Non-linear σ-model — lower critical dimension

Let us conside the Landau free energy for the n-componet case

$$F = \frac{1}{2}\sum_{i=1}^{N}(\vec{\nabla}\phi_i)^2 + \frac{\alpha}{2}\sum_{i=1}^{N}\phi_i^2 + \frac{\beta}{4}\left(\sum_{i=1}^{N}\phi_i^2\right)^2 .$$

say $n=3 \rightarrow$ Heisenberg magnet.

In the low temperature case $\alpha<0$, we have

$$\sqrt{\sum_{i=1}^{N}\phi_i^2} = \sqrt{\frac{|\alpha|}{\beta}} = |\bar{\phi}|$$

As we explained before, the fluctuations of the magnitude of ϕ field is suppressed : $\chi_{\perp}(k) = \frac{1}{2|\alpha| + k^2}$, but its transverse fluctuation, or, direction of ϕ is large and powerlaw

$$\chi_{\parallel} = \frac{1}{k^2} \quad \text{— Gold-stone theorem.}$$

Now the question appears : we have already known that below $d_u = 4$, fluctuations can generate the non-trival Wilson-Fisher fixed point at $d=3$, but nevertheless it still corresponds to a transition to long-range ordered phase. But if we have a continous symmetry, such as the $O(n)$ symmetry, if d is low, shall we be able to have long-range ordering at all? Even $\vec{\phi}$ can develop an non-zero amplitude, but the transverse fluctuations could disorder the configuration. We will show that there exist a lower critical dimension $d_c' = 2$ for continuous symmetry. Thermal fluctuations forbid long-range ordering at any $T>0$. at $d \le d_c = 2$.

An evidence is the divergence of transverse fluctuations. Let us consider the $n=3$ case for example. Suppose we have a long range order along x-direction then $\chi_{\parallel}(k) = 1/k^2$. Let's ask the fluctuations along y-direction:

$$\boxed{\langle \phi_y^2(x) \rangle \sim \int_{1/L}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \sim \int_{\frac{1}{L}}^{\Lambda} k^{d-3} dk = \begin{cases} \frac{1}{d-2}\left[\Lambda^{d-2} - \left(\frac{1}{L}\right)^{d-2}\right] & (d \ge 3) \\ \ln \Lambda L & (d=2) \\ L - 1/\Lambda & (d=1) \end{cases}}$$

at $d \ge 3$, the integral is convergent in the infra-red direction. But the transverse fluctuations diverge as $L \to \infty$ in 2D (logrithmically) and 1D (linearly). This means that our assumption of long-range ordering is not self-consistent.

Actually, the above result can be exactly proved known as Mermin-Wagner theorem. It have a quantum mechanical version that at $T=0$, the 1d system with continuous symmetry cannot have long-range ordering. (There is a subtle case of ferro-magnetism in which there's no quantum fluctuation). But the ground states of 2D quantum system can possess long-range ordering.

(Quantum d-dimensional $\longrightarrow$ classical $d+1$-dimensional systems)

Now, we will also confirm through RG. Suppose that $T \ll T_{MF}$, i.e $\alpha < 0$ and large, such that we can freeze the amplitude fluctuation. For the case of $O(n)$ symmetry, we use the symbol $\vec{n}$ whose magnitude is already normalized. We write down the partition function as

$$\boxed{Z = \int Dn \; e^{-\int d^dx \, \frac{1}{2g} (\partial_\mu \vec{n})^2}}$$ with the constraint $|\vec{n}|^2 = 1$.

the unit of g is a_0^{d-2} where a_0 is the microscopic cut off, we define $g = u a_0^{d-2}$, and u is dimensionless. This model is not free because of the constraint $|\vec{n}|^2 = 1$. If $g \to 0$, the fluctuation is suppressed, which is called the weak coupling limit. If $g \to \infty$, fluctuation is enhanced, which is called the strong coupling limit. The RG flow shows that at $d \le 2$, there're only strong coupling fixed point ($g \to \infty$), which means that no-long range order can exist.

Now we do RG transformation: separate $\hat{n}$ into slow and fast degrees: (local frame are slow modes, ϕ^a are fast modes)

$$\hat{n}(x) = n^0(x) \sqrt{1 - \vec{\phi}^2(x)} + \sum_{a=1}^{n-1} \phi^a \hat{e}^a(x)$$

where $(\overset{0}{n}(x), \hat{e}^a(x))$ are local frame. define $\hat{e}^0(x) \equiv n^0(x)$
then $\hat{e}^\alpha(x) \cdot \hat{e}^\beta(x) = \delta_{\alpha\beta}$, $\alpha, \beta = 0, 1, \cdots n-1$.

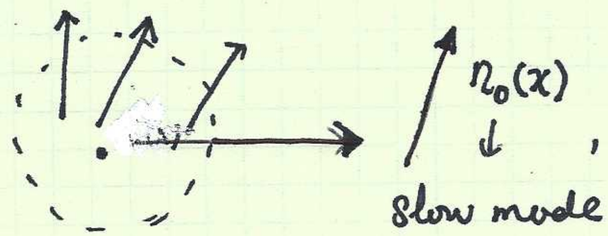
$$\partial_\mu \hat{n}^0 = \sum_a \left(\hat{e}^a \cdot \partial_\mu \hat{n}^0\right) \hat{e}^a = \sum_\alpha \tilde{A}_\mu^{a0} \hat{e}^a$$

define $\tilde{A}_\mu^{\alpha\beta} \equiv \hat{e}^\alpha \cdot \partial_\mu \hat{e}^\beta$, and thus $\tilde{A}_\mu^{\alpha\beta} = -\tilde{A}_\mu^{\beta\alpha}$

Similarly $\partial_\mu \hat{e}^\alpha = \sum_b \tilde{A}_\mu^{ba} \hat{e}^b - \tilde{A}_\mu^{a0} \hat{n}^0$

we also have $\boxed{\sum_a \left(\tilde{A}_\mu^{a0}\right)^2 = \sum_a \left(\hat{e}^a \cdot \partial_\mu \hat{n}^0\right)^2 = \left(\partial_\mu \hat{n}^0\right)^2}$

We determine an intermediate momentum scale $\hat{\Lambda} = \Lambda/\ell$,



, $\phi^a(x)$ are transverse flutuetions: fast modes

$$\partial_\mu \hat{n} = n^0(x)\, \partial_\mu \sqrt{1-\Phi^2(x)} + \sqrt{1-\Phi^2(x)} \sum_a \tilde{A}_\mu^{a0} \hat{e}^a$$

$$+ \sum_a \phi^a \Big(\sum_b \tilde{A}_\mu^{ba} \hat{e}^b - \tilde{A}_\mu^{a0} \hat{n}^0 \Big) + \sum_a \partial_\mu \phi^a \hat{e}^a$$

$\tilde{A}^{ab} = -\tilde{A}^{ba}$

$$= n^0(x) \Big[\partial_\mu \sqrt{1-\Phi^2(x)} - \sum_a \phi^a A_\mu^{a0} \Big) + \sum_a \Big[\tilde{A}_\mu^{a0} \sqrt{1-\Phi^2(x)} + \sum_b \tilde{A}_\mu^{ab} \phi^b \Big] + \partial_\mu \phi^a \Big] \hat{e}^a$$

$$F = \frac{1}{2u a_0^{d-2}} \int d^d x \Big(\partial_\mu \sqrt{1-\Phi^2(x)} - \sum_a \phi^a A_\mu^{a0} \Big)^2 + \sum_a \Big(\partial_\mu \phi^a + \sum_b \phi^b \tilde{A}_\mu^{ab} + \tilde{A}_\mu^{a0} \sqrt{1-\Phi^2(x)} \Big)^2$$

expand to ϕ's second order: $\partial_\mu \sqrt{1-\phi^2} = \partial_\mu (1 - \frac{1}{2}\phi^2) = \phi_a \partial_\mu \phi_a$

