Quantum Entanglement and Topological Order

Ashvin Vishwanath

UC Berkeley

Yi Zhang

Berkeley->
Stanford

Ari Turner

Amsterdam

M. Oshikawa

ISSP

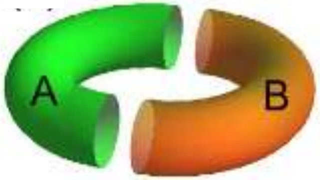
Tarun Grover

Berkeley->KITP

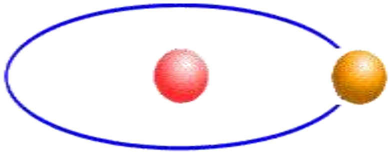
OUTLINE

- Part 1: Introduction
  - Topological Phases, Topological entanglement entropy.
  - Model wave-functions.

- Part 2: Topological Entropy of nontrivial bipartitions.
  - Ground state dependence and Minimum Entropy States.
  - Application: Kagome spin liquid in DMRG.



- Part 3: Quasi particle statistics (modular S-Matrix) from Ground State Wave-functions.



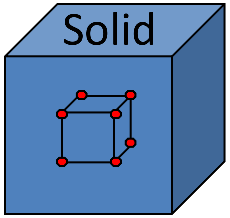
Ref: Zhang, Grover, Turner, Oshikawa, AV: ***arXiv:1111.2342***

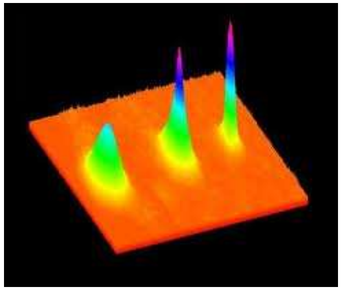
Conventional (Landau) Phases

Distinguished by spontaneous symmetry breaking.

Can be diagnosed in the ground state wave-function by a *local* order parameter.

- **Solid** (broken translation)
- **Superfluids** ψ
- **Magnets** (broken spin symmetry) **M**





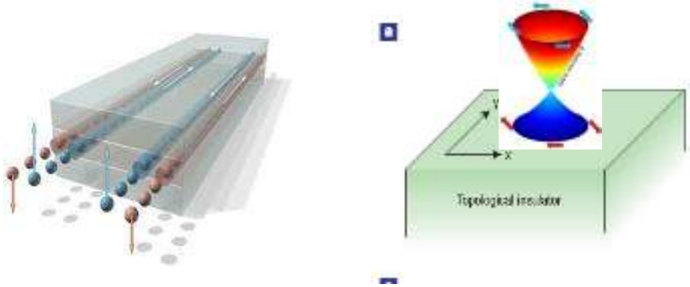
In contrast –topological phases…

Topological Phases

Integer topological phases

- Integer Quantum Hall & Topological insulators
- Haldane (AKLT) S=1 phase
- Interacting analogs in D=2,3 (Kitaev, Chen-Gu-Wen, Lu&AV)

Non-trivial surface states



Fractional topological phases

- Fractional Quantum Hall
- Gapped spin liquids

Topological Order:

1. Fractional statistics excitations (anyons).
2. Topological degeneracy on closed manifolds.

How to tell – given ground state wave-function(s)?
Entanglement as topological `order parameter'.

Topological Order – Example 1

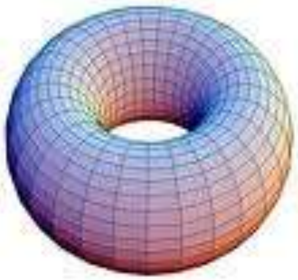
- Laughlin state (ν=1/2 bosons) [`Chiral spin Liquid']

$$\Psi[\{z_i\}] = \prod_{i<j}(z_i - z_j)^2 e^{-\sum_i |z_i|^2} \longrightarrow \boxed{\Psi(r_1, r_2 \ldots, r_N) = \Phi^2_{C=1}(r_1, r_2 \ldots, r_N)}$$

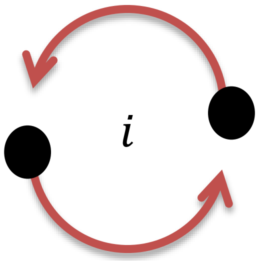
Lattice Version

- Ground State Degeneracy (N=2^g):

N=1



N=2



***s* is a semion**

Quasiparticle Types: {*1*, *s*}.

