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Spin(7)-subgroups of SO(8) and Spin(8)

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The observations made here are prompted by the paper [**1**] of De Sapio in which he gives an exposition of the principle of triality and related topics in the context of the Octonion algebra of Cayley. However many of the results discussed there are highly group theoretic and it seems desirable to have an exposition of them from a purely group theoretic point of view not using the Cayley numbers. In what follows this is what we do. In particular we discuss some properties of the imbeddings of Spin(7) in SO(8) and Spin(8). The techniques used here are well-known to specialists in representation theory and so this paper has a semi–expository character. We take for granted the basic properties of Spin(n) and its spin representations and refer to [**2**] for a beautiful account of these. For general background in representation theory of compact Lie groups we refer to [**3**]. The main results are Theorems 1.3 and 1.5, Theorem 2.3, and Theorems 3.4 and 3.5.

1. Conjugacy classes of Spin(7)–subgroups in SO(8) and Spin(8). We work over $\mathbf{R}$ and in the category of compact Lie groups. Spin(n) is the universal covering group of SO(n). We begin with a brief review of the method of constructing the groups Spin(n) by the theory of Clifford algebras.

Let n be an integer ≥ 3. By the *Clifford algebra* C_n (over the field $\mathbf{R}$ of real numbers) we mean the algebra over $\mathbf{R}$ with n generators $x_i (1 \leq i \leq n)$ such that

$$x_i^2 = -1, \qquad x_i x_j + x_j x_i = 0 \ (i \neq j)$$

It is of dimension 2^n and the elements

$$1,\ x_{i_1} x_{i_2} \ldots x_{i_k} \qquad (1 \leq i_1 < i_2 < \ldots < i_k \leq n)$$

form a basis of C_n. The elements $x_i x_j$ generate the subalgebra C_n^0 which is linearly spanned by

$$1,\ x_{i_1} x_{i_2} \ldots x_{i_{2r}} \qquad (1 \leq i_1 < i_2 < \ldots < i_{2r} \leq n)$$

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This is the so-called *even subalgebra* of C_n. The algebra C_n has a unique antiautomorphism β such that $\beta(x_i) = x_i$ for all i. Spin(n) is the group of all elements u of C_n^0 such that

(i) u is invertible

(ii) $uV_nu^{-1} = V_n$ where V_n is the linear span of the x_i

(iii) $\beta(u)u = 1$.

Actually one should work with a vector space and a nondegenerate quadratic form of arbitrary signature over it, and associate a Clifford algebra to such data, but for our purposes this narrower definition will suffice.

Let us now equip V_n with the metric for which the x_i form an orthonormal basis. In this case Spin(n) is a *connected* Lie group. The action $\rho(u)$ of any element $u \in$ Spin(n) on V_n by $v \longmapsto uvu^{-1}$ is an orthogonal transformation and $\rho(u \longmapsto \rho(u))$ is a surjective morphism Spin$(n) \longrightarrow$ SO(n) with kernel $\{\pm 1\}$. Spin(n) is the universal covering group of SO(n). Spin(n) may also be described as the group of all elements of C_n^0 of the form $u_1u_2\ldots u_{2k}$ where the u_i are elements of V_n with $||u_i|| = 1$ for all i. If $n-k \geq 3$ and we identify SO$(n-k)$ with the subgroup of SO(n) that fixes the x_i $(1 \leq i \leq k)$, the preimage of SO$(n-k)$ in Spin(n) through the covering map is Spin$(n-k)$; this is because -1 is in the connected component of the preimage, as may be seen by the fact that for any i, j $(i \neq j, i, j \geq k+1)$, the path $e^{tx_ix_j}$ $(0 \leq t \leq \pi)$ lies entirely in the connected component of the preimage and connects 1 and -1 . Thus Spin$(m)\subset$Spin(n) if $3 \leq m < n$. If $3 \leq m < n$ and C_m is the subalgebra of C_n generated by the x_i $(1 \leq i \leq m)$, then C_m is the Clifford algebra with m generators. The two descriptions given above of the Spin groups then lead to the formula

$$\mathrm{Spin}(m) = \mathrm{Spin}(n) \cap C_m^0$$

An irreducible representation of Spin(n) is said to be of *spin type* if it is nontrivial on the kernel of the covering map Spin$(n) \longrightarrow$ SO(n). The *spin representations* are the irreducible representations that have the smallest dimension among the spin type representations of Spin(n). It can be shown that these are the irreducible representations corresponding to the right extreme nodes of the Dynkin diagram of SO(n).

In what follows we make essential use of the structure of *real* representations of certain compact Lie groups. To make our arguments self-contained we collect here those properties that are of importance for us. If K, H are compact Lie groups, by a K–subgroup of H we mean a closed subgroup of H which is the image of an imbedding $K \hookrightarrow H$.

