OCTONIONS AND TRIALITY

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Complex numbers and quaternions form special cases of lower-dimensional Clifford algebras, their even subalgebras and their ideals

$$\mathbb{C} \simeq \mathcal{C}\ell_{0,1} \simeq \mathcal{C}\ell_2^+ \simeq \mathcal{C}\ell_{0,2}^+,$$
$$\mathbb{H} \simeq \mathcal{C}\ell_{0,2} \simeq \mathcal{C}\ell_3^+ \simeq \mathcal{C}\ell_{0,3}^+ \simeq \tfrac{1}{2}(1 \pm \mathbf{e}_{123})\mathcal{C}\ell_{0,3}.$$

In this chapter, we explore another generalization of $\mathbb{C}$ and $\mathbb{H}$, a non-associative real algebra, the Cayley algebra of octonions, $\mathbb{O}$. Like complex numbers and quaternions, octonions form a real division algebra, of the highest possible dimension, 8. As an extreme case, $\mathbb{O}$ makes its presence felt in classifications, for instance, in conjunction with exceptional cases of simple Lie algebras.

Like $\mathbb{C}$ and $\mathbb{H}$, $\mathbb{O}$ has a geometric interpretation. The automorphism group of $\mathbb{H}$ is $SO(3)$, the rotation group of $\mathbb{R}^3$ in $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$. The automorphism group of $\mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7$ is not all of $SO(7)$, but only a subgroup, the exceptional Lie group G_2. The subgroup G_2 fixes a 3-vector, in $\bigwedge^3 \mathbb{R}^7$, whose choice determines the product rule of $\mathbb{O}$.

The Cayley algebra $\mathbb{O}$ is a tool to handle an esoteric phenomenon in dimension 8, namely triality, an automorphism of the universal covering group $\mathbf{Spin}(8)$ of the rotation group $SO(8)$ of the Euclidean space $\mathbb{R}^8$. In general, all automorphisms of $SO(n)$ are either inner or similarities by orthogonal matrices in $O(n)$, and all automorphisms of $\mathbf{Spin}(n)$ are restrictions of linear transformations $\mathcal{C}\ell_n \to \mathcal{C}\ell_n$, and project down to automorphisms of $SO(n)$. The only exception is the triality automorphism of $\mathbf{Spin}(8)$, which cannot be linear while it permutes cyclically the three non-identity elements $-1, \mathbf{e}_{12\ldots8}, -\mathbf{e}_{12\ldots8}$ in the center of $\mathbf{Spin}(8)$.

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We shall see that triality is a restriction of a polynomial mapping $\mathcal{C}\ell_8 \to \mathcal{C}\ell_8$, of degree 2. We will learn how to compose trialities, when they correspond to different octonion products. We shall explore triality in terms of classical linear algebra by observing how eigenplanes of rotations transform under triality.

1. Division Algebras

An algebra A over $\mathbb{R}$ is a linear (that is a vector) space A over $\mathbb{R}$ together with a bilinear map $A \times A \to A$, $(a,b) \to ab$, the algebra product. Bilinearity means distributivity $(a+b)c = ac + bc$, $a(b+c) = ab + ac$ and $(\lambda a)b = a(\lambda b) = \lambda(ab)$ for all $a, b, c \in A$ and $\lambda \in \mathbb{R}$. An algebra is without *zero-divisors* if $ab = 0$ implies $a = 0$ or $b = 0$. In a *division algebra* $\mathbb{D}$ the equations $ax = b$ and $ya = b$ have unique solutions x, y for all non-zero $a \in \mathbb{D}$. A division algebra is without zero-divisors, and conversely, every finite-dimensional algebra without zero-divisors is a division algebra. If a division algebra is associative, then it has unity 1 and each non-zero element has a unique inverse (on both sides).

An algebra with a unity is said to *admit inverses* if each non-zero element admits an inverse (not necessarily unique). An algebra is *alternative* if $a(ab) = a^2 b$ and $(ab)b = ab^2$, and *flexible* if $a(ba) = (ab)a$. An alternative algebra is flexible. An alternative division algebra has unity and admits inverses, which are unique. The only alternative division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

An algebra A with a positive-definite quadratic form $N\colon A \to \mathbb{R}$, is said to *preserve norm,* or *admit composition,* if for all $a, b \in A$, $N(ab) = N(a)N(b)$. The dimension of a norm-preserving division algebra $\mathbb{D}$ over $\mathbb{R}$ is 1, 2, 4 or 8; if furthermore $\mathbb{D}$ has unity, then it is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

Examples. 1. Define in $\mathbb{C}$ a new product $a \circ b$ by $a \circ b = a\bar{b}$. Then $\mathbb{C}$ becomes a non-commutative and non-alternative division algebra over $\mathbb{R}$, without unity.
2. Consider a 3-dimensional algebra over $\mathbb{R}$ with basis $\{1, i, j\}$ such that 1 is the unity and $i^2 = j^2 = -1$ but $ij = ji = 0$. The algebra is commutative and flexible, but non-alternative. It admits inverses, but inverses of the elements of the form $xi + yj$ are not unique, $(xi + yj)^{-1} = \lambda(yi - xj) - \dfrac{ix + jy}{x^2 + y^2}$, where $\lambda \in \mathbb{R}$. It has zero-divisors, by definition, and cannot be a division algebra, although all non-zero elements are invertible.
3. Consider a 3-dimensional algebra over $\mathbb{Q}$ with basis $\{1, i, i^2\}$, unity 1 and multiplication table

	i	i^2
i	i^2	3
i^2	3	$-6i$

The algebra is commutative and flexible, but non-alternative. Each non-zero element has a unique inverse. Multiplication by $x + iy + i^2 z$ has determinant $x^3 + 3y^3 - 18z^3$, which has no non-zero rational roots (Euler 1862). Thus, the algebra is a division algebra, $3D$ over $\mathbb{Q}$. ∎

2. The Cayley–Dickson Doubling Process

Complex numbers can be considered as pairs of real numbers with componentwise addition and with the product

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2).$$

Quaternions can be defined as pairs of complex numbers, but this time the product involves complex conjugation

$$(z_1, w_1)(z_2, w_2) = (z_1 z_2 - w_1 \bar{w}_2, z_1 w_2 + w_1 \bar{z}_2).$$

Octonions can be defined as pairs of quaternions, but this time order of multiplication matters

$$(p_1, q_1) \circ (p_2, q_2) = (p_1 p_2 - \bar{q}_2 q_1, q_2 p_1 + q_1 \bar{p}_2).$$

This doubling process, of Cayley-Dickson, can be repeated, but the next algebras are not division algebras, although they still are simple and flexible (Schafer 1954). Every element in such a Cayley-Dickson algebra satisfies a quadratic equation with real coefficients.

Example. The quaternion $q = w + ix + jy + kz$ satisfies the quadratic equation

$$q^2 - 2wq + |q|^2 = 0.$$ ∎

The Cayley-Dickson doubling process

$$\begin{aligned} \mathbb{C} &= \mathbb{R} \oplus \mathbb{R}i \\ \mathbb{H} &= \mathbb{C} \oplus \mathbb{C}j \\ \mathbb{O} &= \mathbb{H} \oplus \mathbb{H}\ell \end{aligned}$$

provides a new imaginary unit ℓ, $\ell^2 = -1$, which anticommutes with i, j, k. The basis $\{1, i, j, k\}$ of $\mathbb{H}$ is complemented to a basis $\{1, i, j, k, \ell, i\ell, j\ell, k\ell\}$ of $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$. Thus, $\mathbb{O}$ is spanned by $1 \in \mathbb{R}$ and the 7 imaginary units $i, j, k, \ell, i\ell, j\ell, k\ell$, each with square -1, so that $\mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7$. Among subsets of 3 imaginary units, there are 7 triplets, which associate and span the imaginary part of a quaternionic subalgebra. The remaining 28 triplets anti-associate.

The multiplication table of the unit octonions can be summarized by the Fano plane, the smallest projective plane, consisting of 7 points and 7 lines, with orientations. The 7 oriented lines correspond to the 7 quaternionic/associative triplets.

3. Multiplication Table of $\mathbb{O}$

Denote the product of $a, b \in \mathbb{O}$ by $a \circ b$. Let $1, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_7$ be a basis of $\mathbb{O}$. Define the product in terms of the basis by

$$\mathbf{e}_i \circ \mathbf{e}_i = -1, \quad \text{and } \mathbf{e}_i \circ \mathbf{e}_j = -\mathbf{e}_j \circ \mathbf{e}_i \quad \text{for } i \neq j,$$

and by the table

$$\begin{array}{lll} \mathbf{e}_1 \circ \mathbf{e}_2 = \mathbf{e}_4, & \mathbf{e}_2 \circ \mathbf{e}_4 = \mathbf{e}_1, & \mathbf{e}_4 \circ \mathbf{e}_1 = \mathbf{e}_2, \\ \mathbf{e}_2 \circ \mathbf{e}_3 = \mathbf{e}_5, & \mathbf{e}_3 \circ \mathbf{e}_5 = \mathbf{e}_2, & \mathbf{e}_5 \circ \mathbf{e}_2 = \mathbf{e}_3, \\ \vdots & \vdots & \vdots \\ \mathbf{e}_7 \circ \mathbf{e}_1 = \mathbf{e}_3, & \mathbf{e}_1 \circ \mathbf{e}_3 = \mathbf{e}_7, & \mathbf{e}_3 \circ \mathbf{e}_7 = \mathbf{e}_1. \end{array}$$

The table can be condensed into the form

$$\mathbf{e}_i \circ \mathbf{e}_{i+1} = \mathbf{e}_{i+3}$$

where the indices are permuted cyclically and translated modulo 7.

If $\mathbf{e}_i \circ \mathbf{e}_j = \pm\mathbf{e}_k$, then $\mathbf{e}_i$, $\mathbf{e}_j$, $\mathbf{e}_k$ generate a subalgebra isomorphic to $\mathbb{H}$. The sign in $\mathbf{e}_i \circ \mathbf{e}_j = \pm\mathbf{e}_k$ can be memorized by rotating the triangle in the following picture by an integral multiple of $2\pi/7$:

Example. The product $\mathbf{e}_2 \circ \mathbf{e}_5 = -\mathbf{e}_3$ corresponds to a triangle obtained by rotating the picture by $2\pi/7$. ▮

In the Clifford algebra $\mathcal{C}\ell_{0,7}$ of $\mathbb{R}^{0,7}$, octonions can be identified with paravectors, $\mathbb{O} = \mathbb{R} \oplus \mathbb{R}^{0,7}$, and the octonion product may be expressed in terms of the Clifford product as

$$a \circ b = \langle ab(1 - \mathbf{v}) \rangle_{0,1},$$

where $\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713} \in \bigwedge^3 \mathbb{R}^{0,7}$. In $\mathcal{C}\ell_{0,7}$, the octonion product can be also written as

$$a \circ b = \langle ab(1 + \mathbf{w})(1 - \mathbf{e}_{12\ldots7}) \rangle_{0,1} \quad \text{for } a, b \in \mathbb{R} \oplus \mathbb{R}^{0,7}$$

where $\frac{1}{8}(1 + \mathbf{w})\frac{1}{2}(1 - \mathbf{e}_{12\ldots7})$ is an idempotent, $\mathbf{w} = \mathbf{v}\mathbf{e}_{12\ldots7}^{-1} \in \bigwedge^4 \mathbb{R}^{0,7}$ and $\mathbf{e}_{12\ldots7}^{-1} = \mathbf{e}_{12\ldots7}$.