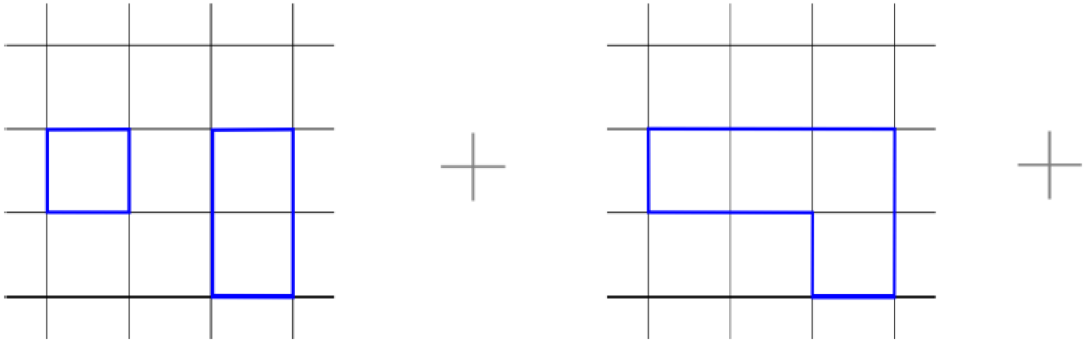
= Torus degeneracy

Topological Order – Example 2
Ising (Z_2) Electrodynamics

- Here E=0,1; B=0,π

$\nabla \cdot E = 0$ (mod 2) (E field loops do not end)

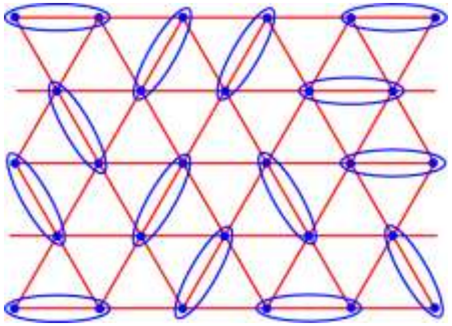
$$\Psi =$$



- Degeneracy on torus=4.
- Degeneracy on cylinder=2

  (no edge states)

Topological Order – Example 2
Z_2 Quantum Spin Liquids

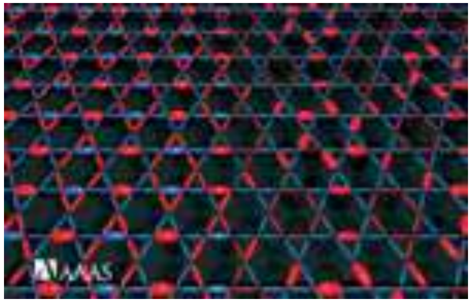


$$\Psi = P_G(\Psi_{BCS})$$

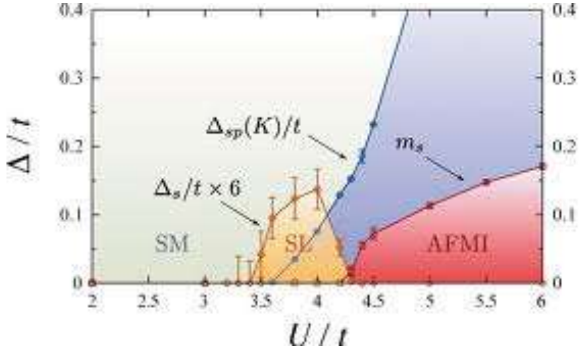
RVB spin liquid: (Anderson ’73). Effective Theory: Z_2 Gauge Theory.

Recently, a number of candidates in numerics with no conventional order.

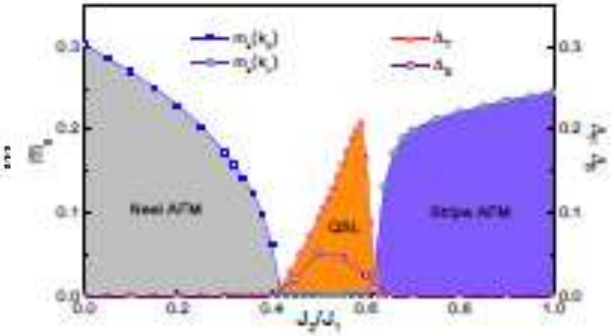
Definitive test: identify topological order.



Kagome (Yan et al)

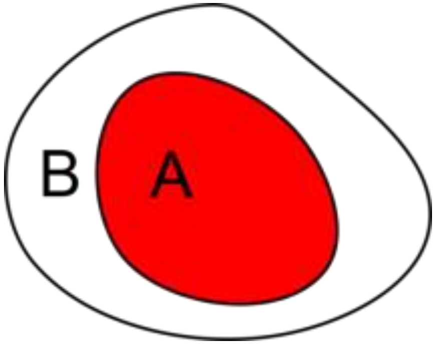


Honeycomb Hubbard (Meng et al)



Square J1_J2
(Jiang et al, Wang et al)

Entanglement Entropy



- Schmidt Decomposition:

$$\Psi = \sum_i \sqrt{p_i}\,|Ai\rangle \otimes |Bi\rangle$$

Entanglement Entropy (von-Neumann):

$$S_A = -\sum_i p_i \log p_i$$

Note: 1) $S_A = S_B$

2) Strong sub-additivity (for von-Neumann entropy)