Let G be a connected compact Lie group. By a real representation of G we mean an action of G by orthogonal linear transformations in a real euclidean space V or some $\mathbf{R}^n$ with the usual scalar product. It may also be viewed as a morphism of G into $\mathrm{SO}(n)$. If such a representation L is irreducible, the commutant of L, namely the $\mathbf{R}$–algebra of endomorphisms of V commuting with L, is a division algebra and so is one of $\mathbf{R}$, $\mathbf{C}$ (as a $\mathbf{R}$–algebra), $\mathbf{H}$, the quaternion algebra. Accordingly we shall say L is of type $\mathbf{R}, \mathbf{C}, \mathbf{H}$. The complexification of a real irreducible representation L is already irreducible if L is of type $\mathbf{R}$, and splits over $\mathbf{C}$ as $M \oplus \overline{M}$ where M is irreducible and $\overline{M}$ is the conjugate representation to M, in the other cases. If L is of type $\mathbf{C}$, M and $\overline{M}$ are not equivalent, while for L of type $\mathbf{H}$, M and $\overline{M}$ are equivalent. If $L_i (i = 1, 2)$ are two real irreducible representations of type $\mathbf{R}$, they are unitarily equivalent over $\mathbf{C}$ if and only if they are orthogonally equivalent over $\mathbf{R}$.

We shall list now the irreducible real representations of dimension ≤ 8 of various groups of importance for us. The statements below are easily proved using the standard theory of representations. We shall write $\mathbf{1}$ for the trivial representation. An irreducible representation of dimension k is denoted by $\mathbf{k}$. Generally this notation will be unambiguous in the context in which it is used. By $\overline{\mathbf{k}}$ is meant the conjugate of the representation $\mathbf{k}$ when it is not equivalent to $\mathbf{k}$.

Spin(8): The three fundamental representations attached to the three extreme nodes of the Dynkin diagram–the vector representation $\mathbf{8}$ and the two spin representations $\mathbf{8}_1$, $\mathbf{8}_2$–are all of dimension 8 and are of type $\mathbf{R}$; for the $\mathbf{8}_i$ this is due to the fact that the signature of the quadratic form on $\mathbf{R}^8$ is $\equiv 0 \bmod 8$ [**2**]. All nontrivial irreducible representations (real or complex) have dimension ≥ 8. Therefore there is no nontrivial real representation in dimension < 8, and in dimension 8 a real nontrivial representation is either the vector or one of the two spin representations.

Spin(7): The vector representation is denoted by $\mathbf{7}$. The spin representation is of dimension 8 and is denoted by $\mathbf{8}$; both are of type $\mathbf{R}$, and for $\mathbf{8}$ this follows from the fact that the signature is $\equiv 7 \bmod 8$ [**2**]. The fundamental representations of Spin (7) over $\mathbf{C}$ are $\mathbf{7}$, $\mathbf{8}$, and $\mathbf{21}$, the adjoint representation. So the only real irreducible representations of Spin(7) of dimension ≤ 8 are $\mathbf{1}$, $\mathbf{7}$, $\mathbf{8}$, and all are of type $\mathbf{R}$. $\mathbf{21}$ is also of type $\mathbf{R}$, while $\mathbf{1}$ and $\mathbf{7}$ descend to $\mathrm{SO}(7)$.

SO(7): Since **8** is a faithful representation of Spin(7), the only real irreducible representations of SO(7) of dimension ≤ 8 are **1**, **7**.

Spin (6) $\simeq$ SU(4): The fundamentals are **4**, **6**, $\overline{\mathbf{4}}$. The representations **4** and $\overline{\mathbf{4}}$ are the spin representations; they are not real and do not descend to SO(6), while **6**, the vector representation of SO(6), is real and of type **R**. All irreducibles over **C** other than the fundamentals are of dimension ≥ 10. So the real irreducible representations of dimension ≤ 8 are **1**, **6**, $\mathbf{4} \oplus \overline{\mathbf{4}}$, of types $\mathbf{R}, \mathbf{R}, \mathbf{C}$ respectively.

SO(6): Since $\mathbf{4} \oplus \overline{\mathbf{4}}$ is faithful on Spin(6), for SO(6) the only real irreducibles of dimension ≤ 8 are **1**, **6**.

Ad(SO(6)): Since **6** is faithful on SO(6), **1** is the only real irreducible of dimension ≤ 8.

$\mathbf{G_2}$: The fundamental representations are **7**, **14**. The representation **7** is thus self–dual and so admits an invariant nondegenerate bilinear form which is either symmetric or anti-symmetric. But the antisymmetric case cannot occur as the dimension is odd. So we have a nontrivial morphism of the complex group corresponding to G_2 into SO(7,**C**). This morphism maps G_2 into a maximal compact of SO(7,**C**) which is conjugate to SO(7). By the minimality of the dimension, **7** has to be of type **R**. The real irreducibles of G_2 of dimension ≤ 8 are thus **1**, **7**. G_2 has trivial center and so there is no other group with the same Lie algebra.

$\mathbf{B_2} \simeq$ **Spin(5):** The fundamentals are **4**, **5**. **4** is the spin representation and is not real but is conjugate to itself; **5**, the vector representation, is real and of type **R**. All other irreducibles have dimension ≥ 10. The real irreducibles of Spin(5) of dimension ≤ 8 are thus **1**, **5**, $\mathbf{4} \oplus \mathbf{4}$, of type $\mathbf{R}, \mathbf{R}, \mathbf{H}$ respectively.

SO(5): As before, the real irreducibles of SO(5) of dimension ≤ 8 are **1** and **5**, both of type **R**.

