Uemura Report Notes: Phase Stiffness, Soft Modes, and Competing Orders in High-T_c Superconductors

Lecture note compiled from Uemura's talk

摘要

The central message of Uemura's report is that unconventional superconductors, especially underdoped cuprates, should not be understood only from the pairing gap Δ. Their transition temperature is strongly constrained by phase stiffness, superfluid density, and the proximity to competing ordered phases. In this language, μSR measures

$$\sigma_{\mu\mathrm{SR}} \propto \lambda^{-2} \propto \frac{n_s}{m^*},$$

and the empirical Uemura relation says that, in many underdoped cuprates and other exotic superconductors,

$$T_c \propto \frac{n_s}{m^*}.$$

This note also connects Uemura's viewpoint to the Emery–Kivelson theory of phase fluctuations, stripes, and spin-gap proximity effects.

1 The basic contrast: BCS, superfluid helium, and HTSC

In conventional BCS superconductors, the superfluid density is large and phase fluctuations are weak. The main small energy scale is the pairing gap,

$$2\Delta(0) \simeq 3.52\, k_{\mathrm{B}} T_c,$$

so one often says, schematically,

$$T_c \sim \Delta.$$

This does not mean that carrier density is mathematically irrelevant. It means that, in ordinary metals, n_s/m^* is so large that phase stiffness is not the bottleneck.

For superfluid ^{4}He the situation is different. The particles are already bosons, so the problem is not pair formation but the onset of macroscopic phase coherence. A useful ideal-gas estimate is

$$k_B T_{BE} = 3.31 \frac{\hbar^2 n^{2/3}}{m},$$

which gives a scale of order 3.2 K for liquid helium density. The real lambda transition is lower,

$$T_\lambda \simeq 2.2\,\mathrm{K},$$

because liquid helium is a strongly interacting quantum liquid. Phonon-roton collective modes and strong correlations reduce the real ordering temperature from the ideal Bose condensation estimate.

Uemura's analogy is then

underdoped HTSC is neither weak-coupling BCS nor ideal BEC, but lies in a BEC–BCS crossover reg

In underdoped cuprates, the pairing scale may be large, but the superfluid stiffness is small. Therefore

$$T^* \sim \text{pairing or pseudogap scale}, \qquad T_c \sim \text{phase-ordering scale}.$$

2 μSR, penetration depth, and the Uemura plot

In a transverse-field μSR experiment, implanted muons precess in the local magnetic field. In a vortex lattice, the field is spatially inhomogeneous, so different muons precess at slightly different frequencies. The muon spin signal decays with a Gaussian relaxation rate

$$\sigma_{\mu SR} = \gamma_\mu \sqrt{\langle \Delta B^2 \rangle}.$$

For an ideal vortex lattice,

$$\langle \Delta B^2 \rangle \propto \frac{\Phi_0^2}{\lambda^4}, \qquad \Rightarrow \qquad \sigma_{\mu SR} \propto \lambda^{-2}.$$

London theory gives

$$\lambda^{-2} = \mu_0 \frac{n_s e^2}{m^*},$$

so

$$\boxed{\sigma_{\mu SR} \propto \lambda^{-2} \propto \frac{n_s}{m^*}.}$$

The 1989 Uemura relation found an approximately linear relation in underdoped cuprates,

$$\boxed{T_c \propto \sigma_{\mu SR}(T \to 0) \propto \frac{n_s}{m^*}.}$$

The 1991 PRL extended this observation to a broad family of exotic superconductors, including cuprates, bismuthates, organics, Chevrel phases, and heavy fermion systems.

The point is not that all these materials share the same microscopic pairing glue. The point is that they share a phenomenological property: high T_c relative to their small superfluid density.

A useful mental picture is

$$\text{BCS metal :} \quad \Delta \text{ small,} \quad \frac{n_s}{m^*} \text{ huge,}$$

while

$$\text{underdoped HTSC :} \quad \Delta \text{ large,} \quad \frac{n_s}{m^*} \text{ small.}$$

Thus, in the second case, T_c is limited by phase coherence rather than by pair formation.

3 Phase stiffness as the bottleneck

The superconducting order parameter is

$$\Psi = |\Psi| e^{i\theta}.$$

There are two distinct problems:

$$|\Psi| \neq 0 \quad \text{and} \quad \theta \text{ has long-range coherence.}$$

The phase stiffness is the energy cost of twisting the phase,

$$F_\theta = \frac{1}{2}\rho_s \int d^d r \, (\nabla\theta)^2.$$

In a clean continuum superfluid,

$$\rho_s \sim \frac{\hbar^2 n_s}{m^*}.$$

Small n_s/m^* means that the phase is soft. Thermal vortices, disorder, stripe fluctuations, or competing orders can easily destroy long-range phase coherence even when local pairing remains strong.