$$\mathbf{S_A + S_B + S_C - S_{BC} - S_{AC} - S_{AB} + S_{ABC} \leq 0}$$

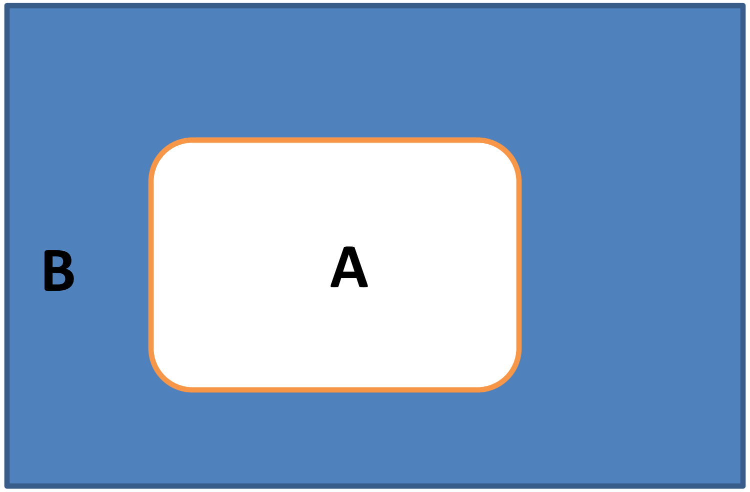
Topological Entanglement Entropy

- Gapped Phase with topological order.
  - Smooth boundary, circumference L_A:

$$S_A = aL_A - \gamma \longleftarrow$$

Topological Entanglement Entropy

(Levin-Wen;Kitaev-Preskill)



$\Upsilon=Log\ D$. (D : total quantum dimension).

Abelian phases:

$$D = \surd\{Torus\ Degeneracy\}$$

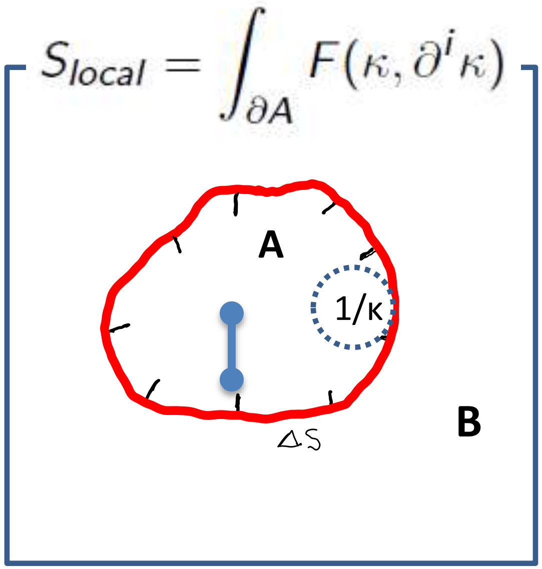
Z_2 gauge theory: $\Upsilon=Log\ 2$

Constraint on boundary – no gauge charges inside.

Lowers Entropy by 1 bit of information.

Entanglement Entropy of Gapped Phases

- Trivial Gapped Phase:
  - Entanglement entropy: sum of local contributions.



$$S_{local} = \int_{\partial A} F(\kappa, \partial^i \kappa)$$

Curvature Expansion (smooth boundary):

$$F = \mathbf{a_0} + \mathbf{a}_{\cancel{1}}\cancel{\kappa} + \mathbf{a_2}\kappa^2 + \mathbf{a_4}(\partial_l \kappa)^2 + \ldots$$

Z_2 symmetry of Entanglement Entropy:

$S_A = S_B$ AND κ→- κ. So $a_1=0$

No constant in 2D for trivial phase.

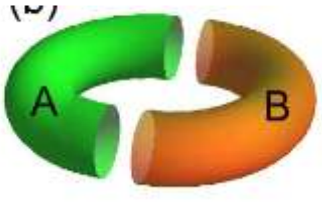
$$S_A = \oint dl[a_0 + a_2\kappa^2 + \ldots]$$

$$= a_0 L_A + \frac{A_2}{L_A} + \ldots$$

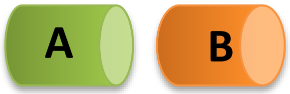
Grover, Turner, AV: PRB 84, 195120 (2011)

Extracting Topological Entanglement Entropy

- Smooth partition boundary on lattice?



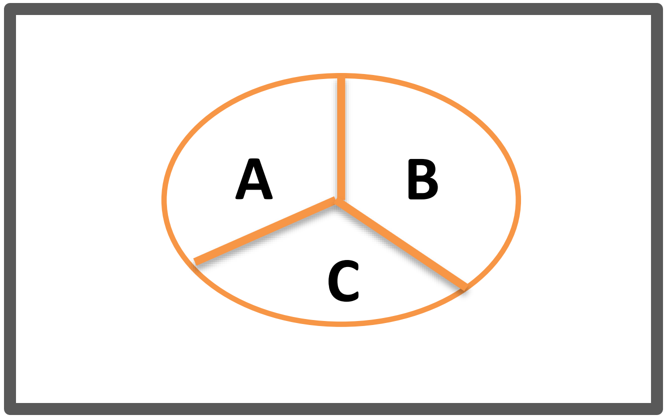
OR



- Problem:

`topological entanglement entropy' depends on ground state. (Dong et al, Zhang et al)

