Lecture 23 - The Magic Square

April 22, 2013

References:
The Octonions. J Baez (2001)
Spin(8), Triality, F4 and all that, F. Adams (1981)
Trialities and the Exceptional Lie Algebras: Deconstructing the Magic Square. J. Evans (2009)
Geometries, the principle of duality, and algebraic groups. M. Carr, S. Garibaldi (2005)
Magic Squares of Lie Algebras. C. Barton, A. Sudbery (2008)

1 Duality and Triality

A_n duality, D_n duality, D_4 triality, E_6 duality.

2 Triality and F_4

2.1 Review of $\mathfrak{h}_3(\mathbb{O})$ and F_4

Recall that $\mathfrak{h}_3(\mathbb{O})$ is the octonionic-Hermitian matrices of the form

$$\begin{pmatrix} \alpha & x & z \\ \bar{x} & \beta & y \\ \bar{z} & \bar{y} & \gamma \end{pmatrix} \tag{1}$$

We saw it was useful to decompose this Jordan algebra in any of the following ways

$$\begin{aligned} \mathfrak{h}_3(\mathbb{O}) &\approx \mathfrak{h}_2(\mathbb{O}) \oplus \mathbb{O}^2 \\ &\approx J(\mathbb{O} \oplus \mathbb{R}) \oplus \triangle_9 \\ &\approx \mathbb{R}^{9,1} \oplus \triangle_9 \end{aligned} \tag{2}$$

The use of $\triangle_9 \approx \mathbb{O}^8$ instead of $\triangle_{9,1}$ is used to emphasize that the Jordan product does *not* come form the $\mathbb{R}^{9,1} \subset Cl_{9,1}$ action, but rather from the $\mathbb{R}^9 \subset Cl_9$ action, with the additional $\mathbb{R}$ factor acting by scalar multiplication. These isomorphisms (2) are nicely summed up by

$$\begin{aligned} \big(\alpha,\, (\psi_1,\, \psi_2)^T,\, M\big) \;&\longrightarrow\; \begin{pmatrix} \alpha & \overline{\psi_1} \quad \overline{\psi_2} \\ \psi_1 & \multirow{2}{*}{M} \\ \psi_2 & \end{pmatrix} \\ \Big(\alpha,\, (\psi_1,\, \psi_2)^T,\, (\vec{V}_9,\, \beta)\Big) \;&\longrightarrow\; \begin{pmatrix} \alpha & \overline{\psi_1} \qquad \overline{\psi_2} \\ \psi_1 & \multirow{2}{*}{$\beta \cdot I_{2\times 2} + \vec{V}_9$} \\ \psi_2 & \end{pmatrix} \end{aligned} \tag{3}$$

From here, it is easily seen that

$$\mathbb{O}P^2 \;\approx\; F_4/Spin(9) \tag{4}$$

so that as vector spaces we have

$$\mathfrak{f}_4 \;=\; \mathfrak{so}(9) \,\oplus\, \triangle_9. \tag{5}$$

With $\mathfrak{so}(9) \approx \mathfrak{so}(8) \oplus \mathbb{R}^8$ and with $\triangle_9 \approx \triangle_8^+ \oplus \triangle_8^-$ after reduction to $\mathfrak{so}(8)$, we have

$$\mathfrak{f}_4 \;=\; \mathfrak{so}(8) \,\oplus \mathbb{R}^8 \,\oplus\, \triangle_8^+ \,\oplus\, \triangle_8^-. \tag{6}$$

The bracket on the $\mathfrak{so}(8) \oplus \mathbb{R}^8$ piece comes from $\mathfrak{so}(9)$. Two things are of note. First, under this bracket, $\mathfrak{so}(8)$ acts on $\mathbb{R}^8$ via the vector representation; second, this provides a natural bracket

$$[\mathbb{R}^8,\, \mathbb{R}^8] \;\longrightarrow\; \mathfrak{so}(8) \tag{7}$$

It is possible to build brackets $[\triangle^+, \triangle^+] \to \mathfrak{so}(8)$, $[\triangle^-, \triangle^-] \to \mathfrak{so}(8)$ as well. To do so, consider an orthonormal bases $\{S_i^+\}$, $\{S_i^-\}$ of $\triangle^+$, $\triangle^-$, and an orthonormal basis $\{g^k\}$ of $\mathfrak{so}(8)$ with associated matrices (g_{ij}^{k+}), (g_{ij}^{k-}). Then define

$$[S_i^\pm,\, S_j^\pm] \;=\; \sum_k g_{ij}^{k\pm} g^k. \tag{8}$$

One must check that the Jacobi identity holds. The bracket on the $\mathbb{R}^8 \,\oplus\, \triangle_8^+ \,\oplus\, \triangle_8^-$ piece, aside from the brackets we have just described, is the triality map! Also, this gives a beautiful description of F_4 as two copies of D_4.

