Compact Boson, Duality, Gauging, and Anomalies

A Summer School Handout

Prepared from the shared discussion on SPT edge modes, compact bosons, and noninvertible defects

Abstract

This note gives a self-contained pedagogical route through the compact boson in 1+1 dimensions. The goal is to make precise the two ordinary-looking symmetries of the compact scalar, the momentum and winding U(1) symmetries, and then show how duality, discrete gauging, emergent dual symmetries, and noninvertible defects naturally arise. A central lesson is that the momentum and winding symmetries have a mixed 't Hooft anomaly: turning on background flux for one symmetry obstructs conservation of the other.

Contents

1 Motivation

The compact boson is one of the simplest quantum field theories that already contains many of the structures that appear throughout modern quantum many-body physics and quantum field theory:

- quantized zero modes and winding sectors;
- ordinary global symmetries and topological currents;
- T-duality;
- discrete gauging and orbifold sectors;
- emergent dual symmetries;

- noninvertible symmetry defects.

This makes it an ideal summer-school example: it is solvable enough to compute with, but rich enough to expose the organizing ideas behind anomalies and duality.

Teaching note. A good classroom strategy is to keep returning to the state labels $|n, w\rangle$. The integer n measures momentum charge, while w measures winding charge. Most abstract statements become concrete when translated into how they act on these states.

2 The Compact Boson

Consider a scalar field

$$\phi \sim \phi + 2\pi$$

in 1+1 dimensions. The Euclidean action is

$$S[\phi] = \frac{R^2}{8\pi} \int_M \mathrm{d}^2 x \, (\partial_\mu \phi)^2,$$

where R is the compactification radius.

The phrase *compact* means that the target space of the field is a circle, not the real line:

$$\phi : M \longrightarrow S^1.$$

As a result, field configurations on a spatial circle can wind around the target circle.

2.1 Winding Sectors

Place the theory on a spatial circle of length L, so $x \sim x + L$. The allowed boundary condition is

$$\phi(x + L, t) = \phi(x, t) + 2\pi w, \qquad w \in \mathbb{Z}.$$

The integer w labels the homotopy class of the map from the spatial circle to the target circle:

$$\pi_1(S^1) = \mathbb{Z}.$$

3 Canonical Quantization

In Lorentzian signature, take

$$S = \frac{R^2}{8\pi} \int \mathrm{d}t \, \mathrm{d}x \, \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 \right].$$

The canonical momentum is

$$\Pi(x) = \frac{R^2}{4\pi} \, \partial_t \phi(x),$$

with canonical commutator

$$[\phi(x), \Pi(y)] = \mathrm{i}\delta(x - y).$$

3.1 Zero Modes

A useful decomposition is

$$\phi(x,t) = \phi_0 + \frac{2\pi w}{L}x + \frac{4\pi n}{R^2 L}t + \phi_{\text{osc}}(x,t),$$

where $w \in \mathbb{Z}$ is the winding number and $n \in \mathbb{Z}$ is the momentum quantum number. The quantization of n follows because the zero mode itself is compact:

$$\phi_0 \sim \phi_0 + 2\pi, \qquad \psi(\phi_0 + 2\pi) = \psi(\phi_0).$$

Thus the Hilbert space decomposes into sectors

$$|n, w\rangle, \qquad n, w \in \mathbb{Z}.$$

3.2 Hamiltonian Spectrum

Ignoring oscillator excitations, the zero-mode energy has the schematic form

$$E_{n,w} = \frac{2\pi}{L}\left(\frac{n^2}{R^2} + \frac{R^2 w^2}{4}\right) + E_{\text{osc}}.$$

Different conventions move factors of 2 and π between the radius and the field normalization, but the important structure is invariant: there are two integer quantum numbers, momentum n and winding w.

4 Momentum and Winding Symmetries

4.1 Momentum $\mathrm{U}(1)_m$

The compact boson has a shift symmetry

$$\mathrm{U}(1)_m : \qquad \phi \mapsto \phi + \alpha.$$

The associated charge acts on states as

$$Q_m |n, w\rangle = n |n, w\rangle.$$

Local vertex operators

$$V_n(x) = e^{in\phi(x)}$$

carry momentum charge:

$$V_n \mapsto e^{in\alpha} V_n.$$

4.2 Winding $\mathrm{U}(1)_w$

There is also a topological current

$$J_w = \frac{1}{2\pi}\mathrm{d}\phi.$$

In components this can be written as

$$j_w^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi.$$

It is conserved because

$$\mathrm{d}J_w = \frac{1}{2\pi}\mathrm{d}^2\phi = 0$$

away from singular operator insertions. The corresponding charge measures winding:

$$Q_w|n, w\rangle = w|n, w\rangle.$$

Teaching note. The winding current is topological: its conservation follows from the identity $\mathrm{d}^2 = 0$, not from the scalar equation of motion. This is why it is the cleanest entry point into generalized symmetry language.

5 Dual Field and T-Duality

Introduce a one-form v and temporarily treat it as independent from $\mathrm{d}\phi$. A first-order Euclidean action is

$$S[v, \widetilde{\phi}] = \frac{R^2}{8\pi}\int v \wedge *v + \frac{\mathrm{i}}{2\pi}\int \widetilde{\phi}\,\mathrm{d}v.$$

The field $\widetilde{\phi}$ is a Lagrange multiplier.

Varying $\widetilde{\phi}$ gives

$$\mathrm{d}v = 0,$$

so locally $v = \mathrm{d}\phi$, and one recovers the original compact boson.

Varying v instead gives a relation of the schematic form

$$v \propto \frac{1}{R^2} * \mathrm{d}\widetilde{\phi}.$$

Substituting back gives a dual compact boson with inverse radius:

$$R \longleftrightarrow \frac{1}{R}.$$

This is T-duality. Under this duality, momentum and winding quantum numbers are exchanged:

$$n \longleftrightarrow w.$$

6 Mixed Anomaly Between Momentum and Winding

The cleanest way to see the mixed 't Hooft anomaly is to couple one symmetry to a background gauge field and observe that the other current is no longer conserved.

Couple the momentum symmetry $\mathrm{U}(1)_m$ to a background gauge field A_m by replacing

$$\mathrm{d}\phi \longrightarrow D\phi = \mathrm{d}\phi - A_m.$$

The gauge-invariant version of the winding current is then

$$J_w = \frac{1}{2\pi} D\phi = \frac{1}{2\pi}(\mathrm{d}\phi - A_m).$$

Taking an exterior derivative gives

$$\mathrm{d}J_w = -\frac{1}{2\pi}\mathrm{d}A_m = -\frac{1}{2\pi}F_m.$$

Thus, in a background with nonzero $\mathrm{U}(1)_m$ flux, winding charge is not conserved. Physically:

momentum flux creates winding charge.

This is the mixed anomaly between $\mathrm{U}(1)_m$ and $\mathrm{U}(1)_w$.

6.1 Anomaly Inflow

The same anomaly can be encoded by a 2+1-dimensional inflow action

$$S_{\text{anom}} = \frac{\mathrm{i}}{2\pi}\int_{M_3} A_m \wedge \mathrm{d}A_w,$$

where A_w is a background gauge field for the winding symmetry. The existence of this bulk term means the two symmetries cannot both be gauged in a purely 1+1-dimensional theory without adding extra structure.

Teaching note. This is often the most important conceptual slide: an anomaly is not a failure of the theory. It is a precise obstruction to making all symmetry backgrounds simultaneously gauge invariant in the same dimension.

7 Gauging a $\mathbb{Z}_2$ Shift Symmetry

Now gauge the subgroup

$$\mathbb{Z}_2 \subset \mathrm{U}(1)_m, \qquad \phi \mapsto \phi + \pi.$$

Introduce a flat $\mathbb{Z}_2$ gauge field

$$a \in H^1(M, \mathbb{Z}_2).$$

Schematically, the gauged action is obtained by replacing

$$\mathrm{d}\phi \longrightarrow \mathrm{d}\phi + \pi a,$$

so

$$S_{\text{gauged}} = \frac{R^2}{8\pi} \int (\mathrm{d}\phi + \pi a) \wedge *(\mathrm{d}\phi + \pi a).$$

Under a $\mathbb{Z}_2$ gauge transformation

$$\phi \mapsto \phi + \pi\lambda, \qquad a \mapsto a - \mathrm{d}\lambda,$$

the combination $\mathrm{d}\phi + \pi a$ is invariant.

7.1 Twisted Sectors

If the spatial holonomy of a is nontrivial,

$$\oint_{S^1} a = 1 \mod 2,$$

then the scalar obeys the twisted boundary condition

$$\phi(x + L) = \phi(x) + 2\pi w + \pi.$$

Equivalently,

$$w \in \mathbb{Z} + \frac{1}{2}.$$

The gauged theory therefore includes both untwisted and twisted sectors. In orbifold language, the partition function has the schematic form

$$Z_{\text{orb}} = \frac{1}{2} \sum_{a,b \in \mathbb{Z}_2} Z[a, b],$$

where a is a spatial twist and b is a temporal twist.

7.2 Rescaled Boson Viewpoint

There is another very useful way to describe the same gauged theory. Once the shift

$$\phi \sim \phi + \pi$$

has become a gauge redundancy, the physical target circle of ϕ has half the original circumference. To write the result again as a standard 2π-periodic compact boson, define

$$\varphi = 2\phi, \qquad \varphi \sim \varphi + 2\pi.$$

This immediately explains the change in momentum quantization. A local operator $e^{in\phi}$ is gauge-invariant only when n is even:

$$n = 2m, \qquad e^{in\phi} = e^{im\varphi}.$$

Equivalently, in the Hilbert-space language, the generator of the gauged $\mathbb{Z}_2$ acts on a momentum eigenstate as

$$|n, w\rangle \mapsto (-1)^n |n, w\rangle.$$

Gauging includes a projection onto gauge-invariant states. Therefore, in the untwisted sector, only $\mathbb{Z}_2$-even states survive:

$$n \in 2\mathbb{Z}.$$

The surviving old momentum states can then be relabeled as ordinary momentum states of the rescaled field φ:

$$n = 2m, \qquad |n, w\rangle_\phi \longleftrightarrow |m, \widetilde{w}\rangle_\varphi.$$

The twisted sectors are also absorbed into the rescaled description. If

$$\phi(x + L) = \phi(x) + 2\pi w + \pi,$$

then

$$\varphi(x + L) = \varphi(x) + 2\pi(2w + 1).$$

Hence untwisted and twisted sectors of ϕ become even and odd winding sectors of φ:

$$\widetilde{w} = \begin{cases} 2w, & \oint a = 0, \\ 2w + 1, & \oint a = 1. \end{cases}$$

Teaching note. This is often the most intuitive picture: gauging the half-shift projects out odd old momentum operators and turns the old $\mathbb{Z}_2$ twisted sectors into odd winding sectors of the new boson $\varphi = 2\phi$.

8 Emergent Dual $\mathbb{Z}_2$ Symmetry

After gauging the original $\mathbb{Z}_2$ shift symmetry, a new dual $\mathbb{Z}_2$ symmetry appears. It measures the gauge flux:

$$\eta = (-1)^{\oint a}.$$

In the rescaled boson language, this is simply winding parity:

$$\eta = (-1)^{\widetilde{w}}.$$

It acts differently on the even- and odd-winding sectors of φ. This is the simplest example of the general principle:

gauging a finite abelian symmetry produces a dual symmetry.

8.1 Background Field for the Dual Symmetry

The response-theory language makes the same point in a compact form. Before gauging, a is a background $\mathbb{Z}_2$ gauge field that records where half-shift twists are inserted. Gauging means making a dynamical:

$$Z_{\text{gauged}} = \sum_{a \in Z^1(M, \mathbb{Z}_2)} Z_\phi[a].$$

To couple the emergent dual $\mathbb{Z}_2$ symmetry to a background field

$$A \in Z^1(M, \mathbb{Z}_2),$$

one inserts the discrete BF factor

$$Z_{\text{gauged}}[A] = \sum_{a \in Z^1(M,\mathbb{Z}_2)} Z_\phi[a] \, \exp\left(\mathrm{i}\pi \int_M a \cup A \right).$$

The cup product $a \cup A$ counts, modulo two, intersections between the original $\mathbb{Z}_2$ twist lines and the dual-symmetry background lines.

Equivalently, A couples to the flux of the dynamical gauge field a. Thus the BF term is the response-theory statement that the new dual symmetry measures the original gauge flux.

8.2 Dual Field Rescaling

This viewpoint also has a clean dual-field interpretation. If

$$\varphi = 2\phi,$$

then the dual field rescales oppositely:

$$\widetilde{\varphi} = \frac{\widetilde{\phi}}{2}.$$

This is the field-theory expression of the fact that shrinking the original circle enlarges the dual circle. A half-period shift of the rescaled dual field

$$\widetilde{\varphi} \mapsto \widetilde{\varphi} + \pi$$

is precisely the emergent dual $\mathbb{Z}_2$ symmetry. To see why, recall that a shift of the dual field measures winding of the original field. If a state has winding number $\widetilde{w}_\varphi$ for the rescaled boson φ, then

$$\widetilde{\varphi} \mapsto \widetilde{\varphi} + \alpha \qquad \Longrightarrow \qquad |\widetilde{w}_\varphi\rangle \mapsto e^{\mathrm{i}\alpha \widetilde{w}_\varphi} |\widetilde{w}_\varphi\rangle.$$

For the half-period shift $\alpha = \pi$, this becomes

$$|\widetilde{w}_\varphi\rangle \mapsto (-1)^{\widetilde{w}_\varphi} |\widetilde{w}_\varphi\rangle.$$

Thus the half-period shift only remembers winding parity. It is therefore an order-two transformation, namely a $\mathbb{Z}_2$ symmetry.

This is exactly the same $\mathbb{Z}_2$ that appears after gauging. Before gauging, the $\mathbb{Z}_2$ gauge field a records half-shift twists. On a spatial circle,

$$\oint a = 0, 1$$

distinguishes untwisted and twisted sectors. After the rescaling

$$\varphi = 2\phi,$$

a nontrivial old twist becomes an odd winding sector of φ. Therefore

$$(-1)^{\oint a} = (-1)^{\widetilde{w}_\varphi}.$$

The dual gauge-flux operator and the half-period shift of $\widetilde{\varphi}$ act on every sector by the same sign. They are the same symmetry written in two languages.