thus

$$\big(\overset{\text{neglected}}{\partial_\mu\sqrt{1-\bar\phi^2}} - \sum_a \phi^a A_\mu^{ao}\big)^2 + \sum_a \big(\partial_\mu\phi^a + \sum_b \phi^b \tilde A_\mu^{ab} + \tilde A_\mu^{ao}\sqrt{1-\bar\phi^2(x)}\big)^2$$

$$\simeq \underbrace{\big(\sum_a \phi^a A_\mu^{ao}\big)^2 + \sum_a\big(\partial_\mu\phi^a + \sum_b \phi^b \tilde A_\mu^{ab}\big)^2}_{①} + \underbrace{\sum_a \tilde A_\mu^{ao\,2}(1-\bar\phi^2(x))}_{②}$$

$$+ \underbrace{2\sum_a \partial_\mu\phi^a \tilde A_\mu^{ao} + 2\sum_{ab}\phi^b \tilde A_\mu^{ab} \tilde A_\mu^{ao}}_{\text{odd in } \phi,\ \text{average to zero with respect to fast modes}}$$

$$① + ② = \sum_a \big(\partial_\mu\phi^a + \sum_b \phi^b \tilde A_\mu^{ab}\big)^2 + \sum_{ba}\big[\phi^a\phi^b - \delta_{ab}\bar\phi^2(x)\big]\tilde A^{ao}\tilde A^{bo}$$

$$+ \sum_a \big(\tilde A_\mu^{ao}\big)^2$$

and $\sum_a \big(\tilde A^{ao}\big)^2 = \big(\partial_\mu \hat n^o\big)^2$ ← the slow variable

$$\Rightarrow Z \simeq \int_{\tilde\Lambda} D\hat n^o \exp\Big(-\int_{\tilde\Lambda} d^d x\ F(\partial_\mu \hat n^o)^2\Big)$$

$$\cdot \int_\Lambda D\phi \exp\Big[-\int_\Lambda dx^d\ F^{(2)}(\hat n; \phi)\Big]$$

where $F^{(2)} = \frac{1}{2ua_0^{d-2}}\sum_a \big(\underbrace{\partial_\mu\phi^a + \sum_b \phi^b \tilde A_\mu^{ab}}_{\text{gauge covariant}}\big)^2 + \sum_{ab}\big(\phi^a\phi^b - \delta_{ab}\bar\phi^2(x)\big)\tilde A^{ao}\tilde A^{bo}$

let us expand $F^{(2)}$

① free part $\sum_a \big(\partial_\mu\phi^a\big)^2 \cdot \frac{1}{2ua_0^{d-2}}$

② interaction with the gauge fields

$2\,\partial_\mu \phi^a \phi^b \tilde{A}_\mu^{ab} \longrightarrow \langle \partial_\mu \phi^a \phi^b \rangle = 0$ over fast field

$(\tilde{A}_\mu^{ab})^2 (\phi^b)^2 \longrightarrow$ only contains field of $\hat{e}^a$ and $\hat{e}^b$
not in the original model

$$\tilde{A}_\mu^{a0} \tilde{A}_\mu^{b0} (\phi^a\phi^b - \delta_{ab}\phi^2) \xrightarrow[a=b]{\text{average}} (\tilde{A}^{a0})^2 (\phi^{2a} - \bar{\phi}^2)$$

we only need to average to the last term

$$\left\langle e^{-(\tilde{A}_\mu^{a0})^2 \frac{1}{2ua_0^{d-2}} (\phi^{2,a} - \bar{\phi}^2)} \right\rangle = e^{-(\tilde{A}_\mu^{a0})^2 \frac{1}{2ua_0^{d-2}} \langle \phi^{2a} - \bar{\phi}^2 \rangle}$$

$$\frac{1}{2ua_0^{d-2}} \langle \phi^{a,2} - \bar{\phi}^2 \rangle = \frac{\langle \phi^{a,2} \rangle (1-(n-1))}{2ua_0^{d-2}} = (2-n) \int_{\tilde{\Lambda}<k<\Lambda} \frac{dk}{k^2}$$

please notice the free field is $\sum_a (\partial_\mu \phi^a)^2 \frac{1}{2ua_0^{d-2}}$

$$(2-n) \int_{\tilde{\Lambda}<k<\Lambda} \frac{dk}{k^2} = (2-n)\, K_d\, \Lambda^{d-2} \begin{cases} (\tilde{\Lambda}/\Lambda)^{-1} - 1 & d=1 \\ -\ln \tilde{\Lambda}/\Lambda & d=2 \\ \frac{1}{d-2}\left[1 - \left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{d-2}\right] & d=3 \end{cases}$$

$\Lambda \sim 1/a_0$

$$\equiv \frac{(2-n)}{a_0^{d-2}} K_d \Delta_d$$

In the above calculation, we have assumed the $O(n-1)$ symmetry where do average over fast mode. Since $\sum_a (\tilde{A}_\mu^{a,0})^2 = (\partial_\mu n^0(x))^2$,

we sum together:

$$F' = \int_{\tilde{\Lambda}} \left(\frac{1}{2u_0 a_0^{d-2}} - \frac{(n-2) K_d \Delta_d}{a_0^{d-2}} \right) (\partial_\mu n^0)^2\, d^d X$$

restore $\tilde{\Lambda} \to \Lambda$, $\displaystyle F' = \int_\Lambda d^dx' \left(\Lambda/\tilde{\Lambda}\right)^{d-2} \left(\frac{1}{2u_0 a_0^{d-2}} - \frac{(n-2)C\,\Delta d}{a_0^{d-2}}\right)(\partial'_\mu n)^2$

$x \to x' = \frac{\Lambda}{\tilde{\Lambda}} x$

$$= \int_\Lambda d^dx' \, \frac{1}{2a_0^{d-2}} \left[\frac{1}{u_0} - (n-2) K_d \Delta d\right] \left(\Lambda/\tilde{\Lambda}\right)^{d-2} (\partial'_\mu n^0)^2$$

$$\Rightarrow \frac{1}{u} = \left[\frac{1}{u_0} - (n-2) K_d \Delta d\right] \left(\Lambda/\tilde{\Lambda}\right)^{d-2}, \quad \text{where } \frac{\Lambda}{\tilde{\Lambda}} = l$$

$$\Delta d = \begin{cases} (\tilde{\Lambda}/\Lambda)^{-1} - 1 \to \ln l \\ -\ln \tilde{\Lambda}/\Lambda \to \ln l \\ \frac{1}{d-2}\left[1 - \left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{d-2}\right] \to \ln l \end{cases} \qquad \left(\Lambda/\tilde{\Lambda}\right)^{d-2} \simeq 1 + (d-2)\ln l$$

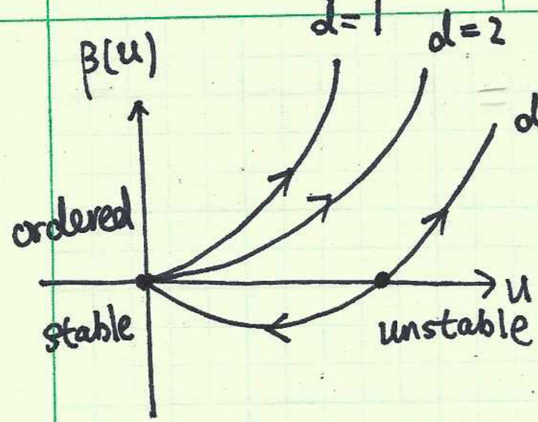
$$\Rightarrow \frac{1}{u} = \left[\frac{1}{u_0} - (n-2) K_d \ln l\right]\left[1 + (d-2)\ln l\right] = \frac{1}{u_0} + \frac{(d-2)t}{u_0} - (n-2) K_d \ln l$$

$$\Rightarrow -\frac{du}{u_0^2} = \frac{1}{u_0}(d-2) K_d \ln l - (n-2)C \cdot \ln l$$

$$\boxed{\Rightarrow \frac{du}{d\ln l} = (2-d)u + K_d(n-2)u^2}$$

$2-d$ is the naive dimension of u. Usullay in the φ^4-theory, interactions are $u\varphi^4$-term, thus the naive dimension of u is $4-d$. Now due to the constrain $|\vec{n}^2| = 1$, $\frac{1}{2g}(\partial_\mu n)^2$ is already interacting, but its dimension is $2-d$. If no constraint, g cannot be interpreated as interaction strength!