- General Partition



$$-\gamma = S_A + S_B + S_C - S_{BC} - S_{AC} - S_{AB} + S_{ABC}$$

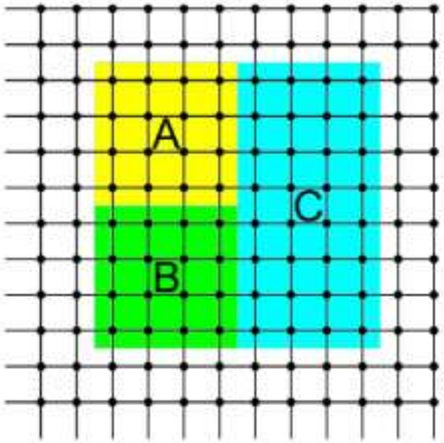
(LevinWen;Preskill Kitaev)

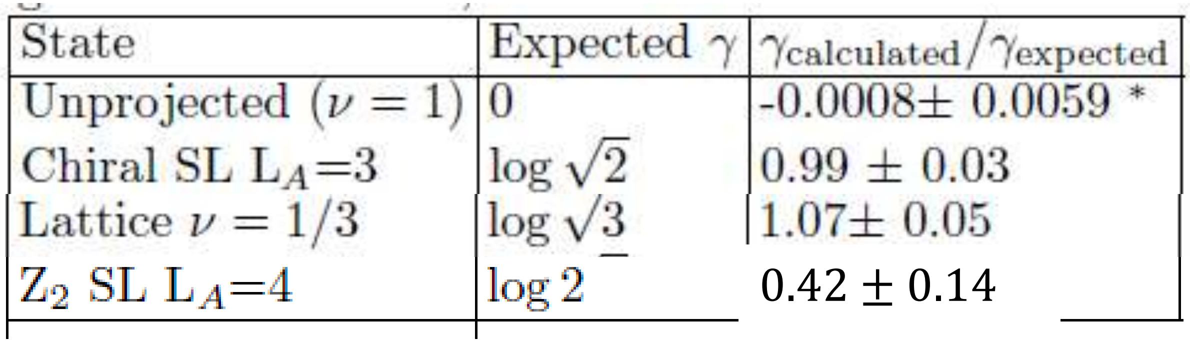
Strong subadditivity implies: $\gamma \geq 0$
Identical result with Renyi entropy

How does this work with generic states?

Topological Entropy of Lattice Wavefunctions

$\boldsymbol{\Upsilon}$ – From MonteCarlo Evaluation of Gutzwiller Projected Lattice wave-functions. $e^{-S_2} = \langle SWAP_A \rangle$. (Y. Zhang, T. Grover, AV Phys. Rev. B 2011.)





State	Expected γ	$\gamma_{\text{calculated}}/\gamma_{\text{expected}}$
Unprojected ($\nu = 1$)	0	-0.0008± 0.0059 *
Chiral SL L_A=3	$\log\sqrt{2}$	0.99 ± 0.03
Lattice $\nu = 1/3$	$\log\sqrt{3}$	1.07± 0.05
Z_2 SL L_A=4	$\log 2$	0.42 ± 0.14

Good agreement for chiral spin liquid.

Z2 not yet in thermodynamic limit(?)

Alternate approach to diagnosing topological order:

Entanglement spectrum (Li and Haldane, Bernevig et al.).

Closely related to edge states

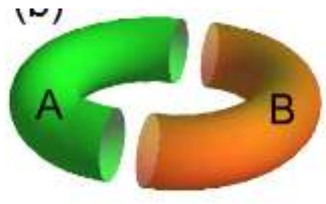
Does not diagnose Z2 SL

Cannot calculate with Monte Carlo.

Part 2: Ground State Dependence of Topological Entropy

Topological Entanglement in Nontrivial bipartitions

- Nontrivial bipartition - entanglement cut is not contractible. Can `sense’ degenerate ground states.



- Result from Chern-Simons field theory: (Dong et al.)
- Abelian topological phase with N ground states on torus.

There is a special basis of ground states for a cut, such that:

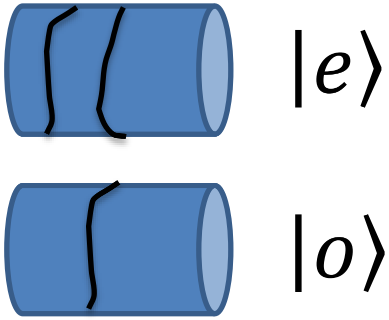
- $\Psi = \sum_{n=1}^{N} c_n |\phi_n\rangle$ $(p_n = |c_n|^2)$

$$\gamma = 2\gamma_0 - \sum_{n=1}^{N} p_n \log \frac{1}{p_n}$$

Topological entropy in general *reduced*. $0 \leq \gamma \leq 2\gamma_0$

For the *special states* $|\phi_n\rangle$, equal to usual value ($\gamma = 2\gamma_0$=2log D).
These Minimum Entropy States correspond to *quasiparticles* in cycle of the torus

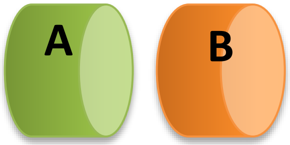
Eg. Z_2 Spin Liquid on a Cylinder

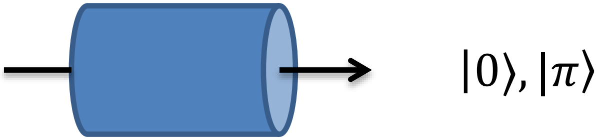


- Degenerate sectors: even and odd E winding around cylinder.