$\mathbf{A_2} \simeq$ **SU(3):** The fundamentals are **3** and $\overline{\mathbf{3}}$. The adjoint representation is real and of type **R** and has dimension 8; we denote it by **8**. In addition there is a pair **6** and $\overline{\mathbf{6}}$ of irreducibles of dimension 6. The real irreducibles of dimension ≤ 8 are **1**, **8**, $\mathbf{3} \oplus \overline{\mathbf{3}}$.

Ad(SU(3)): For Ad(SU(3)) the real irreducibles of dimension ≤ 8 are **1**, **8**.

Lemma 1. *Let f_1, f_2 be two imbeddings of a compact connected Lie group K into a compact Lie group H. Suppose that all automorphisms of K are inner. Then f_1 and f_2 are conjugate by an element of H if and only if their images in H are conjugate as subgroups of H. In particular this is true for $K = G_2$,* Spin(7), SO(7).

Proof. The only nontrivial point is to show that if $f_1(K)$ and $f_2(K)$ are conjugate then f_1 and f_2 are conjugate. We may assume that $f_1(K) = f_2(K)$ and write S for this common image. Then S is a closed Lie subgroup of H and the f_i are isomorphisms of K with S. Hence $f_2^{-1}f_1$ is an automorphism of K and so must be inner, say $f_2^{-1}(f_1(x)) = axa^{-1}(x \in K)$ for some $a \in K$. This gives $f_1(x) = f_2(a)f_2(x)f_2(a)^{-1}$ for all $x \in K$.

Lemma 2. *There is no imbedding of* Spin(7) *into the adjoint group of* SO(8).

Proof. Let H be the adjoint group of SO(8) and K be a closed subgroup isomorphic to Spin(7). Let $\mathfrak{k}, \mathfrak{h}$ be the Lie algebras of K, H respectively. The action of K on $\mathfrak{h}$ splits over $\mathbf{C}$ into irreducible components, and as K acts faithfully, its nontrivial central element must act as -1 in one of the components and so that component must have dimension ≥ 8, the dimension of the spin representation of Spin(7). But the action of K on $\mathfrak{k}$ is the adjoint representation of Spin(7) and so is irreducible over $\mathbf{C}$ and has dimension 21. As the dimension of $\mathfrak{h}$ is 28 there is no room for an irreducible component of dimension ≥ 8, a contradiction.

Theorem 3. *The* Spin(7)*-subgroups of* SO(8) *form exactly* 2 *conjugacy classes. If* $\Sigma_i (i = 1, 2)$ *denote these, then any outer automorphism of* SO(8) *maps* Σ_1 *to* Σ_2*. Finally, any* Spin(7)*-subgroup of* SO(8) *contains* -1 *and* -1 *is the nontrivial central element of that subgroup.*

Proof. The spin representation **8** of Spin(7) may be viewed as an imbedding of Spin(7) into SO(8). Let u be an element of O(8) of determinant -1 and let f be an imbedding of Spin(7) into SO(8); write $f' = ufu^{-1}$. We claim that f cannot be conjugate to f'. If f and f' are conjugate there is x in SO(8) such that $f = yfy^{-1}$ where $y = xu$. This means that y commutes with the image of f and hence that y is a scalar, hence ± 1, and hence $\det(y) = (\pm 1)^8 = 1$. This is a contradiction. To complete the proof of the first statement we must show that if g is any imbedding of Spin(7) into SO(8), g is conjugate either to f or to f'. But from the list of real representations of Spin(7) discussed above we conclude that g must be the spin representation, hence is equivalent to f, and so $g = vfv^{-1}$ where $v \in \mathrm{O}(8)$. If v is in SO(8), then g is conjugate to f; otherwise, it is conjugate to f'. For the last statement, if a Spin(7)–subgroup of SO(8) does not contain -1, it imbeds into the adjoint group of SO(8) which contradicts Lemma 2.

Lemma 4. *Any two* SO(7)*–subgroups of* SO(8) *are conjugate.*

Proof. Let S be an SO(7) sitting inside SO(8). The action of S on $\mathbf{R}^8$ must split as the direct sum of **1** and **7**. Hence S must fix a unit vector which we may move by SO(8) to u_0 where $u_i (0 \leq i \leq 7)$ is the standard basis of $\mathbf{R}^8$. So S must be conjugate to the SO(7) that fixes u_0.

We now come to imbeddings of Spin(7) into Spin(8). We begin with some remarks on Spin(8). Let Z be the center of Spin(8). Then Z is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and so we can write $Z = \{1, e_0, e_1, e_2\}$ where $e_i^2 = 1$ and $e_i = e_j e_k$ where ijk is any permutation of 012. If $\sigma_i (i = 0, 1, 2)$ are the vector and spin representations of Spin(8) (all of dimension 8), we can arrange the notation so that the kernel of σ_i is $\{1, e_i\}$. The representations σ_i may be viewed as morphisms of Spin(8) *onto* SO(8) with $\mathbf{R}\cdot 1$ as their commutants. The inner automorphisms of Spin(8) fix each element of Z but the outer automorphisms permute the e_i. An automorphism is inner if and only if it fixes the e_i; indeed, if it fixes each e_i, it preserves each of the σ_i, and so induces the identity automorphism on the Dynkin diagram, and so must be inner. The group Aut(Spin(8))/Spin(8) is isomorphic to the permutation group of $\{e_0, e_1, e_2\}$.