In the Clifford algebra $\mathcal{C}\ell_8$ of $\mathbb{R}^8$, we represent octonions by vectors, $\mathbb{O} = \mathbb{R}^8$. As the identity of octonions we choose the unit vector $\mathbf{e}_8$ in $\mathbb{R}^8$. The octonion product is then expressed in terms of the Clifford product as

$$\mathbf{a} \circ \mathbf{b} = \langle \mathbf{a}\mathbf{e}_8\mathbf{b}(1 + \mathbf{w})(1 - \mathbf{e}_{12\ldots8}) \rangle_1 \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^8$$

where $\frac{1}{8}(1 + \mathbf{w})\frac{1}{2}(1 - \mathbf{e}_{12\ldots8})$ is an idempotent, $\mathbf{w} = \mathbf{v}\mathbf{e}_{12\ldots7}^{-1} \in \bigwedge^4 \mathbb{R}^8$, $\mathbf{e}_{12\ldots7}^{-1} = -\mathbf{e}_{12\ldots7}$ and $\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713} \in \bigwedge^3 \mathbb{R}^8$.

4. The Octonion Product and the Cross Product in $\mathbb{R}^7$

A product of two vectors is linear in both factors. A vector-valued product of two vectors is called a **cross product**, if the vector is orthogonal to the two factors and has length equal to the area of the parallelogram formed by the two vectors. A cross product of two vectors exists only in $3D$ and $7D$.

The cross product of two vectors in $\mathbb{R}^7$ can be constructed in terms of an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_7$ by antisymmetry, $\mathbf{e}_i \times \mathbf{e}_j = -\mathbf{e}_j \times \mathbf{e}_i$, and

$$\begin{array}{lll} \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_4, & \mathbf{e}_2 \times \mathbf{e}_4 = \mathbf{e}_1, & \mathbf{e}_4 \times \mathbf{e}_1 = \mathbf{e}_2, \\ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_5, & \mathbf{e}_3 \times \mathbf{e}_5 = \mathbf{e}_2, & \mathbf{e}_5 \times \mathbf{e}_2 = \mathbf{e}_3, \\ \vdots & \vdots & \vdots \\ \mathbf{e}_7 \times \mathbf{e}_1 = \mathbf{e}_3, & \mathbf{e}_1 \times \mathbf{e}_3 = \mathbf{e}_7, & \mathbf{e}_3 \times \mathbf{e}_7 = \mathbf{e}_1. \end{array}$$

The above table can be condensed into the form

$$\mathbf{e}_i \times \mathbf{e}_{i+1} = \mathbf{e}_{i+3}$$

where the indices are permuted cyclically and translated modulo 7.

This cross product of vectors in $\mathbb{R}^7$ satisfies the usual properties, that is,

$$\begin{array}{ll} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0, \ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0 & \text{orthogonality} \\ |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 & \text{Pythagorean theorem} \end{array}$$

where the second rule can also be written as $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \sphericalangle(\mathbf{a}, \mathbf{b})$. Unlike the 3-dimensional cross product, the 7-dimensional cross product does not satisfy the Jacobi identity, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} \neq 0$, and so it does not form a Lie algebra. However, the 7-dimensional cross product satisfies the Malcev identity, a generalization of Jacobi, see Ebbinghaus et al. 1991 p. 279.

In $\mathbb{R}^3$, the direction of $\mathbf{a} \times \mathbf{b}$ is unique, up to two alternatives for the orientation, but in $\mathbb{R}^7$ the direction of $\mathbf{a} \times \mathbf{b}$ depends on a 3-vector defining the cross product; to wit,

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b}) \,\lrcorner\, \mathbf{v} \quad [\neq -(\mathbf{a} \wedge \mathbf{b})\mathbf{v}]$$

depends on $\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713} \in \bigwedge^3 \mathbb{R}^7$. In the 3-dimensional space $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ implies that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are in the same plane, but for the cross product $\mathbf{a} \times \mathbf{b}$ in $\mathbb{R}^7$ there are also other planes than the linear span of $\mathbf{a}$ and $\mathbf{b}$ giving the same direction as $\mathbf{a} \times \mathbf{b}$.

The 3-dimensional cross product is invariant under all rotations of $SO(3)$, while the 7-dimensional cross product is not invariant under all of $SO(7)$, but only under the exceptional Lie group G_2, a subgroup of $SO(7)$. When we let $\mathbf{a}$ and $\mathbf{b}$ run through all of $\mathbb{R}^7$, the image set of the simple bivectors $\mathbf{a} \wedge \mathbf{b}$ is a manifold of dimension $2 \cdot 7 - 3 = 11 > 7$ in $\bigwedge^2 \mathbb{R}^7$, $\dim(\bigwedge^2 \mathbb{R}^7) = \frac{1}{2} 7(7-1) = 21$, while the image set of $\mathbf{a} \times \mathbf{b}$ is just $\mathbb{R}^7$. So the mapping

$$\mathbf{a} \wedge \mathbf{b} \to \mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b}) \,\lrcorner\, \mathbf{v}$$

is not a one-to-one correspondence, but only a method of associating a vector to a bivector.

The 3-dimensional cross product is the vector part of the quaternion product of two pure quaternions, that is,

$$\mathbf{a} \times \mathbf{b} = \mathrm{Im}(\mathbf{ab}) \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset \mathbb{H}.$$

In terms of the Clifford algebra $\mathcal{C}\ell_3 \simeq \mathrm{Mat}(2, \mathbb{C})$ of the Euclidean space $\mathbb{R}^3$ the cross product could also be expressed as

$$\mathbf{a} \times \mathbf{b} = -\langle \mathbf{abe}_{123} \rangle_1 \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset \mathcal{C}\ell_3.$$

In terms of the Clifford algebra $\mathcal{C}\ell_{0,3} \simeq \mathbb{H} \times \mathbb{H}$ of the negative definite quadratic space $\mathbb{R}^{0,3}$ the cross product can be expressed not only as

$$\mathbf{a} \times \mathbf{b} = -\langle \mathbf{abe}_{123} \rangle_1 \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^{0,3} \subset \mathcal{C}\ell_{0,3}$$

but also as[1]

$$\mathbf{a} \times \mathbf{b} = \langle \mathbf{ab}(1 - \mathbf{e}_{123}) \rangle_1 \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^{0,3} \subset \mathcal{C}\ell_{0,3}.$$

Similarly, the 7-dimensional cross product is the vector part of the octonion product of two pure octonions, that is, $\mathbf{a} \times \mathbf{b} = \langle \mathbf{a} \circ \mathbf{b} \rangle_1$. The octonion algebra $\mathbb{O}$ is a norm-preserving algebra with unity 1, whence the vector part $\mathbb{R}^7$ in $\mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7$ is an algebra with cross product, that is, $\mathbf{a} \times \mathbf{b} = \frac{1}{2}(\mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a})$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^7 \subset \mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7$. The octonion product in turn is given by

$$a \circ b = \alpha\beta + \alpha\mathbf{b} + \mathbf{a}\beta - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

for $a = \alpha + \mathbf{a}$ and $b = \beta + \mathbf{b}$ in $\mathbb{R} \oplus \mathbb{R}^7$. If we replace the Euclidean space $\mathbb{R}^7$ by the negative definite quadratic space $\mathbb{R}^{0,7}$, then not only

$$a \circ b = \alpha\beta + \alpha\mathbf{b} + \mathbf{a}\beta + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

for $a, b \in \mathbb{R} \oplus \mathbb{R}^{0,7}$, but also

$$a \circ b = \langle ab(1 - \mathbf{v}) \rangle_{0,1}$$

where $\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713} \in \bigwedge^3 \mathbb{R}^{0,7}$.

[1] This expression is also valid for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset \mathcal{C}\ell_3$, but the element $1 - \mathbf{e}_{123}$ does not pick up an ideal of $\mathcal{C}\ell_3$. Recall that $\mathcal{C}\ell_3$ is simple, that is, it has no proper two-sided ideals.

5. Definition of Triality

Let $n \geq 3$. All automorphisms of $SO(n)$ are of the form $U \to SUS^{-1}$ where $S \in O(n)$. All automorphisms of $\mathbf{Spin}(n)$, $n \neq 8$, are of the form $u \to sus^{-1}$ where $s \in \mathbf{Pin}(n)$. The group $\mathbf{Spin}(8)$ has exceptional automorphisms, which permute the non-identity elements $-1, \mathbf{e}_{12\ldots8}, -\mathbf{e}_{12\ldots8}$ in the center of $\mathbf{Spin}(8)$:

$$\begin{array}{ccc} -1 & \longrightarrow & \mathbf{e}_{12\ldots8} \\ \nwarrow & & \swarrow \\ & -\mathbf{e}_{12\ldots8} & \end{array} \qquad \rho(\pm\mathbf{e}_{12\ldots8}) = -I.$$

Such an automorphism of $\mathbf{Spin}(8)$, of order 3, is said to be a *triality automorphism,* denoted by $\text{trial}(u)$ for $u \in \mathbf{Spin}(8)$.

Regard $\mathbf{Spin}(8)$ as a subset of $\mathcal{C}\ell_8$. In $\mathcal{C}\ell_8$, triality sends the lines through $1, -\mathbf{e}_{12\ldots8}$ and $-1, \mathbf{e}_{12\ldots8}$, which are parallel, to the lines through $1, -1$ and $\mathbf{e}_{12\ldots8}, -\mathbf{e}_{12\ldots8}$, which intersect each other. Thus, a triality automorphism of $\mathbf{Spin}(8)$ cannot be a restriction of a linear mapping $\mathcal{C}\ell_8 \to \mathcal{C}\ell_8$.