• disordered



① at $d=1,2$ and $n \geq 3$ there are only strong coupling fixed point

② at $d \geq 3$, $n \geq 3$, there is a unstable fixed point.

and there's a order-disorder transition. we do have a long-range ordering phase at low T.

③ if $n=2$, if $d \geq 3$, there's also a low T long-range order phase. High order calcalations will give rise to a up-turn of the RG curve. At large u (or large T, we will ultimatly go to discordered phase).

④ if $n=2$ and $d=2$. $\longrightarrow$ KT transition (see later lectures).

The $\beta(u)$ also provided a way to calculate the correlation length.

$$\xi[u(\Lambda), \Lambda] = \xi[u(\tilde{\Lambda}), \tilde{\Lambda}]$$

where we do $\Lambda \to \tilde{\Lambda}$, ξ doesn't change, but the ratio ξ / Λ^{-1} changes. And the change of Λ, is compensated by varing $u(\tilde{\Lambda})$.

$$\Lambda \frac{\partial}{\partial \Lambda} \xi[u(\Lambda), \Lambda] = 0 \Rightarrow \left.\frac{\partial \xi}{\partial \ln \Lambda}\right|_{\text{fix } u} + \Lambda \frac{\partial \xi}{\partial u} \cdot \frac{\partial u}{\partial \Lambda} = 0$$

dimensional analysis shows — we do not have other parameters such as r_0

$$\xi[u(\Lambda), \Lambda] \sim \Lambda^{-1} \phi(u) \quad \text{if fixed } u, \ \xi \sim \Lambda^{-1}$$

thus $\boxed{\left.\frac{\partial \xi}{\partial \ln \Lambda}\right|_{\text{fix } u} = -\xi}$, and $\Lambda \frac{\partial u}{\partial \Lambda} = -\frac{\partial u}{\partial \ln l} = -\beta(u)$

$$\Rightarrow \quad \xi(u, \Lambda) + \beta(u) \left.\frac{\partial \xi}{\partial u}(u, \Lambda)\right|_{\text{fixed } \Lambda} = 0$$

$$\Rightarrow -\frac{du}{\beta(u)} = \frac{d\xi}{\xi}$$

$$\Rightarrow \int_{\xi_0}^{\xi} \frac{d\xi}{\xi} = -\int_{u_0}^{u} \frac{du}{\beta(u)} \Rightarrow \ln\left(\frac{\xi}{\xi_0}\right) = -\int_{u_0}^{u} \frac{du}{\beta(u)}$$

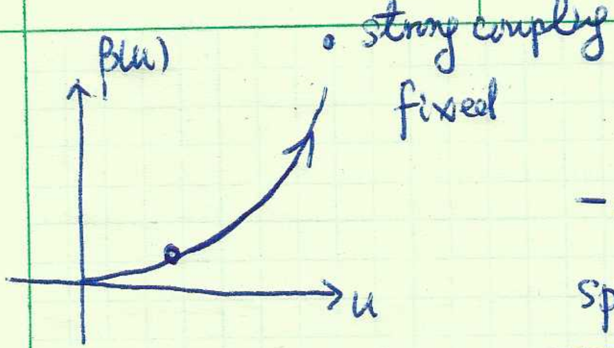
or $\boxed{\xi = \xi_0 \exp\left[-\int_{u_0}^{u} \frac{du}{\beta(u)}\right]}$ ← fix Λ.

for $d=2$: $\beta(u) = \frac{(n-2)}{2\pi} u^2$

let us take $u_0 = \infty$, and at this case $\xi_0 \simeq a_0 = 1/\Lambda$

$$\xi(u) = \xi_0 \exp\left[-\int_{+\infty}^{u} \frac{2\pi\, du}{(n-2) u^2}\right] = \xi_0\, e^{\frac{2\pi}{n-2} \int_{u_0}^{+\infty} \frac{du}{u^2}}$$

$$\boxed{\xi(u) = \xi_0\, e^{\frac{2\pi}{n-2} \frac{1}{u}}}$$



$d=2$ non-liner $O(3)$ σ moder

– Quantum spin chain with integal spin

Haldane: $$\boxed{\mathcal{L}_E = \frac{1}{2g}\left[v_s(\partial_0\vec{m})^2 + \frac{1}{v_s}(\partial_x\vec{m})^2\right] + \frac{i\theta}{8\pi}\epsilon_{ij}\vec{m}\cdot(\partial_i\vec{m}\times\partial_j\vec{m})}$$

where $g = 2/S$, $v_S = 2a_0 JS$

The second term is called θ-term.

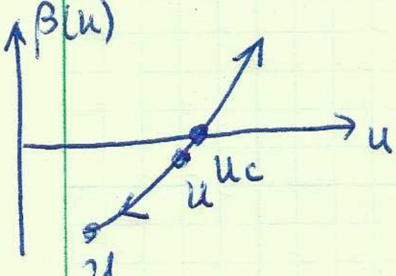
① half integer. θ-term: (-1) for odd winding number

Quantum effect! → criticality (Haldane conjecture).

② integer spin — usual non-linear σ-model

for $d=3$. $$\beta(u) = -u + \frac{u^2}{2\pi^2}, \text{ and } u_c = 2\pi^2$$

$$\simeq (u-u_c) \text{ with } u_c = 2\pi^2$$

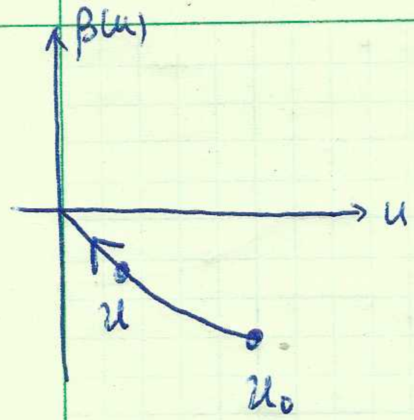


for $u \sim u_c$, we have

$$\xi(u) = \xi(u_0)\exp\left[-\int_{u_0}^{u}\frac{du}{u-u_c}\right] \text{ as } u\to u_c$$

$$= \xi(u_0)\, e^{-\ln\left|\frac{u-u_c}{u_0-u_c}\right|}$$

$$\xi(u) = \xi(u_0)\left|\frac{u-u_c}{u_0-u_c}\right|^{-1} \text{ as } u\to u_c$$



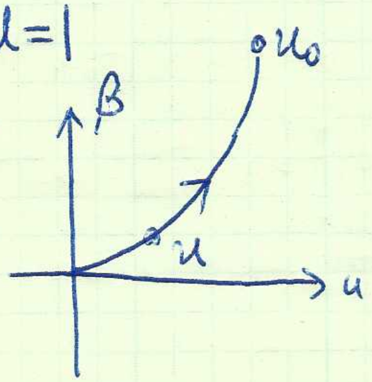
for fixed point arond $u=0$

$$\beta(u) = -u$$

$$\xi(u) = \xi(u_0)\, e^{-\int_{u_0}^{u} \frac{du}{-u}} = \xi(u_0)\, e^{\ln u/u_0}$$

$$= \xi(u_0)\frac{u}{u_0} \longrightarrow 0.$$

For $d=1$



$$\beta(u) = u$$

$$\xi(u) = \xi(u_0\to\infty)\, e^{-\int_{u_0}^{u}\frac{du}{u}}$$

$$= \xi(u_0\to\infty)\left(\frac{u_0}{u}\right)$$

$$\sim \frac{a_0}{u}$$ set $u_0 \sim 1$

Lect 8. Superfluid at finite temperatures, low dimensions

— we will use imaginary path integral

$$Z = \int D^2\varphi \ \exp\left[-\int_0^\beta d^dx\, d\tau \ \left[\frac{1}{2}(\varphi^*\partial_\tau\varphi - \varphi\partial_\tau\varphi^*) + \frac{1}{2m}\partial_x\varphi^*\partial_x\varphi - \mu|\varphi|^2 + \frac{V_0}{2}|\varphi|^4\right]\right.$$

in the time domain, we have a finite size $\beta = \frac{1}{k_B T}$, thus if the correlation length in the time domain ξ_T is much larger than β, then we can neglect the fluctuation in the τ-domain. (The relation between $\xi_T \sim \xi_0^z$, where z is called dynamic critical exponent.) In this case, we arrive at the ~~tot~~ classic partition function

— $$Z = \int D^2\varphi \ \exp\left[-\beta\int d^dx \ \left(\frac{|\varphi_0|^2}{2m}(\partial_x\theta)^2\right]\right\}$$ where only phase fluctuations & are kept.

$$\rightarrow \ S_{eff} = \int d^dx \ \frac{\eta}{2}(\partial_x\theta)^2, \quad \eta = \frac{|\varphi_0|^2}{mTk_B},$$ $T\eta$ is the phase rigidity.