Theorem 5. *There exist* Spin(7)*–subgroups of* Spin(8). *Each of these contains exactly one of the e_i. The subgroups that contain e_i form a single conjugacy classes Σ_i. If α is any automorphism of* Spin(8) *that moves e_i to e_j, then it takes Σ_i to Σ_j. In particular, there are exactly* 3 *conjugacy classes of* Spin(7)*–subgroups of* Spin(8) *and* Aut(Spin(8))/Spin(8) *acts transitively on them.*

Proof. If g is a morphism of Spin(7) into SO(8), it can be lifted to a morphism f of Spin(7) into Spin(8) since Spin(8) is the universal covering group of SO(8). If g is already an imbedding, it is immediate that f is also an imbedding. Since imbeddings of Spin(7) into SO(8) exist by Theorem 3, it is clear that imbeddings of Spin(7) into Spin(8) also exist.

If S is a Spin(7)–subgroup of Spin(8) that does not contain any of the e_i, we have $S \cap Z = \{1\}$. Hence S imbeds as a Spin(7)–subgroup of the adjoint group of Spin(8), contradicting Lemma 2. S must therefore contain at least one e_i. But if it contains two, it contains all of Z. This cannot be true since the center of Spin(7) contains only 2 elements. Since inner automorphisms of Spin(8) fix the e_i, Spin(7)–subgroups that contain different e_i cannot be conjugate.

Fix i and let $S_j (j = 1, 2)$ be two Spin(7)–subgroups of Spin(8) that contain e_i. We claim that S_1 and S_2 are conjugate. We have remarked that we may view the representation σ_i as a morphism of Spin(8) onto SO(8) with kernel $\{1, e_i\}$. Then $\sigma_i(S_j)(j = 1, 2)$ are two SO(7)–subgroups of SO(8) and so are conjugate by Lemma 4, say by an element x' of SO(8). Let $x \in$ Spin(8) be above x'. Since $S_j = \sigma_i^{-1}(\sigma_i(S_j))$, S_1 and S_2 are conjugate by x. This finishes the proof.

Definition. Two Spin(7)–subgroups in Spin(8) or SO(8) are called **like** if they belong to the same conjugacy class; otherwise they are called **unlike.**

2. Conjugacy of G_2 and D_3–subgroups inside Spin(7). Our goal is to study the intersection properties of Spin(7)–subgroups inside SO(8) and Spin(8). This will need some preparation, in fact a study of conjugacy classes of subgroups of Spin(7) which are isomorphic to G_2 or D_3. Here by D_3 we mean any compact connected Lie group whose Lie algebra is isomorphic to $\mathfrak{so}(6)$. i.e., one of Spin(6), SO(6), or Ad(SO(6))=SO(6)/$\{\pm 1\}$.

Lemma 1. *There exist imbeddings of* Spin(6) *in* Spin(7), *and of* G_2 *in* Spin(7) *and* SO(7).

Proof. From our remarks at the beginning of §1 on the construction of the spin groups based on the theory of the Clifford algebras we see that we have imbeddings Spin(m) $\hookrightarrow$ Spin(n) if $3 \leq m < n$. To obtain imbeddings of G_2 in Spin(7) and SO(7) we argue as follows.

First of all the adjoint group of G_2 is already simply connected and so we may use the symbol G_2 to refer to the unique compact group which has trivial center, simply connected, and has the corresponding Dynkin diagram. The irreducible representation **7** is real and of type **R** as we have seen earlier, and so may be viewed as a morphism of G_2 into SO(7). Since G_2 has trivial center, this morphism is an imbedding. It lifts to an imbedding of G_2 in Spin(7).

Lemma 2. *Let H be* Spin(7) *or* SO(7) *and let K be a proper closed connected subgroup of H. Assume that* $\dim(K) \geq 14$. *Then K is either G_2 or D_3. Both* *possibilities exist and in either case K is maximal among closed proper connected subgroups of H.*

Proof. The rank of K is ≤ 3. By classification we see that the only possibilities for K with dimension ≥ 14 and < 21 are $G_2, G_2{\cdot}T$ where T is a circle group commuting with G_2, and D_3. In view of Lemma 1 it is enough to exclude $G_2{\cdot}T$.

Let $K = G{\cdot}T$ where T is a circle group commuting with $G \simeq G_2$. If $H = \mathrm{SO}(7)$, the action of G_2 on $\mathbf{R}^7$ is nontrivial and so G already acts as $\mathbf{7}$. Thus T acts trivially. Hence $T = 1$, a contradiction. If $H = \mathrm{Spin}(7)$, we use the spin representation of Spin(7) in $\mathbf{R}^8$. The action of G is $\mathbf{1} \oplus \mathbf{7}$ and so T acts trivially in each of the two components. Hence $T = 1$ again, a contradiction.

For the maximality we need only check that we cannot have $G_2 \subset D_3$. Suppose there is such an inclusion. The Lie algebra of D_3 has an irreducible faithful complex representation of dimension 6, and its restriction to the Lie algebra of G_2 is nontrivial. This is impossible.