This is precisely the Emery–Kivelson lesson: in superconductors with small superfluid density, T_c is controlled not only by the pairing amplitude but also, and often dominantly, by phase fluctuations. In their language, one should distinguish a pairing scale T_{pair} from a phase-ordering scale T_θ:

$$T_c \sim \min(T_{\text{pair}}, T_\theta).$$

For conventional superconductors,

$$T_{\text{pair}} \ll T_\theta, \qquad T_c \sim T_{\text{pair}} \sim \Delta.$$

For underdoped cuprates,

$$T_\theta \ll T_{\text{pair}}, \qquad T_c \sim T_\theta \sim \frac{n_s}{m^*}.$$

4 Soft modes toward competing phases

Uemura's phrase "T_c is determined by the soft mode energy toward a competing phase" means that superconductivity is not an isolated instability. It occurs near other nearly degenerate states.

A soft mode is a collective excitation whose energy becomes small,

$$\Omega_{\text{soft}} \to 0.$$

This usually means the system is close to a competing ordered phase. In liquid helium, the roton minimum can be viewed as a soft tendency toward density modulation or solidification. In cuprates, analogous competing tendencies include

$$\text{AF order}, \quad \text{spin stripe}, \quad \text{charge stripe}, \quad \text{CDW}, \quad \text{Mott localization}, \quad \text{spin glass}.$$

The important point is two-sided:

$$\text{soft competing fluctuations can help pairing,}$$

but also

$$\text{soft competing fluctuations can reduce phase stiffness and suppress } T_c.$$

Therefore a high pairing scale alone is insufficient. The actual transition temperature is set by how successfully the superconducting phase gains coherence in the presence of competing low-energy modes.

Uemura's broader "phase and mode" viewpoint emphasizes that many exotic superconductors have a characteristic collective mode energy scaling with $k_B T_c$, such as a magnetic resonance energy of order a few times $k_B T_c$. This supports the idea that T_c is tied to collective mode physics rather than to a simple single-particle gap alone.

5 Nernst effect: evidence for vortex-like fluctuations above T_c

The Nernst effect is the transverse electric field generated by a temperature gradient in a magnetic field:

$$\nabla_x T + B_z \quad \Rightarrow \quad E_y, \qquad e_N = \frac{E_y}{-\nabla_x T}.$$

In a vortex liquid, a temperature gradient drives vortex motion. Moving vortices generate an electric field,

$$\boldsymbol{E} \sim \boldsymbol{B} \times \boldsymbol{v}_{\text{vortex}}.$$

A large Nernst signal above T_c in hole-doped cuprates is therefore often interpreted as evidence that vortex-like superconducting fluctuations survive above the resistive transition. The physical message is

$$\boxed{T > T_c \text{ may still contain local pairing and vorticity, but lacks global phase coherence.}}$$

This fits the Uemura–Emery–Kivelson picture:

$$T^* \text{ or } T_{\text{pair}} > T_c, \qquad T_c \text{ is the loss or onset of long-range phase coherence.}$$

One should be cautious: quasiparticle contributions, Fermi-surface reconstruction, and Berry-curvature effects can also contribute to a Nernst signal. Thus, Nernst data are strong evidence for superconducting fluctuations only when combined with diamagnetism, magnetization, and other probes.

6 Photo-induced transient superconductivity and melting stripes

In stripe-ordered La-based cuprates near 1/8 doping, charge and spin stripes compete strongly with bulk three-dimensional superconductivity. Schematically,

$$\rho(\boldsymbol{r}) = \rho_0 + \rho_Q \cos(\boldsymbol{Q}_c \cdot \boldsymbol{r}), \qquad \boldsymbol{S}(\boldsymbol{r}) = \boldsymbol{S}_Q \cos(\boldsymbol{Q}_s \cdot \boldsymbol{r}).$$

These modulations can frustrate interlayer Josephson coupling and suppress coherent superconductivity.

Ultrafast mid-infrared pulses can resonantly drive a lattice vibration, often an in-plane Cu–O stretching mode. Experiments then observe that charge stripe order can melt on sub-picosecond time scales and that optical signatures resembling superconductivity may transiently appear. The symbolic logic is

$$\text{drive phonon resonance} \Rightarrow \text{melt charge/spin stripe order} \Rightarrow \text{release hidden superconducting coherence.}$$

The transient superconducting response is usually inferred from THz optical conductivity, for example

$$\sigma_2(\omega) \sim \frac{1}{\omega},$$

or from the appearance or enhancement of a Josephson plasma resonance.

This is not simply "light creates Cooper pairs from nothing." A better interpretation is

$$\boxed{\text{pairing correlations already exist, but competing stripe order suppresses phase coherence.}}$$

Light temporarily weakens the competing order, allowing superconducting coherence to become visible. Recent work also warns that melting stripes alone may not always be sufficient to produce true transient three-dimensional superconductivity, so this remains an active and subtle question.