A non-linear automorphism of $\mathbf{Spin}(8)$ might also interchange -1 with either of $\pm\mathbf{e}_{12\ldots8}$. Such an automorphism of $\mathbf{Spin}(8)$, of order 2, is said to be a *swap automorphism*, denoted by $\text{swap}(u)$ for $u \in \mathbf{Spin}(8)$.

On the Lie algebra level, triality acts on the space of bivectors $\bigwedge^2 \mathbb{R}^8$, of dimension 28. Triality stabilizes point-wise the Lie algebra $\mathcal{G}_2$ of G_2, which is the automorphism group of $\mathbb{O}$. The dimension of $\mathcal{G}_2$ is 14. In the orthogonal complement $\mathcal{G}_2^\perp$ of $\mathcal{G}_2$, triality is an isoclinic rotation, turning each bivector by the angle 120°. A swap stabilizes point-wise not only $\mathcal{G}_2$ but also a 7-dimensional subspace of $\mathcal{G}_2^\perp$, and reflects the rest of the Lie algebra $so(8) \simeq D_4$, that is, another 7-dimensional subspace of $\mathcal{G}_2^\perp$. For a bivector $\mathbf{F} \in \bigwedge^2 \mathbb{R}^8$, we denote triality by $\text{Trial}(\mathbf{F})$ and swap by $\text{Swap}(\mathbf{F})$.

On the level of representation spaces, triality could be viewed as permuting the vector space $\mathbb{R}^8$ and the two even spinor spaces, that is, the minimal left ideals $\mathcal{C}\ell_8^+ \frac{1}{8}(1+\mathbf{w})\frac{1}{2}(1 \pm \mathbf{e}_{12\ldots8})$, which are sitting in the two-sided ideals $\mathcal{C}\ell_8^+ \frac{1}{2}(1 \pm \mathbf{e}_{12\ldots8}) \simeq \text{Mat}(8,\mathbb{R})$ of $\mathcal{C}\ell_8^+ \simeq {}^2\text{Mat}(8,\mathbb{R})$. This means a 120° rotation of the Coxeter–Dynkin diagram of the Lie algebra D_4 :

Rather than permuting the representation spaces, triality permutes elements of $\mathbf{Spin}(8)$, or their actions on the vector space and the two spinor spaces.

Because of its relation to octonions, it is convenient to view triality in terms of the Clifford algebra $\mathcal{C}\ell_{0,7} \simeq {}^2\text{Mat}(8,\mathbb{R})$, the paravector space $\$R^8 = \mathbb{R} \oplus \mathbb{R}^{0,7}$, having an octonion product, the spin group

$$\$pin(8) = \{u \in \mathcal{C}\ell_{0,7} \mid u\bar{u} = 1;\ \text{for all } x \in \$R^8 \text{ also } ux\hat{u}^{-1} \in \$R^8\},$$

the minimal left ideals $\mathcal{C}\ell_{0,7}\frac{1}{8}(1+\mathbf{w})\frac{1}{2}(1\mp\mathbf{e}_{12\ldots7})$ of $\mathcal{C}\ell_{0,7}\simeq{}^2\mathrm{Mat}(8,\mathbb{R})$, and the primitive idempotents

$$f=\tfrac{1}{8}(1+\mathbf{w})\tfrac{1}{2}(1-\mathbf{e}_{12\ldots7}),\quad \hat{f}=\tfrac{1}{8}(1+\mathbf{w})\tfrac{1}{2}(1+\mathbf{e}_{12\ldots7}).$$

For $u\in \$pin(8)$, define two linear transformations U_1,U_2 of $\$R^8$ by

$$U_1(x)=16\langle uxf\rangle_{0,1},\quad U_2(x)=16\langle ux\hat{f}\rangle_{0,1}.$$

The action of u on the left ideal $\mathcal{C}\ell_{0,7}\frac{1}{8}(1+\mathbf{w})$ of $\mathcal{C}\ell_{0,7}$ results in the matrix representation [2] [3]

$$\$pin(8)\ni u\simeq\begin{pmatrix}U_1&0\\0&U_2\end{pmatrix}\qquad\text{where } U_1,U_2\in \$O(8).$$

For $U\in SO(8)$, [4] define the *companion* $\check{U}$ by

$$\check{U}(x)=\widehat{U(\hat{x})}\quad\text{for all}\quad x\in \$R^8.$$

[2] Choose the bases $(\mathbf{e}_1f,\mathbf{e}_2f,\ldots,\mathbf{e}_7f,f)$ for $\mathcal{C}\ell_{0,7}f$ and $(\mathbf{e}_1\hat{f},\mathbf{e}_2\hat{f},\ldots,\mathbf{e}_7\hat{f},\hat{f})$ for $\mathcal{C}\ell_{0,7}\hat{f}$, where $f=\frac{1}{8}(1+\mathbf{w})\frac{1}{2}(1-\mathbf{e}_{12\ldots7})$ and $\hat{f}=\frac{1}{8}(1+\mathbf{w})\frac{1}{2}(1+\mathbf{e}_{12\ldots7})$. Then the matrices of U_1 and U_2 are the same as in the basis $(\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_7,1)$ of $\$R^8$. Denoting $f_i=\mathbf{e}_if$, $i=1,2,\ldots,7$, and $f_8=f$, $(U_1)_{ij}=16\langle\bar{f}_iuf_j\rangle_0$, and denoting $g_i=\mathbf{e}_i\hat{f}$, $i=1,2,\ldots,7$, and $g_8=\hat{f}$, $(U_2)_{ij}=16\langle\bar{g}_iug_j\rangle_0$.

[3] If we had chosen the bases $(f_1,f_2,\ldots,f_7,f)$ for $\mathcal{C}\ell_{0,7}f$ and $(\hat{f}_1,\hat{f}_2,\ldots,\hat{f}_7,\hat{f})$ for $\mathcal{C}\ell_{0,7}\hat{f}$, where $f_i=\mathbf{e}_if$ and $\hat{f}_i=-\mathbf{e}_i\hat{f}$ for $i=1,2,\ldots,7$, then we would have obtained the following matrix representation

$$u\simeq\begin{matrix}U_1&0\\0&\check{U}_2\end{matrix}\quad,$$

where $U_1(x)=16\langle uxf\rangle_{0,1}$ as before but $\check{U}_2(x)=16\langle\hat{u}xf\rangle_{0,1}$. This representation is used by Porteous 1995.

[4] Or, for $U\in\mathrm{Mat}(8,\mathbb{R})$.

The *companion* $\check{u}$ of $u \in \$pin(8)$ is just its main involution, $\check{u} = \hat{u}$, [5] and corresponds to the matrix

$$\check{u} \simeq \begin{pmatrix} \check{U}_2 & 0 \\ 0 & \check{U}_1 \end{pmatrix}.$$

For a paravector $a \in \$R^8$, define the linear transformation A of $\$R^8$ by [6]

$$A(x) = 16\langle axf \rangle_{0,1}, \qquad \text{that is,} \quad A(x) = a \circ x,$$

making $\$R^8$ the Cayley algebra $\mathbb{O}$. Since $\check{A}^\top(x) = 16\langle ax\hat{f}\rangle_{0,1}$, we have the correspondence

$$a \simeq \begin{pmatrix} A & 0 \\ 0 & \check{A}^\top \end{pmatrix}, \qquad \text{abbreviated as} \quad a \sim A.$$

Computing the matrix product

$$U(a) = ua\hat{u}^{-1} \simeq \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \check{A}^\top \end{pmatrix} \begin{pmatrix} \check{U}_2^{-1} & 0 \\ 0 & \check{U}_1^{-1} \end{pmatrix},$$

we find the correspondence $U(a) \sim U_1 A \check{U}_2^{-1}$. Denote $U_0 = \check{U}$, and let $\check{U}_0(a)$ operate on $x \in \$R^8$, to get

$$\check{U}_0(a) \circ x = (U_1 A \check{U}_2^{-1})(x) = U_1(a \circ \check{U}_2^{-1}(x)).$$

The ordered triple (U_0, U_1, U_2) in $SO(8)$ is called a *triality triplet* with respect to the octonion product of $\mathbb{O}$.

[5] Recall that for $x \in \$R^8 = \mathbb{R} \oplus \mathbb{R}^{0,7}$, $U(x) = ux\hat{u}^{-1}$, and so $\check{U}(x) = \hat{u}\hat{x}u^{-1}$.

[6] The matrix of A can be computed as $A_{ij} = 16\langle \bar{f}_i a f_j \rangle_0$. The paravector $a = a_0 + a_1\mathbf{e}_1 + \cdots + a_7\mathbf{e}_7$ has the matrix

$$A = \begin{pmatrix} a_0 & -a_4 & -a_7 & a_2 & -a_6 & a_5 & a_3 & a_1 \\ a_4 & a_0 & -a_5 & -a_1 & a_3 & -a_7 & a_6 & a_2 \\ a_7 & a_5 & a_0 & -a_6 & -a_2 & a_4 & -a_1 & a_3 \\ -a_2 & a_1 & a_6 & a_0 & -a_7 & -a_3 & a_5 & a_4 \\ a_6 & -a_3 & a_2 & a_7 & a_0 & -a_1 & -a_4 & a_5 \\ -a_5 & a_7 & -a_4 & a_3 & a_1 & a_0 & -a_2 & a_6 \\ -a_3 & -a_6 & a_1 & -a_5 & a_4 & a_2 & a_0 & a_7 \\ -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 & a_0 \end{pmatrix}.$$

If (U_0, U_1, U_2) is a triality triplet, then also (U_1, U_2, U_0), (U_2, U_0, U_1) and $(\check{U}_2, \check{U}_1, \check{U}_0)$ are a triality triplets. This results in

$$\check{U}_0(x \circ y) = U_1(x) \circ U_2(y) \quad \text{for all} \quad x, y \in \mathbb{O} = \$R^8,$$

referred to as *Cartan's principle of triality.* Conversely, for a fixed $U_0 \in \$O(8)$, the identity $\check{U}_0(x \circ y) = U_1(x) \circ U_2(y)$ has two solutions U_1, U_2 in $\$O(8)$, resulting in the triality triplet (U_0, U_1, U_2) and its *opposite* $(U_0, -U_1, -U_2)$. Thus, U_0 corresponds to two triality triplets (U_0, U_1, U_2) and $(U_0, -U_1, -U_2)$, while $-U_0$, corresponds to $(-U_0, -U_1, U_2)$ and $(-U_0, U_1, -U_2)$.