Minimum Entropy States:

$$|0,\pi\rangle = (|e\rangle \pm |o\rangle)/\sqrt{2}$$

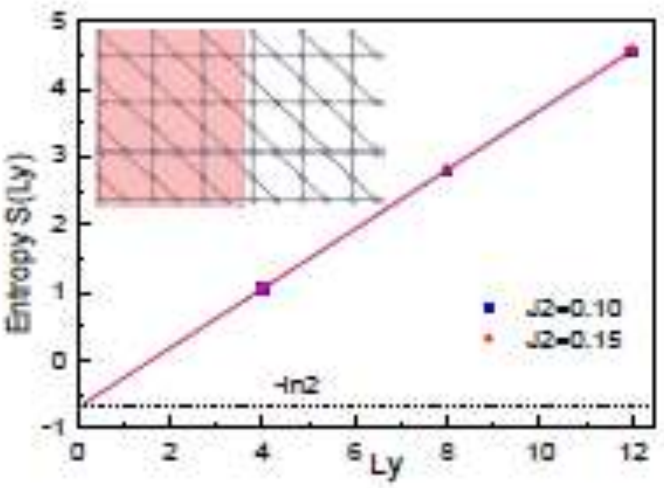




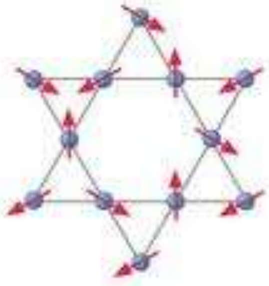
- The minimum entropy states ($\gamma = \log 2$) are `vison' states – magnetic flux through the cylinder that entanglement surface can measure.
- State $|e\rangle$ has $\gamma = 0$. Cancellation from:

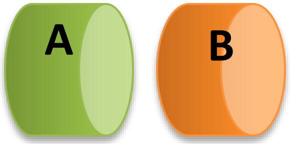
$$\Psi = \frac{|A, even\rangle\big||B, even\rangle + |A, odd\rangle\big||B, odd\rangle}{\sqrt{2}}$$

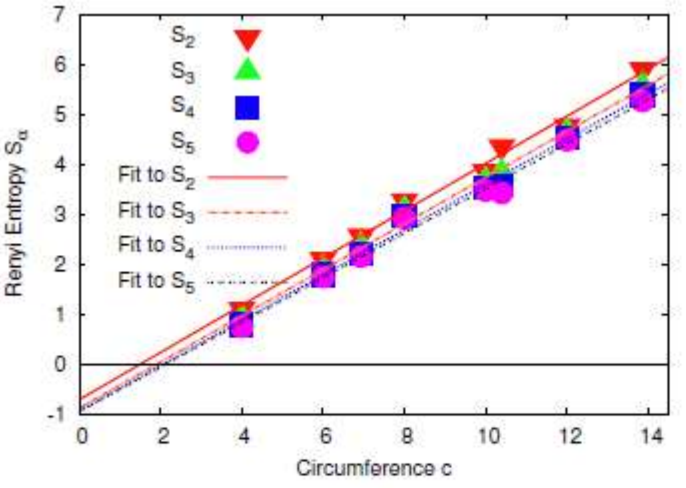
Application: DMRG on Kagome Antiferromagnet