This also explains the BF pairing. Coupling the emergent dual symmetry to a background A inserts

$$\exp\left(\mathrm{i}\pi\int a\cup A\right).$$

When a dual-symmetry defect line A crosses a dynamical a-flux line, the cup product gives

$$\int a\cup A = 1 \mod 2,$$

and the path integral receives the phase -1. But an a-flux line is precisely an odd-winding sector of φ, and the half-period shift of $\widetilde{\varphi}$ also acts on such a sector by -1. Thus

$$\exp\left(\mathrm{i}\pi\int a\cup A\right) \quad \text{is the spacetime version of} \quad \widetilde{\varphi}\mapsto\widetilde{\varphi}+\pi.$$

The same object is therefore visible in three equivalent ways:

$$\text{dual gauge-flux symmetry} \quad\Longleftrightarrow\quad \text{winding parity of } \varphi \quad\Longleftrightarrow\quad \text{half-period shift of } \widetilde{\varphi}.$$

8.3 Four Sectors and Kramers-Wannier Duality

A useful next step is to keep both pieces of $\mathbb{Z}_2$ data before fully summing or projecting. On a spatial circle, one can organize the theory into four sectors labeled by

$$(q,t)\in\mathbb{Z}_2\times\mathbb{Z}_2,$$

where

$$q=0,1 \quad \text{labels } \mathbb{Z}_2 \text{ charge parity}$$

and

$$t=0,1 \quad \text{labels untwisted/twisted boundary condition.}$$

In the compact-boson language,

$$q=n \mod 2, \qquad t=\oint a.$$

Thus the four sectors are

	$t=0$ untwisted	$t=1$ twisted
$q=0$ even	$(0,0)$	$(0,1)$
$q=1$ odd	$(1,0)$	$(1,1)$

Gauging the original $\mathbb{Z}_2$ amounts to projecting onto the even charge sector and summing over twists:

$$Z_{\text{gauged}}=\sum_{t=0,1} Z_{q=0,t}.$$

Equivalently, it keeps $\mathbb{Z}_2$-even states but includes both untwisted and twisted sectors.

Kramers-Wannier duality can be understood as a discrete Fourier transform on these $\mathbb{Z}_2$ labels. It exchanges the role of charge parity and twist:

$$\text{charge parity} \quad \longleftrightarrow \quad \text{twist/flux sector.}$$

More explicitly, an insertion of the original $\mathbb{Z}_2$ charge operator is Fourier-dual to a choice of twisted boundary condition, while the dual $\mathbb{Z}_2$ charge measures the original twist. This is the sector-level meaning of the BF pairing

$$(-1)^{\int a \cup A}.$$

Let us make this precise. Put the theory on a Euclidean torus with time and space cycles. Write the original $\mathbb{Z}_2$ background as

$$a = (a_t, a_x), \qquad a_t, a_x \in \mathbb{Z}_2.$$

Here

$$a_x = t$$

is the spatial twist. The temporal component a_t means that we insert the original $\mathbb{Z}_2$ symmetry operator in the trace. The charge label q is the Fourier-dual variable to this insertion. Thus the original charge–twist sector is

$$Z_{\text{orig}}^{(q,t)} = \frac{1}{2} \sum_{a_t=0,1} (-1)^{q a_t} Z_{\text{orig}}[a_t, t].$$

This formula simply says that the even and odd charge sectors are obtained by projecting with and without a sign under the temporal symmetry insertion.

After gauging, the emergent dual $\mathbb{Z}_2$ has its own background

$$A = (A_t, A_x).$$

On the torus, the BF pairing becomes

$$\int a \cup A = a_t A_x + a_x A_t \mod 2.$$

Therefore

$$Z_{\text{dual}}[A_t, A_x] = \frac{1}{2} \sum_{a_t, a_x=0,1} (-1)^{a_t A_x + a_x A_t} Z_{\text{orig}}[a_t, a_x].$$

Now define the dual sector labels in the same way:

$$T = A_x$$

is the dual spatial twist, while the dual charge Q is Fourier-dual to the dual temporal insertion A_t:

$$Z_{\text{dual}}^{(Q,T)} = \frac{1}{2} \sum_{A_t=0,1} (-1)^{Q A_t} Z_{\text{dual}}[A_t, T].$$

Substituting the BF expression gives

$$Z_{\text{dual}}^{(Q,T)} = \frac{1}{4} \sum_{A_t, a_t, a_x} (-1)^{Q A_t} (-1)^{a_t T + a_x A_t} Z_{\text{orig}}[a_t, a_x].$$

Combine the signs:

$$(-1)^{QA_t+a_tT+a_xA_t} = (-1)^{a_tT}(-1)^{(Q+a_x)A_t}.$$

The sum over A_t gives a Kronecker delta:

$$\sum_{A_t=0,1}(-1)^{(Q+a_x)A_t} = 2\,\delta_{a_x,Q}.$$

Thus only the original twist sector

$$a_x = Q$$

survives, and

$$Z^{(Q,T)}_{\text{dual}} = \frac{1}{2}\sum_{a_t=0,1}(-1)^{Ta_t}Z_{\text{orig}}[a_t,Q].$$

But the right-hand side is exactly

$$Z^{(T,Q)}_{\text{orig}}.$$

Therefore

$$\boxed{Z^{(Q,T)}_{\text{dual}} = Z^{(T,Q)}_{\text{orig}}}.$$

Equivalently, under gauging/Kramers-Wannier duality,

$$(q,t)_{\text{orig}} \longmapsto (Q,T)_{\text{dual}} = (t,q).$$

In words, original charge becomes dual twist, and original twist becomes dual charge. This is exactly what the two terms in the BF pairing say:

$$a_tA_x \quad \text{pairs original charge insertion with dual spatial twist,}$$

while

$$a_xA_t \quad \text{pairs original spatial twist with dual charge insertion.}$$

Thus the BF pairing implements the exchange of charge and twist labels.

Teaching note. This four-sector bookkeeping makes Kramers-Wannier duality concrete: KW duality does not merely map one Hamiltonian to another; it reorganizes the Hilbert space by exchanging $\mathbb{Z}_2$ charge data with $\mathbb{Z}_2$ flux data.

9 Gauging Duality on Gapped Phases

Gauging is not only a manipulation of partition functions. It also gives a duality between gapped phases. In the simplest $\mathbb{Z}_2$ case, it exchanges the ordinary $\mathbb{Z}_2$ order/disorder distinction:

$$\text{original } \mathbb{Z}_2\text{-SSB phase} \quad \longleftrightarrow \quad \text{dual symmetric phase,}$$

and conversely

$$\text{original symmetric phase} \quad \longleftrightarrow \quad \text{dual } \mathbb{Z}_2\text{-SSB phase.}$$

This is the phase-level meaning of Kramers-Wannier duality. The compact-boson orbifold gives a very explicit realization.

The rescaled-boson viewpoint also gives a useful check on the physical meaning of gauging. Add a relevant potential to the original free boson:

$$V(\phi) = -\lambda \cos(2\phi), \qquad \lambda > 0.$$

This potential is invariant under the half-shift symmetry

$$\phi \mapsto \phi + \pi,$$

because

$$\cos(2(\phi + \pi)) = \cos(2\phi + 2\pi) = \cos(2\phi).$$

However, it has two minima on the original 2π circle:

$$\phi = 0,\ \pi \mod 2\pi.$$

These two vacua are exchanged by the $\mathbb{Z}_2$ shift. Choosing one of them therefore spontaneously breaks the original $\mathbb{Z}_2$ symmetry:

$$\text{two vacua exchanged by } \mathbb{Z}_2 \quad \Longrightarrow \quad \mathbb{Z}_2\text{-SSB phase.}$$

Now gauge the same $\mathbb{Z}_2$ symmetry. Since

$$\phi \sim \phi + \pi$$

is now a gauge redundancy, the two old vacua are no longer distinct physical states. In the rescaled variable

$$\varphi = 2\phi,$$

the same potential becomes

$$V(\varphi) = -\lambda \cos\varphi.$$

But φ is a standard 2π-periodic boson, so this potential has a single minimum:

$$\varphi = 0 \mod 2\pi.$$

Thus the gauged theory is a symmetric massive phase rather than an SSB phase.

This is the bosonic analogue of the familiar Kramers-Wannier lesson:

$$\boxed{\text{gauging the broken } \mathbb{Z}_2 \text{ phase} \quad \longleftrightarrow \quad \text{a symmetric gapped phase of the dual theory.}}$$

Gauging identifies the two old vacua. They differed only by the global $\mathbb{Z}_2$ action. In the dual language, the same operation turns the order-parameter potential into an ordinary symmetric mass term:

$$\cos(2\phi) \quad \longrightarrow \quad \cos\varphi.$$

9.1 Partition-Function Proof

The same phase map can be proved directly from the infrared partition functions. Let $A \in H^1(M, \mathbb{Z}_2)$ be the background gauge field for the original $\mathbb{Z}_2$ symmetry. Up to an overall A-independent normalization, the two basic trivial gapped phases are

$$Z_{\mathrm{SSB}}[A] \simeq 2\,\delta(A), \qquad Z_{\mathrm{sym}}[A] \simeq 1.$$

Here $\delta(A)$ is one for the trivial background and zero for a nontrivial background. The factor of 2 counts the two vacua in the broken phase. The second equation says that a trivial symmetric gapped phase is insensitive to flat $\mathbb{Z}_2$ backgrounds in the infrared.

Gauging is the discrete Fourier transform of this function of the background. After gauging, the old background is summed over as a dynamical gauge field a, and the emergent dual $\mathbb{Z}_2$ symmetry is probed by a background $\widehat{A}$:

$$Z_{\mathrm{gauged}}[\widehat{A}] = \frac{1}{2} \sum_{a \in H^1(M,\mathbb{Z}_2)} (-1)^{\int_M a \cup \widehat{A}}\, Z[a].$$

The sign $(-1)^{\int a \cup \widehat{A}}$ is the discrete BF pairing. It is the character pairing between $\mathbb{Z}_2$ flux a and dual $\mathbb{Z}_2$ background $\widehat{A}$.

Now insert the SSB partition function:

$$\begin{aligned} Z_{\mathrm{gauged\ SSB}}[\widehat{A}] &\simeq \frac{1}{2} \sum_a (-1)^{\int a \cup \widehat{A}}\, 2\delta(a) \\ &= 1. \end{aligned}$$

Thus gauging the original broken phase gives a partition function independent of $\widehat{A}$. This is the infrared signature of a dual symmetric gapped phase:

$$\mathbb{Z}_2\text{-SSB} \quad \xrightarrow{\text{gauge}} \quad \text{dual symmetric phase.}$$

Physically, the factor $\delta(A)$ encodes the fact that nontrivial twists of a broken symmetry force a domain wall, whose contribution is exponentially suppressed in the gapped SSB phase.

Now insert the symmetric partition function:

$$\begin{aligned} Z_{\mathrm{gauged\ sym}}[\widehat{A}] &\simeq \frac{1}{2} \sum_a (-1)^{\int a \cup \widehat{A}} \\ &= 2\,\delta(\widehat{A}), \end{aligned}$$

where the last equality is the $\mathbb{Z}_2$ character orthogonality relation, with the same infrared normalization used above. The gauged theory is therefore sensitive to a dual twist: only the trivial $\widehat{A}$ background contributes in the deep infrared. This is precisely the partition-function signature of spontaneous breaking of the dual $\mathbb{Z}_2$ symmetry:

$$\text{symmetric phase} \quad \xrightarrow{\text{gauge}} \quad \text{dual } \mathbb{Z}_2\text{-SSB.}$$

So, in the IR partition-function basis,

$$2\delta(A) \quad \longleftrightarrow \quad 1,$$

and gauging exchanges order and disorder. The potential example $-\lambda\cos(2\phi) \mapsto -\lambda\cos\varphi$ is one concrete realization of this Fourier-transform statement.

The reverse direction is equally important. If the original theory is in a unique $\mathbb{Z}_2$-symmetric gapped phase, then gauging produces a theory with an emergent dual $\mathbb{Z}_2$ symmetry. In 1+1 dimensions, this dual symmetry can be spontaneously broken, because the gauged theory has distinct topological flux sectors. The dual $\mathbb{Z}_2$ charge measures the old gauge flux:

$$\eta = (-1)^{\oint a}.$$

Thus the old symmetric phase becomes a phase with two sectors distinguished by the dual order parameter. In slogan form,

$$\boxed{\text{gauging exchanges order and disorder.}}$$

In the compact-boson example, the phase map can be summarized as

before gauging	after gauging / dual description
$-\lambda\cos(2\phi):$ $\mathbb{Z}_2$-SSB	$-\lambda\cos\varphi:$ symmetric massive phase
symmetric massive phase	dual $\mathbb{Z}_2$-SSB phase

This is exactly the phase analogue of the sector exchange discussed above: charge data and twist/flux data are Fourier-dual, so gauging maps order parameters of the original symmetry to disorder/topological data of the dual symmetry.

9.2 Extension to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and SPT Phases

For a single $\mathbb{Z}_2$ symmetry, the simplest symmetric gapped phase is trivial. For

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2,$$

there are already two distinct symmetric short-range-entangled phases in $1+1$ dimensions: the trivial symmetric phase and the nontrivial G-SPT phase. Let

$$A = (A_1, A_2), \qquad A_i \in H^1(M, \mathbb{Z}_2),$$

be the background fields for the two $\mathbb{Z}_2$ factors. In the infrared partition-function language, the two symmetric phases can be written as

$$Z_{\text{triv}}[A_1, A_2] \simeq 1,$$

and

$$Z_{\text{SPT}}[A_1, A_2] \simeq (-1)^{\int_M A_1 \cup A_2}.$$

The second expression is the nontrivial class in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2$. It means that the SPT phase is still fully symmetric and has a unique bulk ground state, but it has a nontrivial response to background symmetry twists.

These two symmetric phases are related by a decorated-domain-wall unitary, which we denote by T. Conceptually, T decorates a domain wall of one $\mathbb{Z}_2$ symmetry by a charge of the other $\mathbb{Z}_2$ symmetry. In the path integral, this means

$$T: \quad Z[A_1, A_2] \longmapsto (-1)^{\int_M A_1 \cup A_2} Z[A_1, A_2].$$

Therefore

$$T: \quad Z_{\mathrm{triv}} \longleftrightarrow Z_{\mathrm{SPT}}, \qquad T^2 = 1.$$

This is not a continuous symmetric deformation between the two phases. Rather, T is a unitary transformation that changes the SPT response by attaching symmetry charge to symmetry domain walls.

Now gauge the full $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Let $a = (a_1, a_2)$ be the dynamical gauge field and let

$$\widehat{A} = (\widehat{A}_1, \widehat{A}_2)$$

be the background for the emergent dual

$$\widehat{G} = \widehat{\mathbb{Z}_2 \times \mathbb{Z}_2} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The gauging formula is the two-variable Fourier transform

$$Z_{\mathrm{gauged}}[\widehat{A}_1, \widehat{A}_2] = \frac{1}{4} \sum_{a_1, a_2} (-1)^{\int_M a_1 \cup \widehat{A}_1 + \int_M a_2 \cup \widehat{A}_2} Z[a_1, a_2].$$

For the trivial symmetric phase,

$$\begin{aligned} Z_{\mathrm{gauged,triv}}[\widehat{A}_1, \widehat{A}_2] &\simeq \frac{1}{4} \sum_{a_1, a_2} (-1)^{\int a_1 \cup \widehat{A}_1 + \int a_2 \cup \widehat{A}_2} \\ &\propto \delta(\widehat{A}_1)\, \delta(\widehat{A}_2). \end{aligned}$$

Thus the trivial symmetric phase gauges to a phase with full dual $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking.