The rotations $U_1, U_2 \in \$O(8)$ are represented by $\pm u_1, \pm u_2 \in \$pin(8)$. We choose the signs so that

$$\hat{u}_0 \simeq \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \quad \hat{u}_1 \simeq \begin{pmatrix} U_2 & 0 \\ 0 & U_0 \end{pmatrix}, \quad \hat{u}_2 \simeq \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix},$$

where $u_0 = \hat{u}$ and $U_0 = \check{U}$. Using the notion of triality triplets,

$$u_0 \simeq (U_0, U_1, U_2), \quad u_1 \simeq (U_1, U_2, U_0), \quad u_2 \simeq (U_2, U_0, U_1).$$

The rotation U_0 in $\$O(8)$ corresponds to $u_0 \simeq (U_0, U_1, U_2)$ and its opposite $-u_0 \simeq (U_0, -U_1, -U_2)$ in $\$pin(8)$, and the opposite rotation $-U_0$ corresponds to $\mathbf{e}_{12\ldots7} u_0 \simeq (-U_0, -U_1, U_2)$ and $-\mathbf{e}_{12\ldots7} u_0 \simeq (-U_0, U_1, -U_2)$. **Triality** is defined as the mapping

$$\text{trial} : \$pin(8) \to \$pin(8),\ u_1 \simeq \begin{pmatrix} \check{U}_0 & 0 \\ 0 & \check{U}_2 \end{pmatrix} \to u_2 \simeq \begin{pmatrix} \check{U}_1 & 0 \\ 0 & \check{U}_0 \end{pmatrix}.$$

Triality is an automorphism of $\$pin(8)$; it is of order 3 and permutes the non-identity elements $-1, \mathbf{e}_{12\ldots7}, -\mathbf{e}_{12\ldots7}$ in the center of $\$pin(8)$.

Example. Take a unit paravector $a \in \$R^8 = \mathbb{R} \oplus \mathbb{R}^{0,7}$, $|a| = 1$. The action $x \to ax\hat{a}^{-1}$ is a simple rotation of $\$R^8$. [7] Thus, $a \in \$pin(8)$. Denote $a_0 = \hat{a}$, $a_1 = \text{trial}(a_0)$ and $a_2 = \text{trial}(a_1)$ so that $16\langle \hat{a}_1 x f\rangle_{0,1} = a_2 x \hat{a}_2^{-1}$, $16\langle \hat{a}_2 x \hat{f}\rangle_{0,1} = a_1 x \hat{a}_1^{-1}$. Then

$$a \circ x = a_1 x \hat{a}_1^{-1} \quad \text{and} \quad x \circ a = a_2 x \hat{a}_2^{-1}$$

represent isoclinic rotations of $\$R^8$. Left and right multiplications by $a \in S^7 \subset \mathbb{O}$ are positive and negative isoclinic rotations of $\$R^8 = \mathbb{O}$. [8] The Moufang

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[7] Note that $a \circ x \circ a = ax\tilde{a}$, $\tilde{a} = \hat{a}^{-1}$ and $a \circ x \circ a^{-1} = sxs^{-1}$ where $s = a_1 a_2^{-1} \in \$pin(7)$.

[8] Any four mutually orthogonal invariant planes of an isoclinic rotation of $\$R^8$ induce the same orientation on $\$R^8$.

identity

$$a \circ (x \circ y) \circ a = (a \circ x) \circ (y \circ a)$$

results in a special case of Cartan's principle of triality [9]

$$\hat{a}_0 (x \circ y) a_0^{-1} = (a_1 x \hat{a}_1^{-1}) \circ (a_2 y \hat{a}_2^{-1}).$$

In this special case, a_0, a_1, a_2 commute, $a_2 = \tilde{a}_1$ $(= \hat{a}_1^{-1})$ and $a_0 a_1 a_2 = 1$ implying $a = a_1 a_2 = a_1 \hat{a}_1^{-1} = a_2 \hat{a}_2^{-1}$. ∎

Triality sends a simple rotation to a positive isoclinic rotation and a positive isoclinic rotation to a negative isoclinic rotation. The isoclinic rotations can be represented by octonion multiplication having neutral axis in the rotation plane of the simple rotation:

$$\text{simple rotation} \begin{array}{c} \nearrow \\ \\ \nwarrow \end{array} \begin{array}{c} \text{positive isoclinic rotation} \\ \downarrow \\ \text{negative isoclinic rotation} \end{array}$$

6. Spin(7)

Let $u_0 \in \mathbf{Spin}(7) \subset \mathcal{C}\ell_8$, and $u_1 = \text{trial}(u_0)$, $u_2 = \text{trial}(u_1)$. Then $u_2 = \check{u}_1$, that is, $\text{trial}(\text{trial}(u_0)) = \mathbf{e}_8 \text{trial}(u_0) \mathbf{e}_8^{-1}$. [10] Thus, $u_1 \check{u}_2^{-1} = 1$ and $u_1 u_2^{-1} = u_1 \mathbf{e}_8 u_1^{-1} \mathbf{e}_8^{-1} \in \mathbb{R} \oplus \mathbb{R}^7 \mathbf{e}_8$, being a product of two vectors, represents a simple rotation. [11] [12] Since $\check{U}_0 = U_0$, $U_2 = \check{U}_1$,

$$u_0 \simeq \begin{pmatrix} U_1 & 0 \\ 0 & \check{U}_1 \end{pmatrix}, \quad u_1 \simeq \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix}, \quad u_2 \simeq \begin{pmatrix} U_2 & 0 \\ 0 & U_0 \end{pmatrix}.$$

Comparing matrix entries of $u_1 u_0^{-1} u_2$, we find $u_1 u_0^{-1} u_2 \in \mathbf{Spin}(7)$ and so $U_1 U_0^{-1} U_2 \in SO(7)$.

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[9] To prove Cartan's principle of triality, in the general case, iterate the Moufang identity, like $b \circ a \circ (x \circ y) \circ a \circ b = (b \circ (a \circ x)) \circ ((y \circ a) \circ b)$. Observe the nesting $b \circ (a \circ x) = s x \hat{s}^{-1}$, where $s = \text{trial}(\hat{b})\text{trial}(\hat{a}) = \text{trial}(\widehat{ba})$.

[10] Note that $\text{trial}(\text{trial}(u_1)) \neq \mathbf{e}_8 \text{trial}(u_1) \mathbf{e}_8^{-1}$, $\text{trial}(\text{trial}(u_2)) \neq \mathbf{e}_8 \text{trial}(u_2) \mathbf{e}_8^{-1}$.

[11] In $\mathcal{C}\ell_{0,7}$, $u_2 = \hat{u}_1$, and so $u_1 \hat{u}_2^{-1} = 1$, but $u_1 u_2^{-1} \in \mathbb{R} \oplus \mathbb{R}^{0,7}$.

[12] Recall that for $u \in \mathbf{Spin}(8)$, $u^{-1} = \tilde{u}$, and for $u \in \$pin(8)$, $u^{-1} = \bar{u}$.

Let the rotation angles of $U_0 \in SO(7)$ be $\alpha_0, \beta_0, \gamma_0$ so that $\alpha_0 \geq \beta_0 \geq \gamma_0 \geq 0$. Then the rotation angles of $U_1 \in SO(8)$ are

$$\begin{cases} \alpha_1 = \frac{1}{2}(\alpha_0 + \beta_0 + \gamma_0) \\ \beta_1 = \frac{1}{2}(\alpha_0 + \beta_0 - \gamma_0) \\ \gamma_1 = \frac{1}{2}(\alpha_0 - \beta_0 + \gamma_0) \\ \delta_1 = \frac{1}{2}(\alpha_0 - \beta_0 - \gamma_0). \end{cases}$$

Since eigenvalues change in $U_0 \to U_1$, triality cannot be a similarity, $U_1 \neq SU_0S^{-1}$. Represent the rotation planes of $U_0 \in SO(7) \subset SO(8)$ by unit bivectors $\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0$, and choose the orientation of $\mathbf{D}_0 = \mathbf{u}\mathbf{e}_8$, $\mathbf{u} \in \mathbb{R}^7$, $|\mathbf{u}| = 1$ so that $\mathbf{A}_0 \wedge \mathbf{B}_0 \wedge \mathbf{C}_0 \wedge \mathbf{D}_0 = \mathbf{e}_{12\ldots8}$. The rotation planes of U_1 can be expressed as unit bivectors [13]

$$\begin{cases} \mathbf{A}_1 = \frac{1}{2}\mathrm{Trial}(\mathbf{A}_0 + \mathbf{B}_0 + \mathbf{C}_0 - \mathbf{D}_0) \\ \mathbf{B}_1 = \frac{1}{2}\mathrm{Trial}(\mathbf{A}_0 + \mathbf{B}_0 - \mathbf{C}_0 + \mathbf{D}_0) \\ \mathbf{C}_1 = \frac{1}{2}\mathrm{Trial}(\mathbf{A}_0 - \mathbf{B}_0 + \mathbf{C}_0 + \mathbf{D}_0) \\ \mathbf{D}_1 = \frac{1}{2}\mathrm{Trial}(\mathbf{A}_0 - \mathbf{B}_0 - \mathbf{C}_0 - \mathbf{D}_0). \end{cases}$$

The rotation angles and planes of U_2 are

$$\begin{aligned} &\alpha_2 = \alpha_1,\ \beta_2 = \beta_1,\ \gamma_2 = \gamma_1,\ \delta_2 = -\delta_1 \\ &\mathbf{A}_2 = \check{\mathbf{A}}_1,\ \mathbf{B}_2 = \check{\mathbf{B}}_1,\ \mathbf{C}_2 = \check{\mathbf{C}}_1,\ \mathbf{D}_2 = -\check{\mathbf{D}}_1. \end{aligned}$$

The rotation planes of U_0, U_1, U_2 induce the same orientation on $\mathbb{R}^8$, that is,

$$\mathbf{A}_0 \wedge \mathbf{B}_0 \wedge \mathbf{C}_0 \wedge \mathbf{D}_0 = \mathbf{A}_1 \wedge \mathbf{B}_1 \wedge \mathbf{C}_1 \wedge \mathbf{D}_1 = \mathbf{A}_2 \wedge \mathbf{B}_2 \wedge \mathbf{C}_2 \wedge \mathbf{D}_2.$$

For $u_0 \in \mathbf{Spin}(7)$, $u_1, u_2 \in \mathbf{Spin}(8)$, so that

$$\begin{aligned} &u_0 = \exp(\tfrac{1}{2}(\alpha_0\mathbf{A}_0 + \beta_0\mathbf{B}_0 + \gamma_0\mathbf{C}_0)) \\ &u_1 = \exp(\tfrac{1}{2}(\alpha_1\mathbf{A}_1 + \beta_1\mathbf{B}_1 + \gamma_1\mathbf{C}_1 + \delta_1\mathbf{D}_1)) \\ &u_2 = \exp(\tfrac{1}{2}(\alpha_1\mathbf{A}_2 + \beta_1\mathbf{B}_2 + \gamma_1\mathbf{C}_2 - \delta_1\mathbf{D}_2)). \end{aligned}$$