Jiang, Wang, Balents: arXiv:1205.4289





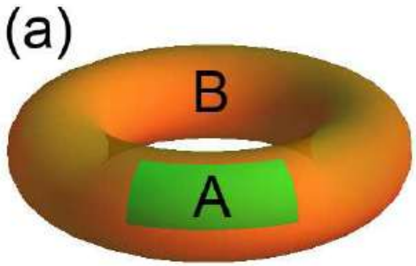


Depenbrock,McCulloch, Schollwoeck (arxiv:1205:4858). Log base 2

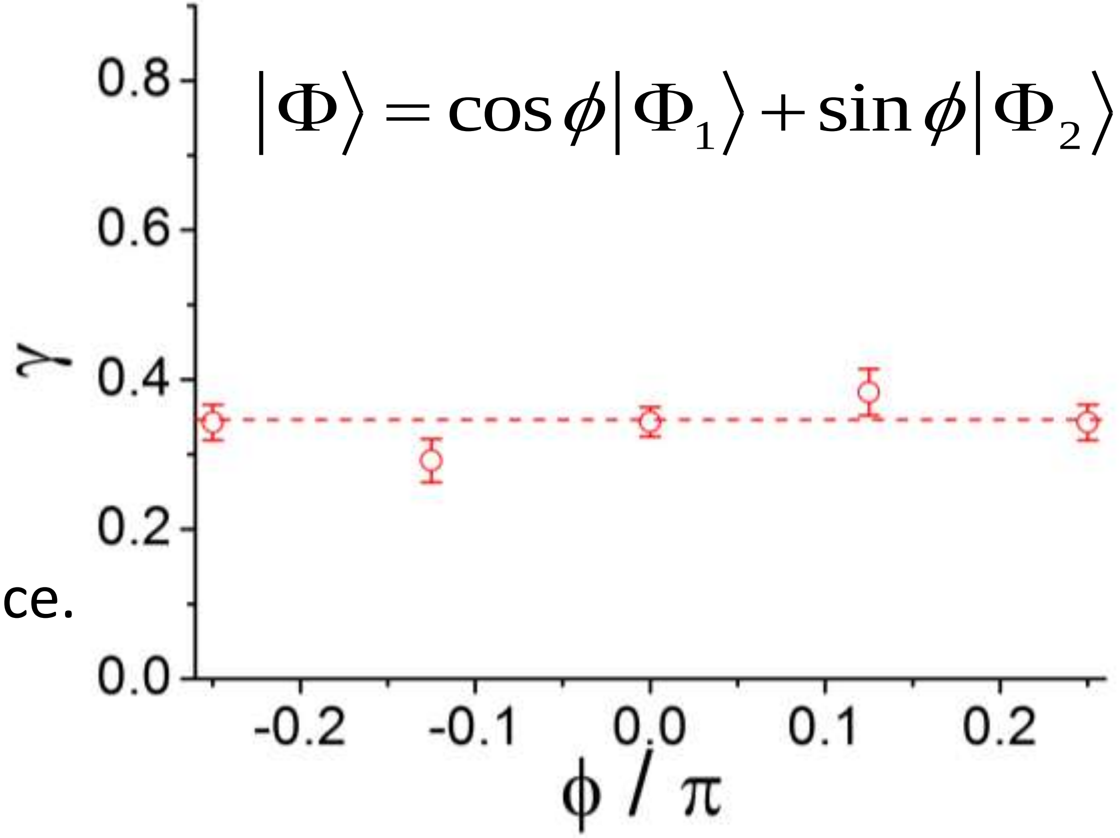
- Topological entanglement entropy found by extrapolation within 1% of *log 2*.
- Minimum entropy state is selected by DMRG (low entaglement).
- Possible reason why only one ground state seen.

Ground State Dependence of Entanglement Entropy

- Chiral spin liquid on Torus: $\Psi(r_1,r_2\ldots,r_N)=\Phi^2_{C=1}(r_1,r_2\ldots,r_N)$
  - Degenerate ground states from changing boundary conditions on Slater det. $\Phi_{C=1}$



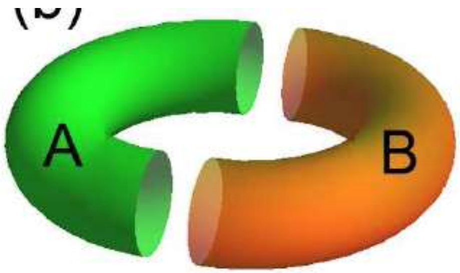
Trivial Bipartition:
No ground state dependence.



$$|\Phi\rangle=\cos\phi|\Phi_1\rangle+\sin\phi|\Phi_2\rangle$$

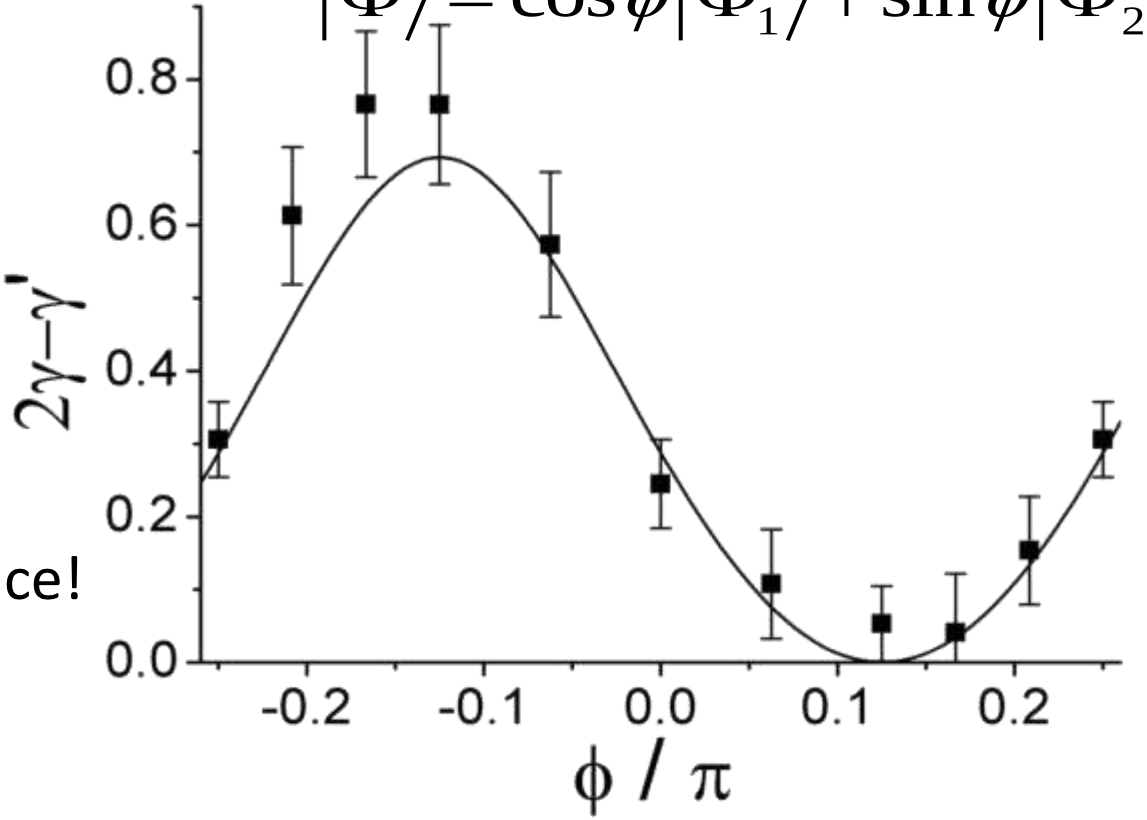
Ground State Dependence of Entanglement Entropy