7 Emery–Kivelson: phase fluctuations with small superfluid density

The 1995 Nature paper by Emery and Kivelson provides the theoretical language behind much of Uemura's phenomenology. The superconducting transition is separated into amplitude formation and phase ordering:

$$\Psi = |\Psi| e^{i\theta}.$$

The amplitude scale is roughly the pair formation scale, while the phase scale is controlled by superfluid stiffness:

$$T_\theta \sim \rho_s \sim \frac{n_s}{m^*}.$$

Their key claim is that high-T_c materials often have a high pairing scale but anomalously low phase stiffness. Therefore phase fluctuations can strongly suppress T_c below the mean-field pairing temperature.

This gives a clean interpretation of the underdoped cuprate phase diagram:

$$\text{underdoping}: \quad \Delta\uparrow, \qquad \rho_s\downarrow, \qquad T_c \text{ forms a dome.}$$

The pairing strength may increase as the system approaches the Mott insulator, but mobile carrier density and phase stiffness decrease. The observed T_c is the compromise between these two tendencies.

8 Emery–Kivelson–Zachar: stripes and spin-gap proximity effect

Emery, Kivelson, and Zachar further developed a stripe-based theory of underdoped cuprates. The key ingredients are:

$$\text{doped holes self-organize into metallic stripes,} \qquad \text{the surrounding regions remain Mott-like and spin-}$$

and superconductivity emerges by communication between these components.

A stripe is quasi-one-dimensional and metallic, so it provides mobile charge. Its environment is antiferromagnetic and can develop a spin gap or pseudogap. Pair hopping between the stripe and the spin-gapped environment gives the stripe a spin gap. This is called a magnetic analog of the superconducting proximity effect:

$$\text{metallic stripe} + \text{spin-gapped Mott environment} \Rightarrow \text{local pairing tendency on the stripe.}$$

Long-range superconductivity then requires Josephson coupling between stripes:

$$H_J = -\sum_{\langle ij \rangle} J_{ij} \cos(\theta_i - \theta_j).$$

Thus the stripe picture naturally produces two separated scales:

$$\text{local pairing/spin-gap scale high,} \qquad \text{global phase-ordering scale low.}$$

This is exactly compatible with the Uemura plot.

9 A unified interpretation of the report

The entire report can be condensed into the following chain:

$$\mu\text{SR} \Rightarrow \sigma_{\mu\text{SR}} \Rightarrow \lambda^{-2} \Rightarrow \frac{n_s}{m^*} \Rightarrow \rho_s \Rightarrow T_c.$$

In ordinary BCS systems,

$$T_c \sim \Delta.$$

In underdoped cuprates and many exotic superconductors,

$$T_c \sim \rho_s \sim \frac{n_s}{m^*}.$$

Nernst effect, diamagnetism, and photo-induced transient superconductivity all support the idea that local pairing or superconducting correlations can exist above the bulk T_c, while global phase coherence is destroyed by vortices, low stiffness, disorder, and competing stripe/CDW/spin orders.

The most compact final message is

High-T_c superconductivity is not simply strong pairing; it is strong pairing plus sufficient phase stiffn

References

1. Y. J. Uemura et al., "Universal Correlations between T_c and n_s/m^* (Carrier Density over Effective Mass) in High-T_c Cuprate Superconductors," *Phys. Rev. Lett.* **62**, 2317–2320 (1989).

2. Y. J. Uemura et al., “Basic similarities among cuprate, bismuthate, organic, Chevrel-phase, and heavy-fermion superconductors shown by penetration-depth measurements,” *Phys. Rev. Lett.* **66**, 2665–2668 (1991).

3. V. J. Emery and S. A. Kivelson, “Importance of phase fluctuations in superconductors with small superfluid density,” *Nature* **374**, 434–437 (1995).

4. V. J. Emery, S. A. Kivelson, and O. Zachar, “Spin-gap proximity effect mechanism of high-temperature superconductivity,” *Phys. Rev. B* **56**, 6120–6147 (1997).

5. V. J. Emery, S. A. Kivelson, and J. M. Tranquada, “Stripe phases in high-temperature superconductors,” arXiv:cond-mat/9907228.

6. Y. J. Uemura, “Superconductivity: Exotic commonalities in phase and mode,” *Nature Materials* **8**, 253–255 (2009).

7. Y. Wang, L. Li, and N. P. Ong, “Nernst effect in high-T_c superconductors,” *Phys. Rev. B* **73**, 024510 (2006).

8. D. Fausti et al., “Light-Induced Superconductivity in a Stripe-Ordered Cuprate,” *Science* **331**, 189–191 (2011).

9. M. Först et al., “Melting of Charge Stripes in Vibrationally Driven $La_{1.875}Ba_{0.125}CuO_4$: Assessing the Respective Roles of Electronic and Lattice Order in Frustrated Superconductors,” *Phys. Rev. Lett.* **112**, 157002 (2014).

10. S. J. Zhang et al., “Light-Induced Melting of Competing Stripe Orders without Introducing Superconductivity in $La_{2-x}Ba_xCuO_4$,” *Phys. Rev. X* **14**, 011036 (2024).