7. The Exceptional Lie Group G_2

A rotation $U \in SO(7)$ such that

$$U(\mathbf{x} \circ \mathbf{y}) = U(\mathbf{x}) \circ U(\mathbf{y}) \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in \mathbb{O}$$

[13] Trial : $\bigwedge^2 \mathbb{R}^8 \to \bigwedge^2 \mathbb{R}^8$ sends negative isoclinic bivectors to simple bivectors.

is an automorphism of the Cayley algebra $\mathbb{O}$. The automorphisms form a group G_2 with Lie algebra $\mathcal{G}_2 \subset \bigwedge^2 \mathbb{R}^8$, $\dim \mathcal{G}_2 = 14$. A bivector $\mathbf{U} \in \mathcal{G}_2$ acts on the octonion product as a derivation

$$\mathbf{U} \llcorner (\mathbf{x} \circ \mathbf{y}) = (\mathbf{U} \llcorner \mathbf{x}) \circ \mathbf{y} + \mathbf{x} \circ (\mathbf{U} \llcorner \mathbf{y}) \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in \mathbb{O} = \mathbb{R}^8.$$

The double cover of $G_2 \subset SO(7)$ in $\mathbf{Spin}(7)$ consists of two components, $\mathbf{G}_2$ and $-\mathbf{G}_2$. The groups $\mathbf{G}_2$ and $G_2 = \rho(\mathbf{G}_2)$ are isomorphic, $G_2 \simeq \mathbf{G}_2$. [14]

A rotation $U_0 \in G_2 \subset SO(7)$ has only one preimage in $\mathbf{G}_2 \subset \mathbf{Spin}(7)$, say u_0, $\rho(u_0) = U_0$. Since $\text{trial}(u_0) = u_0$, $u_1 = \text{trial}(u_0)$ equals u_0, and $U_1 = \rho(u_1)$ equals U_0. The rotation angles $\alpha_0, \beta_0, \gamma_0$ of U_0, such that $\alpha_0 \geq \beta_0 \geq \gamma_0 \geq 0$, satisfy the identities

$$\begin{cases} \alpha_1 = \frac{1}{2}(\alpha_0 + \beta_0 + \gamma_0) = \alpha_0 \\ \beta_1 = \frac{1}{2}(\alpha_0 + \beta_0 - \gamma_0) = \beta_0 \\ \gamma_1 = \frac{1}{2}(\alpha_0 - \beta_0 + \gamma_0) = \gamma_0 \\ \delta_1 = \frac{1}{2}(\alpha_0 - \beta_0 - \gamma_0) = 0 \end{cases}$$

each of which implies

$$\alpha_0 = \beta_0 + \gamma_0.$$

This can also be expressed by saying that the signed rotation angles α, β, γ of $U \in G_2$ satisfy

$$\alpha + \beta + \gamma = 0.$$

Represent the rotation planes of U by unit bivectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and choose orientations so that $u = \exp(\frac{1}{2}(\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}))$, when $U = \rho(u)$. Then $\mathbf{A} \lrcorner \mathbf{w} = \mathbf{B} + \mathbf{C}$. Conversely, for an arbitrary rotation $U \in SO(7)$ to be in G_2 it is sufficient that

$$\mathbf{A} \lrcorner \mathbf{w} = \mathbf{B} + \mathbf{C} \quad \text{and} \quad \alpha + \beta + \gamma = 0.$$

In order to construct a bivector $\mathbf{U} \in \mathcal{G}_2$, pick up a unit bivector $\mathbf{A} \in \bigwedge^2 \mathbb{R}^7$, $\mathbf{A}^2 = -1$, decompose the bivector $\mathbf{A} \lrcorner \mathbf{w}$ into a sum of two simple unit bivectors $\mathbf{B}+\mathbf{C}$ (this decomposition is not unique), choose $\alpha, \beta, \gamma \in \mathbb{R}$ so that $\alpha+\beta+\gamma = 0$, and write $\mathbf{U} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$.

For $u \in \mathbf{G}_2$, $\text{trial}(u) = u$, and for $\mathbf{U} \in \mathcal{G}_2$, $\text{Trial}(\mathbf{U}) = \mathbf{U}$, in other words, triality stabilizes point-wise $\mathbf{G}_2$ and $\mathcal{G}_2$. Multiplication by $u \in \mathbf{G}_2$ stabilizes the idempotent $\frac{1}{8}(1+\mathbf{w})$, $u\frac{1}{8}(1+\mathbf{w}) = \frac{1}{8}(1+\mathbf{w})u = \frac{1}{8}(1+\mathbf{w})$, while a bivector

[14] Note that $-I \notin G_2$ because $-I \notin SO(7)$ and $-1 \notin \mathbf{G}_2$ because triality stabilizes point-wise $\mathbf{G}_2$ but sends -1 to $\pm\mathbf{e}_{12\ldots8}$.

$\mathbf{U} \in \mathcal{G}_2$ annihilates $\frac{1}{8}(1+\mathbf{w})$, $\mathbf{U}\frac{1}{8}(1+\mathbf{w}) = \frac{1}{8}(1+\mathbf{w})\mathbf{U} = 0$, and thus $\mathbf{U}\frac{1}{8}(7-\mathbf{w}) = \frac{1}{8}(7-\mathbf{w})\mathbf{U} = \mathbf{U}$. Conversely, a rotation $U \in SO(7)$ is in G_2 if it fixes the 3-vector

$$\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713}$$

for which $\mathbf{w} = \mathbf{v}\mathbf{e}_{12\ldots7}^{-1} = \mathbf{e}_{1236} - \mathbf{e}_{1257} - \mathbf{e}_{1345} + \mathbf{e}_{1467} + \mathbf{e}_{2347} - \mathbf{e}_{2456} - \mathbf{e}_{3567}$.

A bivector $\mathbf{F} \in \bigwedge^2 \mathbb{R}^8$, $\dim(\bigwedge^2 \mathbb{R}^8) = 28$, can be decomposed as

$$\mathbf{F} = \mathbf{G} + \mathbf{H} \quad \text{where} \quad \mathbf{G} \in \mathcal{G}_2 \quad \text{and} \quad \mathbf{H} = \frac{1}{3}\mathbf{w} \lrcorner (\mathbf{w} \wedge \mathbf{F}) \in \mathcal{G}_2^{\perp}.$$

Under triality, $\mathbf{F}$ goes to $\mathrm{Trial}(\mathbf{F}) = \mathbf{G} + \mathrm{Trial}(\mathbf{H})$, $\mathrm{Trial}(\mathbf{H}) \in \mathcal{G}_2^{\perp}$, where the angle between $\mathbf{H}$ and $\mathrm{Trial}(\mathbf{H})$ is $120°$. In particular, triality is an isoclinic rotation when restricted to $\mathcal{G}_2^{\perp}$, $\dim(\mathcal{G}_2^{\perp}) = 14$.

A bivector $\mathbf{F} \in \bigwedge^2 \mathbb{R}^7$ can be decomposed as $\mathbf{F} = \mathbf{G} + \mathbf{H}$, where $\mathbf{G} \in \mathcal{G}_2$,

$$\mathbf{H} \in \mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7, \quad \dim(\mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7) = 7.$$

For a vector $\mathbf{a} \in \mathbb{R}^7$, $\mathbf{v} \llcorner \mathbf{a} \in \mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7$. The mapping $\mathbf{a} \to \mathbf{v} \llcorner \mathbf{a}$ is one-to-one, since $\mathbf{a} = \frac{1}{3}\mathbf{v} \llcorner (\mathbf{v} \llcorner \mathbf{a})$. The element $u = \exp(\mathbf{v} \llcorner \mathbf{a}) \in \mathbf{Spin}(7)$ induces a rotation of $\mathbb{R}^7$, which has $\mathbf{a}$ as its axis and is isoclinic in $\mathbf{a}^{\perp} = \{\mathbf{x} \in \mathbb{R}^7 \mid \mathbf{x} \cdot \mathbf{a} = 0\}$. A miracle happens when $|\mathbf{a}| = 2\pi/3$. Then the rotation angles of $U = \rho(u)$ are $4\pi/3$, which is the same as $4\pi/3 - 2\pi = -2\pi/3$ in the opposite sense of rotation. For the signed rotation angles we can choose $\alpha = 4\pi/3$, $\beta = \gamma = -2\pi/3$ which satisfy $\alpha + \beta + \gamma = 0$. Since also $\mathbf{A} \lrcorner \mathbf{w} = \mathbf{B} + \mathbf{C}$, it follows that $u \in \mathbf{G}_2$. Therefore, $u = \exp(\mathbf{v} \llcorner \mathbf{a})$, where $\mathbf{a} \in \mathbb{R}^7$ and $|\mathbf{a}| = 2\pi/3$, belongs to

$$\exp(\mathcal{G}_2) \cap \exp(\mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7) \simeq S^6, \quad \text{where} \quad \exp(\mathcal{G}_2) = \mathbf{G}_2.$$

Note that $\alpha+\beta+\gamma = 0$ in $\mathbf{G}_2$ while $\alpha = \beta = \gamma$ in $\exp(\mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7)$. An element $u \in \mathbf{G}_2 \cap \exp(\mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7)$ can be also constructed by choosing a unit bivector $\mathbf{A} \in \bigwedge^2 \mathbb{R}^7$, $\mathbf{A}^2 = -1$, decomposing $\mathbf{A} \lrcorner \mathbf{w} = \mathbf{B} + \mathbf{C}$, constructing bivectors

$$\frac{2\pi}{3}(2\mathbf{A} - \mathbf{B} - \mathbf{C}) = \mathbf{G} \in \mathcal{G}_2 \quad \text{and} \quad \frac{2\pi}{3}(-\mathbf{A} - \mathbf{B} - \mathbf{C}) = \mathbf{H} \in \mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7$$

and exponentiating

$$u = e^{\mathbf{G}} = e^{\mathbf{H}} = -\tfrac{1}{8} + \cdots$$

The elements u are extreme elements in $\mathbf{G}_2$ in the sense that $\langle u\rangle_0 = -\frac{1}{8}$, while for all other $g \in \mathbf{G}_2$, $\langle g\rangle_0 > -\frac{1}{8}$.