3 Differentiation and F_4

A second way of seeing $\mathfrak{f}_4$ can be seen, which is useful in that it lends itself to generalization. Define the 3×3 *special anti-Hermitian* matrices over the division algebra $\mathbb{K}$

$$\mathfrak{sa}_3(\mathbb{K}) \;=\; \Big\{ X \in \mathbb{K}(3) \;\big|\; \overline{X}^T = -X \;\; and \;\; Tr(X) = 0 \Big\}. \tag{9}$$

Given $A \in \mathfrak{sa}_3(\mathbb{K})$ we have

$$ad_X : \mathfrak{h}_3(\mathbb{K}) \ \rightarrow \ \mathfrak{h}_3(\mathbb{K}) \tag{10}$$

is in fact a derivation. However $\mathfrak{sa}_3(\mathbb{K})$ is not a Lie algebra unless $\mathbb{K}$ is commutative and associative! However there is a bracket

$$[ad_X,\, ad_Y] \ \mapsto \ \mathfrak{der}(\mathbb{K}) \ \oplus \ \mathfrak{sa}_3(\mathbb{K}). \tag{11}$$

To describe this, first we define $D_{x,y}$ for $x, y \in \mathbb{K}$. We define

$$D_{x,y}a \ = \ [[x, y],\, a] \ + \ [x,\, y,\, a]. \tag{12}$$

One easily verifies $D_{x,y} \in \mathfrak{der}(\mathbb{K})$. Obviously if $\mathbb{K}$ is commutative and associative then $D_{x,y} = 0$. We set

$$[ad_X,\, ad_Y] \ = \ ad_{[X,Y]_0} \ + \ \frac{1}{3}\sum_{i,j=1}^{3} D_{X_{ij},Y_{ij}}. \tag{13}$$

and if $D, D' \in \mathfrak{der}(\mathbb{K})$ we set

$$\begin{aligned} [D,\, D'] \ &= \ DD' \ - \ D'D \\ [D,\, ad_X] \ &= \ ad_{D(X)}. \end{aligned} \tag{14}$$

This bracket makes

$$\mathfrak{der}(\mathbb{K}) \ \oplus \ \mathfrak{sa}_3(\mathbb{K}) \tag{15}$$

into a Lie algebra. We have

$$\begin{aligned} \mathfrak{so}(3) \ &\approx \ \mathfrak{der}(\mathbb{R}) \ \oplus \ \mathfrak{sa}_3(\mathbb{R}) \\ \mathfrak{su}(3) \ &\approx \ \mathfrak{der}(\mathbb{C}) \ \oplus \ \mathfrak{sa}_3(\mathbb{C}) \\ \mathfrak{sp}(3) \ &\approx \ \mathfrak{der}(\mathbb{H}) \ \oplus \ \mathfrak{sa}_3(\mathbb{H}) \\ \mathfrak{f}_4 \ &\approx \ \mathfrak{der}(\mathbb{O}) \ \oplus \ \mathfrak{sa}_3(\mathbb{O}) \end{aligned} \tag{16}$$

The first three isomorphisms are easy to prove. The fourth requires some calculation to show that all derivations of $\mathfrak{h}_3(\mathbb{O})$ are of the form $\mathfrak{der}(\mathbb{O}) \oplus \mathfrak{sa}_3(\mathbb{O})$.

4 The Magic Square

The so-called "magic square" (or Freudenthal-Tits magic square) is the generalization of the definition of $\mathfrak{f}_4$ from division algebras $\mathbb{K}$ to products $\mathbb{K} \otimes_{\mathbb{R}} \mathbb{K}'$. With $\mathfrak{der}(\mathbb{K} \otimes \mathbb{K}') \approx \mathfrak{der}(\mathbb{K}) \oplus \mathfrak{der}(\mathbb{K})$, we consider the Lie algebra

$$\mathfrak{der}(\mathbb{K}) \ \oplus \ \mathfrak{der}(\mathbb{K}') \ \oplus \ \mathfrak{sa}_3(\mathbb{K} \otimes \mathbb{K}'). \tag{17}$$

The only trouble is defining the bracket on the final summand. For $a \otimes a', b \otimes b' \in \mathbb{K} \otimes \mathbb{K}'$, we set

$$D_{a\otimes a',b\otimes b'} \;=\; (a',\, b')\; D_{a,b} \;+\; (a,\, b)\; D_{a',b'} \tag{18}$$

then

$$[ad_X,\, ad_Y] \;=\; =\; ad_{[X,Y]_0} \;+\; \frac{1}{3}\sum_{i,j=1}^{3} D_{X_{ij},Y_{ij}}. \tag{19}$$

as before. The wonderful thing about this construction is that (17) becomes a semi-simple Lie algebra.

Now when $\mathbb{K}, \mathbb{K}'$ are commutative and associative, the derivation algebras vanish and we obtain the rather trivial little square

	$\mathbb{R}$	$\mathbb{C}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$

(20)

In the next instance, the algebras $\mathfrak{der}(\mathbb{H})$ do not vanish. We obtain the medium square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$

(21)

Finally when we include the octonions we have the magic square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

(22)

5 E_6

We have already seen two good descriptions of E_6. The first is that it is the determinant-preservation group of $\mathfrak{h}_3(\mathbb{O})$. The second is that it preserves the Jordan algebra $\mathfrak{h}_3(\mathbb{C} \otimes \mathbb{O})$, and can therefore be seen as the isometry group of the associated projective space, the *bi-octonionic plane.*

6 E_7

The Lie algebra $\mathfrak{e}_7$ has a representation of lower dimension than its adjoint representation. This representation can be seen as follows. The group E_7 is the group of automorphisms of the Fruedenthal triple system.

7 E_8

The Lie algebra $\mathfrak{e}_8$ has a triality description:

$$\begin{aligned} \mathfrak{e}_8 &\approx \mathfrak{so}(\mathbb{R}^8) \oplus \mathfrak{so}(\mathbb{R}^8) \oplus \mathfrak{end}(\mathbb{R}^8) \oplus \mathfrak{end}(\triangle_8^+) \oplus \mathfrak{end}(\triangle_8^-) \\ &\approx \mathfrak{so}(\mathbb{O}) \oplus \mathfrak{so}(\mathbb{O}) \oplus (\mathbb{O}\otimes\mathbb{O}) \oplus (\mathbb{O}\otimes\mathbb{O}) \oplus (\mathbb{O}\otimes\mathbb{O})\,. \end{aligned} \tag{23}$$