For the nontrivial SPT phase,

$$Z_{\mathrm{gauged,SPT}}[\widehat{A}_1, \widehat{A}_2] \simeq \frac{1}{4} \sum_{a_1, a_2} (-1)^{\int a_1 \cup \widehat{A}_1 + \int a_2 \cup \widehat{A}_2 + \int a_1 \cup a_2}.$$

Using character orthogonality, the sum over a_2 imposes $a_1 = \widehat{A}_2$ up to the conventional cup-product ordering. Substituting back gives

$$Z_{\mathrm{gauged,SPT}}[\widehat{A}_1, \widehat{A}_2] \propto (-1)^{\int_M \widehat{A}_2 \cup \widehat{A}_1}.$$

Up to relabeling the two dual $\mathbb{Z}_2$ factors, this is again the nontrivial SPT response, now for the dual symmetry $\widehat{G}$.

So for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, gauging does not only exchange "symmetric" and "broken". It also sees the SPT decoration:

original phase	after gauging
trivial symmetric	full dual $\mathbb{Z}_2 \times \mathbb{Z}_2$-SSB
nontrivial G-SPT	nontrivial dual $\widehat{G}$-SPT

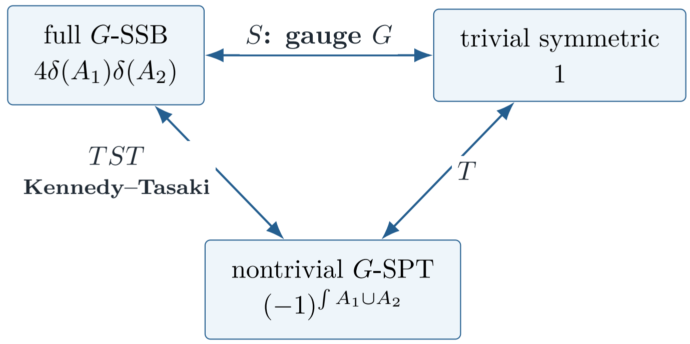
The decorated-domain-wall transformation T is therefore visible in the partition function as multiplication by the cocycle $(-1)^{\int A_1 \cup A_2}$, and gauging acts by Fourier transforming this cocycle into the corresponding dual response.

It is useful to name full $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ gauging as an operation S. Then S acts by the finite Fourier transform above, while T acts by decorating domain walls:

$$S: \quad Z[A_1, A_2] \mapsto \frac{1}{4} \sum_{a_1, a_2} (-1)^{\int a_1 \cup A_1 + \int a_2 \cup A_2} Z[a_1, a_2],$$

$$T: \quad Z[A_1, A_2] \mapsto (-1)^{\int A_1 \cup A_2} Z[A_1, A_2].$$

These two operations organize the three basic gapped phases into a triangle:



The reason is simple in the partition-function basis. The full SSB phase has

$$Z_{\mathrm{SSB}}[A_1, A_2] \simeq 4\,\delta(A_1)\delta(A_2),$$

so the cocycle in T evaluates to one on its support. Hence T leaves the full SSB partition function invariant. Meanwhile

$$S: \quad 4\,\delta(A_1)\delta(A_2) \longleftrightarrow 1,$$

and

$$T: \quad 1 \longleftrightarrow (-1)^{\int A_1 \cup A_2}.$$

Therefore the composite

$$TST$$

maps the full G-SSB phase to the nontrivial G-SPT phase, and maps the nontrivial G-SPT phase back to the full SSB phase. This composite is the partition-function expression of the Kennedy-Tasaki duality: it is a duality between hidden symmetry breaking and an SPT phase, with the decorated-domain-wall unitary providing the SPT decoration.

10 Lattice Models: Kramers–Wannier and Kennedy–Tasaki Dualities

The continuum gauging formulas have exact lattice counterparts. We first review ordinary Kramers–Wannier (KW) duality as $\mathbb{Z}_2$ gauging, and then construct Kennedy–Tasaki (KT) duality by combining KW gauging with domain-wall decoration.

10.1 The $\mathbb{Z}_2$ Spin Chain and Its Four Sectors

Consider a spin-$\frac{1}{2}$ chain with Pauli operators X_i, Z_i and global spin-flip symmetry

$$U = \prod_{i=1}^{L} X_i.$$

On a ring, a state belongs to a charge sector $u \in \mathbb{Z}_2$ and a spatial-twist sector $t \in \mathbb{Z}_2$:

$$U = (-1)^u, \qquad Z_{i+L} = (-1)^t Z_i.$$

Thus the finite-size theory naturally contains four sectors

$$(u, t) \in \mathbb{Z}_2 \times \mathbb{Z}_2.$$

10.2 Direct Gauging with Link Qubits

Introduce a gauge qubit on every link, with Pauli operators $\tau^x_{i+\frac{1}{2}}, \tau^z_{i+\frac{1}{2}}$. We use the convention in which τ^x is the parallel transporter and τ^z is the electric operator. The local gauge transformation at site i is generated by

$$G_i = X_i \tau^z_{i-\frac{1}{2}} \tau^z_{i+\frac{1}{2}}, \qquad G_i |\text{phys}\rangle = |\text{phys}\rangle.$$

Indeed, G_i flips Z_i and the two adjacent τ^x operators. Therefore the minimally coupled Ising bond is gauge invariant:

$$Z_i Z_{i+1} \quad \longrightarrow \quad Z_i \tau^x_{i+\frac{1}{2}} Z_{i+1}.$$

For example, gauging the transverse-field Ising chain gives

$$H_{\text{gauge}} = -J \sum_i Z_i \tau^x_{i+\frac{1}{2}} Z_{i+1} - h \sum_i X_i, \qquad G_i = 1.$$

Because the matter field has unit gauge charge, one may choose unitary gauge $Z_i = 1$. The gauged bond then becomes $\tau^x_{i+\frac{1}{2}}$, while Gauss law solves the matter transverse spin as

$$X_i = \tau^z_{i-\frac{1}{2}} \tau^z_{i+\frac{1}{2}}.$$

Defining the dual spins by

$$\widehat{X}_{i+\frac{1}{2}} = \tau^x_{i+\frac{1}{2}}, \qquad \widehat{Z}_{i+\frac{1}{2}} = \tau^z_{i+\frac{1}{2}},$$

one obtains directly

$$\boxed{Z_i Z_{i+1} \longmapsto \widehat{X}_{i+\frac{1}{2}}, \qquad X_i \longmapsto \widehat{Z}_{i-\frac{1}{2}} \widehat{Z}_{i+\frac{1}{2}}.}$$

Thus KW duality is simply the gauge-fixed description of the gauged spin chain. Interchanging τ^x and τ^z by an on-link Hadamard gate gives the other common convention in the literature.

10.3 State-Level Gauging and the Measurement Circuit

Let $s_i \in \mathbb{Z}_2$ label a matter configuration in the Z basis. Its gauge-invariant domain-wall variable is

$$a_{i+\frac{1}{2}} = s_i + s_{i+1} \mod 2.$$

Fourier transforming each link variable gives

$$|a_{i+\frac{1}{2}}\rangle = \frac{1}{\sqrt{2}} \sum_{\widehat{s}_{i+\frac{1}{2}}=0,1} (-1)^{a_{i+\frac{1}{2}} \widehat{s}_{i+\frac{1}{2}}} |\widehat{s}_{i+\frac{1}{2}}\rangle.$$

Taking the product over links therefore produces the KW wavefunction kernel

$$\mathcal{N}_{\text{KW}} |s\rangle \propto \sum_{\widehat{s}} (-1)^{\sum_i (s_i + s_{i+1}) \widehat{s}_{i+\frac{1}{2}}} |\widehat{s}\rangle.$$

There is a useful measurement-based realization of precisely this map. Prepare every link qubit in $|+\rangle_\tau$, apply controlled-Z gates between each matter spin and its two adjacent links,

$$C = \prod_i CZ_{\sigma_i,\tau_{i-\frac{1}{2}}} CZ_{\sigma_i,\tau_{i+\frac{1}{2}}},$$

and project the matter spins onto the $X = +1$ state:

$$\mathcal{N}_{\mathrm{KW}} = {}_\sigma\langle +|^{\otimes L} C |+\rangle_\tau^{\otimes L}.$$

Acting on $|s\rangle$, the circuit contributes the phase

$$\sum_i s_i \left(\widehat{s}_{i-\frac{1}{2}} \widehat{s}_{i+\frac{1}{2}}\right) = \sum_i (s_i + s_{i+1}) \widehat{s}_{i+\frac{1}{2}},$$

so it gives exactly the domain-wall Fourier transform above.

The same circuit also explains Gauss law. Conjugation by the entangler gives

$$C X_i C^\dagger = X_i \tau^z_{i-\frac{1}{2}} \tau^z_{i+\frac{1}{2}}.$$

Consequently, projecting the matter spin to $X_i = +1$ imposes $G_i = 1$ and removes the gauge-redundant matter degree of freedom. Generic measurement outcomes produce known Pauli or string byproducts; postselecting all + outcomes gives the clean KW map, while other records may be interpreted as background gauge-charge data.

10.4 KW as a Bilinear Fourier Transform

In the Z basis, write a spin configuration as $s = (s_1, \dots, s_L) \in \mathbb{Z}_2^L$. A useful representation of KW is the bilinear phase map

$$\mathcal{N}_A |s\rangle = 2^{-L/2} \sum_{\widehat{s}} (-1)^{s^T A \widehat{s}} |\widehat{s}\rangle,$$

where A is the nearest-neighbor incidence matrix over $\mathbb{Z}_2$. For ordinary KW,

$$s^T A \widehat{s} = \sum_j (s_{j-1} + s_j) \widehat{s}_j \mod 2.$$

The matrix kernel is

$$\ker A = \ker A^T = \langle (1, 1, \dots, 1) \rangle.$$

Consequently,

$$\mathcal{N}_A |s\rangle = \mathcal{N}_A |s + (1, 1, \dots, 1)\rangle :$$

KW identifies configurations related by the global $\mathbb{Z}_2$ symmetry. This is the microscopic meaning of gauging and the source of noninvertibility on a closed chain. More generally,

$$G = \ker A^T, \qquad \mathcal{N}_A^\dagger \mathcal{N}_A = \sum_{g \in G} g.$$

For ordinary KW this gives schematically

$$\mathcal{N}^\dagger \mathcal{N} = 1 + U.$$

10.5 Derivation of the Bond-Dimension-Two MPO

The bilinear KW map has a bond-dimension-two matrix-product-operator representation. Introduce the projectors

$$P_0^j = \frac{1+Z_j}{2} = |0\rangle_j\langle 0|, \qquad P_1^j = \frac{1-Z_j}{2} = |1\rangle_j\langle 1|.$$

The local Hadamard–controlled-Z building block can be decomposed as

$$U_j^D = CZ_{j,j+1}H_j = P_0^{j+1}H_j P_1^{j+1} Z_j H_j = \begin{pmatrix} P_0^{j+1} & P_1^{j+1} \end{pmatrix} \begin{pmatrix} H_j \\ Z_j H_j \end{pmatrix}.$$

The alternatives selected by P_0 and P_1 form a two-state virtual space. Absorbing the projectors on site j into the local tensor gives

$$\boxed{\mathbb{D}_j = \begin{pmatrix} H_j \\ Z_j H_j \end{pmatrix} \begin{pmatrix} P_0^j & P_1^j \end{pmatrix} = \begin{pmatrix} H_j P_0^j & H_j P_1^j \\ Z_j H_j P_0^j & Z_j H_j P_1^j \end{pmatrix}.}$$

Using

$$H|0\rangle = |+\rangle, \qquad H|1\rangle = |-\rangle,$$
$$Z|+\rangle = |-\rangle, \qquad Z|-\rangle = |+\rangle,$$

this becomes

$$\boxed{\mathbb{D}_j = \begin{pmatrix} |+\rangle\langle 0|_j & |-\rangle\langle 1|_j \\ |-\rangle\langle 0|_j & |+\rangle\langle 1|_j \end{pmatrix}.}$$

Closing the virtual index on a ring gives

$$\boxed{D = \mathrm{Tr}_{\mathrm{virt}}\left(\mathbb{D}_L \mathbb{D}_{L-1} \cdots \mathbb{D}_1\right),}$$

so the global KW operator is an MPO of bond dimension two.

To recover the bilinear kernel explicitly, label the virtual indices by $a_j, a_{j-1} \in \mathbb{Z}_2$. The local tensor is

$$(\mathbb{D}_j)_{a_j,a_{j-1}} = H_j|a_j + a_{j-1}\rangle_j\langle a_{j-1}|_j.$$

For physical input and output bits σ_j, τ_j,

$$\langle \tau_j|(\mathbb{D}_j)_{a_j,a_{j-1}}|\sigma_j\rangle = 2^{-1/2}(-1)^{\tau_j(a_j+a_{j-1})}\delta_{\sigma_j,a_{j-1}}.$$

Contracting and tracing all virtual indices yields

$$\begin{aligned}\langle \tau|D|\sigma\rangle &= \sum_{\{a_j\}}\prod_j 2^{-1/2}(-1)^{\tau_j(a_j+a_{j-1})}\delta_{\sigma_j,a_{j-1}} \\ &= 2^{-L/2}(-1)^{\sum_j \tau_j(\sigma_j+\sigma_{j+1})}.\end{aligned}$$

Here the Kronecker deltas set $a_{j-1} = \sigma_j$ and $a_j = \sigma_{j+1}$. Thus the MPO contraction reproduces exactly the KW/BPM kernel. It also gives the intertwining relations

$$X_j D = D Z_j Z_{j+1}, \qquad Z_{j-1}Z_j D = D X_j.$$

The bilinear Fourier transform, local KW operator map, and bond-dimension-two MPO are therefore three equivalent descriptions of the same transformation.

10.6 Local Operator Map

Put dual spins $\widehat{X}_{i+\frac{1}{2}}, \widehat{Z}_{i+\frac{1}{2}}$ on links. The standard local KW rules are

$$X_i \longmapsto \widehat{Z}_{i-\frac{1}{2}} \widehat{Z}_{i+\frac{1}{2}}, \qquad Z_i Z_{i+1} \longmapsto \widehat{X}_{i+\frac{1}{2}}.$$

Hence

$$Z_i Z_{i+r} \longmapsto \prod_{m=i}^{i+r-1} \widehat{X}_{m+\frac{1}{2}}.$$

An individual Z_i becomes a disorder string, while finite neutral products such as $Z_i Z_j$ have local dual images. This is the lattice statement that gauging preserves the local algebra of symmetry-invariant operators.