Theorem 3. *The action, via the spin representation, of* Spin(7) *on the unit sphere in* $\mathbf{R}^8$, *is transitive, the stabilizer of any unit vector is connected and is a* G_2*–subgroup of* Spin(7), *and all* G_2*–subgroups of* Spin(7) *are obtained in this manner. In particular, for any* G_2*–subgroup* $G \subset \mathrm{Spin}(7)$, *we have* $\mathrm{Spin}(7)/G \simeq S^7$, *and all* G_2*–subgroups in* Spin(7) *are conjugate.*

Proof. Let G be a G_2–subgroup of Spin(7). The spin representation of Spin(7) restricted to G is a faithful nontrivial representation of G in $\mathbf{R}^8$ and so splits as $\mathbf{1} \oplus \mathbf{7}$. So G fixes a unit vector u_0 in $\mathbf{R}^8$. Let H be the stabilizer of u_0 in Spin(7). Then H^0, the connected component containing the identity of H, is a closed, connected, proper subgroup of Spin(7) containing G. By Lemma 2 we must have $H^0 = G$. So $\dim(H) = \dim(H^0) = 14$. Hence $\mathrm{Spin}(7)/H$ has dimension 7, showing that Spin(7) acts transitively on the unit sphere in $\mathbf{R}^8$. Thus $\mathrm{Spin}(7)/H \simeq S^7$. If $H \neq H^0$, we would have $\mathrm{Spin}(7)/H^0$ as a nontrivial cover of $\mathrm{Spin}(7)/H \simeq S^7$ which is simply connected. Hence $H = H_0$, i.e., H is connected. Thus G is the stabilizer of u_0. If G_1 is another G_2–subgroup of Spin(7), the above argument applies to G_1 equally, and so there is a unit vector u_1 such that G_1 is the stabilizer of u_1. If $x \in \mathrm{Spin}(7)$ is such that x moves u_1 to u_0, it is clear that xG_1x^{-1} is the stabilizer of u_0. Hence $xG_1x^{-1} = G$. The theorem is completely proved.

Lemma 4. *Any two* G_2*–subgroups of* SO(7) *are conjugate.*

Proof. Let G be a G_2–subgroup of SO(7). Then G acts as **7** on $\mathbf{R}^7$. This implies that if G, G_1 are two G_2–subgroups of SO(7), there is $x \in O(7)$ such that $xGx^{-1} = G_1$. If $\det(x) = -1$, we have $\det(-x) = 1$ and so we may assume that $x \in SO(7)$.

Lemma 5. *Let H be either* SO(7) *or* Spin(7). *Let $G \subset H$ be a G_2–subgroup. If $G \subset K \subset H$ where K is a closed proper subgroup of H, then $K = G$ if $H =$* SO(7), *while K is either G or $G \cup eG$ if $H =$* Spin(7), *e being the nontrivial central element of* Spin(7). *In this case $L = G \cup eG$ is a closed subgroup of H containing G as a subgroup of index 2.*

Proof. We know already that $K^0 = G$. Suppose that $x \in K$. Since all automorphisms of G are inner we can find $y \in G$ such that $z = y^{-1}x$ centralizes G. If $H = SO(7)$, z must be ± 1 as G acts as **7**, hence $z = 1$ as it lies in SO(7). So $x = y \in G$ and therefore $K = G$ in this case. Let now $H = \mathrm{Spin}(7)$. Imbed H in SO(8) via the spin representation. We know that G splits as $\mathbf{1} \oplus \mathbf{7}$. Then z is ± 1 in each of the irreducible components and so, as $det(z) = 1$, we must have $z = \pm 1$. So $G \subset K \subset G \cup eG$. Since G has trivial center, $e \notin G$, and so $L = G \cup eG$ is a closed subgroup of H containing G as a subgroup of index 2.

We take up now the structure and conjugacy properties of D_3–subgroups in Spin(7) and SO(7).

Lemma 6. *Let H be* Spin (7) *or* SO(7) *and let K be a connected D_3–subgroup of H. If $H =$* Spin(7), *then $K =$* Spin(6) *and contains the center of* Spin(7). *If $H =$* SO(7), *then $K =$* SO(6). *If N is the normalizer of K in H (in either case), then $N^0 = K$ and $N/K \simeq \mathbf{Z}_2$. In particular, the only closed proper subgroups of H containing K are K and N.*

Proof. Suppose $K = SO(6)$ and there is a nontrivial morphism of K into Spin(7). For the action of K in $\mathbf{R}^8$ via the spin representation the only possibility is $\mathbf{6} \oplus \mathbf{1} \oplus \mathbf{1}$. So K must fix a unit vector. But then, by Theorem 3, the image of K, of dimension 15, must be contained in a G_2–subgroup of dimension 14, which is impossible. So any D_3–subgroup of Spin(7) must be a Spin(6).

Suppose that $K = \mathrm{Spin}(6) \subset SO(7)$. Then K acts as $\mathbf{1} \oplus \mathbf{6}$ on $\mathbf{R}^7$ and so $K \hookrightarrow SO(6)$, hence $K \simeq SO(6)$ which is impossible because K is simply connected and SO(6) is not. If $K = \mathrm{Ad}(SO(6))$, K acts trivially on $\mathbf{R}^7$ which is impossible. So any D_3–subgroup of SO(7) is a SO(6). This also proves that any

Spin(6)–subgroup of Spin(7) must contain the center of Spin(7) as otherwise it will imbed into SO(7).

