

$$\omega(L_n, L_{-n}) = \tilde{\mu}(L_n, L_{-n}) \tag{211}$$

$$\to \frac{n(n^2-1)}{12} = \mu([L_n, L_{-n}]) \tag{212}$$

$$\to \frac{n(n^2-1)}{12} = 2n\mu(L_0) \tag{213}$$

$$\mu(L_0) = \frac{1}{24}(n^2-1) \tag{214}$$

The last equation cannot hold for every $n \in N$. So the assumption was wrong, which implies $\omega not \in B^2(W, C)$.

We neglect the prove of step 3.

At last, we have proved $H^2(W, C) \cong C$. The Virasoro algebra is the central extension of the Witt algebra W by C defined by $p(m, n)$.

Remark. Since $p(m, n) = 0, n, m = -1, 0, 1$, it relates to L_{-1}, L_0, L_1 are generators of $SL(2, C)/Z_2$ transformations.

L. Highest weight states

Let us define an asymptotic state

$$|h\rangle = \phi(0)|0\rangle \tag{215}$$

which is created by a holomorphic primary field $\phi(z)$ of weight h. In addition, in this system the vacuum, $T(z)|0\rangle$ for $z \to 0$ must be well defined, which implies that (see Eq. 172): $L_n|0\rangle = 0$ for all $n > -2$.

From the operator product expansion between the stress-energy and a primary field we find

$$[L_n, \phi(w)] = \oint \frac{dz}{2\pi i} z^{n+1} \mathcal{R}(T(z)\phi(w)) \tag{216}$$

$$= h(n+1)w^n \phi(w) + w^{n+1}\partial_w \phi(w) \tag{217}$$

Thus we see that $[L_n, \phi(w = 0)] = 0$ for all $n > 0$. And the state $|h\rangle$ thus satisfies

$$[L_0, \phi(0)] = h\phi(0) \to L_0|h\rangle = h|h\rangle, L_{n>0}|h\rangle = 0 \tag{218}$$

Similarly, we have $\bar{L}_0|\bar{h}\rangle = \bar{h}|\bar{h}\rangle$. At last , we see $|h, \bar{h}\rangle$ is the energy eigenstate of the Hamiltonian $H = L_0 + \bar{L}_0$ (see Eq. 111). Please note that, here we have clarified that, the reason to define the primary field (see Eq. 34).

Next, the Virasoro raising operators $L_{-m}(m > 0)$ acting on $|h, \bar{h}\rangle$ yield

$$[L_0, L_{-m}] = mL_{-m} \to L_0 L_{-m}|h, \bar{h}\rangle = (m + h)L_{-m}|h, \bar{h}\rangle \tag{219}$$

thus, $L_{-m}|h\rangle$ takes the dimension of $m + h$. In contrast, $L_m(m > 0)$ is lowing operators due to $L_{m>0}|h, \bar{h}\rangle = 0$ since $L_m|h\rangle = L_m\phi(0)|0\rangle = ([L_m, \phi(0)] - \phi(0)L_m)|0\rangle = 0$.

Any state satisfying Eq. 218 is known as a highest weight state. States of the form $L_{-n_1}...L_{-n_k}|h\rangle(n>0)$ are known as descendant states. The Hilbert space of states arising from $|h,\bar{h}\rangle$ is called a Verma modules. It forms a representation of the Virasoro algebra (see Tab. I). Please note that $L_{-1}\phi=\partial\phi$. Here the number $P(N)$ is defined by

level	dimension	fields
0	h	ϕ
1	$h+1$	$L_{-1}\phi$
2	$h+2$	$L_{-2}\phi$, $L_{-1}^2\phi$
3	$h+3$	$L_{-3}\phi$, $L_{-1}L_{-2}\phi$, $L_{-1}^3\phi$
N	$h+N$	$P(N)$

TABLE I: Verma module: decendant states generated from a highest weight state.

$$\prod_{n=1}^{\infty}\frac{1}{1-q^n}=(1+q+q^2+q^3+...)(1+q^2+q^4+...)(1+q^3+q^6+...) \tag{220}$$

$$=1+q+2q^2+3q^3+5q^4+...=\sum_{N=0}^{\infty}P(N)q^N \tag{221}$$

The normalization condition requires

$$\langle h|L_{-n}^{\dagger}L_{-n}|h\rangle=\langle h|[L_{-n}^{\dagger},L_{-n}]|h\rangle=2n\langle h|L_0|h\rangle+\frac{c}{12}(n^3-n)\langle h|h\rangle=[2nh+\frac{c}{12}(n^3-n)]\langle h|h\rangle \tag{222}$$

For $n=1$, we have $h\geq 0$. For larger n, we need require $c\geq 0$.