Another important question is whether long range order can survive under thermal fluctuations.

Using the result in the last lecture, we have at $d \geq 3$, thermal fluctuations does not always destroys long range order.

—

For $d < 2$, thermal fluctuations always destroy superfluidity:

$$\mathcal{Z}[J] = \int D\theta \; e^{-\int d^d x \, S \; + i\int d^d x \; J(x)\,\theta(x)} \qquad \text{where } J(x) = \delta(x) - \delta(x_0)$$

$$\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle = \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \frac{1}{\mathcal{Z}(0)} \int D\theta \; \exp\Big[-\frac{1}{2}\int d^d x \, d^d x' \; \theta(x)\, G^{-1}(x,x')\, \theta(x') + i J(x)\,\theta(x) \Big]$$

$$= \exp\Big[-\frac{1}{2}\int dx\, dx' \; J(x)\, G(x,x')\, J(x') \Big], \quad \text{where } G \text{ is the inverse of } -\partial_x^2$$

$\Rightarrow$.

$$\langle \theta(x_1)\, \theta(x_2) \rangle = \int D\theta \; \theta(x_1)\theta(x_2)\, e^{-\frac{1}{2}\int d^d x \; \theta(x)\, G^{-1}(x\,x')\, \theta(x')} \Big/ \int D\theta\, e^{-\int d \cdots}$$

$$= G(x_1, x_2)$$

$$\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle = e^{-[G(0,0) - G(x,0)]} = e^{\langle \theta(x)\theta(0)\rangle - \langle \theta(0)\theta(0)\rangle}$$

$$\langle \theta(x)\,\theta(0) - \theta(0)\,\theta(0) \rangle = -\int \frac{d^d k}{(2\pi)^d} \; \frac{1 - e^{ikx}}{\eta\, k^2}$$

$$= \begin{cases} -\dfrac{1}{\eta} \Big[\dfrac{K_d \, \Lambda^{d-2}}{(d-2)} \Big] & \text{for } x \to \infty \; (d > 2) \quad \text{where } \Lambda = 1/\ell \\[2ex] \dfrac{-1}{\eta} \; \dfrac{1}{2\pi} \ln\Big[\dfrac{|x|}{\Lambda^{-1}} \Big] & \text{for } x \to \infty \; (d = 2) \text{ — the wave-vector cut off} \\[2ex] \dfrac{-1}{\eta} \; \dfrac{1}{2} |x| & \text{for } x \to \infty \; (d = 1) \end{cases}$$

at $d=1 \Rightarrow \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \sim e^{-\frac{1}{2\eta}|x|}$, which is exponentlally decaying, with decay length $\underline{2\eta}$.

at $d=2 \Rightarrow \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \sim e^{-\frac{1}{2\pi\eta} \ln \frac{|x|}{\Lambda^{-1}}} = \left(\frac{\Lambda^{-1}}{|x|}\right)^{\frac{1}{2\pi\eta}}$

which has power-law decaying, with exponent $\frac{1}{2\pi\eta}$

for $d>2$, there's no infra-red divergence.

§2. K-T tranistion.

So far we have completely neglected the compactness of θ. we will see at 2D, it gives rise to the topological excitation of vortices, and its effect to change the transition to K-T type.

The action of a single vortex: $\varphi(x,y) = f(r)\, e^{i\phi}$, $f(r) = \begin{cases} 0 & r\to 0 \\ \varphi_0 & r\to\infty \end{cases}$

$$S_v = \int d^2 r \; \frac{1}{2}\eta \frac{1}{r^2} + S_c = \eta\pi \ln\frac{L}{\ell} + S_c \leftarrow \text{core energy}$$

The interaction between two vortices with the same vorticity can be calculated by the analogy with 2D electro-statics.

$$\frac{1}{2}\eta(\nabla\theta)^2 \leftrightarrow \frac{E^2}{8\pi} \Rightarrow E = \sqrt{4\pi\eta}\; \nabla\theta = \sqrt{4\pi\eta}\; \frac{2\pi}{2\pi r}$$

$$q = \frac{2\pi r}{4\pi} E = \frac{2\pi \cdot \sqrt{4\pi\eta}}{4\pi} = \sqrt{\pi\eta}$$

$$\Rightarrow \Delta E = \sqrt{4\pi\eta} \int_\ell^r dr \frac{1}{r} \cdot \sqrt{\pi\eta} = \underline{2\pi\eta \ln\frac{r}{\ell}}$$

To calculate the partition function, we need to include the vortex configuration. For a fixed vortex configuration $\varphi_c = e^{i\theta_c}$ and other fluctuations contribute as

$$Z = \int D\delta\theta \; e^{-\int d^2x \; \frac{\eta}{2}(\partial_x(\theta_c+\delta\theta))^2} = \int D\delta\theta \; e^{-\int d^2x \; \frac{\eta}{2}(\partial_x\theta_c)^2 + \frac{\eta}{2}(\partial_x\delta\theta)^2 + \eta(\partial_x\theta_c)(\partial_x\delta\theta)}$$

$$= e^{-S_{eff}(\theta_c)} \cdot \int D\delta\theta \; e^{-\int dx \frac{\eta}{2}(\partial_x\delta\theta)^2} \quad + \int dx \; \eta \; \partial_x^2\theta_c \cdot \delta\theta \approx 0$$

$$= e^{-S_{eff}(\theta_c)} \cdot Z_0 \leftarrow \text{contribution from vortex free configuration}$$

($\because$ the vortex current is divergence free). $\nabla\cdot(j) = \partial^2\theta_c = 0$

$\uparrow$ given by the long-range interaction between vortices.

$$Z = Z_0 \sum_n \frac{1}{n!n!} \int \prod_{i=1}^{2n} d^2r_c \; e^{-2nS_c} \cdot e^{\sum_{i<j}^{2n} (2\eta\pi)\, q_i q_j \ln\frac{r_{ij}}{\ell}}$$

we must have equal number of vortices and anti-vortices, to avoid energy divergence. $q_{i,j} = \pm 1$, satisfying $\sum q_i = 0$.

Let us estimate the effective action $Z = e^{-S_{eff}}$.

$$S_{eff} \sim 2n\ln n + n\left(2\eta\pi \ln\frac{l_n}{\ell} + 2S_c\right) - 2n\ln\frac{L^2}{\ell^2}$$

$\nearrow$ from $\frac{1}{n!n!}$ $\qquad\uparrow\qquad$ $\uparrow$ integration area

$$= L^2 \cdot \frac{2}{l_n^2}\left[(\eta\pi - 2)\ln\frac{l_n}{\ell} + S_c\right]$$

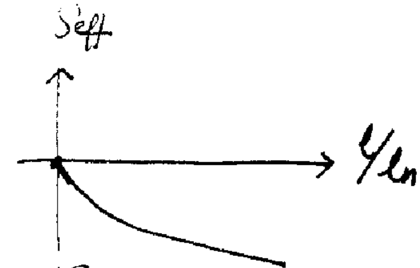
where l_n is the average inter-vortex distance.

let us plot S_{eff} v.s ~~to~~ l/l_n.