It is enough to determine the normalizer in SO(7) as the normalizer in Spin(7) is its preimage. We may assume that K is the SO(6) fixing u_0, $u_i (0 \leq i \leq 6)$ being the standard basis of $\mathbf{R}^7$. If $n \in N$, we have $nu_0 = \pm u_0$ and n acts on U_0, the orthogonal complement of u_0. If $nu_0 = u_0$, then $n \in K$. Fix $n_0 \in$SO(7) such that $n_0 u_0 = -u_0$; then $n_0 \in N$. If $n \in N$ and $nu_0 = -u_0$, then $n_0^{-1} n \in K$. So $N = K \cup n_0 K$, showing that $N/K \simeq \mathbf{Z}_2$.

Lemma 7. *Any two* Spin(6) (resp. SO(6))*–subgroups of* Spin(7) (resp. SO(7)) *are conjugate.*

Proof. Since the Spin(6) subgroups contain the center of Spin(7) we may come down to SO(7) and consider SO(6)–subgroups of SO(7). As we argued above, any SO(6)–subgroup of SO(7) must fix a unit vector. As SO(7) acts transitively on S^6, the conjugacy of any two SO(6)–subgroups of SO(7) is clear.

Lemma 8. *There exists an imbedding of* SU(3) *in* G_2*, while there is no imbedding of the adjoint group of* SU(3) *in* G_2*.*

Proof. From the Dynkin diagram it is clear that there is a nontrivial morphism f of SU(3) into G_2. Since the real irreducibles of Ad(SU(3)) are **1** and **8**, it is clear that Ad(SU(3)) does not imbed into G_2 which has a faithful representation, namely **7**, in dimension 7. In particular, f must be an imbedding.

Lemma 9. *Let G be a G_2–subgroup and D a* Spin(6)*–subgroup of $S =$* Spin(7)*. Then* $(G \cap D)^0 \simeq$ SU(3).

Proof. We note first that if L_1, L_2 are two closed Lie subgroups of a Lie group L, then

$$\dim(L_1 \cap L_2) \geq \dim(L_1) + \dim(L_2) - \dim(L).$$

In fact, if $\mathfrak{l}, \mathfrak{l}_1, \mathfrak{l}_2$ are the Lie algebras of L, L_1, L_2 respectively, $\mathfrak{l}_1 \cap \mathfrak{l}_2$ is the Lie algebra of $L_1 \cap L_2$ and $\dim(\mathfrak{l}_1 + \mathfrak{l}_2) \leq \dim(\mathfrak{l})$. The result is immediate from

$$\dim(\mathfrak{l}_1 + \mathfrak{l}_2) + \dim(\mathfrak{l}_1 \cap \mathfrak{l}_2) = \dim(\mathfrak{l}_1) + \dim(\mathfrak{l}_2).$$

It follows from this that $\dim(G \cap D) \geq 14 + 15 - 21 = 8$. The group $H = (G \cap D)^0$ has rank ≤ 2. Since G is not contained in D, H is a proper subgroup of G and so $8 \leq \dim(H) < 14$. From classification we see that the only possibilities are $H = B_2, A_2$. Suppose $H = B_2$. Then $B_2 \subset G \cap D \subset S$. Let us now look at the image of these groups in the spin representation of S. The action of Spin(6) on $\mathbf{R}^8$ splits either as $\mathbf{4} \oplus \overline{\mathbf{4}}$ or as $\mathbf{6} \oplus \mathbf{1} \oplus \mathbf{1}$ over $\mathbf{C}$. Since Spin(6) acts faithfully, it has to be $\mathbf{4} \oplus \overline{\mathbf{4}}$. Let us look into the action of B_2 in one of these pieces of dimension 4. From the dimensions of the irreducibles of B_2 we see that B_2 must act irreducibly in both components. Hence B_2 also splits as the direct sum of two representations of dimension 4. But $B_2 \subset G$ and G splits as $\mathbf{1} \oplus \mathbf{7}$. So B_2 contains the trivial representation, a contradiction. So $H = A_2 \simeq \mathrm{SU}(3)$, by Lemma 8.

3. Intersection properties of Spin(7)–subgroups in SO(8) and Spin(8). We have the following lemma.

Lemma 1. *Let S_1, S_2 be two distinct* Spin(7)*-subgroups of* SO(8)*. If the S_i are like,*

$$(S_1 \cap S_2)^0 = \mathrm{Spin}(6).$$

Moreover $S_1 \cap S_2$ has at most 2 connected components.

Proof. It follows from the inequality

$$\dim(L_1 \cap L_2) \geq \dim(L_1) + \dim(L_2) - \dim(L).$$

proved above that $\dim(S_1 \cap S_2) \geq 21 + 21 - 28 = 14$. Since $S_1 \neq S_2$, $S_1 \cap S_2$ is a proper subgroup of S_1, and so, from Lemmas 2.2 and 2.6 we know that $G := (S_1 \cap S_2)^0$ is either a G_2 or a Spin(6) subgroup.