- Chiral spin liquid on Torus:
  - Degenerate ground states from changing boundary conditions on Slater det. $\Phi_{C=1}$



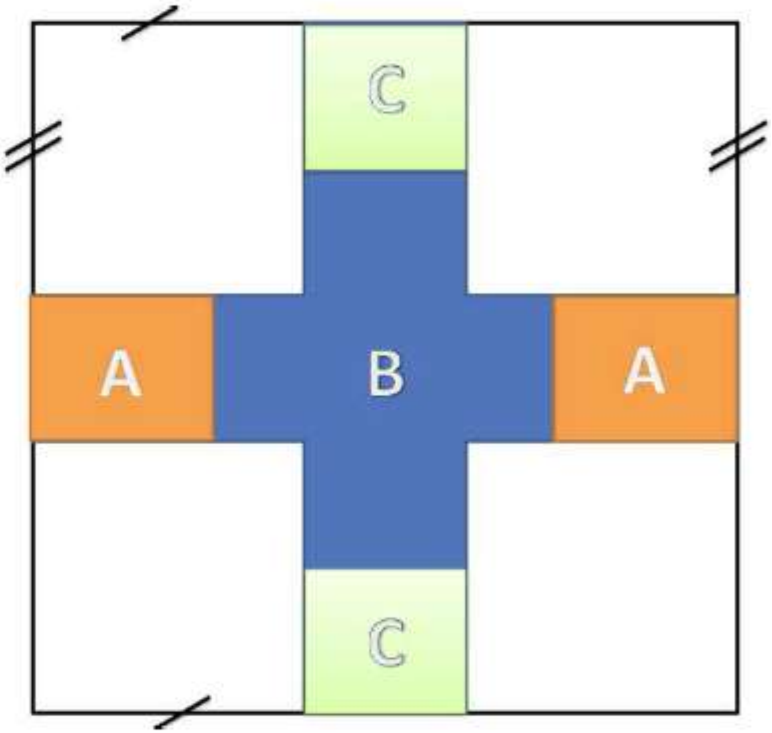
***Non* trivial Bipartition:**
Ground state dependence!

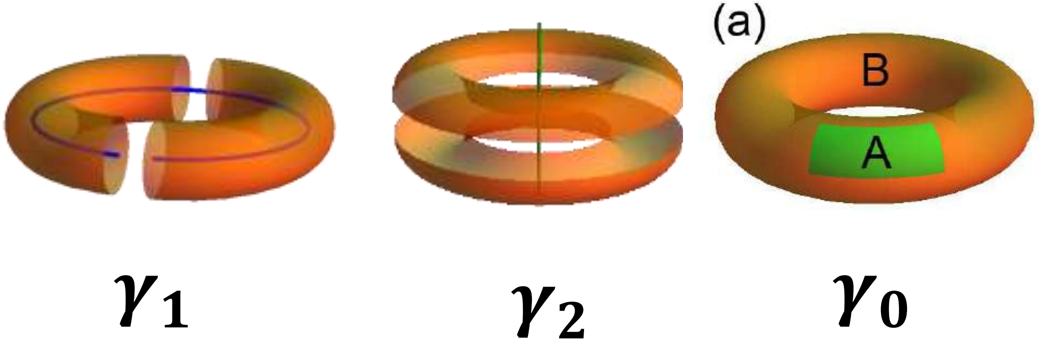
$$\left|\Phi\right\rangle = \cos\phi\left|\Phi_1\right\rangle + \sin\phi\left|\Phi_2\right\rangle$$



Ground State Dependence of Topological Entropy from Strong Sub-additivity

- Strong subadditivity: $S_{ABC} + S_B - S_{AB} - S_{BC} \leq 0$





Obtain `uncertainty' relation:

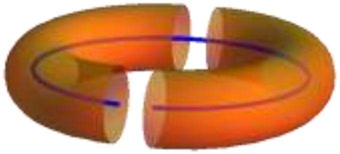
$$\boxed{\gamma_1 + \gamma_2 \leq 2\gamma_0}$$

Naïve result, $\gamma_1 = \gamma_2 = 2\gamma_0$ *cannnot* hold from general quantum information requirement.

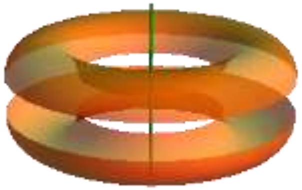
True even without topological field theory .