— If $S_c << -1$, thus it's cheep to creat vortices, we minimize S_{eff}

$$S_{eff} = 2\left(\frac{L}{l}\right)^2 \cdot \left(\frac{l}{l_n}\right)^{+2} \left[S_c - (\eta\pi - 2) \ln \frac{l}{l_n} \right]$$

if $\eta\pi < 2$,



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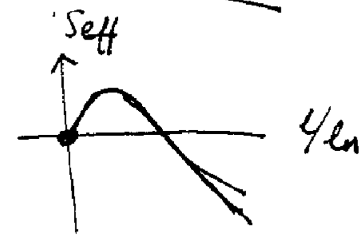
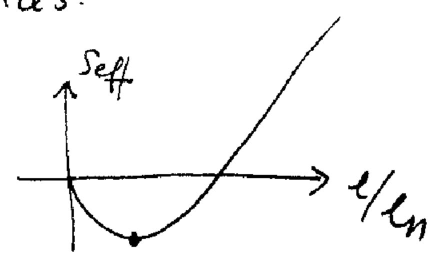


$\Rightarrow$ proliferation of vortices. $l_n \to l$

$\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \sim e^{-|x|/\xi}$, where $\xi \to l$.

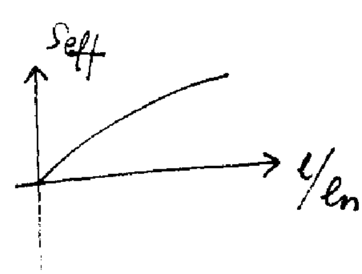
— if $S_c >> 1$, it's expensive to make vortices.



if $\eta\pi < 2$, high ~~temperature~~

the optimal $l/l_n \sim e^{-S_c/(2-\eta\pi)}$

i.e. $l_n \sim e^{\frac{S_c}{2-\eta\pi}} \cdot l$, a finite number of vortices.

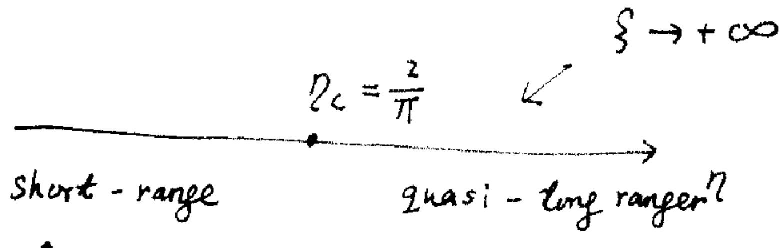


if $\eta\pi > 2$. low temperture $T < \frac{\pi}{2} \rho_s$

no free vortices.

—

Thus when vortices are expensive ($S_c \gg 1$), ~~the~~ we have a transition as increasing temperature.



$$\boxed{\xi \sim l_n \sim l\, e^{\frac{S_c}{2-\eta_c \pi}}} \qquad \text{and} \qquad \eta = \frac{\rho_s}{T} \sim \frac{2}{\pi}$$

$$T_K = \frac{\pi}{2} \rho_s .$$

naively, we would expect

$$\xi \sim l\, e^{\frac{S_c}{2 - \rho_s \pi / T}} \sim l\, e^{\frac{S_c T_K/2}{T - \frac{\pi}{2}\rho_s}} \propto l\, e^{\frac{E_s}{T - T_K}}.$$

but as $T \to T_K$, ρ_s changes. actually, $2 - \eta_c^* \pi \propto (T - T_{TK})^{1/2}$.

and $\xi(T) \sim l\, e^{\left(\frac{E_s}{T - T_{KT}}\right)^{1/2}}$ (we will analysis it later).

§ Renormalization group & Scaling dimensions

Relevant perterbation can change the long distance behavior of the system. How to decide a perterbation is relevant or not can be learned from the scaling dimension analysis.

Say, at $e^{-S_c} \ll 1$, i.e. it seems that vortex is costly. But we know at $\eta\pi < 2$, now matter how small e^{-S_c} is, vortex will destory the algebric correlation of $\langle e^{i\theta(x)} e^{i\theta(y)} \rangle$. Then it's a relevant perturbation. On the other hand, at $\eta\pi > 2$, vortex is an irrelevant perturbation.

Let us consider a theory of $S = S_0 + \int d^d\left(\frac{x}{\ell}\right) g\, O(x)$, in which

$$\langle O(x) O(y) \rangle \sim \frac{1}{\left|\frac{x-y}{\ell}\right|^{2\Delta}}$$, where ℓ is the short ~~energy~~ length scale.

At second order perturbation

$$Z = \int D\varphi \; e^{-S_0 + g\int d^d\left(\frac{x}{\ell}\right) O(x)} = \int D\varphi \; e^{-S_0} \left(1 + g\int d^d\left(\frac{x}{\ell}\right) O(x) + \frac{g^2}{2}\int d\frac{x}{\ell}\, d\frac{y}{\ell} \langle O(x) O(y) \rangle \ldots\right.$$

if $\langle O(x) \rangle = 0$

$$= \int D\varphi \; e^{-S_0} \; e^{\frac{g^2}{2}\int \frac{dx}{\ell}\frac{dy}{\ell} \langle O(x) O(y) \rangle}$$

$\Rightarrow$

$$\Delta S_{eff} = -\ln Z + \ln Z_0 = \ln\left[\frac{g^2}{2} \int \frac{d^d x}{\ell^d} \int \frac{d^d y}{\ell^d} \langle O(x) O(y) \rangle\right]$$

$$= -2\ln g - \ln \int \frac{d^d R}{\ell^d} \cdot \int \frac{d^d r}{\ell^d} \frac{1}{(r/\ell)^{2\Delta}}$$

$$= -2\ln g - \left[\ln\left(\frac{L}{\ell}\right)^d + \ln\left(\frac{L}{\ell}\right)^{d-2\Delta}\right] = -2\ln g - 2(d-\Delta)\ln\frac{L}{\ell}$$

if $\Delta < d$ and $L > \xi = g^{\frac{-1}{d-\Delta}}$ $\Rightarrow$ $\Delta S_{eff} < 0$, the system prefers to have two "O"-operators appear at short-distance (at the order of ξ). Thus if we are interested at long distance behavior, perturbation of O is always ~~important~~. On the other hand if $\Delta > d$, it's irrelavant for long range correlations.

§ The duality between 2D xy-model and the 2D clock model

Let us conside a generic clock model

$$S = \int d^2x \left[\frac{K}{2} (\partial_x \varphi)^2 - g \cos(n\varphi)\right]$$

$\mathbb{Z}_n$ - symmetry
$\varphi \to \varphi + \frac{2\pi}{n}$.

hear φ is non-compact.

say, at $n=1$.

$$Z = \int D\varphi \; e^{-\int d^2x \frac{K}{2} (\partial_x\varphi)^2} \; e^{-g\int \cos\varphi \, dx}$$

$$= \int D\varphi \; e^{-\int d^2x \frac{K}{2} (\partial_x\varphi)^2} \cdot \sum_k \frac{g^k}{k!} \left[\int d^2x \; \frac{e^{i\theta} + e^{-i\theta}}{2}\right]^k$$

$$= Z_0 \sum_k \frac{1}{k!k!} \int \prod_{i=1}^{2k} \underbrace{\left\langle e^{+i\theta(r_1)} e^{i\theta(r_2)} \cdots e^{i\theta(r_k)} \, e^{-i\theta(r_{k+1})} \cdots e^{-i\theta(r_{2k})} \right\rangle}_{\left(\frac{g}{2}\right)^{2k}}$$

how to calculate $\left\langle e^{i\left(\sum_{i=1}^{k} \theta(r_i) - \sum_{i=k+1}^{2k} \theta(r_i)\right)} \right\rangle$.