Lastly, we emphasize that the operator state correspondence is defined as: Conformal families have one-to-one correspondence with Verma Modules.

MINIMAL MODELS

A minimal model conformal field theory is a conformal field theory with a finite number of conformal families. For example, the 2d Ising model at criticality.

1. Reducible Verma Modules & Singular Vectors

A singular vector is a descendant state $|\chi\rangle$ with $L_n|\chi\rangle = 0$ for all $n > 0$, that this, $|\chi\rangle$ is a highest weight state. A singular vector and its descendants are orthogonal to any other state.

Starting from a highest weight state $|h\rangle$, we build the set of states (see Tabel. I) known as a Verma module. We are not guaranteed however that all the above states are linearly independent. That depends on the structure of the Virasoro algebra for given values of h and c. A linear combination of states that vanishes is known as a null state, and the representation of the Virasoro expressed as a linear combination of primary and descendant states. Suppose algebra with highest weight $|h\rangle$ is constructed from the above Verma module by removing the null states.

Let us now consider the consequences of a linear combination of states that vanish. At level 1, the only possibility is that

$$L_{-1}|h\rangle = 0 \tag{223}$$

The only possibility is $|h\rangle = |0\rangle, h = 0$. At level 2, on the other hand, it may happen that

$$L_{-2}|h\rangle + aL_{-1}^2|h\rangle = 0 \tag{224}$$

for some value of a. By applying L_1 to the above equation, we derive a consistency condition,

$$0 = [L_1, L_{-2}]|h\rangle + a[L_1, L_{-1}^2]|h\rangle = [3L_{-1} + 2aL_{-1}L_0 + 2aL_0L_{-1}]|h\rangle = [3 + 2a(2h+1)]L_{-1}|h\rangle \tag{225}$$

which requires that $a = -3/2(2h+1)$. By applying L_2, we have

$$0 = [L_2, L_{-2}]|h\rangle + a[L_2, L_{-1}^2]|h\rangle = [4L_0 + \frac{c}{12}6 + 3aL_1L_{-1}]|h\rangle = [4h + c/2 + 6ah]|h\rangle \tag{226}$$

so that the central charge must satisfy $c = 2(-6ah - 4h) = 2h(5-8h)/(2h+1)$. Thus we conclude the state $L_{-2}|h\rangle + aL_{-1}^2|h\rangle$ is a null decendant state at level 2, is also called degenerate at level 2.

2. Kac determinant

At any given level in Tab I, the quantity to calculate to determine more generally whether there are any non-trivial linear relations among the states is the Gram matrix of inner products at that level. A zero eigenvector of this matrix gives a linear combination with zero norm. The Kac determinants are the determinants of the Gram Matrices. At level 2, for example, the Gram matrice is

$$M^{(2)}(c,h) = \begin{bmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_2L_{-1}^2|h\rangle \\ \langle h|L_1^2L_{-2}|h\rangle & \langle h|L_1^2L_{-1}^2|h\rangle \end{bmatrix} = \begin{bmatrix} 4h+c/2 & 6h \\ 6h & 4h(1+2h) \end{bmatrix} \tag{227}$$

We can write the determinant of this matrix as

$$det[M^{(2)}(c,h)] = 32h(h^2 - \frac{5h}{8} + \frac{hc}{8} + \frac{c}{16}) = 32(h-h_{1,1})(h-h_{1,2})(h-h_{2,1}) \tag{228}$$

$$h_{1,1} = 0 \tag{229}$$

$$h_{1,2} = \frac{1}{16}(5 - c - \sqrt{(1-c)(25-c)}) \tag{230}$$

$$h_{2,1} = \frac{1}{16}(5 - c + \sqrt{(1-c)(25-c)}) \tag{231}$$

Here the trivial solution of $h_{1,1} = 0$ has been in the level-1.