For Pauli monomials

$$X^r = \prod_i X_i^{r_i}, \qquad Z^v = \prod_i Z_i^{v_i},$$

a general bilinear phase map obeys

$$\mathcal{N}_A X^r = \widehat{Z}^{A^T r} \mathcal{N}_A,$$

and, when $v = Aw$,

$$\mathcal{N}_A Z^v = \widehat{X}^w \mathcal{N}_A.$$

The condition $v \in \operatorname{im} A$ is precisely neutrality under $G = \ker A^T$.

10.7 Charge–Twist Exchange and Boundary Terms

On the sector-extended Hilbert space, KW acts as

$$\boxed{(u, t) \longmapsto (\widehat{u}, \widehat{t}) = (t, u).}$$

Thus original charge becomes dual twist, while original twist becomes dual charge:

$(u,t)_{\text{original}}$	$(\widehat{u},\widehat{t})_{\text{dual}}$
$(0,0)$	$(0,0)$
$(1,0)$	$(0,1)$
$(0,1)$	$(1,0)$
$(1,1)$	$(1,1)$

If one fixes a single boundary condition, the bilinear pairing is degenerate and KW is nonunitary. The correct closed-chain kernel contains twist-dependent boundary terms. Explicitly,

$$\mathcal{N}_{t,\widehat{t}} |s; t\rangle = 2^{-L/2} \sum_{\widehat{s}} (-1)^{\Phi_{t,\widehat{t}}(s,\widehat{s})} |\widehat{s}; \widehat{t}\rangle,$$
$$\Phi_{t,\widehat{t}} = \sum_{i=1}^{L-1} (s_i + s_{i+1}) \widehat{s}_{i+\frac{1}{2}} (s_L + s_1 + t) \widehat{s}_{\frac{1}{2}} \widehat{t}\, s_L.$$

The first two terms are the Fourier pairing with all domain walls, including the twisted boundary wall. Their total parity is t, so the dual state has

$$\prod_i \widehat{X}_{i+\frac{1}{2}} = (-1)^t, \qquad \widehat{u} = t.$$

Under the original global spin flip $s_i \mapsto s_i + 1$, all domain-wall terms are unchanged, but the final boundary term contributes $(-1)^{\widehat{t}}$. Hence

$$\mathcal{N}_{t,\widehat{t}} U = (-1)^{\widehat{t}} \mathcal{N}_{t,\widehat{t}}, \qquad \widehat{t} = u.$$

This proves the charge–twist exchange directly from the BF-like bilinear pairing.

Equivalently, one enlarges the Hilbert space to include both twisted and untwisted sectors. KW then becomes a unitary permutation of the four sectors. On a fixed sector,

$$\mathcal{N}^\dagger \mathcal{N} = 1 + U,$$

so the map is noninvertible; on the sector-extended Hilbert space the missing twist information is restored and the transformation is invertible. On an open chain, the global kernel is absent and KW can be represented by an ordinary unitary map.

10.8 Transverse-Field Ising Model

For

$$H_{\text{Ising}} = -J \sum_i Z_i Z_{i+1} - h \sum_i X_i,$$

KW gives

$$H_{\text{Ising}} \longmapsto -J \sum_i \widehat{X}_{i+\frac{1}{2}} - h \sum_i \widehat{Z}_{i-\frac{1}{2}} \widehat{Z}_{i+\frac{1}{2}}.$$

After relabeling dual sites, J and h are exchanged. Therefore the ordered $\mathbb{Z}_2$-SSB phase is dual to the symmetric paramagnet, and conversely. At the self-dual point $J = h$, KW becomes a noninvertible symmetry of the critical Ising Hamiltonian.

10.9 The Local KW Defect and Its Motion

At the self-dual point, KW can be inserted on one bond rather than applied to the whole chain. For

$$H_{\text{crit}} = -\sum_j \left(Z_{j-1} Z_j + X_j \right),$$

a defect on the bond $(J, J+1)$ is represented by replacing

$$-Z_J Z_{J+1} - X_{J+1} \quad \longrightarrow \quad -Z_J X_{J+1}.$$

The resulting defect Hamiltonian may be written

$$H_D^{(J,J+1)} = - \sum_{k \neq J+1} (Z_{k-1} Z_k + X_k) - Z_J X_{J+1}.$$

The defect is topological on the lattice because it can be moved by a local unitary. Define

$$U_j^D = CZ_{j+1,j} H_j, \qquad H_j = \frac{X_j + Z_j}{\sqrt{2}}.$$

Its conjugation rules are

operator	$U_j^D(\text{operator})(U_j^D)^\dagger$
X_j	Z_j
Z_j	X_jZ_{j+1}
X_{j+1}	Z_jX_{j+1}
Z_{j+1}	Z_{j+1}.

Consequently,

$$-Z_{j-1}X_j \mapsto -Z_{j-1}Z_j, \qquad -Z_jZ_{j+1} \mapsto -X_j, \qquad -X_{j+1} \mapsto -Z_jX_{j+1},$$

and therefore

$$H_D^{(j,j+1)} = U_j^D H_D^{(j-1,j)}(U_j^D)^\dagger.$$

No spectrum changes when the defect is displaced.

10.10 Half-Line KW Transformation on Operators

To create, rather than merely move, a KW defect at $(J, J+1)$, cut the bond-dimension-two MPO and retain its open virtual qubit ℓ. Separate the strict half-line product from its endpoint tensor:

$$U_{<J}^D = \cdots U_{J-2}^D U_{J-1}^D, \qquad U_j^D = CZ_{j,j+1}H_j \quad (j < J),$$

$$U_J^D = CZ_{J,\ell}H_J, \qquad U_{\le J}^D = U_J^D U_{<J}^D.$$

On an infinite chain these are formal half-infinite products, but their conjugation action on every local operator is well defined because only finitely many nearby factors contribute.

For $j < J$, the local tensor acts as

$$X_j \mapsto Z_j, \qquad Z_j \mapsto X_jZ_{j+1},$$

$$X_{j+1} \mapsto Z_jX_{j+1}, \qquad Z_{j+1} \mapsto Z_{j+1}.$$

At the endpoint, replace $j+1$ by ℓ:

$$X_J \mapsto Z_J, \qquad Z_J \mapsto X_JZ_\ell,$$

$$X_\ell \mapsto Z_JX_\ell, \qquad Z_\ell \mapsto Z_\ell.$$

Composing the movement tensors gives the half-line transformation rules

$$\boxed{U_{\le J}^D X_j (U_{\le J}^D)^\dagger = Z_{j-1}Z_j, \qquad j \le J.}$$

For a $\mathbb{Z}_2$-odd operator, the result is necessarily nonlocal:

$$\boxed{U_{\le J}^D Z_j (U_{\le J}^D)^\dagger = \left(\prod_{k=j}^{J} X_k\right) Z_\ell, \qquad j \le J.}$$

This follows recursively. The first movement maps $Z_j \mapsto X_jZ_{j+1}$; every subsequent movement pushes the endpoint one site to the right, and the last tensor replaces Z_J by X_JZ_ℓ.

All physical operators strictly to the right are unchanged:

$$U^D_{\le J} X_j (U^D_{\le J})^\dagger = X_j, \qquad U^D_{\le J} Z_j (U^D_{\le J})^\dagger = Z_j, \qquad j > J.$$

The endpoint qubit transforms as

$$X_\ell \mapsto Z_J X_\ell, \qquad Z_\ell \mapsto Z_\ell.$$

The strings cancel for symmetry-even local operators. For $j < J$,

$$\begin{aligned} Z_j Z_{j+1} &\mapsto \left(\prod_{k=j}^{J} X_k\right) Z_\ell \left(\prod_{k=j+1}^{J} X_k\right) Z_\ell \\ &= X_j. \end{aligned}$$

Therefore, in the bulk of the left half-chain,

$$\boxed{X_j \mapsto Z_{j-1} Z_j, \qquad Z_j Z_{j+1} \mapsto X_j.}$$

The nonlocal string is visible only for charged operators and must terminate on the defect endpoint qubit. This is the operator-algebra meaning of the open KW line.

10.11 Half-Line KW Transformation of the Hamiltonian

Apply the map directly to the ordinary critical transverse-field Ising model

$$H_{\mathrm{TFIM}} = -\sum_j \left(Z_j Z_{j+1} + X_j\right).$$

For $j < J$, the two bulk terms are exchanged:

$$Z_j Z_{j+1} \mapsto X_j, \qquad X_j \mapsto Z_{j-1} Z_j.$$

For $j > J$, both terms are unchanged. At the endpoint,

$$\begin{gathered} X_J \mapsto Z_{J-1} Z_J, \\ Z_J Z_{J+1} \mapsto X_J Z_\ell Z_{J+1}, \qquad X_{J+1} \mapsto X_{J+1}. \end{gathered}$$

Thus the half-line image of the TFIM Hamiltonian is

$$\boxed{H_D^{(J,J+1)} = -\sum_{j<J} (X_j + Z_{j-1} Z_j) - \sum_{j>J} (Z_j Z_{j+1} + X_j) - Z_{J-1} Z_J - X_J Z_\ell Z_{J+1} - X_{J+1}.}$$

The sums omit the three endpoint terms displayed separately. At the self-dual point, the transformed left bulk is the same critical Ising Hamiltonian, up to the conventional half-site relabeling. The only essential change is therefore localized near $(J, J+1)$ and involves the open-MPO qubit ℓ.

This enlarged-algebra Hamiltonian should be distinguished from the minimal one-qubit defect Hamiltonian derived by moving an already existing defect. A local fusion or basis choice that absorbs ℓ into the neighboring physical degree of freedom reduces the endpoint region to the familiar form

$$-Z_J X_{J+1}.$$

The two formulas encode the same KW defect in different local representations: the first keeps the open virtual Hilbert space explicit, while the second has fused it into the physical chain.

10.12 Open MPO and the Defect Endpoint Qubit

If the virtual trace is not taken, the open virtual index is a two-state Hilbert space; denote it by ℓ. Replacing the neighboring physical spin in the movement tensor by this virtual qubit gives

$$U^{D}_{j,\ell} = CZ_{\ell,j} H_j.$$

Its local action is

$$X_j \mapsto Z_j, \qquad Z_j \mapsto X_j Z_\ell,$$
$$X_\ell \mapsto Z_j X_\ell, \qquad Z_\ell \mapsto Z_\ell.$$

The endpoint coupling is therefore

$$-Z_J X_\ell.$$

Identifying ℓ with the first physical spin to the right of the cut gives $-Z_J X_{J+1}$. Keeping ℓ open describes a local defect endpoint; contracting it around the ring produces the global noninvertible operator D. This is why the MPO bond degree of freedom is naturally interpreted as the Hilbert space carried by the end of the open KW defect line.

10.13 D-Preserving Deformations

On the periodic chain, the noninvertible KW operator obeys schematically

$$D^2 = (1+U)T^{-1}, \qquad UD = DU = D,$$

where T is one-site translation. Its local action on $\mathbb{Z}_2$-even operators is

$$DX_j = Z_{j-1} Z_j D, \qquad DZ_{j-1} Z_j = X_{j-1} D.$$

A translation-invariant local deformation preserves D when it is invariant under this exchange.

For example,

$$\delta H_1 = \lambda_1 \sum_j (X_j X_{j+1} + Z_j Z_{j+2})$$

is D invariant. Indeed,

$$X_j X_{j+1} \longmapsto (Z_{j-1} Z_j)(Z_j Z_{j+1}) = Z_{j-1} Z_{j+1},$$

which is the second term after shifting j, while

$$Z_j Z_{j+2} = (Z_j Z_{j+1})(Z_{j+1} Z_{j+2}) \longmapsto X_j X_{j+1}.$$

A second family is

$$\delta H_2 = \lambda_2 \sum_j (X_j Z_{j+1} Z_{j+2} + Z_j Z_{j+1} X_{j+2}).$$

Writing the two summands as A_j and B_j, respectively, KW gives

$$A_j \longmapsto B_{j-1}, \qquad B_j \longmapsto A_j,$$

so the translation-invariant sum is again exactly D preserving.

10.14 Majorana Translation and Stability of the Ising CFT

The Jordan–Wigner Majorana operators

$$\chi_{2j-1} = \left(\prod_{k<j} X_k\right) Z_j, \qquad \chi_{2j} = \left(\prod_{k<j} X_k\right) Y_j$$

give a compact interpretation:

$$\boxed{\text{KW on the Ising chain} \quad \longleftrightarrow \quad \text{translation by one Majorana site.}}$$

Thus the apparently unusual spin interactions above are simply translation-invariant four-Majorana interactions. For example, δH_1 maps locally to terms of the form $\chi_\ell \chi_{\ell+1} \chi_{\ell+2} \chi_{\ell+3}$. The spin and Majorana chains nevertheless differ globally on a ring because Jordan–Wigner bosonization reorganizes charge and boundary-condition sectors.

At the Ising CFT, the relevant scalar primaries are the energy operator ε with $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ and the spin operator σ with $(h, \bar{h}) = (\frac{1}{16}, \frac{1}{16})$. Both transform nontrivially under the continuum KW defect. There is therefore no relevant KW-preserving deformation; the first invariant scalar is of the $T\bar{T}$ type. Small λ_1 and λ_2 are consequently irrelevant near the fixed point, producing a finite gapless region within the space of D-preserving Hamiltonians.

10.15 An LSM-Type Constraint from Noninvertible KW Symmetry

A finite-range Hamiltonian preserving D cannot have a unique, trivially gapped ground state. It must either remain gapless or enter a gapped phase in which D is spontaneously broken. In the latter case, the number of superselection sectors is

$$\boxed{N_{\text{sectors}} \in 3\mathbb{Z}.}$$

The origin of this counting is the order–disorder exchange. Enlarge the Hilbert space to include both the untwisted Hamiltonian H and the U-twisted Hamiltonian H_U:

$$H_{1\oplus U} = H \oplus H_U.$$

Besides the ordinary charge U, introduce

$$\widetilde{U} = \begin{cases} +1, & \text{untwisted sector,} \\ -1, & U\text{-twisted sector.} \end{cases}$$

The sector-extended KW operator is invertible and exchanges these labels:

$$D_{1\oplus U} U D_{1\oplus U}^{-1} = \widetilde{U}, \qquad D_{1\oplus U} \widetilde{U} D_{1\oplus U}^{-1} = U.$$

In an ordered phase, the untwisted problem has two low-energy states,

$$(U, \widetilde{U}) = (+1, +1), \quad (-1, +1),$$

whereas the twist forces a finite-energy domain wall. In a disordered phase, both boundary conditions have low-energy states,

$$(U, \widetilde{U}) = (+1, +1), \quad (+1, -1).$$

KW exchanges these two patterns. Hence a gapped collection closed under D must contain, for each disordered sector, two ordered sectors:

$$m \text{ disordered} + 2m \text{ ordered} = 3m \text{ sectors}.$$

This is the lattice LSM-type obstruction associated with noninvertible KW symmetry.