Again, we introduce $\overbrace{\text{field}}^{\text{source}}$ $J(x) = \sum_{i=1}^{k} \delta(x-r_i) - \sum_{i=k+1}^{2k} \delta(x-r_i)$

$$\left\langle e^{i\left(\sum_{i=1}^{k} \theta(r_i) - \sum_{i=k+1}^{2k} \theta(r_i)\right)} \right\rangle = \exp\left[-\frac{1}{2}\int dx\,dx'\, J(x)\, G(x-x')\, J(x')\right],$$

where G is the inverse of $-K\partial_x^2$

$$= \exp\left[-\sum_{i<j} q_i q_j \langle \theta(r_i)\theta(r_j)\rangle - \frac{1}{2}\sum_{i=1}^{2K} q_i^2 \langle\theta(r_i)\,\theta(r_i)\rangle\right]$$

$$= \exp\left[+\sum_{i<j} q_i q_j \frac{1}{2\pi K} \ln\frac{|r_i - r_j|}{L}\right] \cdot \exp\left[+\frac{1}{2}\underset{\times\frac{1}{2\pi K}}{\sum_{i=1}^{2K}} q_i^2 \ln\frac{\ell}{L}\right]$$

$$= \exp\left[+\sum_{i<j} q_i q_j \frac{1}{2\pi K}\ln\frac{|r_i - r_j|}{\ell}\right] \qquad \exp\left[-\underset{\frac{1}{2\pi K}}{\frac{1}{2}}\left(\sum_{i=1}^{2K} q_i\right)^2 \ln\frac{\ell}{L}\right]$$

$$= \left[\frac{|r_i - r_j|}{\ell}\right]^{+\frac{q_i q_j}{2\pi K}}$$

$$\Rightarrow Z = Z_0 \sum_k \frac{1}{k!\,k!}\int \prod_{i=1}^{2K} e^{-2kS_c} \cdot e^{\sum_{i<j}^{2K} \frac{q_i q_j}{2\pi K}\ln\frac{r_{ij}}{\ell}}$$

which is the same as vortex partition function, if

$$\boxed{e^{-S_c} = g/2, \quad \text{and} \quad 2\eta\pi = \frac{1}{2\pi K}.}$$

Then the vortex in the xy model, maps into $e^{i\varphi(x)}$ operator

non-local object   local

in the cola k model.

(dual representation).

xy model $\xleftrightarrow{\text{dual to}}$ Clock model

$$S = \int \frac{\eta}{2} (\partial_x \theta)^2 \, dx^2 \qquad \boxed{\begin{array}{c} 2\eta\pi = \frac{1}{2\pi K} \\ e^{-S_c} = g/2 \end{array}} \qquad S' = \int dx^2 \, \frac{K}{2} (\partial_x \varphi)^2 + g \cos\varphi$$

compact, allow vortex $(\theta + 2\pi = \theta)$ — non-compact, no vortex $(\varphi + 2\pi \neq \varphi)$

$U(1)$ symmetry. no vertex operator — vertex operator $e^{\pm i\varphi}$

non-local excitation : vortex $\longleftrightarrow$ vertex operator (local operator)

how about compact clock model

$$S = \int \frac{\eta}{2} (\partial_x \theta) + g \cos\theta \quad \longleftrightarrow \quad S = \int \frac{K}{2} (\partial_x \varphi) + g \cos\varphi$$

φ is also compact

$g\cos\theta$: vertex of the $\underbrace{\text{field}}_{\text{dual}}$ φ.

$g\cos\varphi$: vortex fluctuating in the original xy model

Lect 9: RG analysis to K-T transition

Let us consider the dual theory of XY model: the non-compact clock model

$$S = \int d^2x \; \frac{K}{2} (\partial_x \varphi)^2 + g \cos n\varphi$$

where we set "n" to a general number.

as we know $\langle e^{in\theta(x)} e^{-in\theta(0)} \rangle = \left(\frac{\ell}{|x|} \right)^{\frac{n^2}{2\pi K}}$ ⇒ the value of the correlation function depends on the short length scale! In order to have a well-defined field theory, we must explicitly specify a short length scale! i.e. we would like to write

$$S = \int d^2x \; \frac{K_\ell}{2} (\partial_x \theta_\ell)^2 - g_\ell \cos \theta_\ell, \quad \text{where } \theta_\ell(x) = \int_{|k| < \frac{2\pi}{\ell}} d^2k \; \theta_k e^{i\vec{k}\vec{x}}$$

we have three parameters K_ℓ, g_ℓ and "ℓ", and we can make two dimensionless parameters K_ℓ, and $\bar{g}_\ell = g_\ell \ell^2$.

We would think there are two different phases: ~~At~~ ~~large~~ small K and small $\bar{g}_\ell$, fluctuations are large and pinning potentials are weak, we have a Z_n-symmetric state. On the other hand, if K_ℓ ~~are~~ $\bar{g}_\ell$ large, pinning potentials are strong and fluctuations are weak, we have a symmetry breaking ground state.

Let us take $\cos n\varphi$ term as perturbation. As we explained before the scaling dimension $\Delta = \frac{n^2}{4\pi X_\ell}$, thus at $\frac{n^2}{4\pi X_\ell} > 2$ (i.e $X_\ell < \frac{n^2}{8\pi}$), the clock term is irrelevant; at $\frac{n^2}{4\pi X_\ell} < 2$ (i.e. $X_\ell > \frac{n^2}{8\pi}$), the clock term is relevant. Thus $X_\ell \gtrless \frac{n^2}{8\pi}$, we suppose to have two different phases. We'll use RG to confirm it.

let us change the short range cut off ~~$X \to X' = X + \delta X$~~ $\ell \to \ell' = \ell + \Delta\ell$, and we will see how these coupling constant changes. We seperate the fast and slow modes

$\theta_\ell = \theta_{\ell'} + \delta\theta$, where $\delta\theta$ is the fast mode, containing modes with wavelengths between ℓ and $\ell + \Delta\ell$.

$$(\partial_x\theta_\ell)^2 = (\partial_x\theta_{\ell'})^2 + (\partial_x\delta\theta)^2 \quad \text{(no mixing after integration } \int d^2x)$$

$$\cos n\theta_\ell = \cos(n\theta_{\ell'} + n\delta\theta) = \cos n\theta_{\ell'} - n\sin n\theta_{\ell'}\,\delta\theta - \frac{n^2}{2}\cos n\theta_{\ell'}(\delta\theta)^2$$

$$\Rightarrow S = \int d^2x \;\frac{X_\ell}{2}(\partial_x\theta_{\ell'})^2 - g_\ell \cos n\theta_{\ell'} + \frac{X_\ell}{2}(\partial_x\delta\theta)^2$$

$$\int d^2x \; + n g_\ell \sin n\theta_{\ell'}\,\delta\theta + \frac{n^2}{2} g_\ell \cos n\theta_{\ell'}(\delta\theta)^2$$

we will treat $\theta_{\ell'}$ as background field and integrate out the fast field of $\delta\theta$.

This will generate a number of terms such as $(\partial_x \theta_{\ell'})^2$, $\cos 2n\theta_{\ell'} (\partial_x \theta_{\ell'})^2$, $\cos 2n\theta_{\ell'}$, $\cdots$, but we only look at the term that we've already had.

$$+\int d^2x\, d^2y\ \frac{n^2 g_\ell^2}{8} (\partial_x\theta_{\ell'})^2 \ \langle n\delta\theta(x)\, n\delta\theta(y)\rangle \cdot (\vec{x} - \vec{y})^2$$

$$\Rightarrow \quad \chi_{\ell+\delta\ell} = \chi_\ell + \frac{n^2}{8} g_\ell^2\, K_2 , \quad \text{where } K_2 = \int d^2x\ |x|^2 \cdot \langle n\delta\theta(x)\, n\delta\theta(0)\rangle$$

$$K_2 = \int_{\frac{2\pi}{\ell+\delta\ell}<|k|<\frac{2\pi}{\ell}} \frac{d^2k}{(2\pi)^2} \int d^2x\ \frac{n^2 x^2}{\chi_\ell k^2}\, e^{ikx\cdot\cos\theta - 0^+|x|}$$