We claim that $G \simeq \mathrm{Spin}(6)$. Otherwise $G \simeq G_2$. Now $S_2 = yS_1y^{-1}$ for some $y \in \mathrm{SO}(8)$ and so G and $y^{-1}Gy$ are both $\subset S_1$. By Theorem 2.3 we can find $y_1 \in S_1$ such that $y_1^{-1}y^{-1}Gyy_1 = G$. Since all automorphisms of G are inner, we can find $z \in G$ such that yy_1z centralizes G. But the action of G on $\mathbf{R}^8$ splits as the direct sum $\mathbf{1} \oplus \mathbf{7}$ and so yy_1z is ± 1 in each of the two components; as it is in SO(8), $yy_1z = \pm 1$. Since $\pm 1, y_1, z$ are all in S_1, we see that $y \in S_1$. But then $S_2 = S_1$. The last statement is immediate from Lemma 2.6 since $S_1 \cap S_2$ normalizes its connected component which is $\simeq \mathrm{Spin}(6)$.

We consider next the case of unlike Spin(7)–subgroups. The argument for this case is a little more involved. We begin with:

Lemma 2. *Let $u_0 \in \mathbf{R}^8$ be a unit vector and let S be a* Spin(7)*-subgroup of* SO(8). *Let G be the G_2-subgroup which is the stabilizer of u_0 in S. Then we can find an x in* O(8) *of determinant -1 such that $xu_0 = -u_0$ and*

$$S \cap xSx^{-1} = G \cup (-1)G.$$

Proof. Let $\theta \in \mathrm{O}(8)$ be the reflection in the orthogonal complement U_0 of u_0. Then $\theta S \theta$ is also a Spin(7)–subgroup of SO(8) and $\theta G \theta$ is the stabilizer of u_0 in it. So both G and $\theta G \theta$ may be viewed as G_2–subgroups of $\mathrm{SO}(U_0)$ which is an SO(7), and hence are conjugate by Lemma 2.4. Thus there is $y \in \mathrm{SO}(8)$ which fixes u_0 such that $y\theta G\theta y^{-1} = G$. If $x = y\theta$ then $xGx^{-1} = G$ and so G is contained in $S \cap xSx^{-1}$. Since $\det(x) = -1$, Theorem 2.3 implies that S cannot be conjugate to xSx^{-1} and hence in particular $S \neq xSx^{-1}$. Thus $S \cap xSx^{-1}$ is a proper closed subgroup of S containing G and so must be contained in $G \cup (-1)G$ by Lemma 2.5. As -1 lies in both of them the intersection must be $G \cup (-1)G$.

Lemma 3. *If S_1, S_2 are two unlike* Spin(7)*-subgroups of* SO(8)*, then*

$$S_1 \cap S_2 = G \cup (-1)G, \qquad G \simeq G_2.$$

Proof. Suppose this were not true. Since $(S_1 \cap S_2)^0 \simeq \mathrm{Spin}(6)$ or G_2, it must be a Spin(6)–subgroup. We can find by Lemma 2 a Spin(7)–subgroup S_3 such that S_1 and S_3 are unlike and $(S_1 \cap S_3)^0 = G$ where $G \simeq G_2$. Clearly S_3 is different from S_2. Hence

$$\dim(S_1 \cap S_2) = 15, \ \dim(S_1 \cap S_3) = 14, \ \dim(S_2 \cap S_3) = 15$$

the last relation following from Lemma 1 as S_2 and S_3 are like.

We now claim that

$$\dim((S_1 \cap S_2 \cap S_3)^0) \geq 9.$$

Let $\mathfrak{s}_i$ be the Lie algebra of S_i. Since $\dim(\mathfrak{s}_1 \cap \mathfrak{s}_3) = 14$ it follows that $\dim(\mathfrak{s}_1 + \mathfrak{s}_3) = 28$ and so $\mathfrak{s}_1 + \mathfrak{s}_3 = \mathfrak{so}(8)$. Hence

$$\begin{aligned} 28 = \dim(\mathfrak{s}_1 + \mathfrak{s}_3 + \mathfrak{s}_2) &= 28 + 21 - \dim((\mathfrak{s}_1 + \mathfrak{s}_3) \cap \mathfrak{s}_2) \\ &\leq 28 + 21 - \dim((\mathfrak{s}_1 \cap \mathfrak{s}_2) + (\mathfrak{s}_3 \cap \mathfrak{s}_2)) \\ &= 49 - \dim(\mathfrak{s}_1 \cap \mathfrak{s}_2) - \dim(\mathfrak{s}_3 \cap \mathfrak{s}_2) + \dim(\mathfrak{s}_1 \cap \mathfrak{s}_2 \cap \mathfrak{s}_3) \\ &= 49 - 15 - 15 + \dim(\mathfrak{s}_1 \cap \mathfrak{s}_2 \cap \mathfrak{s}_3). \end{aligned}$$

Hence

$$\dim(\mathfrak{s}_1 \cap \mathfrak{s}_2 \cap \mathfrak{s}_3) \geq 9.$$

We thus have $(S_1 \cap S_3)^0 = G \simeq G_2$, $(S_2 \cap S_3)^0 = D \simeq \text{Spin }(6)$, and

$$\dim(G \cap D) \geq 9.$$

This contradicts Lemma 2.9. Thus $(S_1 \cap S_2)^0 \simeq G_2$. Since $-1 \in S_1 \cap S_2$, Lemma 2.5 finishes the proof.

We have thus proved the following theorem.