Kac suspects that, at level-N, the determinant has a general form, and it can compute a formula for the Kac determinants (generally)

$$det[M^{(N)}(c,h)] = \alpha_N \prod_{r,s>1 rs<N} (h - h_{r,s})^{P(N-rs)} \tag{232}$$

where α_N is some numerical (non-vanishing) constant independent of h and c. The product is over all positive integers s, r whose product is less than or equal to N. Importantly, the $h_{r,s}$ are most easily expressed by reparametrizing c:

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}, \rightarrow c = 1 - \frac{6}{m(m+1)} \tag{233}$$

$$h_{r,s} = \frac{[(m+1)r - sm]^2 - 1}{4m(m+1)} \tag{234}$$

where m could be complex number.

Importantly, the values of c, h determines the distinct types of Virasora algebra. At any level-N, if Kac determinant is negative, which corresponds to odd number of negative eigenvalues of $h_{r,s}$. These negative h (or negative norm) relate to the non-unitary representation

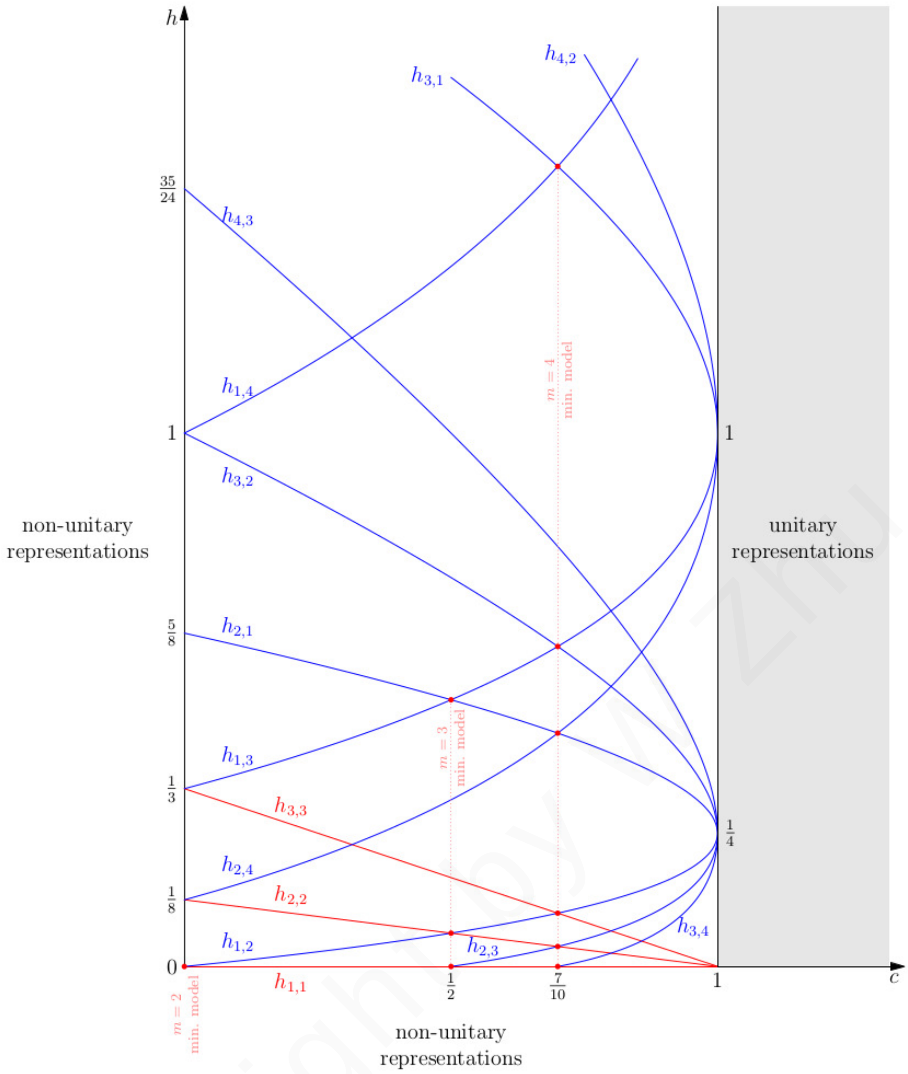

FIG. 8: the vanishing curves $h_{r,s}(c)$ in c, h plane.