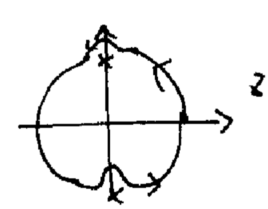
$$= \int d^2x\ \frac{n^2}{\chi_\ell}\cdot x^2 \int_0^{2\pi} d\theta\, e^{i\cdot\frac{2\pi}{\ell} x\cos\theta - 0^+ x} \cdot \frac{1}{(2\pi)^2}\ln\left(\frac{\ell+\delta\ell}{\ell}\right)$$

$$= \frac{n^2}{\chi_\ell}\frac{1}{4\pi^2}\frac{\Delta\ell}{\ell}\cdot\int_0^{2\pi} d\theta \int x^3 dx\ e^{i\frac{2\pi}{\ell}(\cos\theta + i0^+)x}\cdot 2\pi$$

$$= \frac{n^2}{\chi_\ell}\cdot\frac{1}{2\pi}\frac{\Delta\ell}{\ell}\int_0^{2\pi} d\theta\cdot\frac{1}{\left(\frac{2\pi}{\ell}\right)^4(\cos\theta+i0^+)^4}\int_0^{+\infty} x'^3\, dx'\ e^{ix'(1+i0^+)}$$

$$= \frac{\Delta\ell}{\ell}\frac{n^2}{\chi_\ell}\frac{\ell^4}{32\pi^5}\int_0^{2\pi} d\theta\ \frac{1}{(\cos\theta+i0^+)^4}\cdot 6 = \frac{\Delta\ell}{\ell}\ \frac{3n^2\ell^4}{16\pi^5}\int d\theta\ \frac{1}{(\cos\theta+i0^+)^4}$$

$$= \frac{\Delta\ell}{\ell}\ \frac{3n^2\ell^4}{2\pi^4\chi_\ell}$$



$\cos\theta \to \frac{z+z^{-1}}{2}$

$$\Rightarrow \boxed{\frac{d\chi_\ell}{d\ln\ell} = \frac{3n^4(g_\ell a^2)^2}{16\pi^4\chi_\ell}}$$

$$Z = \int D\theta_{\ell'}\, D\delta\theta \; e^{-\int d^2x \left[\frac{K_\ell}{2}(\partial_x\theta_{\ell'})^2 - g_\ell \cos\theta_{\ell'}\right]} \cdot e^{-\int d^2x \frac{K_\ell}{2}(\partial_x \delta\theta)^2}$$

$$\cdot \exp\left(\int d^2x \; - n g_\ell \sin n\theta_{\ell'}\, \delta\theta - \frac{n^2}{2} g_\ell \cos n\theta_{\ell'} (\delta\theta)^2\right]$$

$$= \int D\theta_{\ell'}\; e^{-\int d^2x \left(\frac{K_\ell}{2}(\partial_x\theta_{\ell'})^2 - g_\ell\cos\theta_{\ell'}\right)} \; e^{-\int d^2x \frac{n^2}{2} g_\ell \cos n\theta_{\ell'} \langle(\delta\theta)^2\rangle + \int d^2x\, d^2y \frac{1}{2} g_\ell^2 \sin n\theta_\ell(x) \sin n\theta_\ell(y) \langle \delta\theta_\ell(x)\, \delta\theta(y)\rangle}$$

$$\Rightarrow \quad \delta S = \int d^2x \; \frac{1}{2} g_\ell \cos n\theta_{\ell'} \underbrace{\langle \delta\theta(0)\, \delta\theta(0)\rangle}_{n^2}$$

$$- \int d^2x\, d^2y \; \frac{1}{2}(g_\ell)^2 \sin n\theta_{\ell'}(x) \sin n\theta_{\ell'}(y) \; \langle n\delta\theta(x)\; n\delta\theta(y)\rangle .$$

$$\langle n^2 (\delta\theta(0))^2\rangle = \int_{\frac{2\pi}{\ell+\delta\ell} < k < \frac{2\pi}{\ell}} \frac{d^2k}{(2\pi)^2} \frac{n^2}{\chi_\ell |k|^2} = \frac{n^2}{2\pi\chi_\ell} \ln\left(\frac{\ell+\delta\ell}{\ell}\right)$$

Thus the first term will change $g_\ell \to g_{\ell'} = g_\ell - \frac{1}{2} g_\ell \frac{n^2}{2\pi\chi_\ell} \ln(1+\frac{\delta\ell}{\ell})$

i.e. $\quad dg_\ell = - g_\ell \dfrac{n^2}{4\pi\chi_\ell} \cdot \dfrac{d\ell}{\ell} \quad \Rightarrow \quad \boxed{\dfrac{dg_\ell}{d\ln\ell} = - g_\ell \dfrac{n^2}{4\pi\chi_\ell}}$

$$\downarrow$$

$$\boxed{\frac{d(g_\ell \ell^2)}{d\ln\ell} = \left(2 - \frac{n^2}{4\pi\chi_\ell}\right)(g_\ell \ell^2)}$$

The second term can be represented as

$$\int d^2x\, dy^2 \; \frac{g_\ell^2}{4} \left[\sin n\theta_{\ell'}(x) - \sin n\theta_{\ell'}(y)\right]^2 \langle n\delta\theta(x)\, n\delta\theta(y)\rangle - \int d^2x\, dy^2 \; \frac{g_\ell^2}{2} \sin^2 n\theta(x) \langle n\delta(x)\, n\delta(y)\rangle$$

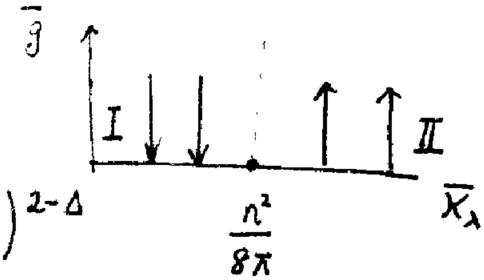
$$= \int d^2x\, d^2y \; \frac{n^2 g_\ell^2}{4} \cos^2 n\theta_{\ell'} (\partial_x \theta_{\ell'})^2 (x-y)^2 \langle n\delta\theta(x)\, n\,\delta\theta(y)\rangle$$

$$- \int dx^2 \; \frac{g_\ell^2}{2} \sin^2 n\theta_{\ell'}(x) \int dy^2 \; \langle n\delta\theta(x)\, n\delta\theta(y)\rangle$$

§ RG flow, fixed point $\bar{g} = g\ell^2$, and $\bar{K} = K$, are dimensionless.

$$\frac{d\bar{g}}{d\ln\ell} = \left(2 - \frac{n^2}{4\pi\bar{K}}\right)\bar{g} \; ; \quad \frac{d\bar{K}}{d\ln\ell} = \frac{3n^4\,\bar{g}^2}{16\pi^4\,\bar{K}}$$

✱ let us first ignore the second equation



$$\bar{g}(\ell) = \bar{g}(\ell_0)\, e^{\left(2 - \frac{n^2}{4\pi\bar{K}}\right)\ln\left(\frac{\ell}{\ell_0}\right)} = \bar{g}(\ell_0)\left(\frac{\ell}{\ell_0}\right)^{2-\Delta}$$

$$\Delta = \frac{n^2}{4\pi\bar{K}}$$

if $2-\Delta > 0$, no matter how small $\bar{g}(\ell_0)$ is, it can become as large as you want. Let us set the length scale of $\bar{g}(\ell) \sim 1 = \bar{g}(\ell_0)\left(\frac{\ell}{\ell_0}\right)^{2-\Delta}$

$$\Rightarrow \quad \ell = \ell_0\left(\frac{1}{\bar{g}_{\ell_0}}\right)^{\frac{1}{2-\Delta}}$$

at this scale RG fails, ~~as~~ the perturbative method does not apply.

This implies we enter the length scale of correlation length $\xi \sim \ell_0\left(\frac{1}{\bar{g}_{\ell_0}}\right)^{\frac{1}{2-\Delta}}$.

(Symmetry breaking state) $\boxed{\langle e^{i\varphi}\rangle \neq 0, \rightarrow \text{vortex condensation.}}$

if $2-\Delta < 0$, then $\bar{g}$ dies exponetially, and we arrive at a symmetric phase.