For the λ_2 deformation, a special point occurs at $\lambda_2 = \frac{1}{2}$, where three ground states are exact even at finite volume. In the thermodynamic limit they may be represented by

$$|+\cdots+\rangle, \qquad |00\cdots 0\rangle, \qquad |11\cdots 1\rangle.$$

The ordinary symmetry fixes the first and exchanges the last two, while

$$D|+\cdots+\rangle = |00\cdots 0\rangle + |11\cdots 1\rangle,$$

$$D|00\cdots 0\rangle = D|11\cdots 1\rangle = |+\cdots+\rangle.$$

Thus no individual vacuum is invariant under D: the three-sector gapped phase realizes spontaneous breaking of the noninvertible symmetry.

10.16 Exchanged-Axis XXZ Chain

KW gauging is not restricted to the Ising universality class. Consider the spin-$\frac{1}{2}$ XXZ chain with the anisotropy axis chosen to be x:

$$H_{x-\mathrm{XXZ}} = J \sum_j \left(Z_j Z_{j+1} + Y_j Y_{j+1} + \Delta X_j X_{j+1} \right).$$

The Hamiltonian has a continuous $U(1)_x$ rotation symmetry. We gauge its $\mathbb{Z}_2$ subgroup generated by

$$\eta_x = \prod_j X_j.$$

Using the KW rules,

$$Z_j Z_{j+1} \longmapsto \widehat{X}_{j+\frac{1}{2}},$$

and

$$X_j X_{j+1} \longmapsto \widehat{Z}_{j-\frac{1}{2}} \widehat{Z}_{j+\frac{3}{2}}.$$

Furthermore,

$$Y_j Y_{j+1} = -X_j X_{j+1} Z_j Z_{j+1},$$

so

$$Y_j Y_{j+1} \longmapsto -\widehat{Z}_{j-\frac{1}{2}} \widehat{X}_{j+\frac{1}{2}} \widehat{Z}_{j+\frac{3}{2}}.$$

The exact lattice dual is therefore

$$\boxed{\widehat{H}_{x-\mathrm{XXZ}} = J \sum_j \left[\widehat{X}_{j+\frac{1}{2}} - \widehat{Z}_{j-\frac{1}{2}} \widehat{X}_{j+\frac{1}{2}} \widehat{Z}_{j+\frac{3}{2}} + \Delta\, \widehat{Z}_{j-\frac{1}{2}} \widehat{Z}_{j+\frac{3}{2}} \right].}$$

Thus KW does not return a nearest-neighbor XXZ Hamiltonian. It produces a local cluster-like chain. This illustrates the general principle: symmetry- invariant local operators remain local, but their dual images may be dressed by nearby dual spins.

10.17 XXZ Continuum Limit and the Half-Shift Orbifold

For

$$-1 < \Delta < 1,$$

the XXZ chain is gapless and flows to a $c = 1$ Luttinger liquid, equivalently a compact free boson. A convenient continuum Hamiltonian is

$$H_{\mathrm{LL}} = \frac{v}{2\pi} \int \mathrm{d}x \left[K(\partial_x \theta)^2 + \frac{1}{K} (\partial_x \varphi)^2 \right].$$

With the anisotropy axis chosen along x, the leading bosonization rules have the form

$$X_j \sim \frac{a}{\pi} \partial_x \varphi + A(-1)^j \cos(2\varphi) + \cdots,$$

$$Z_j + \mathrm{i}Y_j \sim e^{\mathrm{i}\theta} \left[B(-1)^j + C \cos(2\varphi) + \cdots \right].$$

Therefore

$$U(1)_x : \quad \theta \mapsto \theta + \alpha,$$

and the lattice symmetry being gauged acts as

$$\eta_x : \quad \theta \mapsto \theta + \pi.$$

The continuum operation is consequently the freely acting shift orbifold

$$\boxed{\mathcal{B}_R \longmapsto \mathcal{B}_R / \mathbb{Z}_2^{\mathrm{shift}}.}$$

Since gauging identifies

$$\theta \sim \theta + \pi,$$

one may introduce standard 2π-periodic rescaled fields

$$\widehat{\theta} = 2\theta, \qquad \widehat{\varphi} = \frac{\varphi}{2}.$$

The opposite rescaling preserves the canonical pairing. In this convention,

$$H_{\mathrm{LL}} = \frac{v}{2\pi} \int \mathrm{d}x \left[\frac{K}{4} (\partial_x \widehat{\theta})^2 + \frac{4}{K} (\partial_x \widehat{\varphi})^2 \right],$$

so

$$\widehat{K} = \frac{K}{4}.$$

The convention-independent statement is that KW gauges a half-period shift: it projects onto even charge, adds twisted sectors, and changes the compactification lattice.

10.18 Operators and Sectors in the XXZ CFT

Let

$$q = Q_x \mod 2$$

be the $\mathbb{Z}_2$ charge and $t \in \mathbb{Z}_2$ the spatial twist. Since $Z + \mathrm{i}Y \sim e^{\mathrm{i}\theta}$, the twisted boundary condition is

$$\theta(x + L) = \theta(x) + \pi t \mod 2\pi.$$

The closed-chain KW map is again

$$\boxed{(q, t) \longmapsto (\widehat{q}, \widehat{t}) = (t, q).}$$

Thus the parity of the $U(1)_x$ charge becomes the dual spatial twist, while the original twist becomes the dual charge.

The operator map also matches the compact-boson discussion:

$$e^{\mathrm{i}\theta}$$

is odd under the gauged half-shift and is removed from the local untwisted operator algebra. By contrast,

$$e^{2\mathrm{i}\theta} = e^{\mathrm{i}\widehat{\theta}}$$

survives as a local dual vertex operator. New disorder operators arise from the twisted sectors. This is the continuum counterpart of the lattice facts

$$Z_j \text{ is nonlocal after KW}, \qquad Z_j Z_{j+1} \text{ has a local dual image.}$$

10.19 Two Interpenetrating Chains and KT Duality

Now put Pauli spins σ_i on integer sites and $\tau_{i+\frac{1}{2}}$ on links, with

$$U_\sigma = \prod_i \sigma_i^x, \qquad U_\tau = \prod_i \tau_{i+\frac{1}{2}}^x.$$

The symmetry is

$$G = \mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\tau.$$

On an open chain, the KT transformation is a nonlocal unitary. It leaves the transverse operators invariant,

$$\sigma_i^x \longmapsto \sigma_i^x, \qquad \tau_{i+\frac{1}{2}}^x \longmapsto \tau_{i+\frac{1}{2}}^x,$$

and acts on symmetric Ising bonds as

$$\sigma_{i-1}^z \sigma_i^z \longmapsto \sigma_{i-1}^z \tau_{i-\frac{1}{2}}^x \sigma_i^z,$$

$$\tau_{i-\frac{1}{2}}^z \tau_{i+\frac{1}{2}}^z \longmapsto \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z.$$

Each domain wall of one $\mathbb{Z}_2$ is therefore decorated by the charge of the other $\mathbb{Z}_2$.

Microscopically one may write

$$N_{\mathrm{KT}} = N^\dagger U_{\mathrm{DW}} N,$$

where N performs full $\mathbb{Z}_2 \times \mathbb{Z}_2$ KW gauging and U_{DW} decorates domain walls. In standard operator-composition notation,

$$\mathrm{KT} = STS.$$

The earlier phase triangle used TST to follow a path between phase partition functions. Since T fixes the fully broken partition function and S fixes the nontrivial SPT response up to relabeling, both composites describe the same SSB–SPT exchange. The microscopic gauge–decorate–ungauge construction is conventionally denoted by STS.

10.20 Trivial, SSB, and Cluster-SPT Hamiltonians

The trivial symmetric paramagnet is

$$H_{\mathrm{triv}} = -\sum_i \left(\sigma_i^x + \tau_{i-\frac{1}{2}}^x\right).$$

Since KT fixes the transverse fields,

$$N_{\mathrm{KT}} H_{\mathrm{triv}} N_{\mathrm{KT}}^\dagger = H_{\mathrm{triv}}.$$

A representative of the fully broken phase is a pair of decoupled Ising ferromagnets,

$$H_{\mathrm{SSB}} = -\sum_i \left(\sigma_{i-1}^z \sigma_i^z + \tau_{i-\frac{1}{2}}^z \tau_{i+\frac{1}{2}}^z\right).$$

Both $\mathbb{Z}_2$ factors are spontaneously broken, giving four vacua in the thermodynamic limit.

Applying KT gives

$$\begin{aligned} H_{\mathrm{SPT}} = -\sum_i \big(&\sigma_{i-1}^z \tau_{i-\frac{1}{2}}^x \sigma_i^z \\ &+ \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z\big). \end{aligned}$$

This is the commuting-projector $\mathbb{Z}_2 \times \mathbb{Z}_2$ cluster Hamiltonian. On a ring it has a unique symmetric ground state and realizes the nontrivial SPT phase. Thus

$$\boxed{H_{\mathrm{SSB}} \xleftrightarrow{\mathrm{KT}} H_{\mathrm{SPT}}, \qquad H_{\mathrm{triv}} \xleftrightarrow{\mathrm{KT}} H_{\mathrm{triv}}.}$$

10.21 String Order Becomes Local Order

KT maps ordinary ferromagnetic correlators to SPT string correlators:

$$\sigma_i^z \sigma_j^z \quad \xleftrightarrow{\mathrm{KT}} \quad \sigma_i^z \left(\prod_{k=i}^{j-1} \tau_{k+\frac{1}{2}}^x\right) \sigma_j^z,$$

$$\tau_{i+\frac{1}{2}}^z \tau_{j+\frac{1}{2}}^z \quad \xleftrightarrow{\mathrm{KT}} \quad \tau_{i+\frac{1}{2}}^z \left(\prod_{k=i+1}^{j} \sigma_k^x\right) \tau_{j+\frac{1}{2}}^z.$$

Long-range string order in the SPT phase is therefore ordinary long-range ferromagnetic order in the KT-dual variables:

$$\text{SPT string order} \quad \longleftrightarrow \quad \text{local SSB order.}$$

10.22 Open-Chain Edge Modes

For an open cluster chain, stabilizers that would extend beyond the boundaries are absent. Within the ground-state subspace, the global symmetries factorize into left- and right-edge actions:

$$U_\sigma \simeq U_\sigma^L U_\sigma^R, \qquad U_\tau \simeq U_\tau^L U_\tau^R.$$

At either edge the two localized actions anticommute,

$$U_\sigma^L U_\tau^L = -U_\tau^L U_\sigma^L,$$

and similarly on the right. Each edge therefore carries a two-dimensional projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The fourfold open-chain edge degeneracy matches the four bulk vacua of the fully broken phase:

$$\text{SPT edge degeneracy} \quad \longleftrightarrow \quad \text{bulk SSB degeneracy.}$$

10.23 KT on a Ring

On a ring, an ordinary unitary cannot map a unique SPT ground state to four SSB vacua. The closed-chain KT operator is therefore noninvertible in a fixed sector, just like closed-chain KW. Label the sectors by two charges and two twists:

$$(u_\sigma, u_\tau; t_\sigma, t_\tau).$$

The KT action is

$$\boxed{(u_\sigma, u_\tau; t_\sigma, t_\tau) \longmapsto \big(u_\sigma, u_\tau; t_\sigma + u_\tau,\ t_\tau + u_\sigma\big).}$$

KT keeps the charges fixed but shifts each twist by the charge of the other symmetry. This is the sector-level expression of domain-wall decoration. When all twist sectors are included, KT becomes a unitary sector-shuffling map; on a fixed periodic Hilbert space it is noninvertible.

10.24 Lattice Summary

input	duality image
$\mathbb{Z}_2$ charge u	KW dual twist $\widehat{t} = u$
$\mathbb{Z}_2$ twist t	KW dual charge $\widehat{u} = t$
Ising ferromagnet	dual paramagnet
$H_{\text{SSB}}^{\mathbb{Z}_2 \times \mathbb{Z}_2}$	cluster SPT
local ferromagnetic order	SPT string order
bulk SSB degeneracy	open-chain edge degeneracy

Teaching note. This lattice discussion follows the shared conversation and the modern formulations in the spin-chain gauging discussion, Li, Oshikawa, and Zheng, arXiv:2301.07899 and Yan and Li, arXiv:2403.16017, together with the lattice twisted-gauging construction of Lu, Sun, and You, arXiv:2405.14939.

11 Toward Noninvertible Defects

One can package gauging as a codimension-one interface defect $\mathcal{N}$. Crossing $\mathcal{N}$ means performing the $\mathbb{Z}_2$ gauging operation.

Fusing two such interfaces corresponds, schematically, to gauging twice. The second gauging leaves behind a sum over the sectors of the emergent dual symmetry:

$$\mathcal{N} \times \mathcal{N} = 1 + \eta.$$

Since the fusion product is not the identity but a sum of defects, $\mathcal{N}$ is not invertible. This is a prototypical noninvertible symmetry defect.

12 Maxwell Theory and Higher-Form Symmetry

The compact boson is a useful entry point into generalized symmetry in 1+1 dimensions. The standard higher-dimensional example is Maxwell theory in 3+1 dimensions. It shows the same organizing principle in a new form: symmetries are represented by topological defect operators, and the charged objects need not be local operators.

12.1 General Definition of a p-Form Symmetry

An ordinary, or 0-form, global symmetry acts on local operators. Its charged objects are pointlike, and its conserved charge is measured on a codimension-one spatial slice.

A p-form symmetry acts on p-dimensional extended operators. In d spacetime dimensions:

symmetry	charged object	symmetry defect
0-form	local operator	$(d-1)$-dimensional surface
1-form	line operator	$(d-2)$-dimensional surface
2-form	surface operator	$(d-3)$-dimensional surface

Thus, a 1-form symmetry in 3+1 dimensions acts on line operators, and its symmetry operators are supported on closed two-dimensional surfaces.

If J_{p+1} is the $(p+1)$-form current, the conservation law can be written as

$$\mathrm{d} * J_{p+1} = 0$$

in the Noether-current convention, or equivalently as a closed differential form after applying the Hodge star. The associated topological operator is obtained by integrating the conserved flux on a closed codimension-$(p+1)$ surface. Schematically,

$$U_\alpha(M_{d-p-1}) = \exp\left(\mathrm{i}\alpha \int_{M_{d-p-1}} *J_{p+1}\right).$$

The word *topological* means that U_α can be smoothly deformed without changing correlation functions, unless it crosses an operator charged under the symmetry.