Part 3: Mutual Statistics from Entanglement

- Relate minimum entropy states along independent torus cuts. (modular transformation: S matrix)



MES: ϕ_1, ϕ_2



MES: ϕ'_1, ϕ'_2

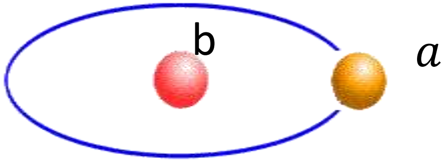
$$\begin{bmatrix} \phi'_1 \\ \phi'_2 \end{bmatrix} = \boldsymbol{S} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

S encodes quasiparticle braiding statisitics:
Chiral Spin Liquid:

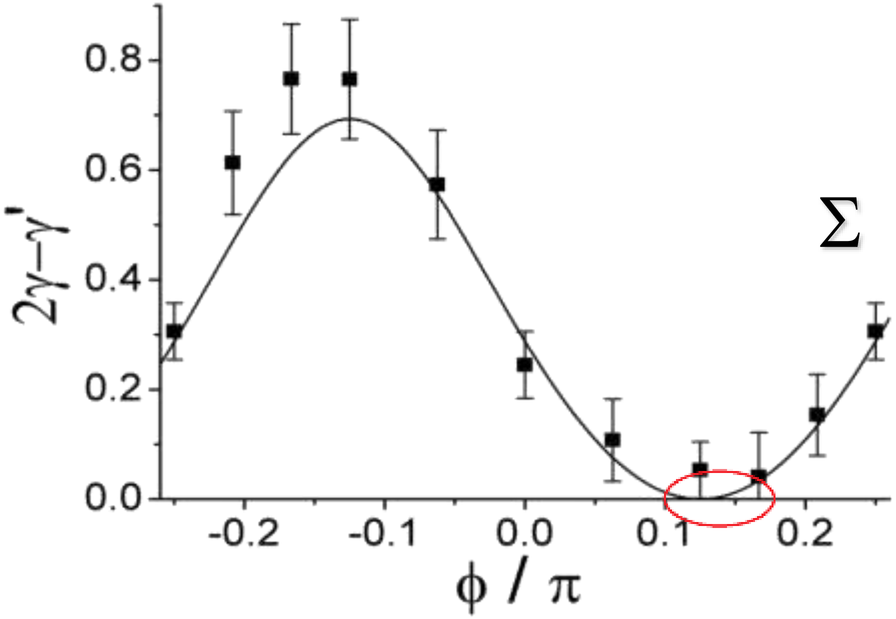
$$\begin{array}{cccc} & \mathbf{e} & \mathbf{s} & \\ S = \frac{1}{\sqrt{2}} \Big[& 1 & 1 \Big] & \mathbf{e} \\ & 1 & -1 & \mathbf{s} \end{array}$$

$$S = \frac{1}{\sqrt{2}} \begin{array}{c} \\ \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right] \end{array} \begin{array}{l} \\ \mathbf{e} \\ \mathbf{s} \end{array}$$

S_{ab}



Statistics from Entanglement – Chiral Spin Liquid



Σ

$$\approx \frac{1}{\sqrt{2}}\begin{pmatrix}1.09 & 0.89\\ 0.89 & -1.09\end{pmatrix}$$

Semion Statistics!

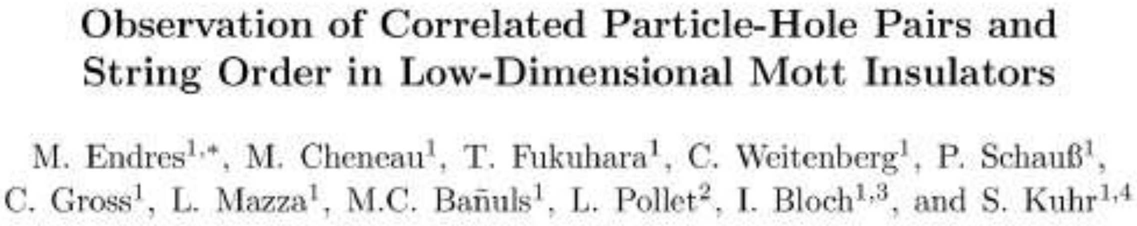
Wavefunction `knows’ about semion exciations;

Zhang, Grover, Turner, Oshikawa, AV (2011).

Conclusions

- Entanglement of non-trivial partitions can be used to define `quasiparticle’ like states, and extract their statistics.
- Useful to distinguish two phases with same D. (eg. Z_2 and doubled chiral spin liquid, no edge states) Less prone to errors.

- Can topological entanglement entropy constrain new types of topological order (eg D=3)?
- Experimental measurement? Need nonlocal probe.



Observation of Correlated Particle-Hole Pairs and String Order in Low-Dimensional Mott Insulators

M. Endres[1,*], M. Cheneau[1], T. Fukuhara[1], C. Weitenberg[1], P. Schauß[1], C. Gross[1], L. Mazza[1], M.C. Bañuls[1], L. Pollet[2], I. Bloch[1,3], and S. Kuhr[1,4]