(see Eq. 222). Non-unitary CFT is not the focus of this lecture (even though it becomes interesting very recently.). If Kac determinant vanishes, then there exists a linear combination of states with zero norm for c, h.

For the requirement of unitary CFT, we need $c, h \geq 0$ (see Eq. 222). Then we have four regimes:

- $1 < c < 25$: if $h > 0$ we have $det[M^{(N)}(c, h)] > 0$;
- $c \geq 25$: $-1 < m < 0$,$h_{r,s} < 0$ we have $det[M^{(N)}(c, h)] > 0$
- $c = 1$: $h = n^2/4$, $det[M^{(N)}(c, h)] = 0$

- $0 < c < 1$: on (c, h) plane, Kac determinat is zero on $h = h_{r,s}(c) = \frac{1-c}{96}[((r+s) \pm (r-s)\frac{25-c}{1-c})^2 - 4]$ (by reexpressing Eq. 233 in the form). Only on these curves, we have unitary CFT, otherwise, the CFT is nonunitary. A more careful analysis of the determinant shows that there is an additional negative norm state everywhere on the vanishing curves except at certain points where they intersect (see Fig. 8)

Since the determinant never vanishes for $c > 1, h \geq 0$, all of the eigenvalues must stay positive in the entire region, so $c \geq 1, h \geq 0$ does not give new constrains on the Kac determinant. If we hope to study this case, we need other methods. (The status of conformal field theories with $c \geq 1$ is not as yet well understood, and much effort is currently being expended to develop more powerful techniques to study them.)

What we can focus on is $0 < c < 1, h > 0$. In this case, the discrete set of points for unitary representations of the Virasoro, occur at values of the central charge

$$c = 1 - \frac{6}{m(m+1)}, m = 2, 3, 4, ... \tag{235}$$

To each such value of c there are $m(m-1)/2$ allowed values of h given by

$$h_{r,s} = \frac{[(m+1)r - sm]^2 - 1}{4m(m+1)} \tag{236}$$

where $1 \leq p \leq m-1, 1 \leq q \leq p$. Now we see that, the necessary conditions for unitary highest weight repre- sentations of the Virasoro algebra are $c \geq 1, h \geq 0$) or Eq. 235.

Conformal grids and null descendants

To prepare for our discussion of the operator content in later sections, we need a convenient way of organizing the allowed highest weights $h_{r,s}$ in the minimal models. As noted, $h_{r,s} = \frac{[(m+1)r-sm]^2-1}{4m(m+1)}$ is invariant under $r \to m-r, s \to m+1-s$. Thus if we extend the range of s to $1 \leq s \leq m$, we will have a total of $m(m-1)$ values of $h_{r,s}$ with each appearing exactly twice. This arranges the extended range in an $(m?1) \times m$ "conformal grid" with columns s and row r. The example of $m = 3$ is shown below. where

$$[\phi_{1,1}] = [\phi_{2,3}], h_{1,1} = 0 \tag{237}$$

$$[\phi_{2,1}] = [\phi_{1,3}], h_{2,1} = 1/2 \tag{238}$$

$$[\phi_{2,2}] = [\phi_{1,2}], h_{2,2} = 1/16 \tag{239}$$

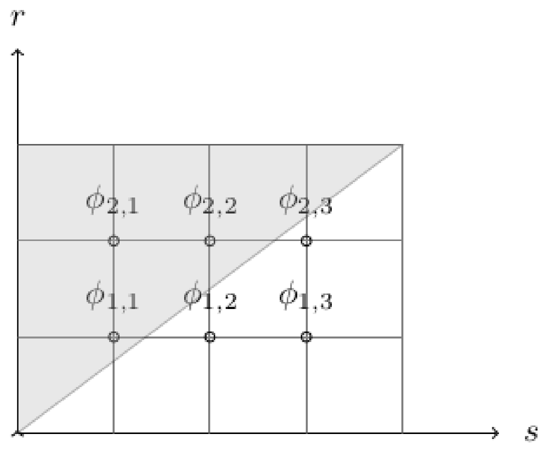

FIG. 9: Conformal grids for the minimal model $m = 3$.