The conservation law and the topological nature of the defect are two expressions of the same structure. Suppose two defect supports M_{d-p-1} and M'_{d-p-1} differ by the boundary of a codimension-p region V_{d-p}. Then

$$\frac{U_\alpha(M'_{d-p-1})}{U_\alpha(M_{d-p-1})} = \exp\left(\mathrm{i}\alpha \int_{V_{d-p}} \mathrm{d} * J_{p+1}\right).$$

If

$$\mathrm{d} * J_{p+1} = 0,$$

the ratio is one, so the defect can move freely. If the deformation crosses a charged extended operator, the integral detects its charge and the correlator changes by the corresponding symmetry phase.

Thus, in the general p-form framework,

$$\text{topological defect} \quad \Longleftrightarrow \quad \text{conservation law} \quad \Longleftrightarrow \quad \text{well-defined charge of extended operators.}$$

Teaching note. This is the modern operational definition: a generalized global symmetry exists when there is a family of topological defect operators. The charged objects are detected by linking with those defects.

12.2 Electric and Magnetic 1-Form Symmetries in Maxwell Theory

Consider pure Maxwell theory

$$S = -\frac{1}{2e^2} \int F \wedge *F, \qquad F = \mathrm{d}A.$$

It has two U(1) 1-form symmetries. With the convention

$$J_m = \frac{1}{2\pi} F, \qquad J_e = \frac{1}{2\pi} * F,$$

the magnetic symmetry follows from the Bianchi identity

$$\mathrm{d}F = 0,$$

while the electric symmetry follows from the equation of motion

$$\mathrm{d} * F = 0.$$

The corresponding topological symmetry operators are supported on closed two-dimensional surfaces Σ:

$$U_m(\alpha, \Sigma) = \exp\left(\frac{\mathrm{i}\alpha}{2\pi} \int_\Sigma F\right), \qquad U_e(\alpha, \Sigma) = \exp\left(\frac{\mathrm{i}\alpha}{2\pi} \int_\Sigma *F\right).$$

These operators are topological because the relevant currents are conserved.

The charged objects are line operators. Wilson lines

$$W_q(C) = \exp\left(\mathrm{i}q \int_C A\right)$$

and 't Hooft lines are charged under the two 1-form symmetries. If a line C links the surface Σ, the surface operator acts by a phase:

$$U(\alpha, \Sigma)\, L(C) = e^{\mathrm{i}\alpha\,\mathrm{Link}(C,\Sigma)} L(C).$$

This is the higher-form analogue of a local charged operator acquiring a phase under an ordinary symmetry transformation.

12.3 Mixed Anomaly of Electric and Magnetic 1-Form Symmetries

Pure Maxwell theory also exhibits a mixed anomaly between the electric and magnetic 1-form symmetries. The clean way to see this is to couple both symmetries to background gauge fields. Since these are 1-form symmetries, their background fields are two-form gauge fields:

$$B_e,\ B_m \in \Omega^2(M).$$

Here B_e is the background for the electric 1-form symmetry and B_m is the background for the magnetic 1-form symmetry.

First couple the electric 1-form symmetry to a background field B_e. The gauge-invariant field strength is

$$\mathcal{F} = \mathrm{d}A - B_e.$$

The background-coupled Maxwell action is schematically

$$S[A; B_e] = -\frac{1}{2e^2} \int \mathcal{F} \wedge *\mathcal{F}.$$

Varying A gives the equation of motion

$$\mathrm{d} * \mathcal{F} = 0.$$

Thus the electric current built from $\mathcal{F}$ is conserved. However, the magnetic 1-form current is now

$$J_m = \frac{1}{2\pi}\mathcal{F} = \frac{1}{2\pi}(\mathrm{d}A - B_e).$$

Taking an exterior derivative gives

$$\mathrm{d}J_m = -\frac{1}{2\pi}\mathrm{d}B_e.$$

Therefore an electric 1-form background sources magnetic 1-form charge. In words: once we turn on a background gauge field for the electric 1-form symmetry, the magnetic 1-form current is no longer conserved.

This already shows one side of the mixed anomaly: the electric background modifies the Bianchi identity for the magnetic current. The dual statement is not obtained by making the same replacement with B_m. Instead, the magnetic background is a source for the magnetic current, and therefore it modifies the equation of motion for A.

Equivalently, we can see directly how a magnetic background makes the electric current fail to be conserved. The magnetic background should not be introduced as a shift of the Maxwell field strength. Instead, it couples as a source for the magnetic current. Schematically, suppressing the conventional factor of i in the Euclidean source term,

$$S[A;B_m] = -\frac{1}{2e^2}\int F\wedge *F + \frac{1}{2\pi}\int B_m\wedge *J_m, \qquad J_m = \frac{1}{2\pi}F.$$

Equivalently, after absorbing the conventional factors into the normalization of B_m, we may write the source term as

$$S_{\text{source}}[A;B_m] = \frac{1}{2\pi}\int B_m\wedge *F.$$

Now vary with respect to the dynamical gauge field A. Since

$$\delta F = \mathrm{d}(\delta A),$$

we have

$$\delta S = -\frac{1}{e^2}\int \mathrm{d}(\delta A)\wedge *F + \frac{1}{2\pi}\int B_m\wedge *\mathrm{d}(\delta A).$$

Using the symmetry of the inner product on two-forms in four dimensions,

$$\int B_m\wedge *\mathrm{d}(\delta A) = \int \mathrm{d}(\delta A)\wedge *B_m,$$

and integrating by parts, this becomes

$$\delta S = \int \delta A\wedge\left(\frac{1}{e^2}\mathrm{d}*F + \frac{1}{2\pi}\mathrm{d}*B_m\right).$$

Therefore the equation of motion is

$$\frac{1}{e^2}\mathrm{d}*F + \frac{1}{2\pi}\mathrm{d}*B_m = 0,$$

or

$$\mathrm{d}*F = -\frac{e^2}{2\pi}\mathrm{d}*B_m.$$

The electric 1-form current is

$$J_e = \frac{1}{2\pi}*F.$$

Thus

$$\mathrm{d}J_e = \frac{1}{2\pi}\mathrm{d}*F = -\frac{e^2}{(2\pi)^2}\mathrm{d}*B_m.$$

So in a magnetic background, the electric current is not conserved. The precise coefficient depends on the normalization of the background field, but the physical statement is invariant:

$$\mathrm{d}J_e \propto \mathrm{d}*B_m.$$

So the two backgrounds diagnose the same mixed anomaly from two dual viewpoints:

$$B_e:\quad \mathrm{d}J_m\neq 0, \qquad B_m:\quad \mathrm{d}J_e\neq 0.$$

This is often the most intuitive diagnostic of a mixed anomaly:

$$\text{turn on a background for one symmetry} \quad\Longrightarrow\quad \text{the other symmetry current is not conserved.}$$

It is directly parallel to the compact-boson relation

$$\mathrm{d}J_w = -\frac{1}{2\pi}F_m,$$

where a background for momentum symmetry sources winding charge.

More formally, the anomaly is captured by a 4+1-dimensional inflow action

$$S_{\text{anom}} = \frac{\mathrm{i}}{2\pi}\int_{M_5} B_e \wedge \mathrm{d}B_m.$$

Under a background gauge transformation

$$B_e \mapsto B_e + \mathrm{d}\Lambda_e,$$

the inflow action changes by a boundary term

$$\delta S_{\text{anom}} = \frac{\mathrm{i}}{2\pi}\int_{\partial M_5} \Lambda_e \wedge \mathrm{d}B_m.$$

This boundary variation is precisely what is needed to cancel the anomalous variation of the 3+1-dimensional Maxwell theory coupled to both backgrounds.

The lesson is the same as for the compact boson mixed anomaly:

background flux for one symmetry creates charge of the dual symmetry.

Equivalently, the electric and magnetic 1-form symmetries are both good global symmetries, but their background fields cannot be made simultaneously gauge invariant in a purely 3+1-dimensional way. The anomaly records the electric-magnetic linking structure of Wilson and 't Hooft lines.

12.4 Coulomb, Higgs/SSB, and Confined Phases

Higher-form symmetry gives a sharp language for confinement because the charged objects of a 1-form symmetry are line operators. In Maxwell theory the basic charged probes are the Wilson and 't Hooft loops

$$W_n(C) = \exp\left(\mathrm{i}n\int_C A\right), \qquad T_m(C) = \text{'t Hooft line of magnetic charge } m.$$

The electric 1-form symmetry generated by

$$U_e(\alpha,\Sigma) = \exp\left(\frac{\mathrm{i}\alpha}{2\pi}\int_\Sigma *F\right)$$

acts on Wilson lines by a linking phase,

$$U_e(\alpha,\Sigma)W_n(C) = e^{\mathrm{i}n\alpha\,\mathrm{Link}(\Sigma,C)}W_n(C),$$

while the magnetic 1-form symmetry acts similarly on 't Hooft lines. Thus the large-loop behavior of W and T is the order parameter for the realization of the corresponding higher-form symmetry.

The two basic asymptotic behaviors are

$$\langle W(C)\rangle \sim \begin{cases} e^{-T\,\mathrm{Area}(C)}, & \text{area law,} \\ e^{-\mu\,\mathrm{Perimeter}(C)}, & \text{perimeter law.} \end{cases}$$

The area law means that the energy cost is proportional to the minimal surface spanned by C. Physically, a pair of external charges is connected by a flux tube with tension T, so the charges are confined. The perimeter law means that the cost is localized near the probe itself; the charge is deconfined.

The higher-form SSB dictionary is:

$$\text{unbroken 1-form symmetry} \quad \Longleftrightarrow \quad \text{area law for charged lines,}$$

whereas

$$\text{broken 1-form symmetry} \quad \Longleftrightarrow \quad \text{perimeter law for charged lines.}$$

This is the higher-form analogue of ordinary SSB, but with line operators replacing local order parameters.

Coulomb phase

In the pure Maxwell Coulomb phase,

$$\mathrm{d}F = 0, \qquad \mathrm{d} * F = 0.$$

Both $\mathrm{U}(1)_e^{(1)}$ and $\mathrm{U}(1)_m^{(1)}$ are exact. They are also both spontaneously broken. The diagnostic is that both Wilson and 't Hooft loops are deconfined:

$$\langle W(C)\rangle \sim e^{-\mu_e\,\mathrm{Perimeter}(C)} \times (\text{Coulomb correction}),$$

$$\langle T(C)\rangle \sim e^{-\mu_m\,\mathrm{Perimeter}(C)} \times (\text{Coulomb correction}).$$

The extra Coulomb correction is the long-range interaction mediated by the gapless photon. This photon is the Goldstone mode of the broken continuous 1-form symmetry. In this sense, the Coulomb phase is a higher-form Goldstone phase:

$$\mathrm{U}(1)_e^{(1)} \text{ and } \mathrm{U}(1)_m^{(1)} \text{ broken} \quad \Rightarrow \quad \text{gapless photon.}$$

Higgs phase and discrete 1-form SSB

Now add dynamical electric matter of charge q and let it condense. This is the charge-q Higgs phase. The condensate screens electric charges in multiples of q, so the continuous electric 1-form symmetry is explicitly reduced to a discrete subgroup:

$$\mathrm{U}(1)_e^{(1)} \longrightarrow \mathbb{Z}_q^{(1)}.$$

The remaining $\mathbb{Z}_q^{(1)}$ can then be spontaneously broken. The charged order parameters are Wilson lines modulo q:

$$W_n(C), \qquad n \in \mathbb{Z}_q.$$

In the deconfined Higgs phase,

$$\langle W_n(C)\rangle \sim e^{-\mu_n\,\mathrm{Perimeter}(C)} \qquad (n \not\equiv 0 \mod q),$$

so the residual $\mathbb{Z}_q^{(1)}$ is broken. Since the symmetry is discrete, this does not produce a Goldstone photon. Instead it produces long-range topological order, described at low energies by a $\mathbb{Z}_q$ gauge theory.

Magnetic flux behaves differently. The Higgs condensate expels magnetic field, so magnetic flux is squeezed into tubes. In a superconductor these are Abrikosov vortices. Thus the Higgs phase is deconfined for the residual electric mod-q probes, but it confines magnetic flux into strings.

Confined phase

The electric confined phase is diagnosed by an area law for Wilson loops:

$$\langle W(C)\rangle \sim e^{-T_e\,\mathrm{Area}(C)}.$$

Equivalently, the electric 1-form symmetry is unbroken. Physically, external electric charges cannot be isolated at finite energy because electric flux forms a tube between them.

The standard dual picture is monopole condensation. Condensing magnetic objects is the magnetic analogue of the charge-condensed Higgs phase. It produces electric flux tubes, so Wilson loops have an area law. The phase structure can be summarized as

phase	$\langle W(C)\rangle$	$\langle T(C)\rangle$
Coulomb	perimeter + Coulomb	perimeter + Coulomb
charge-q Higgs	perimeter for $W_{n \mod q}$	magnetic flux tubes
electric confinement	area	perimeter

In higher-form language, the Coulomb phase breaks the continuous electric and magnetic 1-form symmetries, the charge-q Higgs phase breaks the residual $\mathbb{Z}_q^{(1)}$, and the electric confined phase leaves the electric 1-form symmetry unbroken.

12.5 Symmetry Breaking and Goldstone Mode Counting

Spontaneous breaking of higher-form symmetry parallels ordinary spontaneous symmetry breaking, but the order parameters and Goldstone modes are extended. For an ordinary 0-form symmetry, the charged objects are local operators, and a broken continuous symmetry gives an ordinary scalar Goldstone mode. For a p-form symmetry, the charged objects are p-dimensional operators, and a broken continuous symmetry gives a massless p-form gauge field.

The comparison is:

symmetry	charged object	broken-phase signal	Goldstone field
0-form	local operator	long-range order	scalar
p-form	p-dimensional operator	perimeter-type law	p-form gauge field

There is also a clean degree-of-freedom count. In d spacetime dimensions, a massless p-form gauge field has

$$\binom{d-2}{p}$$

physical polarizations. The reason is that a massless excitation moving in a fixed direction has only $d-2$ transverse directions. Gauge redundancy removes all components with indices along the time direction or the direction of motion, so the physical polarization tensor of a p-form can only carry indices in this transverse $(d-2)$-dimensional space. Since a p-form is antisymmetric, the number of independent transverse components is the number of ways to choose p directions out of $d-2$:

$$N_{\text{pol}} = \#\{\text{antisymmetric } p\text{-indices in } d-2 \text{ transverse directions}\} = \binom{d-2}{p}.$$

Equivalently, the massless little group is $SO(d-2)$, and the Goldstone mode transforms as an antisymmetric p-form representation of this little group.

For $p=0$, this gives one scalar Goldstone mode per broken continuous generator. For $p=1$ in $d=4$, it gives

$$\binom{2}{1} = 2,$$

the two physical polarizations of the photon.

In pure 3+1d Maxwell theory, the gapless photon is therefore the Goldstone mode associated with broken continuous 1-form symmetry.