Theorem 4. *Let $S_i (i = 1, 2)$ be two* Spin(7)*-subgroups of* SO(8)*. If they are like we have*

$$(S_1 \cap S_2)^0 \simeq \text{Spin}(6), \qquad [S_1 \cap S_2 : (S_1 \cap S_2)^0] \leq 2$$

while, if they are unlike, we have

$$S_1 \cap S_2 = G \cup (-1)G \qquad G \simeq G_2.$$

It now remains to lift this result to Spin(8). We have the following theorem.

Theorem 5. *Let $S_i (i = 1, 2)$ be two distinct* Spin(7)*-subgroups of* Spin(8)*. If they are like, then*

$$S_1 \cap S_2 = \text{Spin}(6)$$

If they are unlike, then

$$S_1 \cap S_2 = G \simeq G_2.$$

Moreover, in this case, we can find a unique Spin(7)*-subgroup S_0 such that $S_i (i = 0, 1, 2)$ belong to distinct conjugacy classes and*

$$S_0 \cap S_1 \cap S_2 = G \simeq G_2.$$

If S_0' is another Spin(7)*-subgroup distinct from S_0 such that S_0, S_0' are like, we have*

$$(S_0' \cap S_1 \cap S_2)^0 = A \simeq \text{SU}(3)$$

Proof. Let the center of Spin(8) be $\{1, e_0, e_1, e_2\}$. Let σ_0 be the fundamental irreducible representation with kernel $\{1, e_0\}$. If S_i contain e_0, they map mod $\{1, e_0\}$ into two distinct SO(7)–subgroups of SO(8) whose intersection acts trivially on a plane and so is SO(6). Clearly $S_1 \cap S_2 = \sigma_0^{-1}$ (SO(6)) must be a Spin(6) group, as otherwise we will have an SO(6) inside S_1 which is impossible by Lemma 2.6. Suppose S_i are unlike. We may assume that $e_i \in S_i (i = 1, 2)$. Then σ_0 maps $S_i (i = 1, 2)$ isomorphically onto a Spin(7)–subgroup S'_i of SO(8). Since the S'_i are unlike, $(S'_1 \cap S'_2)^0 \simeq G_2$ and hence $(S_1 \cap S_2)^0 = G \simeq G_2$. But $S_1 \cap S_2 \subset S_i$ normalizes G, and hence, by Lemma 2.6, $S_1 \cap S_2 \subset (G \cup e_1 G) \cap (G \cup e_2 G)$. Since G has trivial center, it cannot contain any of $e_1, e_2, e_1 e_2$, so that $G \cap e_1 G = G \cap e_2 G = e_1 G \cap e_2 G = \emptyset$. Hence

$$S_1 \cap S_2 = G.$$

Under σ_0, S_1 maps to the Spin(7)–subgroup S'_1 of SO(8) and the image G' of G is a G_2–subgroup. It must therefore fix a unit vector u_0 so that we may view it as a subgroup of an SO(7). The preimage of this SO(7) by σ_0 in Spin(8) is a Spin(7)–subgroup S_0 of Spin(8) and $G \subset S_0$. Hence

$$S_0 \cap S_1 \cap S_2 = S_0 \cap G = G$$

Suppose now S'_0 is as in the statement of the theorem. Then

$$S'_0 \cap S_1 \cap S_2 = S'_0 \cap G = G \cap S'_0 \cap S_0$$

Since S_0, S'_0 are like, $S'_0 \cap S_0 = D$ is a Spin(6)–subgroup and

$$(S'_0 \cap S_1 \cap S_2)^0 = (G \cap D)^0$$

By Lemma 2.9, as both G and D are contained in S_0, $(G \cap D)^0 \simeq \mathrm{SU}(3)$. This finishes the proof of the theorem.

Example. It is possible that in Theorem 3.4, $S_1 \cap S_2$ for like S_i is not connected. Let C be the Clifford algebra generated by $x_i (1 \le i \le 8)$ with relations $x_i^2 = -1, x_i x_j + x_j x_i = 0$ $(i \ne j)$. The center of Spin(8) is easily seen to be $\{\pm 1, \pm x_1 x_2 \ldots x_8\}$. Let C_1, C_2 be the respective subalgebras generated by $x_i (2 \le i \le 8), x_i (1 \le i \le 8, i \ne 2)$. They are both the Clifford algebras in dimensions 7. Let S_1, S_2 be the spin groups inside C_1^0, C_2^0. Clearly $-1 \in S_i (i = 1, 2)$. The images mod $\{1, x_1 x_2 \ldots x_8\}$ of S_1 and S_2 are two like Spin(7)–subgroups

S_1', S_2' in SO(8) and $(S_1 \cap S_2)^0$ (which is $S_1 \cap S_2$ by Theorem 3.5 but we do not need this fact) maps onto $(S_1' \cap S_2')^0$. We claim that $S_1' \cap S_2'$ has 2 connected components. To verify this it is enough to exhibit elements $u_i \in S_i$ such that (1) $u_1 \equiv u_2 \bmod x_1 x_2 \ldots x_8$ and (2) u_1 does not map into $(S_1' \cap S_2')^0$, i.e., neither u_1 nor $u_1 x_1 x_2 \ldots x_8$ is in $(S_1 \cap S_2)^0$. These properties can be verified for $u_1 = x_2 x_8$ and $u_2 = x_1 x_3 x_4 \ldots x_7$. Actually, $u_1 \notin C_2$, and $u_1 x_1 x_2 \ldots x_8 \notin C_1$.

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