3. Fusion rules

Singular vectors give rise to selection rules for non-vanishing three point correlators. Applied directly to the 3-point functions, the above method does not determine the structure factor, but does give useful selection rules that determine which are allowed to be non-vanishing.

Two- and three- points correlation function

We start with the two-point function of two chiral quasi-primary fields

$$\langle\phi_1(z)\phi_2(w)\rangle = g(z-w) \tag{240}$$

- The invariance under translations $f(z) = z + a$ generated by L_{-1} requires g to be of the form $g(z, w) = g(z - w)$.
- The invariance under L_0 , i.e. rescalings of the form $f(z) = \lambda z$, implies that $\langle\phi_1(z)\phi_2(w)\rangle \to \lambda^{h_1}\lambda^{h_2}\langle\phi_1(\lambda z)\phi_2(\lambda w)\rangle = \lambda^{h_1+h_2}g(\lambda(z-w))$, from which we conclude $g(z-w) = \frac{d_{12}}{(z-w)^{h_1+h_2}}$, to satisfy the invariance.
- The invariance of the two-point function under L_1 essentially implies the invariance under transformations $f(z) = -\frac{1}{z}$: $\langle\phi_1(z)\phi_2(w)\rangle \to \frac{1}{z^{2h_1}}\frac{1}{w^{2h_2}}\langle\phi_1(-\frac{1}{z})\phi_2(-\frac{1}{w})\rangle = \frac{1}{z^{2h_1}}\frac{1}{w^{2h_2}}\frac{d_{12}}{(-\frac{1}{z}+\frac{1}{w})^{h_1+h_2}}$. The invariance requires that $h_1 = h_2$.

We therefore arrive at the result that, conformal symmetry fixes the two-point function of two chiral quasi-primary fields to be of the form

$$\langle\phi_j(z)\phi_k(w)\rangle = \frac{d_{12}\delta_{h_j,h_k}}{(z-w)^{h_j+h_k}} \tag{241}$$

After having determined the two-point function of two chiral quasi-primary fields up to a constant, let us now consider the three-point function. From the invariance under translations, we can infer that

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = f(z_{12}, z_{23}, z_{13}) \tag{242}$$

The requirement of invariance under dilation can be expressed as

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle \to \lambda^{h_1+h_2+h_3}\langle\phi_1(\lambda z_1)\phi_2(\lambda z_2)\phi_3(\lambda z_3)\rangle \tag{243}$$

$$= \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{13}) \tag{244}$$

from which it follows that $f(\lambda z_{12}, \lambda z_{23}, \lambda z_{13}) = \frac{C_{123}}{z_{12}^a z_{23}^b z_{13}^c}$ and $a+b+c = h_1+h_2+h_3$. Finally, from the Special Conformal Transformations, we obtain the condition

$$\frac{1}{z_1^{2h_1} z_2^{2h_2} z_3^{2h_3}} \frac{(z_1 z_2)^a (z_2 z_3)^b (z_1 z_3)^c}{z_{12}^a z_{23}^b z_{13}^c} \sim \frac{1}{z_{12}^a z_{23}^b z_{13}^c} \tag{245}$$

Solving this expression for a, b, c leads to $a = h_1+h_2-h_3, b = h_2+h_3-h_1, c = h_1+h_3-h_2$. Conformal symmetry fixes the three-point function of chiral quasi-primary fields up to a constant to

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}} \tag{246}$$

Fusion Rules

We work at level 2 and a singular vector (null state)

$$|\chi\rangle = [L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2]|h\rangle \tag{247}$$

Since $|\chi\rangle$ is singular, we have

$$0 = \langle \chi(z) X\rangle, X = \phi(z_1)\phi(z_2)... \tag{248}$$

This implies the following differential equation for the correlation (L_n only apply on z):

$$[L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2]\langle \phi(z)\phi(z_1)\phi(z_2)...\rangle = 0 \tag{249}$$

More explicitly, we have differential equations

$$\{\sum_{j=1}^{N} [\frac{1}{z-z_j}\frac{\partial}{\partial z_j} + \frac{h_j}{(z-z_j)^2}] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\}\langle \phi(z)\phi(z_1)\phi(z_2)...\rangle = 0 \tag{250}$$

Here we used the relation $L_{-n} = \sum_i [\frac{(n-1)h_i}{(w_i-w)^n} - \frac{\partial_{w_i}}{(w_i-w)^{n-1}}]$ (*Question* How to prove it?).