The counting is subtle but beautiful. There are two continuous 1-form symmetries, electric and magnetic, but they are not independent sources of two separate photons. They are related by electric-magnetic duality and by the fact that the photon field strength packages both electric and magnetic fluxes:

$$F \quad \text{and} \quad *F.$$

The result is one massless photon with two physical polarizations in 3+1 dimensions. Thus the Coulomb phase may be viewed as the generalized-symmetry analogue of a Goldstone phase:

$$\text{broken continuous 1-form symmetry} \quad \Rightarrow \quad \text{gapless photon.}$$

12.6 Summary of the Maxwell Lesson

Maxwell theory is a compact laboratory for higher-form symmetry:

symmetry	current	defect	charged object
$\mathrm{U}(1)^{(1)}_m$	$F/2\pi$	$\exp(\frac{\mathrm{i}\alpha}{2\pi}\int_\Sigma F)$	line operator
$\mathrm{U}(1)^{(1)}_e$	$*F/2\pi$	$\exp(\frac{\mathrm{i}\alpha}{2\pi}\int_\Sigma *F)$	line operator

The main lesson is the same one that appeared in the compact-boson gauging story: symmetries are best understood by tracking topological defects, their conservation laws, and the charged objects they link with.

12.7 Adding Charge-q Matter

As a final application, add dynamical electric matter of charge q. The Bianchi identity remains unchanged:

$$\mathrm{d}F = 0.$$

Therefore the magnetic 1-form symmetry remains exact, as long as no magnetic monopoles are included.

The electric equation of motion, however, becomes

$$\mathrm{d} * F = *J_{\text{matter}}.$$

Since all dynamical electric charges are multiples of q,

$$\int_V *J_{\text{matter}} \in q\mathbb{Z}$$

for any three-volume V that counts matter worldlines passing through it. Therefore the continuous electric 1-form symmetry is no longer conserved.

The surviving symmetry is discrete. Deform the electric surface operator from Σ to Σ' through V. The ratio is

$$U_e(\alpha, \Sigma')U_e(\alpha, \Sigma)^{-1} = \exp\left(\mathrm{i}\alpha \int_V *J_{\text{matter}}\right).$$

Because the matter charge is quantized in multiples of q, the surface operator remains topological only if

$$e^{\mathrm{i}q\alpha} = 1.$$

Thus the continuous electric 1-form symmetry is reduced to

$$\mathrm{U}(1)_e^{(1)} \longrightarrow \mathbb{Z}_q^{(1)}.$$

Equivalently, the continuous family of electric surface defects ceases to be topological, but the discrete subgroup still can pass through all dynamical charge-q worldlines without producing a nontrivial phase.

The $\mathbb{Z}_q^{(1)}$-SSB phase

The phrase "$\mathbb{Z}_q^{(1)}$-SSB phase" means that the remaining discrete electric 1-form symmetry is not only exact, but spontaneously broken. The order parameter is not a local field. It is a line operator carrying nontrivial $\mathbb{Z}_q^{(1)}$ charge, for example a Wilson line of charge 1 modulo q:

$$W_1(C) = \exp\left(\mathrm{i}\int_C A\right).$$

The broken and unbroken realizations are distinguished by the long-distance behavior of such charged lines:

$$\mathbb{Z}_q^{(1)} \text{ unbroken} \iff \langle W_1(C)\rangle \sim e^{-T\,\mathrm{Area}(C)},$$

whereas

$$\mathbb{Z}_q^{(1)} \text{ spontaneously broken} \iff \langle W_1(C)\rangle \sim e^{-\mu\,\mathrm{Perimeter}(C)}.$$

Thus a $\mathbb{Z}_q^{(1)}$-SSB phase is a deconfined phase for the remaining mod-q electric probes.

A standard realization is a compact U(1) gauge theory Higgsed by a charge-q field. Write the Higgs field in the condensed phase as

$$\Phi(x) = ve^{\mathrm{i}\theta(x)},$$

where the radial mode is massive and can be ignored at low energy. The phase field is compact:

$$\theta \sim \theta + 2\pi.$$

The low-energy Abelian Higgs action contains

$$S = \int \left[\frac{1}{2e^2} F \wedge *F + \frac{v^2}{2} (\mathrm{d}\theta - qA) \wedge *(\mathrm{d}\theta - qA) \right].$$

The two parameters e and v control different physical scales. With the normalization used here, the Maxwell term is weighted by $1/e^2$, while the Higgs stiffness is v^2. Once the charge-q field condenses, the gauge field mass is

$$m_A \sim qev.$$

The corresponding penetration depth is

$$\lambda \sim \frac{1}{m_A} \sim \frac{1}{qev}.$$

The radial Higgs mode has another mass scale, say m_H, and its inverse

$$\xi \sim \frac{1}{m_H}$$

is the coherence length, namely the size of the region over which the condensate amplitude can vary appreciably. In the simplified phase-only description above, we have already sent the radial mode to high energy and kept only θ.

These scales organize the phases. If $v = 0$, the charge-q field is not condensed and the photon can remain gapless: this is the Coulomb phase of the compact gauge theory, up to effects from monopoles or other nonperturbative objects. If $v \neq 0$, the gauge field is Higgsed and the continuous 1-form Goldstone phase is gone; only the discrete $\mathbb{Z}_q^{(1)}$ structure remains at long distances. In a compact gauge theory, at strong gauge coupling or in a regime where magnetic objects condense, one can instead enter a confined phase, characterized by an area law for Wilson loops. Thus v controls whether electric charge condenses, while e controls the gauge-field fluctuation scale and the penetration depth of magnetic flux.

The gauge-invariant combination is

$$\mathrm{d}\theta - qA.$$

Under an ordinary gauge transformation

$$A \mapsto A + \mathrm{d}\lambda, \qquad \theta \mapsto \theta + q\lambda,$$

this combination is unchanged.

The Higgs mechanism is now visible directly. In unitary gauge one sets

$$\theta = 0,$$

so the second term becomes

$$\frac{q^2 v^2}{2} A \wedge *A.$$

Thus the photon is massive:

$$m_A \sim qev.$$

This is the Meissner effect. The continuous Coulomb photon is gone, so this phase cannot be the Goldstone phase of a continuous 1-form symmetry.

However, because the condensed field has charge q, it only screens electric charges modulo q. A Wilson line of charge n can end on dynamical matter only when

$$n \in q\mathbb{Z}.$$

Therefore line charges are conserved modulo q:

$$n \sim n + q.$$

This is the operator meaning of the residual electric 1-form symmetry

$$\mathbb{Z}_q^{(1)}.$$

Now let us derive the BF theory. Dualize the compact phase θ. Introduce a three-form auxiliary field H and rewrite the Higgs kinetic term in first-order form:

$$S_\theta = \int \left[\frac{1}{2v^2} H \wedge *H + \frac{\mathrm{i}}{2\pi} H \wedge (\mathrm{d}\theta - qA) \right].$$

Integrating out H returns the original $(v^2/2)(\mathrm{d}\theta - qA) \wedge *(\mathrm{d}\theta - qA)$ term. Instead, integrate out θ. Since θ appears through

$$\frac{\mathrm{i}}{2\pi} \int H \wedge \mathrm{d}\theta = -\frac{\mathrm{i}}{2\pi} \int \mathrm{d}H \, \theta,$$

the θ equation of motion imposes

$$\mathrm{d}H = 0.$$

Locally this means

$$H = \mathrm{d}b,$$

where b is a compact two-form gauge field. Substituting back gives

$$S_\theta = \int \frac{1}{2v^2} \mathrm{d}b \wedge *\mathrm{d}b - \frac{\mathrm{i}q}{2\pi} \int \mathrm{d}b \wedge A.$$

Integrating by parts,

$$-\int \mathrm{d}b \wedge A = \int b \wedge \mathrm{d}A,$$

so

$$S_\theta = \int \frac{1}{2v^2} \mathrm{d}b \wedge *\mathrm{d}b + \frac{\mathrm{i}q}{2\pi} \int b \wedge F.$$

At distances much longer than the Higgs scale, the ordinary kinetic terms are irrelevant compared with the topological coupling. The effective low-energy topological action is therefore

$$S_{\mathrm{BF}} = \frac{\mathrm{i}q}{2\pi} \int b \wedge \mathrm{d}A.$$

The integer q is not a coupling that can be continuously changed; it records the residual discrete gauge structure.

The BF theory contains deconfined line and surface operators:

$$W_n(C) = \exp\left(\mathrm{i}n \int_C A\right), \qquad V_m(\Sigma) = \exp\left(\mathrm{i}m \int_\Sigma b\right).$$

The labels are defined modulo q:

$$n, m \in \mathbb{Z}_q.$$

To see the linking phase, insert $V_m(\Sigma)$ into the path integral. It adds

$$\mathrm{i}m \int_\Sigma b = \mathrm{i}m \int b \wedge \delta_\Sigma,$$

where δ_Σ is the two-form Poincare dual to Σ. The equation of motion from varying b becomes

$$\frac{q}{2\pi}\mathrm{d}A + m\,\delta_\Sigma = 0.$$

Therefore, on a disk D whose boundary is C,

$$\int_C A = \int_D \mathrm{d}A = -\frac{2\pi m}{q}\int_D \delta_\Sigma = -\frac{2\pi m}{q}\,\mathrm{Link}(C, \Sigma).$$

Thus a Wilson line linked with the surface operator acquires the phase

$$W_n(C) \longmapsto \exp\left(-\frac{2\pi \mathrm{i} nm}{q}\,\mathrm{Link}(C, \Sigma)\right) W_n(C).$$

Up to orientation conventions, the correlation function is

$$\langle W_n(C) V_m(\Sigma)\rangle = \exp\left(\frac{2\pi \mathrm{i} nm}{q}\,\mathrm{Link}(C, \Sigma)\right) \langle W_n(C)\rangle\langle V_m(\Sigma)\rangle.$$

This is the discrete version of electric-magnetic linking in Maxwell theory.

There is one important contrast with the continuous Coulomb phase. Broken continuous 1-form symmetry gives a gapless photon as a Goldstone mode. Broken discrete 1-form symmetry gives no Goldstone mode. Instead, it gives long-range topological order: deconfined extended operators, topological linking phases, and ground-state degeneracy on spatial manifolds with nontrivial cycles. For example, pure $\mathbb{Z}_q$ gauge theory in $3+1$ dimensions has discrete holonomy sectors on a spatial three-torus.

Relation to type-II superconductors

A conventional superconductor is a particularly physical realization of this story. The Cooper pair has electric charge $q = 2$, so the condensate Higgses

$$\mathrm{U}(1) \longrightarrow \mathbb{Z}_2.$$

In generalized-symmetry language, the continuous electric 1-form symmetry is reduced to a discrete subgroup:

$$\mathrm{U}(1)_e^{(1)} \longrightarrow \mathbb{Z}_2^{(1)}.$$

The photon becomes massive; this is the Meissner effect. The low-energy topological sector is the $q = 2$ BF theory

$$S_{\mathrm{BF}} = \frac{2\mathrm{i}}{2\pi} \int b \wedge \mathrm{d}A.$$

For a charge-q condensate, magnetic flux is quantized in units

$$\Phi = \frac{2\pi}{q}.$$

This follows directly from the Higgs phase winding. Far from a vortex core, the energy density is minimized by

$$\mathrm{d}\theta - qA = 0.$$

Here is the reason. In the Higgs phase, the relevant static energy density contains

$$\mathcal{E}_{\mathrm{Higgs}} \supset \frac{v^2}{2} |\nabla\theta - q\mathbf{A}|^2 + \frac{1}{2e^2} |\mathbf{B}|^2 + V(|\Phi|).$$

Inside the vortex core, the phase θ winds and is singular, so the condensate amplitude must go to zero:

$$|\Phi| \to 0.$$

That makes the core costly but finite. Far outside the core, however, the amplitude returns to its vacuum value $|\Phi| = v$, and the potential energy is already minimized. The remaining way to minimize the energy is to make the covariant phase gradient vanish:

$$\nabla\theta - q\mathbf{A} = 0,$$

or in differential-form notation,

$$\mathrm{d}\theta - qA = 0.$$

This is not saying that θ itself is constant around the vortex. Rather, the gauge field cancels the phase winding away from the core, so the energy density does not grow with distance. Around a loop C encircling the vortex,

$$\oint_C \mathrm{d}\theta = 2\pi k, \qquad k \in \mathbb{Z}.$$

Therefore

$$q \oint_C A = 2\pi k,$$

and by Stokes' theorem

$$\Phi = \int_D F = \oint_C A = \frac{2\pi k}{q}.$$

The minimal vortex has $k = 1$. In a type-II superconductor these quantized flux tubes are the Abrikosov vortices. They are the physical vortex strings associated with the discrete higher-form ordered phase. Their linking with electric quasiparticles produces the characteristic Aharonov–Bohm phase encoded by the BF action.

This is why superconductors make the $\mathbb{Z}_q^{(1)}$-SSB picture concrete: the quantized flux tubes are physical vortex strings associated with the higher-form ordered phase.

One should add a standard caveat. A real laboratory superconductor is coupled to the electromagnetic field outside the material, so it is not literally an isolated intrinsic $\mathbb{Z}_2$ topological order in the same idealized sense as a standalone $\mathbb{Z}_2$ gauge theory. But as an effective Higgs phase of a compact gauge theory, it realizes the same generalized-symmetry structure:

$$\text{charge-}q \text{ Higgs phase} \quad \simeq \quad \mathbb{Z}_q^{(1)}\text{-SSB} \quad \simeq \quad \mathbb{Z}_q \text{ topological gauge theory.}$$

13 Gauging a General $\mathbb{Z}_n^{(p)}$ Symmetry

The compact-boson example is the simplest case of a general principle. In d spacetime dimensions, gauging a finite abelian p-form symmetry

$$\mathbb{Z}_n^{(p)}$$

produces an emergent dual symmetry

$$\widehat{\mathbb{Z}_n}^{(d-p-2)} \simeq \mathbb{Z}_n^{(d-p-2)}.$$

In short,

$$\boxed{\mathbb{Z}_n^{(p)} \text{ gauged} \quad \Longrightarrow \quad \text{dual } \mathbb{Z}_n^{(d-p-2)} \text{ symmetry.}}$$

The degree of the dual symmetry is fixed by topology. A p-form symmetry has a $(p+1)$-form background gauge field. Once this field is summed over, its flux is measured by operators supported on submanifolds of complementary dimension. This complementary dimension is

$$d-(p+1)-1 = d-p-2,$$

which is why the dual symmetry is a $(d-p-2)$-form symmetry.