The elements $u = \exp(\mathbf{v} \lrcorner \mathbf{a})$, where $\mathbf{a} \in \mathbb{R}^{0,7}$ and $|\mathbf{a}| = 2\pi/3$, satisfy $u^3 = 1$, and they are the only non-identity solutions of $u^3 = 1$ in $\mathbf{G}_2$. The octonion $a = e^{\circ\mathbf{a}}$ $(= e^{\mathbf{a}})$ satisfies $a^{\circ 3} = 1$ and $a^{\circ -1} \circ x \circ a = uxu^{-1}$ for all $x \in \$R^8 = \mathbb{R} \oplus \mathbb{R}^{0,7}$. Conversely, the only unit octonions $a \in S^7 \subset \mathbb{O} = \$R^8$ satisfying

$$a^{\circ -1} \circ (x \circ y) \circ a = (a^{\circ -1} \circ x \circ a) \circ (a^{\circ -1} \circ y \circ a) \quad \text{for all} \quad x, y \in \$R^8$$

are solutions of $a^{\circ 3} = \pm 1$.

8. Components of the Automorphism Group of Spin(8)

In general, the only exterior automorphisms of $\mathbf{Spin}(n)$, $n \neq 8$, are of type $u \to sus^{-1}$, where $s \in \mathbf{Pin}(n) \setminus \mathbf{Spin}(n)$. Thus, $\mathrm{Aut}(\mathbf{Spin}(n))/\mathrm{Int}(\mathbf{Spin}(n)) \simeq \mathbb{Z}_2$, when $n \neq 8$. However, in the case $n = 8$, the following sequence is exact

$$1 \longrightarrow \mathrm{Int}(\mathbf{Spin}(8)) \longrightarrow \mathrm{Aut}(\mathbf{Spin}(8)) \longrightarrow S_3,$$

that is, $\mathrm{Aut}(\mathbf{Spin}(8))/\mathrm{Int}(\mathbf{Spin}(8)) \simeq S_3$, a non-commutative group of order 6.

For $u \in \mathbf{Spin}(8)$, denote $\mathrm{swap}_1(u) = \mathbf{e}_8 \mathrm{trial}(u)\mathbf{e}_8^{-1} = \mathrm{trial}(\mathrm{trial}(\mathbf{e}_8 u \mathbf{e}_8^{-1}))$ and $\mathrm{swap}_2(u) = \mathbf{e}_8 \mathrm{trial}(\mathrm{trial}(u))\mathbf{e}_8^{-1} = \mathrm{trial}(\mathbf{e}_8 u \mathbf{e}_8^{-1})$. Then trial, swap_1, swap_2 generate S_3 :

$$\begin{aligned} &\mathrm{trial} \circ \mathrm{trial} \circ \mathrm{trial} = \mathrm{swap}_1 \circ \mathrm{swap}_1 = \mathrm{swap}_2 \circ \mathrm{swap}_2 = \text{ identity} \\ &\mathrm{swap}_1 \circ \mathrm{swap}_2 = \mathrm{trial} \quad \text{and} \quad \mathrm{swap}_2 \circ \mathrm{swap}_1 = \mathrm{trial} \circ \mathrm{trial}. \end{aligned}$$

The automorphism group of $\mathbf{Spin}(8)$ contains 6 components, represented by the identity, trial, $\mathrm{trial} \circ \mathrm{trial}$, swap_1, swap_2 and the companion. [15] In the component of trial, all automorphisms of order 3 are trialities for some octonion product.

[15] The subgroup of linear automorphisms contains 2 components, represented by the identity and the companion, $u \to \mathbf{e}_8 u \mathbf{e}_8^{-1}$.

9. Triality is Quadratic

Triality of $u \in \mathbf{Spin}(8) \subset \mathcal{C}\ell_8$ is a restriction a polynomial mapping $\mathcal{C}\ell_8 \to \mathcal{C}\ell_8$, of degree 2,

$$\begin{aligned}
&\mathrm{trial}(u) = \mathrm{trial}_1(u)\mathrm{trial}_2(u)\\
&\mathrm{trial}_1(u) = \tfrac{1}{2}(1+\mathbf{e}_{12\ldots8})[\langle u(1+\mathbf{w})(1-\mathbf{e}_{12\ldots8})\rangle_{0,6} \wedge \mathbf{e}_8]\mathbf{e}_8^{-1}\\
&\qquad\qquad +\tfrac{1}{2}(1-\mathbf{e}_{12\ldots8})\\
&\mathrm{trial}_2(u) = (\mathbf{w}-3)[(u(1+\mathbf{e}_{12\ldots8}))\wedge \mathbf{e}_8]\mathbf{e}_8^{-1}(\mathbf{w}-3)^{-1}.
\end{aligned}$$

The first factor is affine linear and the second factor is linear. To verify that trial is a triality, it is sufficient to show that it is an automorphism of order 3 sending -1 to $\mathbf{e}_{12\ldots8}$.

In the Lie algebra level, the triality automorphism of a bivector $\mathbf{F} \in \bigwedge^2 \mathbb{R}^8$ is

$$\begin{aligned}
\mathrm{Trial}(\mathbf{F}) &= \mathbf{e}_8\langle \mathbf{F} - \tfrac{1}{2}\mathbf{F}(1+\mathbf{w})(1+\mathbf{e}_{12\ldots8})_2\mathbf{e}_8^{-1}\\
&= \tfrac{1}{2}\mathbf{e}_8(\mathbf{F} - \mathbf{F}\,\lrcorner\,\mathbf{w} - (\mathbf{F}\wedge\mathbf{w})\,\lrcorner\,\mathbf{e}_{12\ldots8})\mathbf{e}_8^{-1}.
\end{aligned}$$

The triality automorphism of a para-bivector $F \in \mathbb{R}^{0,7} \oplus \bigwedge^2 \mathbb{R}^{0,7}$ is

$$\begin{aligned}
\mathrm{Trial}(F) &= \langle F - \tfrac{1}{2}F(1+\mathbf{w})(1-\mathbf{e}_{12\ldots7})\rangle_{1,2}^{\wedge}\\
&= \tfrac{1}{2}(F - \langle F\rangle_2\,\lrcorner\,\mathbf{w} + (F\wedge\mathbf{w})\,\lrcorner\,\mathbf{e}_{12\ldots7})^{\wedge}.
\end{aligned}$$

For $u \in \$pin(8)$, triality is

$$\begin{aligned}
&\mathrm{trial}(u) = \mathrm{trial}_1(u)\mathrm{trial}_2(u)\\
&\mathrm{trial}_1(u) = \tfrac{1}{2}(1-\mathbf{e}_{12\ldots7})\langle u(1+\mathbf{w})(1+\mathbf{e}_{12\ldots7})\rangle_{0,6}\\
&\qquad\qquad +\tfrac{1}{2}(1+\mathbf{e}_{12\ldots7})\\
&\mathrm{trial}_2(u) = (\mathbf{w}-3)\ \mathrm{even}\ (u(1-\mathbf{e}_{12\ldots7}))(\mathbf{w}-3)^{-1}.
\end{aligned}$$

10. Triality in Terms of Eigenvalues and Invariant Planes

Triality can also be viewed classically, without Clifford algebras, by inspection of changes in eigenvalues and invariant planes of rotations. Consider $U_0 \in SO(8)$ and a triality triplet (U_0, U_1, U_2). Let the rotation angles $\alpha_0, \beta_0, \gamma_0, \delta_0$ of U_0 be such that $\alpha_0 \geq \beta_0 \geq \gamma_0 \geq \delta_0 \geq 0$. Represent the rotation planes of U_0 by the unit bivectors $\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0, \mathbf{D}_0$. Then the rotation angles of U_1 and U_2 are

$$\begin{cases}
\alpha_1 = \tfrac{1}{2}(\alpha_0+\beta_0+\gamma_0-\delta)\\
\beta_1 = \tfrac{1}{2}(\alpha_0+\beta_0-\gamma_0+\delta)\\
\gamma_1 = \tfrac{1}{2}(\alpha_0-\beta_0+\gamma_0+\delta)\\
\delta_1 = \tfrac{1}{2}(\alpha_0-\beta_0-\gamma_0-\delta)
\end{cases}
\quad\text{and}\quad
\begin{cases}
\alpha_2 = \tfrac{1}{2}(\alpha_0+\beta_0+\gamma_0+\delta)\\
\beta_2 = \tfrac{1}{2}(\alpha_0+\beta_0-\gamma_0-\delta)\\
\gamma_2 = \tfrac{1}{2}(\alpha_0-\beta_0+\gamma_0-\delta)\\
\delta_2 = \tfrac{1}{2}(-\alpha_0+\beta_0+\gamma_0-\delta)
\end{cases}$$

and the rotation planes are

$$\begin{cases} \mathbf{A}_1 = \frac{1}{2}\mathrm{Trial}(\mathbf{A}_0 + \mathbf{B}_0 + \mathbf{C}_0 - \mathbf{D}_0) \\ \mathbf{B}_1 = \frac{1}{2}\mathrm{Trial}(\mathbf{A}_0 + \mathbf{B}_0 - \mathbf{C}_0 + \mathbf{D}_0) \\ \mathbf{C}_1 = \frac{1}{2}\mathrm{Trial}(\mathbf{A}_0 - \mathbf{B}_0 + \mathbf{C}_0 + \mathbf{D}_0) \\ \mathbf{D}_1 = \frac{1}{2}\mathrm{Trial}(\mathbf{A}_0 - \mathbf{B}_0 - \mathbf{C}_0 - \mathbf{D}_0) \end{cases}$$

and

$$\begin{cases} \mathbf{A}_2 = \frac{1}{2}\mathrm{Trial}(\mathrm{Trial}(\mathbf{A}_0 + \mathbf{B}_0 + \mathbf{C}_0 + \mathbf{D}_0)) \\ \mathbf{B}_2 = \frac{1}{2}\mathrm{Trial}(\mathrm{Trial}(\mathbf{A}_0 + \mathbf{B}_0 - \mathbf{C}_0 - \mathbf{D}_0)) \\ \mathbf{C}_2 = \frac{1}{2}\mathrm{Trial}(\mathrm{Trial}(\mathbf{A}_0 - \mathbf{B}_0 + \mathbf{C}_0 - \mathbf{D}_0)) \\ \mathbf{D}_2 = \frac{1}{2}\mathrm{Trial}(\mathrm{Trial}(-\mathbf{A}_0 + \mathbf{B}_0 + \mathbf{C}_0 - \mathbf{D}_0)). \end{cases}$$

11. Trialities with Respect to Different Octonion Products

An arbitrary 4-vector $\mathbf{w} \in \bigwedge^4 \mathbb{R}^8$, for which $\frac{1}{8}(1+\mathbf{w})$ is an idempotent in $\mathcal{C}\ell_8^+$, is called a calibration. [16] The calibration $\mathbf{w}$ fixes excatly one line of vectors, namely $\{\mathbf{n} \in \mathbb{R}^8 \mid \mathbf{wn} = \mathbf{nw}\}$. This line, together with a chosen orientation, is called the neutral axis of the calibration. A calibration together with its neutral axis can be used to introduce an octonion product on $\mathbb{R}^8$ and a triality of $\mathbf{Spin}(8)$.