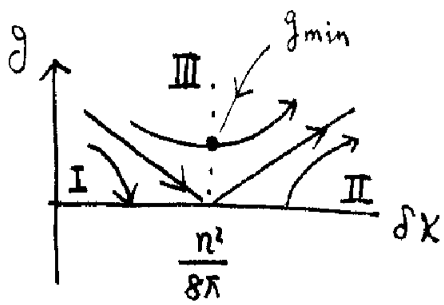
↖ powerlaw superfluid phase.

✮ Near the transition point at $K = \frac{n^2}{8\pi}$, the changes of $\bar{g}$ and $\bar{K}$ are comparable to each other. We need to conside both simultaneously.

Let us linearize the RG around $\bar{X} = \frac{n^2}{8\pi}$, $\bar{g} = 0$. $\Rightarrow$

$$\frac{d\bar{g}}{d\ln l} = \frac{16\pi}{n^2}\,\delta\bar{X}\,\bar{g}\,, \qquad \frac{d\delta\bar{X}}{d\ln l} = \frac{3n^2\bar{g}^2}{2\pi^3}$$

$$\Rightarrow \frac{3n^4}{32\pi^4}\,\bar{g}\,d\bar{g} - \delta\bar{X}\,d\delta\bar{X} = 0 \quad \text{i.e.} \quad (\delta\bar{X})^2 = \frac{3n^4}{32\pi^4}\,\bar{g}^2 + \text{Const}$$



In region I, we have the solution

$$\delta\bar{X}_l = -\sqrt{\frac{3n^4}{32\pi^4}\,\bar{g}^2 + (\delta X_\infty)^2}$$

In region II,

$$\delta\bar{X}_l = \sqrt{\frac{3n^4}{32\pi^4}\,\bar{g}^2 + (\delta X(l_0))^2}$$

In region III

$$\bar{g}_l = \sqrt{\frac{32\pi^4}{3n^4}(\delta\bar{X}_l)^2 + (\bar{g}_{min})^2}\,.$$

Regions I, II are approximately described by neglecting the renormalization of δX.

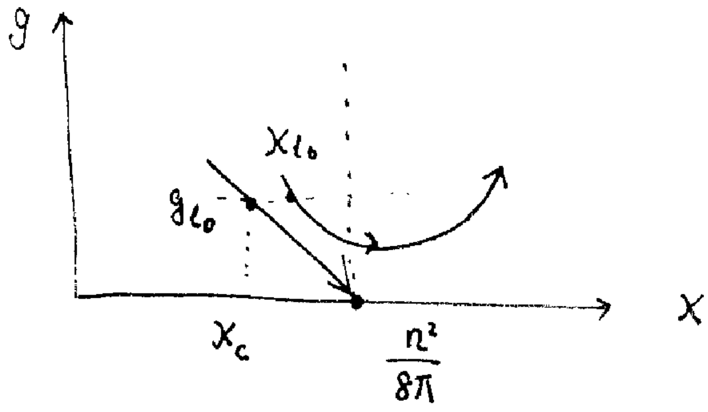
Regions III is the crossover region

$$\frac{d\,\delta\bar{X}}{\frac{32\pi^4}{3n^4}(\delta\bar{X}_l)^2 + (\bar{g}_{min})^2} = n^2\,d\ln l\,, \quad \text{we integrate out from } l \text{ to } \xi$$

$$\Rightarrow \int_{\delta X(l_0)}^{\delta X(\xi)} \frac{d\,\delta\bar{X}}{\frac{32\pi^4}{3n^4}(\delta\bar{X}_l)^2 + (\bar{g}_{min})^2} = n^2\ln\left(\frac{\xi}{l}\right)$$

At ξ, $\bar{g}$ and δX are at the order of 1

$\Rightarrow$ relation between ξ and $\delta X(l_0)$



along the line of

$$\delta x_c = -\sqrt{\frac{3n^4}{32\pi^4}}\ \bar{g}_l ,$$

$$\Rightarrow g_{min} = 0 ,$$

The left hand side diverges as $\delta x(\xi) \to 0$, thus $\xi \to +\infty$.

let set x_{l_0} slightly larger than x_c, ~~then~~ $\cancel{\bar{g}^2_{min} = 2\left(\frac{32\pi^4}{3n^4}\right)^{1/2}}$ and fix g_{l_0}.

$$\bar{g}^2_{l_0} - \frac{32\pi^4}{3n^4}(\delta x_c)^2 = 0$$

$$\bar{g}^2_{l_0} - \frac{32\pi^4}{3n^4}\left[(\delta x_c)^2 + 2\delta x_c\,(x_{l_0} - x_c)\right] = g^2_{min}$$

$$\Rightarrow \bar{g}^2_{min} = 2\left(\frac{32\pi^4}{3n^4}\right)^{1/2} \bar{g}_{l_0}\,(x_{l_0} - x_c)$$

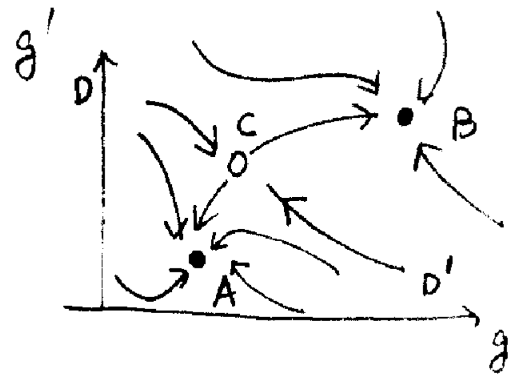
The integration can be performed from ($\delta x = 0$, $g = g_{min}$)

$$\Rightarrow \int_0^{\delta x(\xi)=1} \frac{d\,\delta x}{\frac{32\pi^4}{3n^4}(\delta x)^2 + (g_{min})^2} \simeq \int_{\sqrt{\frac{3n^4}{32\pi^4}}\, g_{min}}^{\delta x(\xi)=1} \frac{d\,\delta x}{\frac{32\pi^4}{3n^4}(\delta x)^2} = n^2 \ln\left(\frac{\xi}{l_0}\right)$$

$$-\frac{3n^4}{32\pi^4}\frac{1}{\delta x}\Bigg|_{\sqrt{\frac{3n^4}{32\pi^4}}\, g_{min}}^{1} \simeq \sqrt{\frac{3}{32}}\ \frac{n^2}{\pi^2}\ \frac{1}{g_{min}} = n^2 \ln\left(\frac{\xi}{l_0}\right)$$

$$\Rightarrow \xi \simeq l_0\, e^{\left[\frac{c}{x_l - x_c}\right]^{1/2}} \qquad C = \frac{n^2}{2\pi\,\bar{g}_{l_0}}\left(\frac{3}{32\pi^2}\right)^{3/2}$$

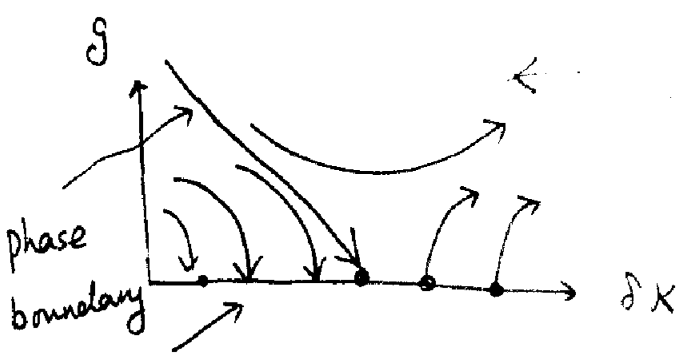
general principle of RG Fixed points / phase transitions



Stable fixed points A. B corresponds to stable phases.

unstable fixed points controls phase transitions C. The line $D \to C \to D'$ is the phase boundary between phase A/B.

K-T transition.



← symmetry breaking phase in the clock model.

(vortex condensation / disordered phase of the XY model).

a line of stable fixed point.

low-~~temperature~~, superfluid phase.

power law - correlation