13.1 Background Field and Gauging

A $\mathbb{Z}_n^{(p)}$ global symmetry couples to a background

$$B_{p+1} \in Z^{p+1}(M,\mathbb{Z}_n),$$

where $Z^{p+1}(M,\mathbb{Z}_n)$ denotes closed $(p+1)$-cochains:

$$\delta B_{p+1} = 0 \mod n.$$

The partition function in the background B_{p+1} is written as

$$Z[B_{p+1}].$$

Gauging means that this background is promoted to a dynamical discrete gauge field. We write it as

$$b_{p+1} \in Z^{p+1}(M,\mathbb{Z}_n)$$

and sum over it:

$$Z_{\text{gauged}} = \frac{1}{|Z^{p+1}(M,\mathbb{Z}_n)|} \sum_{b_{p+1}\in Z^{p+1}(M,\mathbb{Z}_n)} Z[b_{p+1}].$$

The overall normalization is conventional and depends on how one divides by gauge redundancies. The important point is conceptual: gauging is summing over flat $\mathbb{Z}_n$ backgrounds, the higher-form analogue of orbifolding.

13.2 The Emergent Dual Symmetry

After gauging, the dynamical field b_{p+1} has topological flux sectors. The new dual symmetry measures these fluxes. To couple the dual symmetry to a background, introduce

$$\widehat{B}_{d-p-1} \in Z^{d-p-1}(M, \mathbb{Z}_n).$$

This is the correct degree because it is a background for a $(d-p-2)$-form symmetry:

$$(d-p-2)+1 = d-p-1.$$

The universal BF-type response is

$$\boxed{\exp\left[\frac{2\pi\mathrm{i}}{n}\int_M b_{p+1} \cup \widehat{B}_{d-p-1}\right].}$$

The degrees add to d:

$$(p+1)+(d-p-1) = d,$$

so the cup product can be integrated over spacetime. This term says that the dual background detects the flux of the dynamical gauge field produced by gauging the original symmetry.

This general formula reduces to the compact-boson expression. There we had

$$d = 2, \qquad p = 0, \qquad n = 2,$$

so

$$b_{p+1} = a \in Z^1(M, \mathbb{Z}_2), \qquad \widehat{B}_{d-p-1} = A \in Z^1(M, \mathbb{Z}_2),$$

and

$$\frac{2\pi\mathrm{i}}{n}\int b_{p+1} \cup \widehat{B}_{d-p-1} = \mathrm{i}\pi \int a \cup A.$$

13.3 Charge–Flux Fourier Transform

The BF factor implements the exchange between charge labels and flux labels. For an ordinary finite symmetry, this exchange is the familiar statement that orbifolding sums over twisted sectors and projects onto invariant states. For higher-form symmetries, the same logic applies, but the charged objects and twists are extended.

Schematically, before gauging one can label sectors by

$$\text{charge under } \mathbb{Z}_n^{(p)} \quad \text{and} \quad \text{background flux } b_{p+1}.$$

After gauging, the old flux becomes charge under the emergent dual symmetry. The BF pairing is the finite Fourier kernel that implements this exchange:

$$\text{old charge data} \quad \longleftrightarrow \quad \text{new dual twist data,}$$

$$\text{old twist/flux data} \quad \longleftrightarrow \quad \text{new dual charge data.}$$

In the compact-boson $\mathbb{Z}_2$ case, this became the sector exchange

$$(q,t)_{\mathrm{orig}} \longmapsto (t,q)_{\mathrm{dual}}.$$

The general $\mathbb{Z}_n^{(p)}$ statement is the same: gauging trades charged objects of the original symmetry for topological flux sectors measured by the dual symmetry.

13.4 Useful Examples

Ordinary $\mathbb{Z}_n$ symmetry in $1+1$d. Here

$$d = 2, \qquad p = 0.$$

The dual symmetry has degree

$$d - p - 2 = 0.$$

Thus gauging an ordinary $\mathbb{Z}_n$ symmetry in two spacetime dimensions produces another ordinary dual $\mathbb{Z}_n$ symmetry. This is the general version of the compact-boson $\mathbb{Z}_2$ orbifold story.

Ordinary $\mathbb{Z}_n$ symmetry in $2+1$d. Here

$$d = 3, \qquad p = 0.$$

The dual symmetry has degree

$$d - p - 2 = 1.$$

Thus gauging an ordinary finite symmetry in $2+1$d produces an emergent $\mathbb{Z}_n^{(1)}$ one-form symmetry. This is the familiar quantum symmetry of a finite gauge theory: Wilson lines are charged under the emergent one-form symmetry.

$\mathbb{Z}_n^{(1)}$ symmetry in $3+1$d. Here

$$d = 4, \qquad p = 1.$$

The dual symmetry has degree

$$d - p - 2 = 1.$$

Thus gauging a $\mathbb{Z}_n^{(1)}$ one-form symmetry in four spacetime dimensions produces another dual $\mathbb{Z}_n^{(1)}$ one-form symmetry. This is the structure behind four-dimensional BF theory,

$$S_{\mathrm{BF}} = \frac{\mathrm{i}n}{2\pi} \int b_2 \wedge \mathrm{d}A_1,$$

where the two gauge fields encode mutually linked electric and magnetic extended operators.

13.5 Dictionary

The main dictionary is

object	degree/type
$\mathbb{Z}_n^{(p)}$ symmetry	p-form symmetry
background field	$B_{p+1} \in Z^{p+1}(M, \mathbb{Z}_n)$
dynamical field after gauging	$b_{p+1} \in Z^{p+1}(M, \mathbb{Z}_n)$
dual symmetry	$\mathbb{Z}_n^{(d-p-2)}$
dual background	$\widehat{B}_{d-p-1} \in Z^{d-p-1}(M, \mathbb{Z}_n)$
BF pairing	$\frac{2\pi \mathrm{i}}{n} \int b_{p+1} \cup \widehat{B}_{d-p-1}$

The slogan is:

$$\boxed{\text{gauging trades charge sectors of the original symmetry for flux sectors of the dual symmetry.}}$$

14 Two-Dimensional $\mathbb{Z}_2$ Gauging and Toric-Code One-Form Symmetry

14.1 Lattice Gauging: From $\mathbb{Z}_2^{(0)}$ to $\mathbb{Z}_2^{(1)}$

The higher-form structure can be seen directly by gauging an ordinary $\mathbb{Z}_2$ symmetry in $2+1$ dimensions. Put matter qubits σ_v on the vertices of a square lattice, with global symmetry

$$U_{\mathbb{Z}_2^{(0)}} = \prod_v \sigma_v^x.$$

Introduce one gauge qubit τ_e on every edge. We use the convention

$$G_v = \sigma_v^x \prod_{e \ni v} \tau_e^z,$$

where the product runs over the edges incident on v. The physical Hilbert space obeys

$$G_v |\text{phys}\rangle = |\text{phys}\rangle \qquad \text{for every } v.$$

Multiplying all Gauss laws on a closed lattice gives

$$\prod_v G_v = \prod_v \sigma_v^x,$$

because every edge is incident on two vertices and hence every τ_e^z appears twice. Thus the Gauss constraints automatically retain only the original $\mathbb{Z}_2^{(0)}$-even matter sector.

Gauge-Invariant Dressing

A single σ_v^z is charged and is not gauge invariant. A nearest- neighbor Ising bond is gauged by inserting the parallel transporter:

$$\sigma_v^z \sigma_{v'}^z \quad \longmapsto \quad \sigma_v^z \tau_{vv'}^x \sigma_{v'}^z.$$

At either endpoint the two minus signs, one from σ^z and one from τ^x, cancel.

More generally, let S be an even set of vertices and choose an edge 1-chain η whose boundary is S:

$$\partial \eta = S.$$

Then

$$\prod_{v \in S} \sigma_v^z \quad \longmapsto \quad \left(\prod_{v \in S} \sigma_v^z \right) \left(\prod_{e \in \eta} \tau_e^x \right).$$

Different choices of η differ by a closed τ^x loop. Therefore the ambiguity in dressing is precisely the emergent one-form symmetry or, for contractible differences, a product of plaquette-flux operators.

CZ Entangler and Removal of Matter

Define a finite-depth circuit by coupling every edge qubit to its two endpoint matter qubits:

$$U_{\mathrm{CZ}} = \prod_{e=\langle vv'\rangle} CZ_{v,e} CZ_{v',e}.$$

Its Pauli conjugation rules are

$$U_{\mathrm{CZ}} \sigma_v^x U_{\mathrm{CZ}}^\dagger = \sigma_v^x \prod_{e\ni v} \tau_e^z,$$

$$U_{\mathrm{CZ}} \tau_e^x U_{\mathrm{CZ}}^\dagger = \tau_e^x \sigma_{\partial_- e}^z \sigma_{\partial_+ e}^z,$$

while σ_v^z and τ_e^z are unchanged. Since $U_{\mathrm{CZ}} = U_{\mathrm{CZ}}^\dagger$, it follows that

$$U_{\mathrm{CZ}} G_v U_{\mathrm{CZ}}^\dagger = \sigma_v^x.$$

In the entangled basis, Gauss law is simply $\sigma_v^x = 1$. The matter qubits are frozen into $|+\rangle_v$ and may be removed.

This gives the operator dictionary

$$\sigma_v^x \longmapsto A_v \equiv \prod_{e\ni v} \tau_e^z,$$

$$\sigma_v^z \tau_{vv'}^x \sigma_{v'}^z \longmapsto \tau_{vv'}^x.$$

Thus the square-lattice transverse-field Ising Hamiltonian

$$H = -J \sum_{\langle vv'\rangle} \sigma_v^z \sigma_{v'}^z - h \sum_v \sigma_v^x$$

first becomes

$$H_{\text{gauged}} = -J \sum_{\langle vv'\rangle} \sigma_v^z \tau_{vv'}^x \sigma_{v'}^z - h \sum_v \sigma_v^x, \qquad G_v = 1,$$

and, after the entangler and matter projection,

$$\boxed{\widetilde{H}_{\text{gauged}} = -J \sum_e \tau_e^x - h \sum_v A_v.}$$

One may also include the gauge-flux operator

$$B_p = \prod_{e\in\partial p} \tau_e^x$$

through a term $-K \sum_p B_p$.

Emergent One-Form Symmetry

For every closed direct-lattice loop C, define

$$U_m(C) = \prod_{e\in C} \tau_e^x.$$

It commutes with every τ_e^x term. It also commutes with every star term A_v: a closed loop meets a vertex in zero or two incident edges, producing an even number of τ^x–τ^z anticommutations. Hence

$$[U_m(C), \widetilde{H}_{\text{gauged}}] = 0.$$

These closed line operators generate the emergent $\mathbb{Z}_2^{(1)}$ symmetry. An open string

$$W_e(\gamma) = \prod_{e\in\gamma} \tau_e^x$$

is not a symmetry operator because it anticommutes with A_v at its endpoints.

Contractible and noncontractible loops carry different information. If $C = \partial R$ is contractible, then

$$U_m(\partial R) = \prod_{p\subset R} B_p.$$

Thus contractible symmetry loops measure the flux enclosed by R. Noncontractible loops on a torus label independent global one-form sectors.

For an exact gauging map between Hilbert spaces, one commonly restricts to flat gauge fields,

$$B_p = 1,$$

and separately sums over global holonomies. For a dynamical gauge theory, the fluxes are physical excitations and B_p is instead included as a Hamiltonian term. This distinction separates a duality construction from a gauge theory with all flux sectors.

14.2 Toric-Code Limit and the e, m Excitations

At $J = 0$, adding the flux term gives

$$H_{\text{TC}} = -h\sum_v A_v - K\sum_p B_p,$$

$$A_v = \prod_{e\ni v} \tau_e^z, \qquad B_p = \prod_{e\in\partial p} \tau_e^x.$$

This is the toric code with X and Z exchanged relative to another common convention.

An electric gauge charge e is a vertex excitation:

$$A_v = -1.$$

It is created in pairs at the endpoints of an open direct-lattice string

$$W_e(\gamma) = \prod_{e\in\gamma} \tau_e^x.$$

The string commutes with all star operators except at its endpoints.

A magnetic flux, or vison, m is a plaquette excitation:

$$B_p = -1.$$

It is created in pairs by an open dual-lattice string

$$W_m(\gamma^\vee) = \prod_{e \perp \gamma^\vee} \tau_e^z,$$

where the product is over primal edges crossed by the dual path. This string commutes with all plaquette operators except at its two dual endpoints.

If the two open strings cross once, their operators contain τ_e^x and τ_e^z on the same edge. Therefore

$$W_e(\gamma) W_m(\gamma^\vee) = -W_m(\gamma^\vee) W_e(\gamma).$$

Braiding e around m consequently gives the phase -1: e and m have mutual semionic statistics.

The pure toric code has two $\mathbb{Z}_2^{(1)}$ symmetries. A closed direct-lattice string

$$U_m(C) = \prod_{e \in C} \tau_e^x$$

satisfies

$$U_m(\partial R) = \prod_{p \subset R} B_p = (-1)^{N_m(R)},$$

so it detects the number of m fluxes inside R. This is the one-form symmetry produced by gauging the original $\mathbb{Z}_2^{(0)}$ symmetry.

There is also a closed dual-lattice string

$$U_e(C^\vee) = \prod_{e \perp C^\vee} \tau_e^z.$$

For a contractible dual loop,

$$U_e(\partial R^\vee) = \prod_{v \subset R^\vee} A_v = (-1)^{N_e(R^\vee)},$$

so it detects e charges. In summary,

excitation	location	open creation string	detecting closed string
e	vertex	$\prod \tau^x$	$\prod \tau^z$ on a dual loop
m	plaquette	$\prod \tau^z$	$\prod \tau^x$ on a direct loop

15 Exercises

Exercise 1. Starting from the zero-mode decomposition, verify that the winding number is

$$w = \frac{1}{2\pi} \int_0^L \mathrm{d}x\, \partial_x \phi.$$

Exercise 2. Show that the vertex operator $e^{\mathrm{i}n\phi}$ has charge n under the momentum symmetry.

Exercise 3. Couple the shift symmetry to a background gauge field A_m and rederive

$$\mathrm{d}J_w = -\frac{1}{2\pi} F_m.$$

Explain in words why this is a mixed anomaly.

Exercise 4. For the $\mathbb{Z}_2$ orbifold, explain why a nontrivial spatial gauge-field holonomy produces half-integer winding sectors.

Exercise 5. Starting from $\varphi = 2\phi$, show that the untwisted and twisted sectors of ϕ become even and odd winding sectors of φ.

Exercise 6. Explain why the response factor

$$\exp\left(\mathrm{i}\pi \int a \cup A\right)$$

means that the emergent dual $\mathbb{Z}_2$ symmetry measures $\mathbb{Z}_2$ gauge flux.

Exercise 7. Interpret the fusion rule

$$\mathcal{N}^2 = 1 + \eta$$

as a statement that $\mathcal{N}$ is not invertible.

16 Takeaway

The compact boson is a compact classroom for modern symmetry ideas. Its momentum and winding symmetries look elementary, but together they already carry a mixed anomaly. Gauging a discrete subgroup produces twisted sectors and an emergent dual symmetry. The same dual symmetry can be understood as gauge flux, as winding parity after the rescaling $\varphi = 2\phi$, or as a half-period shift of the rescaled dual field. Treating gauging itself as an interface leads naturally to noninvertible fusion.