Let $\mathbf{w}_1, \mathbf{w}_2$ be calibrations, with neutral axes $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{R}^8$. Denote the octonion products by

$$\begin{aligned} \mathbf{x} \circ_{\mathbf{w}_1,\mathbf{n}_1} \mathbf{y} &= \langle \mathbf{x}\mathbf{n}_1\mathbf{y}(1+\mathbf{w}_1)(1-\mathbf{e}_{12\ldots8})\rangle_1, \\ \mathbf{x} \circ_{\mathbf{w}_2,\mathbf{n}_2} \mathbf{y} &= \langle \mathbf{x}\mathbf{n}_2\mathbf{y}(1+\mathbf{w}_2)(1-\mathbf{e}_{12\ldots8})\rangle_1 \end{aligned}$$

and the trialities by $\mathrm{trial}_{\mathbf{w}_1,\mathbf{n}_1}(u)$, $\mathrm{trial}_{\mathbf{w}_2,\mathbf{n}_2}(u)$. Denote the opposite of the composition of the trialities by $\mathrm{trial}_{\mathbf{w}_{12},\mathbf{n}_{12}}(u)$ so that

$$\mathrm{trial}_{\mathbf{w}_{12},\mathbf{n}_{12}}(\mathrm{trial}_{\mathbf{w}_{12},\mathbf{n}_{12}}(u)) = \mathrm{trial}_{\mathbf{w}_1,\mathbf{n}_1}(\mathrm{trial}_{\mathbf{w}_2,\mathbf{n}_2}(u)).$$

Then

$$\begin{aligned} \mathbf{w}_{12} &= \tfrac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2) + \tfrac{1}{2}(-\mathbf{w}_1 + \mathbf{w}_2)\mathbf{e}_{12\ldots8} \\ &= \tfrac{1}{2}(1 - \mathbf{e}_{12\ldots8})\mathbf{w}_1 + \tfrac{1}{2}(1 + \mathbf{e}_{12\ldots8})\mathbf{w}_2 \end{aligned}$$

and

$$\mathbf{n}_{12} = \frac{\mathbf{y}}{|\mathbf{y}|} \quad \text{for a non} - \text{zero} \quad \mathbf{y} = \mathbf{x} - \tfrac{1}{4}\mathbf{w}_{12} \llcorner (\mathbf{w}_{12} \llcorner \mathbf{x}), \quad \text{where} \quad \mathbf{x} \in \mathbb{R}^8.$$

[16] Note that $\mathbf{w}$ satisfies $\mathbf{w}^2 = 7 + 6\mathbf{w}$.

12. Factorization of $u \in \mathbf{Spin}(8)$

Take $u \in \mathbf{Spin}(8) \setminus \mathbf{Spin}(7)$. Denote $s = (u \wedge \mathbf{e}_8)\mathbf{e}_8^{-1}$, and $u_7 = \dfrac{s}{|s|}$. Then $u_7 \in \mathbf{Spin}(7)$, and $u = u_8 u_7 = u_7 u_8'$, where $u_8, u_8' \in \mathbb{R} \oplus \mathbb{R}^7 \mathbf{e}_8$. [17] These factorizations are unique, up to a sign (-1 is a square root of 1 in $\mathbf{Spin}(7)$) :

$$u = u_8 u_7 = (-u_8)(-u_7) = u_7 u_8' = (-u_7)(-u_8').$$

The following factorizations are unique, up to a cube root of 1 in $\mathbf{G}_2$:

$$u_7 = h_0 g_0 = h_1 g_1 = h_2 g_2 = g_0 h_0' = g_1 h_1' = g_2 h_2',$$

where

$$h_0^3 = u_7 \mathrm{trial}(u_7)^{-1} u_7 \mathrm{trial}(\mathrm{trial}(u_7))^{-1},$$
$$h_0'^3 = \mathrm{trial}(u_7)^{-1} u_7 \mathrm{trial}(\mathrm{trial}(u_7))^{-1} u_7,$$

and

$$h_1 = h_0 g,\ h_2 = h_0 g^2, \quad h_1' = g' h_0',\ h_2' = g'^2 h_0',$$
$$g = \exp(\tfrac{2\pi}{3}\mathbf{v} \llcorner \tfrac{\mathbf{h}_0}{|\mathbf{h}_0|}), \quad g' = \exp(\tfrac{2\pi}{3}\mathbf{v} \llcorner \tfrac{\mathbf{h}_0'}{|\mathbf{h}_0'|}),$$
$$\mathbf{h}_0 = (\mathbf{H}_0 \wedge \mathbf{H}_0 \wedge \mathbf{H}_0)\mathbf{e}_{12\ldots7}, \quad \mathbf{h}_0' = (\mathbf{H}_0' \wedge \mathbf{H}_0' \wedge \mathbf{H}_0')\mathbf{e}_{12\ldots7},$$
$$h_0 = e^{\mathbf{H}_0}, \quad h_0' = e^{\mathbf{H}_0'}.$$

In this factorization, $g_0, g_1, g_2 \in \mathbf{G}_2$ and $h_0, h_1, h_2, h_0', h_1', h_2' \in \exp(\mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7)$ and $g, g' \in \mathbf{G}_2 \cap \exp(\mathcal{G}_2^{\perp} \cap \bigwedge^2 \mathbb{R}^7) \simeq S^6$. These factorizations are unique, up to a factor $g, g' \in \mathbf{G}_2$, $g^3 = g'^3 = 1$. [18]

Appendix: Comparison of Formalisms in $\mathbb{R}^8$ and $\mathbb{R} \oplus \mathbb{R}^{0,7}$

We use the 3-vector

$$\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713}$$

in $\bigwedge^3 \mathbb{R}^8$ or $\bigwedge^3 \mathbb{R}^{0,7}$, and the 4-vector $\mathbf{w} = \mathbf{v}\mathbf{e}_{12\ldots7}^{-1}$ in $\bigwedge^4 \mathbb{R}^8$ or $\bigwedge^4 \mathbb{R}^{0,7}$,

$$\mathbf{w} = \mathbf{e}_{1236} - \mathbf{e}_{1257} - \mathbf{e}_{1345} + \mathbf{e}_{1467} + \mathbf{e}_{2347} + \mathbf{e}_{2456} - \mathbf{e}_{3467}.$$

Note that $\mathbf{e}_{12\ldots7}^{-1} = -\mathbf{e}_{12\ldots7}$ in $\mathcal{C}\ell_8$ while $\mathbf{e}_{12\ldots7}^{-1} = \mathbf{e}_{12\ldots7}$ in $\mathcal{C}\ell_{0,7}$.

[17] Note that $u_8 = \sqrt{u(\mathbf{e}_8 u \mathbf{e}_8^{-1})^{-1}}$ and $u_8' = \sqrt{(\mathbf{e}_8 u \mathbf{e}_8^{-1})^{-1} u}$.

[18] Recall that $-1 \notin \mathbf{G}_2$.

We use the octonion product

$$\mathbf{x}\circ\mathbf{y}=\langle\mathbf{x}\mathbf{e}_8\mathbf{y}(1+\mathbf{w})(1-\mathbf{e}_{12\ldots8})\rangle_1 \quad \text{for vectors} \quad \mathbf{x},\mathbf{y}\in\mathbb{R}^8,$$
$$x\circ y=\langle xy(1+\mathbf{w})(1-\mathbf{e}_{12\ldots7})\rangle_{0,1} \quad \text{for paravectors} \quad x,y\in\mathbb{R}\oplus\mathbb{R}^{0,7}.$$

Note that in $\mathcal{C}\ell_{0,7}$, also $x\circ y=\langle xy(1-\mathbf{v})\rangle_{0,1}$.

The bivector $\mathbf{F}=\mathbf{A}+\mathbf{B}\in\bigwedge^2\mathbb{R}^8$, with $\mathbf{B}\in\bigwedge^2\mathbb{R}^7$ and $\mathbf{A}=\mathbf{a}\mathbf{e}_8$, $\mathbf{a}\in\mathbb{R}^7$, corresponds to the para-bivector $F=\mathbf{a}-\mathbf{B}\in\mathbb{R}^{0,7}\oplus\bigwedge^2\mathbb{R}^{0,7}$, with $\mathbf{a}=\mathbf{A}\mathbf{e}_8^{-1}\in\mathbb{R}^{0,7}$. Let $u=u_++u_-\mathbf{e}_8\in\mathbf{Spin}(8)$, where $u_\pm\in\mathcal{C}\ell_7^\pm$. Then $u\in\mathbf{Spin}(8)$ corresponds to

$$(\widetilde{\widehat{u_++u_-}})=(\widehat{u_++u_-})^{-1}\in \$pin(8).$$

The companion $\check{u}$ of u is

$$\check{u}=\mathbf{e}_8u\mathbf{e}_8^{-1} \quad \text{for} \quad u \text{ in } \mathbf{Spin}(8) \text{ or } \mathcal{C}\ell_8^+(\text{ or } \mathcal{C}\ell_8),$$
$$\check{u}=\hat{u} \quad \text{for } u \text{ in } \$pin(8) \text{ or } \mathcal{C}\ell_{0,7}.$$

For u_0, $u_1=\text{trial}(u_0)$, $u_2=\text{trial}(u_1)$, Cartan's principle of triality says

$$\check{u}_0(\mathbf{x}\circ\mathbf{y})\check{u}_0^{-1}=(u_1\mathbf{x}u_1^{-1})\circ(u_2\mathbf{y}u_2^{-1}) \quad \text{in} \quad \mathbf{Spin}(8),$$
$$\hat{u}_0(x\circ y)u_0^{-1}=(u_1x\hat{u}_1^{-1})\circ(u_2y\hat{u}_2^{-1}) \quad \text{in} \quad \$pin(8).$$

In the Lie algebra level, Freudenthal's principle of triality says [19]

$$(\mathbf{x}\circ\mathbf{y})\,\lrcorner\,\check{\mathbf{F}}_0=(\mathbf{x}\,\lrcorner\,\mathbf{F}_1)\circ\mathbf{y}+\mathbf{x}\circ(\mathbf{y}\,\lrcorner\,\mathbf{F}_2) \quad \text{in} \quad \textstyle\bigwedge^2\mathbb{R}^8,$$
$$\langle(x\circ y)\,\lrcorner\,F_0\rangle_{0,1}=\langle x\,\lrcorner\,\hat{F}_1\rangle_{0,1}\circ y+x\circ\langle y\,\lrcorner\,\hat{F}_2\rangle_{0,1} \quad \text{in} \quad \mathbb{R}^{0,7}\oplus\textstyle\bigwedge^2\mathbb{R}^{0,7}.$$