This differential equation should bring nothing new to our knowledge of the two-point function. Indeed, if we let $X = \phi(w)$,

$$\{[\frac{1}{z-w}\frac{\partial}{\partial w} + \frac{h}{(z-w)^2}] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\}\langle \phi(z)\phi(w)\rangle = 0 \tag{251}$$

which is trivially satisfied as $\langle \phi(z)\phi(w)\rangle = \frac{1}{(z-w)^{2h}}$. However, this differential equation has a nontrivial effect on the three-point function. We consider $X = \phi_1(z_1)\phi_2(z_2)$. The three-point function is

$$\langle \phi(z)\phi_1(z_1)\phi_2(z_2)\rangle = \frac{g(h, h_1, h_2)}{(z-z_1)^{h+h_1-h_2}(z_1-z_2)^{h_1+h_2-h}(z-z_2)^{h+h_2-h_1}} \tag{252}$$

Inserting it into the differential equation, we get differential equation imposes constraints on the conformal dimensions :

$$2(2h+1)(h+2h_2-h_1)=3(h-h_1+h_2)(h-h_1+h_2+1) \tag{253}$$

This expression can be solved for h_2 leading to

$$h_2=\frac{1}{6}+\frac{h}{3}+h_1\pm\frac{2}{3}\sqrt{h^2+3hh_1-\frac{h}{2}+\frac{3}{2}h_1+\frac{1}{16}} \tag{254}$$

Next, let us apply the above equation to the primary fields $\phi_{r,s}$ of the rational models with central charges $c(m)$. In particular, if we choose $h=h_{2,1}(m)$ and $h_1=h_{p,q}(m)$ then the two solutions for h_2 are precisely $h_{p-1,q}(m), h_{p+1,q}(m)$. Therefore, at most two of the coefficients $g(h,h_1,h_2)$ of a three-point function will be non-zero. The OPE of $\phi_2=\phi_{2,1}$ with any other primary field $\phi_{r,s}$ in a unitary minimal model is then restricted to be of the form

$$\phi_{2,1}\times\phi_{r,s}=\phi_{r-1,s}+\phi_{r+1,s} \tag{255}$$

$$\Rightarrow\phi_{2,1}\times\phi_{2,1}=\phi_{1,1}\rightarrow\epsilon\times\epsilon=I \tag{256}$$

$$\phi_{2,1}\times\phi_{1,1}=\phi_{2,1}\rightarrow\epsilon\times I=\epsilon \tag{257}$$

$$\phi_{2,1}\times\phi_{2,2}=\phi_{1,2}\rightarrow\epsilon\times\sigma=\epsilon \tag{258}$$

The other fusion rules can be obtained in the similar way:

$$\phi_{2,2}\times\phi_{2,2}=\phi_{1,1}+\phi_{1,2}\rightarrow\sigma\times\sigma=I+\epsilon \tag{259}$$

Thus, the existence of a null vector at level 2 imposes additional constraints on the three-point functions, which are equivalent to constraints imposed on the operator algebra. Fusion rules, on the other hand, summarize the conformal families appearing in the OPE. For a CFT with states ϕ_p, we can define the fusion product:

$$\phi_k\times\phi_j=\sum_l N_{kj}^l\phi_l \tag{260}$$

A dynamical analysis of minimal models yields

$$\phi_{r,s}\times\phi_{r',s'}=\sum_{k=1+|r-r'|,k+r+r'=1\%2}^{r+r'-1}\;\sum_{l=1+|s-s'|,l+s+s'=1\%2}^{s+s'-1}\phi_{k,l} \tag{261}$$

4. Critical statistical mechanical models

We pause here to emphasize the import of Eq. 233. The representation theory of the Virasoro algebra in principle allows us to describe the possible scaling dimensions of fields of two dimensional conformal field theories, and thereby the possible critical indices of two dimensional systems at their second order phase transitions. In the case of unitary systems with $c \leq 1$, this has turned out to give a complete classification of possible two dimensional critical behavior. We shall later see how to identify the particular representations of the Virasoro algebra which occur in the description of a given two dimensional system at its critical point.

While the $c < 1$ discrete series distinguishes a set of representations of the Virasoro algebra, it is not obvious that these should be realized by readily constructed statistical mechanical model at their critical points. The first few members of the series Eq. 233 with $m = 3, 4, 5, 6$, i.e. central charge $c = \frac{1}{2}, \frac{7}{10}, \frac{4}{5}$ were associated in [18] respectively with the critical points of the Ising model, tricritical Ising model, 3-state Potts model, and tricritical 3-state Potts model, by comparing the allowed conformal weights Eq. 233 with known scaling dimensions of operators in these models. The first of these, $m = 3$, we will treat in great detail later. In general, there may exist more than one model at a given discrete value of $c < 1$, corresponding to different consistent subsets of the full unitarity-allowed operator content.

Identification of $m = 3$ with the critical Ising model

We shall now make explicit the identification of the first member of the discrete unitary series, i.e. the case $m = 3$ with $c = 1/2$, with the Ising model at its critical point.

The field content of the Ising model

$$\textit{spinfield}, \sigma(z, \bar{z}) : (h, \bar{h}) = (\frac{1}{16}, \frac{1}{16}) \tag{262}$$

$$\textit{energydensityfield}, \epsilon(z, \bar{z}) : (h, \bar{h}) = (\frac{1}{2}, \frac{1}{2}) \tag{263}$$

$$\textit{identityfield}, I : (h, \bar{h}) = (0, 0) \tag{264}$$

The $m = 3$ minimal model has $c = 1/2$, the simplest possibility to to consider the left-right symmetric fields: $\sigma(z, \bar{z}) = \phi_{22}(z)\phi_{22}(\bar{z})$, $\epsilon(z, \bar{z}) = \phi_{21}(z)\phi_{21}(\bar{z})$. Here we just used

Eq. 233 by $(r,s)=(1,1),(2,1),(2,2)$ where $\phi_{r,s}=\phi_{m-r,m+1-s}$.

Critical exponents

Suppose we have a system with an order parameter σ (such as the spin ($\sigma=\pm 1$) in the Ising model. Suppose further that the system has a 2nd order transition separating a high temperature (disordered) phase with $\langle\sigma\rangle=0$ from a low temperature (ordered) $\langle\sigma\rangle\neq 0$. In the high temperature phase the 2-point function of the order parameter will fall off exponentially $\langle\sigma_n\sigma_0\rangle\sim\exp(-|n|/\xi)$, where the correlation length ξ depends on the temperature. At the critical point the correlation length diverges, where the theory becomes gapless, the 2-point function instead falls off as a power-law

$$\langle\sigma_n\sigma_0\rangle\sim\frac{1}{|n|^{d-2+\eta}} \tag{265}$$

Another exponent, ν, can be defined in terms of the 4-point function at criticality

$$\langle\epsilon_n\epsilon_0\rangle=\langle\sigma_{n+1}\sigma_n\sigma_0\sigma_1\rangle\sim\frac{1}{|n|^{2(d-1/\nu)}} \tag{266}$$

The critical exponents calculated for the two dimensional Ising model are $\eta=1/4,\nu=1$. Thus $\langle\sigma_n\sigma_0\rangle\sim\frac{1}{|n|^{1/4}}\sim 1/r^{2\Delta_\sigma}$, and $\langle\epsilon_n\epsilon_0\rangle\sim\frac{1}{|n|^2}\sim 1/r^{2\Delta_\epsilon}$. (in CFt, we know that $\Delta_\sigma=2h_\sigma=1/8$, and $\Delta_\epsilon=2h_\epsilon=1$.)

Fusion Rule

Let us briefly illustrate these rules for the Ising model, i.e. for m = 3. We label the relevant fields $\phi_{p,q}$ as $\phi_{1,1}=\phi_{2,3}=1$, $\phi_{1,2}=\phi_{2,2}=\sigma$, $\phi_{1,3}=\phi_{2,1}=\epsilon$. The fusion rule is (Eq. 261)

$$\sigma\times\sigma=1+\epsilon,\epsilon\times\epsilon=1,\epsilon\times\sigma=\sigma,1\times x(=1,\sigma,\epsilon)=x \tag{267}$$

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