The non-identity central elements of the Lie groups are permuted as follows

$$\text{trial}(-1)=\mathbf{e}_{12\ldots8},\ \text{trial}(\mathbf{e}_{12\ldots8})=-\mathbf{e}_{12\ldots8} \quad \text{in} \quad \mathbf{Spin}(8),$$
$$\text{trial}(-1)=-\mathbf{e}_{12\ldots7},\ \text{trial}(-\mathbf{e}_{12\ldots7})=\mathbf{e}_{12\ldots7} \quad \text{in} \quad \$pin(8).$$

Exercises

Show that

1. For $\mathbf{U}\in\mathcal{G}_2$, $\mathbf{U}\,\lrcorner\,\mathbf{w}=-\mathbf{U}$, $\mathbf{U}\mathbf{w}=-\mathbf{U}$, $\mathbf{U}\,\lrcorner\,\mathbf{U}=-|\mathbf{U}|^2$, $|\mathbf{U}\wedge\mathbf{U}|=|\mathbf{U}|^2$.
2. For $\mathbf{U}_0\in\bigwedge^2\mathbb{R}^8$, $\mathbf{U}_1=\text{Trial}(\mathbf{U}_0)$, $\mathbf{U}_2=\text{Trial}(\mathbf{U}_1)$, $\mathbf{U}_0+\mathbf{U}_1+\mathbf{U}_2\in\mathcal{G}_2$, $2\mathbf{U}_0-\mathbf{U}_1-\mathbf{U}_2\in\mathcal{G}_2^\perp$.

[19] Note that $\mathbf{x}\,\lrcorner\,\mathbf{F}$ corresponds to $\langle x\,\lrcorner\,\hat{F}\rangle_{0,1}$.

3. In $\mathcal{C}\ell_{0,7}$, $(1+\mathbf{w})(1-\mathbf{e}_{12\ldots7}) = (1-\mathbf{e}_{124})(1-\mathbf{e}_{235})(1-\mathbf{e}_{346})(1-\mathbf{e}_{457})$, $(1+\mathbf{w})(1-\mathbf{e}_{12\ldots7}) = (1-\mathbf{v})(1-\mathbf{e}_{12\ldots7})$.
4. $\mathbf{w}^2 = 7+6\mathbf{w}$.
5. $\mathbf{w}^n = \begin{cases} \frac{1}{8}(7^n-1)+1+\frac{1}{8}(7^n-1)\mathbf{w}, & n \text{ even}, \\ \frac{1}{8}(7^n+1)-1+\frac{1}{8}(7^n+1)\mathbf{w}, & n \text{ odd}. \end{cases}$
6. For $x \in \mathbb{R}$, $f(x\mathbf{w}) = f(-x)\frac{1}{8}(7-\mathbf{w}) + f(7x)\frac{1}{8}(1+\mathbf{w})$. Hint: the minimal polynomial of $\mathbf{w}$, $x^2 = 7+6x$, has roots $x=-1$, $x=7$.
7. For $\mathbf{G} \in \mathcal{G}_2$, $\frac{1}{8}(1+\mathbf{w})\mathbf{G}\frac{1}{8}(1+\mathbf{w}) = 0$, $\frac{1}{8}(7-\mathbf{w})\mathbf{G}\frac{1}{8}(7-\mathbf{w}) = \mathbf{G}$, $e^{\mathbf{G}} = \frac{1}{8}(7-\mathbf{w})e^{\mathbf{G}}\frac{1}{8}(7-\mathbf{w}) + \frac{1}{8}(1+\mathbf{w})$.
8. $\mathbf{v}^2 = -7-6\mathbf{w}$ in $\mathcal{C}\ell_8$, $\mathbf{v}^2 = 7+6\mathbf{w}$ in $\mathcal{C}\ell_{0,7}$.
9. For $\mathbf{v} \in \bigwedge^3 \mathbb{R}^7$, $e^{\pi\mathbf{v}} = -1$. Hint: $\mathbf{v}^4 + 50\mathbf{v}^2 + 49 = 0$, and $x^4 + 50x^2 + 49 = 0$ has roots $\pm i$, $\pm 7i$.
10. For $\mathbf{v} \in \bigwedge^3 \mathbb{R}^{0,7}$, $\cos(\pi\mathbf{v}) = -1$, $\sin(\pi\mathbf{v}) = 0$. Hint: $\mathbf{v}^4 - 50\mathbf{v}^2 + 49 = 0$, and $x^4 - 50x^2 + 49 = 0$ has roots ± 7, ± 1.
11. For $\mathbf{F} \in \bigwedge^2 \mathbb{R}^8$, $\mathbf{F} = \mathbf{G}+\mathbf{H}$, $\mathbf{G} \in \mathcal{G}_2$, $\mathbf{H} \in \mathcal{G}_2^\perp$: $\mathbf{H} = \frac{1}{3}\mathbf{w} \rfloor (\mathbf{w} \wedge \mathbf{F})$.
12. For $\mathbf{H} \in \mathcal{G}_2^\perp$, $\mathbf{H} = \frac{1}{4}\mathbf{w} \llcorner (\mathbf{w} \llcorner \mathbf{H})$.
13. For $u_0 \in \$pin(8)$, $u_1 = \mathrm{trial}(u_0)$, $u_2 = \mathrm{trial}(u_1)$: $u_0\hat{u}_1^{-1}u_2 \in \$pin(7)$. Hint: $a = u_0^{-1}\hat{u}_0 \in \mathbb{R} \oplus \mathbb{R}^{0,7}$ and so $a = \widehat{\mathrm{trial}(a)}\mathrm{trial}(a)^{-1}$.
14. For the opposite product $\mathbf{x} \bullet \mathbf{y} = \mathbf{y} \circ \mathbf{x}$ of $\mathbf{x}, \mathbf{y} \in \mathbb{O} = \mathbb{R}^8$, $\check{u}_0(\mathbf{x} \bullet \mathbf{y})\check{u}_0^{-1} = (u_2\mathbf{x}u_2^{-1}) \bullet (u_1\mathbf{y}u_1^{-1})$.
15. $u_0(\mathbf{x} \circ \mathbf{y})u_0^{-1} = (\check{u}_2\mathbf{x}\check{u}_2^{-1}) \circ (\check{u}_1\mathbf{y}\check{u}_1^{-1})$.
16. $\langle \hat{u}_0 x(1+\mathbf{w})(1-\mathbf{e}_{12\ldots7})\rangle_{0,1} = u_1 x \hat{u}_1^{-1}$, $\langle \hat{u}_0 x(1+\mathbf{w})(1+\mathbf{e}_{12\ldots7})\rangle_{0,1} = u_2 x \hat{u}_2^{-1}$.
17. $\langle \hat{u}_0 xy(1+\mathbf{w})(1-\mathbf{e}_{12\ldots7})\rangle_{0,1} = u_1(x \circ y)\hat{u}_1^{-1}$, $\langle \hat{u}_0 xy(1+\mathbf{w})(1+\mathbf{e}_{12\ldots7})\rangle_{0,1} = u_2(y \circ x)\hat{u}_2^{-1}$.
18. For $\mathbf{B} \in \bigwedge^2 \mathbb{R}^8 \cap \mathbf{Spin}(8)$, $\langle \mathrm{trial}(\mathbf{B})\rangle_0 = \frac{1}{4}$. For $\mathbf{C} \in \bigwedge^4 \mathbb{R}^8 \cap \mathbf{Spin}(8)$, $\langle \mathrm{trial}(\mathbf{C})\rangle_0 = 0$. For $\mathbf{D} \in \bigwedge^6 \mathbb{R}^8 \cap \mathbf{Spin}(8)$, $\langle \mathrm{trial}(\mathbf{D})\rangle_0 = -\frac{1}{4}$.
19. For $\mathbf{C} \in \bigwedge^4 \mathbb{R}^8 \cap \mathbf{Spin}(8)$, $\mathrm{trial}(\mathbf{C}) \in \bigwedge^4 \mathbb{R}^8$.
20. For $u \in \mathbf{Spin}(8)$, inducing a simple rotation $U = \rho(u)$: $\langle \mathrm{trial}(u)\rangle_8 \mathbf{e}_{12\ldots8} \rangle 0$ and $\langle \mathrm{trial}(\mathrm{trial}(u))\rangle_8 \mathbf{e}_{12\ldots8} \langle 0$.
21. $\langle \mathbf{G}_2\rangle_0 \geq -\frac{1}{8}$, $\langle \mathrm{trial}(\mathbf{Spin}(7))\rangle_0 \geq -\frac{1}{4}$.
22. $\mathbf{G}_2 \cap \exp(\mathcal{G}_2^\perp \cap \bigwedge^2 \mathbb{R}^7)$ is homeomorphic but not isometric to S^6.
23. diam $(\mathbf{G}_2) = \frac{3}{2}$, diam $(\mathrm{trial}(\mathbf{Spin}(7))) = \sqrt{\frac{5}{2}}$.
24. Triality does not extend to an automorphism of $\mathbf{Pin}(8)$.
25. $[\frac{1}{4}(\mathbf{w}-3)]^2 = 1$.

Determine

26. The matrix of $\frac{1}{4}(\mathbf{w}-3)$ in the basis $(f_1, f_2, \ldots, f_8)$ of $\mathcal{C}\ell_8^+ f$, where $f_i =$

$\mathbf{e}_i\mathbf{e}_8 f$, $i = 1, 2, \ldots, 8$, and $f = \frac{1}{8}(1+\mathbf{w})\frac{1}{2}(1+\mathbf{e}_{12\ldots8})$.

Solutions

13. $u_0^{-1}\hat{u}_0 = \hat{u}_1^{-1}u_2\hat{u}_2^{-1}u_1$ so $1 = u_0\hat{u}_1^{-1}u_2(\widehat{\hat{u}_2^{-1}u_1}\hat{u}_0^{-1}) = u_0\hat{u}_1^{-1}u_2(\hat{u}_0u_1^{-1}\hat{u}_2)^{-1}$ which implies $u_0\hat{u}_1^{-1}u_2 = \hat{u}_0u_1^{-1}\hat{u}_2 = (\widehat{u_0\hat{u}_1^{-1}u_2})$.

24. Triality sends $-1 \in$ Cen $(\mathbf{Pin}(8))$ to $\mathbf{e}_{12\ldots8} \notin$ Cen $(\mathbf{Pin}(8))$.

26.

$$16\langle\tilde{f}_i\tfrac{1}{4}(\mathbf{w}-3)f_j\rangle_0 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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